https://ntrs.nasa.gov/search.jsp?R=19710018676 2018-07-23T08:38:41+00:00Z

LECTURE NOTES 71-1

Ames ~~FH 71-0259~~
Mr. B. F. Doolin
Code: 210.3

JACK K. HALE

APPLICATIONS OF ALTERNATIVE PROBLEMS

N71-28152

MAY, 1971

CENTER FOR DYNAMICAL SYSTEMS

CASE FILE COPY

Approved for [illegible]
unlimited. [illegible]
[illegible]

DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

APPLICATIONS OF ALTERNATIVE PROBLEMS

by

Jack K. Hale[1]

Division of Applied Mathematics
Center for Dynamical Systems
Brown University
Providence, Rhode Island 02912

[1]This research was supported in part by the National Aeronautics and Space Administration under Grant No. NGL 40-002-015, and in part by the Air Force Office of Scientific Research under Grant No. AF-AFOSR 67-0693D, and by the United States Army - Durham under Grant No. DA-31-124-ARO-D-270.

Table of Contents

		Page
1.	Introduction	1
2.	General Theory	4
3.	Methods of Solution	15
4.	Periodic Solutions	21
5.	Eigenvalues of Nonlinear Operators	34
6.	Analytic Systems, Regular and Irregular Singular Points	43
7.	Partial Differential Equations	50
8.	Admissibility	63
	References	65

APPLICATIONS OF ALTERNATIVE PROBLEMS

Jack K. Hale

1. Introduction. Many problems in analysis and applied mathematics can be reduced to the solution of equations in a function space or functional equations. The equations often arise from the desire to obtain solutions of ordinary or partial differential equations with subsidiary conditions - periodicity or more general boundary conditions, specified asymptotic behavior, analyticity conditions, etc. Many of the problems involve a linear operator and a nonlinear operator which is small when some parameter is small. If the linear operator has an inverse, conceptually there are no difficulties in obtaining approximate solutions although practically there may be many difficulties. At least one can draw on all existing fixed point techniques. When the linear operator has elements in its null space, then new concepts must be introduced in order to proceed. Generally, one relies on some type of "Fredholm" alternative to determine conditions on the elements of the null space of the linear operator in order to obtain solutions of the nonlinear problem. For equations with a small parameter in the nonlinearity, one often expands everything in a power series in the parameter and equates coefficients in the spirit of the method of Poincaré for obtaining periodic solutions of ordinary differential equations. On the other and, experience over the last twenty years has shown that it is advantageous to look at these latter questions in a more general way in order to understand the underlying structure. In fact, proceeding

in this fashion leads not only to new results but also to other iterative methods for obtaining approximations to the solutions.

It is purpose of these lectures to present a general approach to the solution of functional equations and to indicate how to obtain the appropriate functional equations for a variety of applications.

Let X, Z be Banach spaces and $B: X \to Z$ be a mapping from some subset of X into a subset of Z. The range, domain, and null space of B will be denoted by $\mathscr{R}(B)$, $\mathscr{D}(B)$, $\mathfrak{N}(B)$, respectively. A projection R in a Banach space X is a continuous linear mapping taking X into X such that $R^2 = R$. If R is a projection in X, the range of R will be denoted by X_R and the symbol X_R will always denote a subspace of X which is obtained through a projection operator R. The identity operator will be denoted by I. If R is a projection in X, then the space X is a direct sum of X_R and X_{I-R} and $x \in X$ can be written in a unique manner as $x = x_R + x_{I-R}$, with $x_R \in X_R$, $x_{I-R} \in X_{I-R}$.

Let $A: \mathscr{D}(A) \subset X \to Z$ be a linear operator defined in the subspace $\mathscr{D}(A)$ and $N: X \to Z$ be a linear or nonlinear map. The basic problem is to find an $x \in \mathscr{D}(A)$ such that

$$Ax = Nx. \tag{1.1}$$

In later sections, we show how this type of equation arises in the theory of bifurcation in integral equations, partial differential equations and ordinary differential equations, the theory of

differential equations with singular points, the theory of boundary value problems and asymptotics. For the next few pages we will be concerned primarily with methods of solution for (1.1). The spirit of the general presentation in Section 2 has its origin in a paper of Cesari [4]. Personal conversations of the author with Stephen A. Williams led to the particular formulation given below. The author is indebted to Robert Glassey and Orlando Lopez for assistance in the preparation of the notes.

2. General Theory. As we have pointed out, our main purpose is to solve: $Ax = Nx$, where A is a linear operator defined on some subspace $\mathscr{D}(A)$ of a Banach space X with values on Z and N is a linear or nonlinear operator.

Let us assume the following:

(H1) there exist projections U and E such that

$$\mathfrak{N}(A) = X_U \quad \text{and} \quad \mathscr{R}(A) = Z_E$$

Clearly this assumption implies A is one-to-one on $X_{I-U} \cap \mathscr{D}(A)$. Furthermore, for each $x \in \mathscr{D}(A)$, $x = x_U + x_{I-U}$, we have $Ax = Ax_{I-U}$ because $AU = 0$ and so the image of $x_{I-U} \cap \mathscr{D}(A)$ under A is $\mathscr{R}(A)$. This means there exists a linear operator $M: \mathscr{R}(A) \to X$ such that $AM = I$ on $\mathscr{R}(A)$, M is onto $x_{I-U} \cap \mathscr{D}(A)$ and $MA = I$ on $x_{J-U} \cap \mathscr{D}(A)$. Then, for each $x \in \mathscr{D}(A)$, we can write:

$$MAx = MAx_U + MAx_{I-U} = x_{I-U} = (I-U)x \qquad \forall\, x \in \mathscr{D}(A)$$

and so $MA = I - U$ on $\mathscr{D}(A)$.

Obviously, the equation $Ax - Nx = 0$ is equivalent to the system: $(I-E)(A-N)x = 0$ and $E(A-N)x = 0$. Noticing that $(I-E)A = 0$, $EA = A$, $MA = I - U$ we obtain the following:

Lemma 2.1. If (H1) is satisfied, then there exists a linear operator $M: Z_E \to \mathscr{D}(A)$ such that $AM = I$ on Z_E, $MA = I - U$ on $\mathscr{D}(A)$ and the equation

$$Ax = Nx \tag{2.1}$$

is equivalent to the system

$$\begin{aligned} &\text{a)} \quad x = Ux + MENx \\ &\text{b)} \quad (I-E)Nx = 0. \end{aligned} \tag{2.2}$$

Any method for the determination of a solution of (2.1) must take into account Lemma 2.1. In fact, equation (2.2b) says that Nx must be in the range of A and, if Nx is in $\mathscr{R}(A)$, (2.2a) says that the solution of (2.1) is a particular solution $MENx$ plus an element of the null space of A.

Lemma 2.1 is basic for many of the known methods for solving problems which involve a small nonlinearity (say the nonlinearity is continuous in a small parameter and vanishes together with its first derivative for the parameter equal to zero). More specifically, one can fix an arbitrary element $y \in \mathfrak{N}(A)$ and solve the equation:

$$x = y + MENx \tag{2.3}$$

for a function $x = x^*(y)$. The function $x^*(y)$ will be a solution of (2.1) if y can be determined in such a way that

$$(I-E)Nx^*(y) = 0. \tag{2.4}$$

These latter equations are usually referred to as the bifurcation equations or determining equations for (2.1). After careful study, the method of solution indicated in (2.3), (2.4) will be seen to be the underlying principle in the papers of Cesari [1 - 3, 11], Hale [1 - 4], Perelló [1], Friedrichs [1], Cronin [1], Bartle [1], Graves [1], Lewis [1, 2], Bass [1, 2], Nirenberg [1], Vainberg and Tregonin [1], and Antosiewicz [1, 2].

As we shall see later, many different iterative schemes can be devised for the successive determination of approximations to $x^*(y)$ and y.

Another method for attempting to solve equations (2.2) is to write them as

$$\begin{aligned} &\text{a)} \quad x_{I-U} = \mathrm{MEN}(x_U + x_{I-U}) \\ &\text{b)} \quad (I-E)N(x_U + x_{I-U}) = 0. \end{aligned} \tag{2.5}$$

For a fixed $y \in x_{I-U}$ one can try to determine $x_U = x_U^*(y)$ so that

$$(I-E)N(x_U^*(y) + y) = 0 \tag{2.6}$$

and then determine y so that

$$y = \mathrm{MEN}(x_U^*(y) + y). \tag{2.7}$$

This latter approach is very similar to the one used by Rabinowitz [1] and Hall [1] for a problem in partial differential equations.

Let us make our second assumption:

(H2) there exists a projection $S: X \to X$ with $X_S \subset X_{I-U} \cap \mathscr{D}(A)$ and $SU = 0$.

Of course the assumption implies that $US = 0$ and so $R = I - U - S$ is a projection.

Lemma 2.2. Suppose (H1), (H2) are satisfied, $R = I - U - S$, M is the right inverse of A as in Lemma 2.1 and $x \in X$ is written as $x = x_U + x_S + x_R$. Then the equation (2.1) is equivalent to

$$\begin{aligned} &\text{a)} \quad x_R - RMEN(x_U + x_S + x_R) = 0 \\ &\text{b)} \quad x_S - SMEN(x_U + x_S + x_R) = 0 \\ &\text{c)} \quad (I-E)N(x_U + x_S + x_R) = 0 \end{aligned} \tag{2.8}$$

Proof: Lemma 2.1 implies (2.1) is equivalent to (2.2) and $SU = 0$ implies that (2.2a) is equivalent to (2.3a), (2.3b). This proves the result.

Lemma 2.3. Under the hypotheses of Lemma 2.2, there is a constant $k > 0$ such that $|RM| \leq k$ (or $|SM| \leq k|$) if and only if $|Rx| \leq k|Ax|$ (or $|Sx| \leq k|Ax|$) for all $x \in \mathscr{D}(A)$.

Proof: For any $z \in Z_E$ and $x \in \mathscr{D}(A)$ for which $Ax = z$, we have $RMz = RMAx = R(I-U)x = Rx$ because $RU = 0$ (or $SMz = SMAx = S(I-U)x = Sx$ because $SU = 0$). The result is now obvious.

Corollary 2.1. If (H1) is satisfied, then A has a bounded right inverse $M: Z_E \to X_{I-U} \cap \mathscr{D}(A)$ if and only if there is a constant k such that

$$|(I-U)x| \leq k|Ax| \quad \text{for} \quad x \in \mathscr{D}(A). \tag{2.9}$$

Furthermore $|(I-U)M| \leq k$.

Proof: Take $S = 0$ and apply Lemma 2.3.

This corollary shows in particular that the computation of the right inverse is unnecessary since a bound can be obtained using only the known operator A.

The existence of a bounded right inverse for A can often be deduced from the following result.

Lemma 2.4. If (H1) is satisfied and A is a closed operator, then A has a bounded right inverse.

For a proof, see Nirenberg [1].

Lemma 2.2 is the simple observation that equation (2.2a) may be written as two equations by means of the projection operators R, S and is applicable to problems in which N may not necessarily be small.

By choosing the operator S appropriately, for a fixed $y \in X_U$, $z \in X_S$, one can attempt to solve the equation

$$x_R - RMEN(y+z+x_R) = 0 \quad \text{for an} \tag{2.10}$$

for an $x_R = x_R^*(y+z)$ and then determine y, z so that

$$\begin{aligned} &z - SMEN(y+z+x_R^*(y+z)) = 0 \\ &(I-E)N(y+z+x_R^*(y+z)) = 0 \ . \end{aligned} \tag{2.11}$$

A special case of this procedure can be traced to the paper of Cesari [4]. Knowing a priori that $x_R^*(y+z)$ is small for y, z in some bounded set, then a natural first approximation for the solution of (2.1) is to let $x_R^* = 0$ in (2.11) and solve the resulting equations. Retracing the steps through which we arrived at (2.11), one observes that this approximation corresponds to a generalized Galerkin approximation. The general procedure outlined above can actually be used as a theoretical way to justify Galerkin's method (see Cesari [4]).

In spite of the simplicity of Lemma 2.2, it seems to include the method given by Bancroft, Hale and Sweet [1] as far as the applications are concerned. To see this, we give a detailed description of the analogue of Lemma 2.2 in the cited paper. Suppose (H1) and (H2) are satisfied, A has a bounded right inverse $M: Z_E \to X_{I-U} \cap \mathscr{D}(A)$ and let $P = U + S$. Then there is a projection

$Q: Z \to Z$ such that (2.1) is equivalent to the equations

$$\begin{aligned} &\text{a)} \quad x = Px + MQNx \\ &\text{b)} \quad (I-Q)(A-N)x = 0. \end{aligned} \tag{2.12}$$

These equations are very interesting because they have the same form as the equations (2.2) since (2.2b) is equivalent to $(I-E)(A-N)x = 0$. It can also be shown that

$$(I-Q)A = AP, \; I - Q = I - E + \tilde{Q} \quad \text{where} \quad MZ_{\tilde{Q}} = X_S \cap \mathscr{D}(A)$$
$$(I-E)\tilde{Q} = \tilde{Q}(I-E) = 0.$$

Therefore, if $x = x_U + x_S + x_R$, $R = I - U - S = I - P$, then (2.12b) is equivalent to

$$Ax_S - \tilde{Q}N(x_U+x_S+x_R) = 0$$
$$(I-E)N(x_U+x_S+x_R) = 0.$$

Since $MZ_{\tilde{Q}} = X_S \cap \mathscr{D}(A)$, it follows that (2.12) is equivalent to:

$$\begin{aligned} &\text{a)} \quad x_R - MQN(x_U+x_S+x_R) = 0 \\ &\text{b)} \quad x_S - M\tilde{Q}N(x_U+x_S+x_R) = 0 \\ &\text{c)} \quad (I-Q)(A-N)(x_U+x_S+x_R) = 0. \end{aligned} \tag{2.13}$$

These equations have the same form as (2.8) except with $RM = MQ$, $SM = M\tilde{Q}$. On the other hand, Lemma 2.3 shows that the norm of these operators can be estimated in terms of the operator A so that it is unnecessary to discuss the operator Q.

We now give some sufficient conditions for the solvability of (2.2a), and for that we need the following assumptions:

(H3) there is a constant $k > 0$ such that $|Rx| \leq k|Ax|$ for all $x \in \mathscr{D}(A)$, where $R = I - U - S$.

(H4) there are positive continuous nondecreasing functions $\alpha(\rho)$, $\beta(\rho)$, $0 \leq \rho < \infty$ such that

$$|Nx| \leq \beta(\rho)$$
$$|Nx-Ny| \leq \alpha(\rho)|x-y| \quad \text{for} \quad |x| \leq \rho,\ |y| \leq \rho.$$

For any positive constants c, d let

$$V(c) = \{x \in X_{I-R}:\ |x| \leq c\}$$
$$S(d) = \{x \in X_R:\ |x| \leq d\}$$

<u>Theorem 2.1.</u> Suppose (H1) - (H4) are satisfied, M is the operator in Lemma 2.1 and c, d are such that

$$|RME|\alpha(c+d) < 1, \quad |RME|\beta(c+d) < d \tag{2.14}$$

Then for any $\tilde{x}$ in $V(c)$, the map

$$\Delta(\cdot) = RMEN(\tilde{x} + \cdot): \; X_R \to X$$

is a contraction mapping of $S(d)$ into itself with a resulting unique fixed point $F\tilde{x}$. The function $F: V(c) \to X_R$ is continuous. Furthermore, if N has a continuous Frechet derivative, then F has a continuous Frechet derivative.

Proof: Lemma 2.3 implies $|RME| \leq k|E|$ is bounded and the result is now easily verified using the contraction principle.

Theorem 2.2. Suppose the conditions of Theorem 2.1 are satisfied and F is the function given in Theorem 2.1. Then the equation $Ax = Nx$ has a solution if there exist x_S, x_U such that $(x_S+x_U) \in V(c)$ and

$$\begin{aligned} &\text{a)} \quad x_S - SMEN(x_U+x_S+F(x_U+x_S)) = 0 \\ &\text{b)} \quad (I-E)N(x_U+x_S+F(x_U+x_S)) = 0. \end{aligned} \tag{2.15}$$

Conversely, if there exists an x such that $Ax = Nx$, $|x_R| \leq d$, $|x_U+x_S| \leq c$, $x = x_R + x_U + x_S$, then $x_R = F(x_U+x_S)$, where x_U, x_S satisfy (2.15).

Proof: This is obvious from Theorem 1 and Lemma 2.2.

The conditions of Theorem 2.2 imply that equations (2.15)

(the determining equations) are equivalent to the equation $Ax = Nx$ for the solutions $x = x_R + x_U + x_S$ with $|x_R| \leq d$, $|x_U + x_S| \leq c$. It is of interest to determine for given c, d conditions on A which will ensure that the range of the projection operator S is always finite dimensional. Such a result is contained in:

<u>Theorem 2.3</u>. Suppose X is a Hilbert space, A satisfies (H1) and the set $V = \{x \in (I-U)\mathscr{D}(A): |Ax| \leq 1\}$ is relatively compact. Then, for any $k > 0$, there is a projection $S: X \to X$ with finite dimensional range and satisfying (H2), (H3). Consequently, if (H4) is satisfied, then, for any positive constants c, d there always exists an S such that (2.14) is satisfied and, therefore, the conclusions of Theorem 2.2 hold.

<u>Proof</u>: If we call B_1 the unit ball in Z_E, we see that $M(B_j) = V$, and then M is a compact (and hence bounded) operator. Since V is totally bounded, for any $k > 0$, there is a positive integer $n(k)$ and a sequence $\{x_j\} \subset V$, $j = 1,2,\ldots,n(k)$ such that for any z in B_1, there is an x_j with $|Mz - x_j| < k$. Let $S = S(k): X \to X$ be the projection whose range coincides with the finite dimensional subspace spanned by $\{x_j\}$ and whose null space is $[x_j,\ldots,x_{n(k)}]^{\perp} \oplus X_U$, where the orthogonal complement is taken on X_{I-U}. Since an orthogonal projection has norm 1, $|I-S| \leq \alpha$, where α depends only on U. Then, for each z in B_1 we have: $|(I-S)Mz| = |(I-S)(Mz-x_j)| \leq k\alpha$. Clearly $US = SU = 0$. Since $UM = 0$, Lemma 2.3 implies (H3) is satisfied. The remainder of the proof is obvious.

<u>Theorem 2.4</u>. Suppose the conditions of Theorem 2.3 are satisfied and there is a sequence of finite dimensional orthogonal projection operates $S_m: X_{I-U} \to X_{I-U}$ such that $X_{S_m} \subset X_{S_{m+1}}$, $m = 1,2,\ldots$ and for any finite dimensional projection operator $T: X_{I-U} \to X_{I-U}$ and any $\varepsilon > 0$, there is an $m_0(\varepsilon) > 0$ such that $|(I-S_m)T| < \varepsilon$ for $m \geq m_0(\varepsilon)$. For any constants $0 < c < d$, there is an integer $m_j > 0$ such that each S_m for $m \geq m_j$ satisfies (2.14) and, therefore, the conclusion of Theorem 2.2 hold.

<u>Proof</u>: For any k_j satisfying (2.14) there always exists an $S = S(k_1)$ by Theorem 2.3 such that (H2), (H3) are satisfied. Choose m_1 so large that $|(I-S_m)SM| < \eta$ where $|E|(\eta+k_1) < \min\{[\alpha(c+d)]^{-1}, d[\beta(c+d)]^{-1}\}$ for $m \geq m_j$. Then

$$|(I-S_m)M| \leq |(I-S_m)SM| + |(I-S_m)(I-S)M| \leq \eta + |(I-S)M| \leq k_j + \eta.$$

Therefore, S_m satisfies (H2), (H3) with $k = k_1 + \eta$. On the other hand, the choice of η implies that (2.14) is satisfied with $k = k_j + \eta$. The remainder of the proof is clear.

In the case where the determining equations are finite dimensional and are obtained by the application of the contraction principle on the operator Δ in Theorem 2.1, Williams [1] has shown that it is possible to establish a theoretical connection with the Leray-Schauder theory and topological degree.

3. Methods of Solution

Under appropriate hypotheses on A, we have seen in the previous section that the solution of $Ax = Nx$ can be reduced to the solution of equations (2.2) or (2.8). Our goal in this section is to discuss in a very general manner various methods for obtaining approximate solutions of these equations. An abstract version of this problem is as follows: Suppose U, V, W are Banach spaces and $B: U \times V \to V$; $C: U \times V \to W$ are continuous operators. The solution of equations (2.2) or (2.8) is a special case of the following problem: find $u \in U$, $v \in V$ such that the system of equations

$$\begin{aligned} u - B(u,v) &= 0 \\ C(u,v) &= 0 \end{aligned} \tag{3.1}$$

is satisfied.

If B and C have continuous Fréchet derivatives, the classical iterative method for solving (3.1) is Newton's Method. Let $B_u(u,v)$, $C_u(u,v)$, $B_v(u,v)$, $C_v(u,v)$ denote the Fréchet derivatives of $B(u,v)$ $C(u,v)$ with respect to u,v, respectively. If the linear mapping $\mathscr{B}(u,v)$: $U \times V \to V \times W$ given by

$$[\mathscr{B}(u,v)](\varphi,\psi) = [(I-B_u)\varphi - B_v\psi,\ C_u\varphi + C_v\psi]$$

has a bounded inverse for every (u,v) in some bounded set, then Newton's method is given by

$$(u_0, v_0) \quad \text{given;}$$

(3.2)

$$(u_{n+1}, v_{n+1}) = (u_n, v_n) - [\mathscr{D}(u_n, v_n)]^{-1}[u_n - B(u_n, v_n), C(u_n, v_n)] \quad (n = 0, 1, 2, \ldots)$$

Conditions for the convergence of this method as well as error bounds may be found in Antosiewicz [3].

There are many variants of Newton's method that can be used to advantage. For example, we could always evaluate the inverse of $\mathscr{D}$ at (u_0, v_0) in order to eliminate some computations. Another scheme we might choose is

$$(u_0, v_0) \quad \text{given;}$$

(3.3)

$$u_{n+1} = u_n - [I - B_u(u_n, v_n)]^{-1}[u_n - B(u_n, v_n)];$$

$$v_{n+1} = v_n - [C_v(u_n, v_n)]^{-1} C(u_n, v_n) \quad (n = 0, 1, 2, \ldots);$$

provided, of course, that the inverse operators exist. Alternatively, we could attempt the iteration

$$(u_0, v_0) \text{ given;}$$

(3.4)

$$u_{n+1} = B(u_n, v_n);$$

$$v_{n+1} = v_n - [C_v(u_n, v_n)]^{-1} C(u_n, v_n) \quad (n = 0, 1, 2, \ldots).$$

Convergence criteria and error bounds are easily obtained for each of the above methods.

A case of (3.1) that occurs quite frequently in the applications is when $N = \varepsilon\overline{N}$, where ε is a small real parameter satisfying $0 \leq |\varepsilon| \leq \varepsilon_0$. If H_1) and H_3) are satisfied for $S = 0$ and if $\overline{N}$ is locally Lipschitzian, then we can see that the conditions of Theorems 2.1 and 2.2 are satisfied for ε_0 sufficiently small. Therefore, there exists a function

$$F\colon V(c) \times [-\varepsilon_0,\varepsilon_0] \to X_{I-U}$$

such that $F(x_U,\varepsilon)$ is continuous in x_U,ε, and satisfies $F(x_U,0) = 0$. Thus system (2.1) has a solution if there is an $\varepsilon_1 > 0$, $\varepsilon_1 \leq \varepsilon_0$, for which the equation

$$(I-E)\overline{N}(x_U + F(x_U,\varepsilon)) = 0 \tag{3.5}$$

has a solution $x_U = x_U(\varepsilon)$ for $0 \leq |\varepsilon| \leq \varepsilon_1$.

If $\overline{N}$ has a continuous Frechet derivative $\overline{N}_x$, then $F(x_U,\varepsilon)$ has continuous derivatives in x_U,ε. Therefore, if there exists an x_U^o such that

(3.6)
a) $(I-E)\overline{N}(x_U^o) = 0$;

b) $(I-E)\overline{N}_x(x_U^o)$ has a bounded inverse,

then the implicit function theorem implies the existence of a solution $x_U(\varepsilon)$ of (3.5) for $|\varepsilon| \leq \varepsilon_1$, $\varepsilon_1 \leq \varepsilon_0$ sufficiently small, satisfying $x_U(0) = x_U^o$. Thus, equation (2.1) has a solution

$$x = x_U(\varepsilon) + F(x_U(\varepsilon),\varepsilon) \quad \text{for} \quad |\varepsilon| \leq \varepsilon_1.$$

A rather natural way to calculate the solution x for this special case is to use a variant of (3.4); namely,

$$(3.7) \qquad \begin{aligned} & x_n = u_n+v_n;\ u_n \in X_{I-U},\ v_n \in X_U \quad (n = 0,1,2,\ldots); \\ & u_0 = 0,\ v_0 = x_U^0; \\ & u_{n+1} = \varepsilon M E \overline{N}(u_n+v_n); \\ & v_{n+1} = v_n - \varepsilon[(I-E)\overline{N}_x(x_U^0)]^{-1}(I-E)\overline{N}(v_n+u_{n+1}) \quad (n = 0,1,2,\ldots). \end{aligned}$$

It is fairly easy to show that this method converges for $|\varepsilon|$ sufficiently small (see Lazer [1] for a special case).

If N is analytic in some region, then the function $F(x_U,\varepsilon)$ will be analytic in an appropriate region, and condition (3.6) will imply the solution $x_U(\varepsilon)$ is analytic in some region. An obvious way to obtain the power series expansion of $x = x_U(\varepsilon) + F(x_U(\varepsilon),\varepsilon)$ in ε is to simply let

$$x = \sum_{k=0}^{\infty} x_k\varepsilon^k,\ x_k = u_k+v_k,\quad u_k \in X_{I-U};\ v_k \in X_U$$

and try to determine the coefficients successively. These coefficients must satisfy

$$(3.8) \qquad \begin{aligned} & u_0 = 0,\ v_0 = x_U^0; \\ & u_{k+1} = f_k(u_0,\ldots,u_k,\ v_0,\ldots,v_k); \end{aligned}$$

$$(I-E)\overline{N}_x(x_U^o)v_{k+1} = g_k(u_0,\ldots,u_k,\ v_0,\ldots,u_k) \quad (k = 0,1,2,\ldots);$$

where the f_k, g_k are some given polynomial functions of their arguments. Condition (3.6) implies that these equations can be solved. This is the natural generalization of the Poincaré procedure for obtaining periodic solutions of ordinary differential equations.

In general, the method (3.7) is probably a more accurate way to obtain the solution, but (3.8) is certainly very simple.

We conclude this section with a brief description of the notion of <u>quasilinearization</u>. We have seen that the equation $Ax = Nx$ is equivalent to the system

$$\begin{cases} x_{I-U} = MEN(x_U + x_{I-U}); \\ (I-E)N(x_U+x_{I-U}) = 0 \end{cases}$$

If we let $x_U = v$, $x_{I-U} = u$, the system becomes

$$\begin{cases} u-MEN(u+v) = 0; \\ (I-E)N(u+v) = 0. \end{cases}$$

For solution, we use the modified Newton method:

$$\begin{cases} u_{n+1} = u_n - [I-MEN_x(u_n+v_n)]^{-1}[u_n-MEN(u_n+v_n)]; \\ v_{n+1} = v_n - [(I-E)N_x(u_n+v_n)]^{-1}(I-E)N(u_n+v_n) \ ; \end{cases}$$

i.e.,

$$\begin{cases} u_{n+1} = ME\{N_x(u_n+v_n)[u_{n+1}-u_n] + N(u_n+v_n)\}; \\ (I-E)\{[N_x(u_n+v_n)][v_{n+1}-v_n] + N(u_n+v_n)\} = 0. \end{cases}$$

This last system is equivalent to

$$\begin{cases} [A-EN_x(u_n+v_n)]u_{n+1} = E[N(u_n+v_n)-N_x(u_n+v_n)u_n]; \\ (I-E)N_x(u_n+v_n)v_{n+1} = (I-E)[N(u_n+v_n) - N_x(u_n+v_n)v_n]. \end{cases}$$

Thus whenever $I = E$, the pair of equations above reduces to a single equation. In such an event, we have quasilinearization (see Antosiewicz [3]).

4. Periodic Solutions. To illustrate how to formulate specific problems in the above manner, consider the equation

$$\dot{x} = Bx + \varepsilon f(t,x) \tag{4.1}$$

where $f(t+2\pi,x) = f(t,x)$ is an n-vector continuous in (t,x), B is an $n \times n$ constant matrix and ε is a small real paramter. Our problem is to find 2π-periodic solutions of (4.1).

To formulate this in abstract terms, let $X = Z$ be the Banach space of continuous 2π-periodic functions with $|x| = \sup_{0 \leq t \leq 2\pi} |x(t)|$ for $x \in X$. For any $x \in X$ with $\dot{x}$ continuous let

$$(Ax)(t) = \dot{x}(t) - Bx(t), \quad -\infty < t < \infty, \tag{4.2}$$

and for any $x \in X$, let

$$(Nx)(t) = \varepsilon f(t,x(t)), \quad -\infty < t < \infty. \tag{4.3}$$

Finding 2π-periodic solutions of (4.1) is now equivalent to finding an $x \in X$ such that

$$Ax = Nx, \tag{4.4}$$

with A,N defined in (4.2), (4.3).

The set $\mathfrak{N}(A)$ is the set of all 2π-periodic solutions of the

unperturbed equation:

$$\dot{x} - Bx = 0. \tag{4.5}$$

Therefore, it is a finite dimensional subspace of X and admits projection by an operator U, $\mathfrak{N}(A) = X_U$. If J is the set of integers j such that $(B-jI)$ is singular, $e_1,\ldots,e_{p_j}$ are a basis for the eigenspace of j and $x \sim \sum_{k=-\infty}^{\infty} x_k \exp(ikt)$, then $Ux = \sum_{k\in J} x_k^U \exp(ikt)$ where x_k^U is the projection of x_k onto the span of the vectors $e_1,\ldots,e_{p_k}$.

The classical Fredholm alternative states that the equation

$$\dot{x} - Bx = y, \quad y \in X \tag{4.6}$$

has a solution $x \in X$ if and only if

$$\int_0^{2\pi} z(t)y(t)dt = 0 \tag{4.7}$$

for all row-vectors z for which $z' \in X$ ($'$ = transpose) and

$$\dot{z} + zB = 0. \tag{4.8}$$

Since the linear subspace spanned by such z' is finite dimensional, if follows that $\mathscr{R}(A) = X_E$ for some projection E. To check (H3), we see that for any $x \in \mathscr{D}(A)$, $x \sim \sum x_k e^{ikt}$,

$\dot{x} \sim \sum ikx_k e^{ikt}$, and hence $\sum k^2|x_k|^2 < \infty$. Then

$$|(I-U)x| = |\sum_{k\notin J}(ikI-B)^{-1}(ikI-B)x_k^U e^{ikt}| \leq$$

$$(\sum_{k\notin J}|(ikI-B)^{-1}|^2)^{1/2}(\sum|(ikI-B)x_k^U|^2)^{1/2} \leq M(\frac{1}{2\pi}\int_0^{2\pi}|Ax(t)|^2dt)^{1/2} \leq$$

$$\leq M_1|Ax| \quad \text{for some constant } M_1.$$

Consequently, the previous theory implies (if $f(t,x)$ has a continuous first derivative with respect to x) for any $c < d$, there is an $\varepsilon_0 > 0$ such that for any fixed $u \in X_U$, $|u| \leq c$, there is a unique solution $F(u,\varepsilon) \in X_{I-U}$ of

$$Av = EN(u+v)$$

for $0 \leq |\varepsilon| \leq \varepsilon_0$ and $F(u,\varepsilon)$ is continuously differentiable in u,ε. The bifurcation equations are then given by

$$(I-E)N(u + F(u,\varepsilon)) = 0. \tag{4.9}$$

Since $\mathfrak{N}(A) = X_U$ is finite dimensional, it is more convenient to write the bifurcation equations in terms of a variable in some Euclidean space. If $\Phi = (\varphi_1,\ldots,\varphi_d)$ is a basis for $\mathfrak{N}(A)$ then $u = \Phi a$ for some d-vector a, and the bifurcation equations become

$$G(a,\varepsilon) \overset{\text{def}}{=} (I-E)N(\Phi a+F(\Phi a,\varepsilon)) = 0. \tag{4.10}$$

If $f(t,x)$ is analytic in x in some region, then $F(u,\varepsilon)$ is analytic in u,ε. One can show that if there is an a_0 such that

$$\begin{aligned} &G(a_0,0) = 0 \\ &\det[\frac{\partial G}{\partial a}(a_0,0)] \neq 0 \end{aligned} \tag{4.11}$$

then there is a solution $a(\varepsilon)$ of (4.10) for ε small, which is analytic in ε. One could, therefore, obtain the solution in powers of ε by substituting in the original equation

$$x = \sum_{k=0}^{\infty} x_r \varepsilon^k, \quad x_k = \Phi a_k + v_k, \; v_k \in X_{I-U} \tag{4.12}$$

and determining a_k, v_k so that the equation (4.1) is satisfied. This is the classical Poincaré perturbation procedure for obtaining periodic solutions of (4.1).

Of course, any of the other methods mentioned before give other iterative schemes. In particular, the analogue of (3.7) for this special case and hypothesis (4.11) was given by Lazer [1].

In many applications, condition (4.11) may not be satisfied and therefore the above iterative procedures fail. However, it is sometimes possible to discuss the qualitative properties of the determining equations (4.10) without any successive approximations.

We mention only the classical problem of Liapunov on the existence of two parameter families of periodic solutions of Hamiltonian systems. One can show without successive approximations that the bifurcation equations are identically zero (see Hale [4]).

One point about the above procedure that has been neglected is the precise determination of the projection operators U and E. There are many ways to choose these and each gives a different form for the approximating equations. We mention only one way.

If Φ is the $n \times d$ matrix whose columns are a basis for the 2π-periodic solutions of (4.1) and Ψ is a $d \times n$ matrix whose rows are a basis for the 2π-periodic solutions of (4.8), define the nonsingular $d \times d$ matrices C, D by

$$C = \int_0^{2\pi} \Phi'(t)\Phi(t)dt, \qquad D = \int_0^{2\pi} \Psi(t)\Psi'(t)dt$$

and projection operators U, Q on X by:

$$Ux = \Phi a \qquad a = C^{-1} \int_0^{2\pi} \Phi'(t)x(t)dt$$

$$Qx = \Psi' b \qquad b = D^{-1} \int_0^{2\pi} \Psi(t)x(t)dt;$$

then

$$X_U = \mathfrak{N}(A), \; X_{I-Q} = \mathscr{R}(A), \; E = I - Q.$$

To illustrate the application of the above ideas to a problem involving no small parameter, we consider the following problem of Lazer and Sanchez [1]. Suppose x is an n-vector, X is the space of L^2 n-vector functions of period 2π, $p \in X$, $H: E^n \to R$ is a function with continuous second derivatives satisfying:

$$N^2 I < \mu_N I \leq \left(\frac{\partial^2 H(a)}{\partial x_i \partial x_j}\right) \leq \mu_{N+1} I < (N+1)^2 I \quad \text{for all} \quad a \in E^n.$$

A matrix is greater (strictly greater) than zero if it is positive (positive definite). The problem is to discuss the existence of a 2π-periodic solution of

$$\ddot{x} + \text{grad } H(x) = p(t) \tag{4.15}$$

The case $n = 1$ was first treated by Loud [1] and generalized by Lazer and Leach [1]. For the n-dimensional case, we have the following

<u>Theorem</u>. Under the above conditions, the equation (4.15) has a 2π-periodic solution.

<u>Proof</u>: We only indicate the essential ideas of the proof and the reader can consult the original paper for details. Let $A: \mathscr{D}(A) \subset X \to X$ be defined by $(Ax)(t) = \frac{d^2x(t)}{dt^2}$, $t \in (-\infty,\infty)$, and $N: X \to X$ be defined by $Nx(t) = -\text{grad } H(x(t)) + p(t)$, $t \in (-\infty,\infty)$. Then $\mathfrak{N}(A) = X_U$ is the set of constant functions in X.

The operator A obviously has a bounded right inverse M

and without loss in generality, we can assume $UM = 0$. Therefore, finding 2π-periodic solutions of (4.15) is equivalent to $Ax = Nx$ which in turn is equivalent to

$$\begin{aligned} &\text{a)} \quad x = Ux + M(I-U)Nx \\ &\text{b)} \quad UNx = 0. \end{aligned} \tag{4.16}$$

If $x = y + z$, $y \in X_{I-U}$, $z \in X_U$, then (4.16) is equivalent to

$$\begin{aligned} &\text{a)} \quad y = M(I-U)N(y+z) \\ &\text{b)} \quad UN(y+z) = 0. \end{aligned} \tag{4.17}$$

The first step of the proof is to show that for any $z \in X_U$, there is a unique solution $F(z)$ of (4.17a). This is accomplished in the following manner: For any μ such that $(I-\mu M)$ has a bounded inverse, equation (4.17a) is equivalent to

$$y = (I-\mu M)^{-1}M[(I-U)N(y+z) - \mu y] \overset{\text{def}}{=} (I-\mu M)^{-1}MG(y,z).$$

Using methods very similar to those of the Hilbert-Schmitt theory of Fredholm integral equations and for $\mu = \frac{1}{2}(\mu_{N+1} + \mu_N)$, it is then shown that the operator $(I-\mu M)^{-1}MG(\cdot,z)$ is a contraction for each z. This depends upon the fact that M is completely continuous and has a complete orthonormal system of eigenvectors and that the Gateaux derivative $G'(x)$ of G satisfies

$|G'(x)| \leq \frac{1}{2}[(N+1)^2 - N^2]$.

In the process of the proof of the contraction one also obtains that $F(z)$ is globally Lipschitzian in z, with Lipschitz constant L and $|F(z) - z| \leq L\alpha$ for some constant α. Therefore, $UN(F(z) + z)$ is globally Lipschitz in z with Lipschitz constant K.

The next step is to show that the function $\gamma(z) = UN(F(z) + z)$ satisfies $z\cdot\gamma(z) \to \infty$ as $|z| \to \infty$, and for any r for which $z\cdot\gamma(z) > 0$ on $V = \{Z: |Z| = r\}$, there is an $\varepsilon > 0$ such that $|z - \varepsilon\gamma(z)| < |z^2| = r^2$ on V. This implies the map $z - \varepsilon\gamma(z)$ takes $\partial V \to V$ and a known theorem in E^n implies there is $z_0 \in \partial V$ such that $z_0 - \varepsilon\gamma(z_0) = z_0$; that is, $\gamma(z_0) = 0$. This proves the theorem.

We now indicate the method of Cesari [4] for the justification of the Galerkin procedure for finding periodic solutions of ordinary differential equations. This method applies to general nonlinear problems and involves choosing a projection operator S as in Lemma 2.2.

Let X be the Banach space of continuous 2π-periodic function with the topology of uniform convergence and let $A: \mathscr{D}(A) \subset X \to X$ be defined by $(Ax)(t) = \frac{dx}{dt}$, $-\infty < t < \infty$. For a given continuous function $g: R \times E^n \to E^n$, $g(t,x)$ locally Lipschitzian in x, $g(t+2\pi,x) = g(t,x)$ for all t,x, let $N: X \to X$ be defined by $(Nx)(t) = g(t,x(t))$, $-\infty < t < \infty$.

The equation

$$\dot{\mu}(t) = g(t,\mu(t)) \tag{4.18}$$

has a 2π-periodic solution if and only if there is an $x \in X$ such that

$$Ax = Nx. \tag{4.19}$$

It is clear that $\mathfrak{N}(A) = X_U$ is the set of constant functions in X and U may be chosen as $(Ux)(t) = \frac{1}{2\pi}\int_0^{2\pi} x(s)ds$, $-\infty < t < \infty$. Also, $\mathscr{R}(A) = X_{I-U}$. A bounded right inverse M for A is defined for any $z \in X_{I-U}$ as the unique solution of mean value zero of the equation $(Ax)(t) = \dot{x}(t) = z(t)$, $-\infty < t < \infty$. Clearly $M: X_{I-U} \to X_{I-U}$, $UM = 0$.

Any $x \in X$ has a Fourier series

$$x(t) \sim \sum_{k=-\infty}^{\infty} a_k e^{ikt}, \quad a_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}x(t)dt.$$

For any given integer $m > 0$, let $S_m: X \to X$ be defined by

$$(S_m x)(t) = \sum_{0<|k|\leq m} a_k e^{ikt}. \tag{4.20}$$

One easily sees that $S_m M = MS_m$ for all $m > 0$ and Lemma 2.2 implies that equation (4.19) is equivalent to

$$\begin{aligned} &y = M(I-\tilde{P}_m)N(y+z) \\ &\tilde{P}_m[A(y+z)-N(y+z)] = 0 \\ &\tilde{P}_m = U + S_m, \quad y \in X_{I-\tilde{P}_m}, \quad z \in X_{\tilde{P}_m}. \end{aligned} \tag{4.21}$$

Using the definition of $\tilde{P}_m$, M, Parseval's inequality and Schwarz's inequality one easily shows that

$$|M(I-\tilde{P}_m)x| \leq \gamma(m)|x|, \quad \gamma(m) = (2\pi)^{1/2}\Big(\sum_{|k|>m} k^{-2}\Big)^{1/2}.$$

Since $\gamma(m) \to 0$ as $m \to \infty$, one can use the contraction principle to prove for any $c > 0$, $d > 0$ that the conditions of Theorem 2.1 are satisfied for m sufficiently large. Thus for m sufficiently large, the equation (4.18) has a 2π-periodic solution if and only if there exists a $z \in X_{\tilde{P}_m}$ such that

$$\tilde{P}_m(A-N)(z+F(z)) = 0. \tag{4.22}$$

In words, the above remarks imply the following: there is always an integer m such that one can fix

$$z(t) = \sum_{|k|\leq m} z_k e^{ikt}$$

and determine a function $x = F(z) + z$ in such a way that the Fourier series of the function $v(t) = \dot{x}(t) - g(t,x(t))$ contains only the harmonics e^{ikt}, $|k| \leq m$. The determining equations (4.22) involve the determination of the z_k, $|k| \leq m$, in such a way that the remaining Fourier coefficients of $v(t)$ vanish.

Now to determine the existence of a periodic solution using

this method, one must determine first how large m must be in order to apply the contraction principle and then try to show the existence of a solution of the determining equations (4.22) even though they involve the function Fz which can only be known approximately. One usually proceeds in the following manner: first obtain an a priori bound on $F(z)$ as $|F(z)| \leq \delta_m$ for some constant δ_m. Then try to show the equation

$$\tilde{P}_m(A-N)(y+z) = 0$$

has a solution for every function $y \in X_{I-\tilde{P}_m}$ with $|y| \leq \delta_m$.

To show this latter property is satisfied one naturally looks at the problem for $y = 0$,

$$\tilde{P}_m(A-N)(z) = 0,$$

which is the mth Galerkin approximation to the solution. An index argument can sometimes be used to complete the investigation. For some applications see Cesari [4], Locker [1, 2].

In the limited space we have here, we give a simpler application due to Knobloch [1]. Consider the system of second order differential equations

$$\begin{aligned} &\ddot{y} + f(t,y,\dot{y}) = 0 \\ &f(t+2\pi,y,z) = f(t,y,z) \qquad (4.23) \\ &y = (y_1,\ldots,y_m), \quad f = (f_1,\ldots,f_m) \end{aligned}$$

and assume that $f(t,y,z)$ is bounded in (t,y,z) and globally Lipschitzian in y,z and there exist $\alpha_i < \beta_i$ such that

$$f_i(t,y,z) < 0 \quad \text{if} \quad y_i < \alpha_i$$

$$f_i(t,y,z) > 0 \quad \text{if} \quad y_i > \beta_i, \quad i = 1,2,\ldots m.$$

We assert that the above hypotheses imply that <u>(4.23) has at least one 2π-periodic solution</u>.

In fact, repeating the same process as above with $(Ay)(t) = \ddot{y}(t)$, $(Ny)(t) = -f(t,y(t),\dot{y}(t))$, one concludes that there is an integer n_0 and a function $F: X_{\tilde{P}_{n_0}} \to X_{I-\tilde{P}_{n_0}}$ such that F is continuous and equation (4.23) has a 2π-periodic solution if and only if there exists a $u \in X_{\tilde{P}_{n_0}}$ satisfying $\tilde{P}_{n_0}(A-N)u = 0$. These equations are equivalent to

$$g_j(a) \overset{\text{def}}{=} -j^2 a_j - \frac{1}{2\pi}\int_0^{2\pi} f(t,u(t) + Fu(t),\ \dot{u}(t) + F(\dot{u}(t))dt = 0$$

$$0 < |j| \leq n_0$$

$$g_0(a) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} f(t,u(t) + Fu(t),\ \dot{u}(t) + F\dot{u}(t))dt = 0$$

if

$$u(t) = \sum_{|k|\leq n_0} a_k e^{ikt} \quad \text{and} \quad a = (a_{-n_0},\ldots,a_{-1},a_0,a_1,\ldots,a_{n_0}).$$

In the space of vectors a, let us consider now a box around the origin with faces parallel to the planes $a_j = 0$, $0 \leq |j| \leq n_0$ and such that $g_j(a)$, $0 < |j| \leq m$, takes opposite signs on opposite faces to the plane $a_j = 0$. For these faces fixed, move the other faces so far apart that $g_0(a)$ takes opposite signs on these latter faces. Mirando's version of the Brouwer fixed point theorem now implies there is a zero of all the functions g_j, $0 \leq |j| \leq n_0$, in the box. This proves the result.

Further interesting applications of these techniques to periodic solutions of ordinary differential equations may be found in the papers of Mawhin [1-3].

5. Eigenvalues of Nonlinear Operators

Let X be a Banach space and $C: \mathscr{D}(C) \subset X \to X$ be a linear operator; let also $B: X \to X$ be an operator such that $B(0) = 0$ and B has a continuous Frechét derivative B'. Then, $Bx = Dx + \Omega x$, with $D = B'(0)$, $\Omega(0) = 0$, $\Omega'(0)$, and in fact,

$$|\Omega x - \Omega y| \leq \beta(\rho)|x-y| \quad \text{for} \quad |x|, |y| \leq \rho,$$

where $\beta(0) = 0$.

The eigenvalue problem is the following: Does there exist $\mu \neq 0$ and $x \neq 0$ such that $Cx = \mu Bx$?

We say that μ_0 is a bifurcation point if for any $\varepsilon > 0$ and any $\rho > 0$ there is a μ in $|\mu - \mu_0| < \varepsilon$ and $x \neq 0$, $|x| < \rho$ such that $Cx = \mu Bx$.

If we write the equation as $(C-\mu_0 D)x = (\mu-\mu_0)Dx + \Omega x$ and $(C - \mu_0 D)$ has a bounded inverse, then there can be no bifurcation at μ_0. In fact, $x = (C-\mu_0 D)^{-1}[(\mu-\mu_0)Dx + \Omega x]$ and the contraction principle implies a unique solution of this equation for $|\mu-\mu_0| < \varepsilon$, $|x| < \rho$ if ε and ρ are small. Since $x = 0$ is a solution, μ_0 cannot be a bifurcation point. Therefore, to have a bifurcation, $(C-\mu_0 D)$ must not have a bounded inverse.

If we call $A = C-\mu_0 D$, $N = (\mu-\mu_0)D + \mu\Omega$ we are going to assume as before that $\mathfrak{N}(A) = X_U$, $\mathscr{R}(A) = X_E$ for some projections U and E, and that A has a bounded right inverse M, with $UM = 0$. For example, if $C = I$ and B is completely continuous, this hypothesis is always satisfied. For this latter situation ($C = I$ and B completely continuous)

Krasnoselskii [1] has proved by using rotation of a vector field that there exists always a bifurcation at μ_0 if the generalized eigenspace of A is of odd dimension. We show that the techniques mentioned before can be used in the more general situation to attack this problem, but will only be able to conclude that bifurcation exists (under an additional weak hypothesis on A) if $\dim \mathfrak{N}(A) = 1$.

As before, the equation $Cx = \mu Bx$ is equivalent to $Ax = Nx$, and hence it is equivalent to the system

(5.1) a) $x = Ux + MENx$

b) $(I-E)Nx = 0$.

If we call $x = y+z$, $y \in X_{I-U}$, $z \in X_U$, (5.1) can be written as

(5.2) a) $y = MEN(y+z)$

b) $(I-E)N(y+z) = 0$.

For any positive constants c and d let

$$V(c) = \{x \in X_U: |x| \leq c\}, \quad \mathcal{S}(d) = \{x \in X: |x| \leq d,\ Ux = 0\}.$$

<u>Theorem 5.1.</u> There exist $e_0 > 0$, $d_0 > 0$ such that for every $0 < c$, $d \leq d_0$, there exists a unique continuous function $F: V(c) \times \{|\mu-\mu_0| \leq e_0\} \to X$ such that, $F(0,\mu) = 0$,

$$F(\tilde{x},\mu) \in \mathcal{S}(d) \quad \text{for} \quad \tilde{x} \in V(c) \quad \text{and} \quad |\mu-\mu_0| \leq e_0$$

and

$$F(\tilde{x},\mu) = MEN(F(\tilde{x},\mu)+\tilde{x}), \ |\mu-\mu_0| \leq e_0.$$

Furthermore, $x = \tilde{x} + F(\tilde{x},\mu)$ satisfies $Cx = \mu Bx$ if and only if $\tilde{x}$ satisfies

$$(I-E)N(\tilde{x}+F(\tilde{x},\mu)) = 0$$

The function $F(\tilde{x},\mu)$ has a continuous first derivative with respect to $\tilde{x},\mu$ and there is a constant α such that $|F'_{\tilde{x}}(0,\mu)| \leq \alpha|\mu-\mu_0|$.

<u>Proof</u>. From the definition of N, we have $N(0) = 0$ and $|Nx-Ny| \leq |\mu|\beta(c+d)|x-y| + |\mu-\mu_0||D||x-y| \leq [|\mu|\beta(2d_0) + |D||\mu-\mu_0|]|x-y|$ for $|x| \leq c$, $|y| \leq d$, $|x| + |y| \leq c+d \leq 2d_0$, $|\mu-\mu_0| \leq e_0$.

The contradiction principle gives the result that there exist $d_0 > 0$, $e_0 > 0$ and $F(\tilde{x},\mu)$, $\tilde{x} \in V(c)$, $|\mu-\mu_0| \leq e_0$ such that

$$F(\tilde{x},\mu) = MEN(F(\tilde{x},\mu) + \tilde{x}).$$

The continuity and differentiability of $F(\tilde{x},\mu)$ also is obtained from the contraction principle and the differentiability of N with respect to x and μ. The fact that $\tilde{x}$ must satisfy the bifurcation equation is clear.

To prove the last assertion of the theorem observe the equality implies that

$$F(\tilde{x},\mu) = MEN(F(\tilde{x},\mu) + \tilde{x})$$

$$F'_{\tilde{x}}(\tilde{x},\mu) = MEN_x(F(\tilde{x},\mu) + \tilde{x}) \ (F'_{\tilde{x}}(\tilde{x},\mu) + I)$$

But $N_x = (\mu-\mu_0)D + \Omega'(x)$, with $\Omega'(0) = 0$ and so

$$F'_{\underset{\sim}{x}}(0,\mu) = ME(\mu-\mu_0)D(F'_{\underset{\sim}{x}}(0,\mu) + I),$$

and then if $|\mu-\mu_0|$ is small, we have F'

$$F'_{\underset{\sim}{x}}(0,\mu) = (\mu-\mu_0)T(\mu),$$

where $T(\mu)$ is a bounded linear operator for which we can find a bound independent of μ if $|\mu-\mu_0|$ is small.

From the above theorem, the problem remaining is to solve the bifurcation equation. Suppose now that the subspaces $\mathfrak{N}(A)$, and $(I-E)D\,\mathfrak{N}(A)$ are one dimensional. For $C = I$ and D completely continuous, this latter condition is not a restriction. For C a self adjoint partial differential operator, $D = I$ and X one of the usual Sobolev spaces, it is also not a restriction.

Let φ and ψ be a basis for $\mathfrak{N}(A)$ and $(I-E)D\,\mathfrak{N}(A)$, respectively. If $\tilde{x} = a\varphi$, then the bifurcation equations are:

$$0 = (I-E)N(\tilde{x}+F(\tilde{x},\mu)) = (I-E)N(a\varphi+F(a\varphi,\mu))$$

$$= (I-E)[(\mu-\mu_0)D(a\varphi+F(a\varphi,\mu)) + \mu\Omega(a\varphi+F(a\varphi,\mu))]$$

$$= (\mu-\mu_0)pa\psi + (\mu-\mu_0)(I-E)DF(a\varphi,\mu) + \mu(I-E)\Omega(a\varphi+F(a\varphi,\mu)),$$

where $p\psi = (I-E)D\varphi$, $p \neq 0$. If we let

$$(I-E)DF(a\varphi,\mu) = \psi\Gamma(a,\mu)$$

$$\mu(I-E)\Omega(a\varphi+F(a\varphi,\mu)) = \psi\delta(a,\mu)$$

the bifurcation equations are equivalent to

$$(\mu-\mu_0)\beta a + (\mu-\mu_0)\Gamma(a,\mu) + \delta(a,\mu) = 0 \tag{5.3}$$

From Theorem 5.1, there are constants k, α, and a continuous function $\gamma(r)$, $r \geq 0$, $\gamma(0) = 0$, such that the following inequalities hold:

$$|\Gamma(a,\mu)| \leq k|F(a\varphi,\mu)| \leq k[\alpha|a\varphi||\mu-\mu_0| + \gamma(|a\varphi|)\cdot|a\varphi|],$$

$$|\delta(a,\mu)| \leq k[\beta(|a\varphi| + |F(a\varphi,\mu)|)][|a\varphi| + |F(a\varphi,\mu)|]$$

$$\leq k\beta(|a\varphi| + |F(a\varphi,\mu)|)[|a\varphi| + \alpha|a\varphi|\cdot|\mu-\mu_0| + \gamma(|a\varphi|)|a\varphi|]$$

From these properties we have the bifurcation equations are equivalent to (for any $a \neq 0$)

$$\mu = \mu_0 - (\mu-\mu_0)\frac{\Gamma(a,\mu)}{ap} - \frac{\delta(a,\mu)}{ap}$$

where $\frac{\Gamma(a,\mu)}{a}$ and $\frac{\delta(a,\mu)}{a}$ are continuous as $a = 0$ and go to zero as $a \to 0$, $\mu \to \mu_0$ (actually continuously differentiable). The implicit function theorem implies there exists a $\mu(a)$ satisfying the bifurcation equation for a and $|\mu-\mu_0|$ small enough. Also $\mu(a) \to \mu_0$ as $a \to 0$. We thus have the following:

<u>Theorem 5.2.</u> Under the above hypothesis on the simple eigenvalue μ_0, there always is a bifurcation at $\mu = \mu_0$.

We can find the solution iteratively as:

$$y_0 = 0, \quad \mu_0 = \mu_0$$

$$y_{n+1} = MEN(y_n + a\varphi)$$

$$\mu_{n+1} = \mu_0 - (\mu_n - \mu_0)\frac{\Gamma_n(a)}{ap} - \frac{\delta_n(a)}{ap} \quad n = 0,1,2,\ldots$$

where

$$(I-E)D\varphi = \psi p$$

$$(I-E)Dy_n = \psi\Gamma_n(a)$$

$$\mu_n(I-E)\Omega(a\varphi+y_n) = \psi\delta_n(a).$$

These ideas are closely related to the paper of H. Keller [1], where the author studies the equation:

$$Lu + \lambda g(\lambda, x, \mu) = 0, \ x \in D, \ Bu = 0, \ x \in \partial D,$$

L elliptic operator of order $2m$, B linear of order m.

Another application of these ideas can be seen in the work of J. B. Keller [1]. To see this, consider the equation

$$(5.4) \qquad \begin{aligned} &\text{a)} \quad [I(t)u_t]_t + f(u,t,\lambda) = 0 \qquad t_1 \le t \le t_2 \\ &\text{b)} \quad \begin{aligned} &\alpha_1 u_t(t_1) + \beta_1 u(t_1) = 0 \\ &\alpha_2 u_t(t_2) + \beta_2 u(t_2) = 0, \qquad \alpha_1 \ne 0, \ \alpha_2^2 + \beta_2^2 \ne 0. \end{aligned} \end{aligned}$$

where $u(t)$ is a scalar function and $I(t)$ is positive and continuously differentiable while f has derivatives with respect to λ, u up through order three which are continuous in some rectangle $u_1 \le u \le u_2$,

$\lambda_1 \leq \lambda \leq \lambda_2$, $t_1 \leq t \leq t_2$.

Assume (5.4) has a solution $u_0(t,\lambda)$ for $\lambda_1 \leq \lambda \leq \lambda_2$, $u_1 \leq u_0(t,\lambda) \leq u_2$. Also, assume the linear variational problem at λ_0 has a nontrivial solution, i.e., there exists a solution φ of

$$
\begin{aligned}
&(Iv_t)_t + f_u(u_0(t,\lambda_0),t,\lambda_0)v = 0 \\
(5.5) \qquad &\alpha_1 v_t(t_1) + \beta_1 v(t_1) = 0 \\
&\alpha_2 v_t(t_2) + \beta_2 v(t_2) = 0.
\end{aligned}
$$

Then necessarily $\varphi(t_1) = 0$ (since $\varphi(t_1) = 0$ would imply $\varphi_t(t_1) = 0$ and, by uniqueness, $\varphi(t)$ would be zero). Also the first boundary condition says that $\varphi_t(t_1)$ can be determined from $\varphi(t_1)$, and so there is only one linearly independent solution of (5.5).

Let us normalize it by putting $\int_{t_1}^{t_2} \varphi^2 dt = 1$. The problem is to find conditions under which there are solutions of (5.4) near $\lambda = \lambda_0$ other than the given solution $u_0(t,\lambda)$.

To phrase this problem in the language of bifurcation of a nonlinear operator, let X be the Banach space of the continuously differentiable functions on $[t_1,t_2]$ satisfying the boundary conditions (5.4b) with the topology of uniform convergence of the function and its derivative. Let

$$
\begin{aligned}
&C\colon \mathscr{D}(C) \subset X \to X, \quad (Cu)(t) = (Iu_t)_t, \quad t_1 \leq t \leq t_2, \\
(5.6) \qquad &B_\lambda\colon X \to X, \qquad (B_\lambda u)(t) = -f(u(t),t,\lambda), \quad t_1 \leq t \leq t_2.
\end{aligned}
$$

The above hypotheses imply that $u_0(\cdot,\lambda) \in X$ is a solution of

$$Cu = B_\lambda u \tag{5.7}$$

for $\lambda_1 \leq \lambda \leq \lambda_2$ and, for $\lambda = \lambda_0$, the equation

$$(C-B'_{\lambda_0}(u_0(\cdot,\lambda_0))u = 0 \tag{5.8}$$

has a one dimensional subspace of solutions spanned by φ. The problem is to find conditions on B_λ which will ensure that (5.7) has solution $\neq u_0(\cdot,\lambda)$ for each λ in a neighborhood of $\lambda = \lambda_0$.

This is not exactly a special ase of the problem discussed at the beginning of this section, but the ideas used there are easily adapted to this situation. If we let $u = u_0(\cdot,\lambda) + v$, then equation (5.6) becomes

$$[C-B'_{\lambda_0}(u_0(\cdot,\lambda_0))]v = B_\lambda(u_0(\cdot,\lambda)+v) - B_\lambda(u_0(\cdot,\lambda)) - B'_{\lambda_0}(u_0(\cdot,\lambda_0))v \tag{5.9}$$

or v satisfies the differential equation

$$(Iv_t)_t + f_u(u_0(t,\lambda_0),t,\lambda_0)v = -f(u_0(t,\lambda)+v,t,\lambda)) + f(u_0(t,\lambda),t,\lambda) + f_u(u_0(t,\lambda_0),t,\lambda_0)\, v. \tag{5.10}$$

and the boundary conditions (5.4b). If

$$A = C - B'_{\lambda_0}(u_0(\cdot,\lambda_0)),$$

$$Nv = B_\lambda(u_0(\cdot,\lambda)+v) - B_\lambda(u_0(\cdot,\lambda)) - B'_{\lambda_0}(u_0(\cdot,\lambda_0))v$$

then (5.9) is equivalent to $Ax = Nx$. Furthermore, $\mathfrak{N}(A) = X_U =$

$\{u \in X: u = a\varphi$ for some constant $a\}$; $R(A) = X_{I-U} = \{u \in X: \int_{t_1}^{t_2} \varphi u = 0\}$.

The function Nv has the form

$$Nv = [f_{uu}u_{0\lambda} + f_{u\lambda}](\lambda_0-\lambda)v - f_{uu}\frac{v^2}{2}$$

where the functions are evaluated at some intermediate points between $u_0(\cdot,\lambda),\lambda$ and $u_0(\cdot,\lambda_0)$, λ_0. Therefore, N has the same form as the N discussed at the beginning of this section. The self-adjointness of the dif ferential operator implies Theorem 5.1 may be applied to obtain the bifurcation equations in the form of (5.3). Calculating the constants in (5.3) for this particular case, the bifurcation equations are

$$\begin{aligned} &a[p(\lambda-\lambda_0) + \Gamma a + \cdots] = 0 \\ (5.11) \qquad &p = \int_{t_1}^{t_2}(f_{uu}u_{0\lambda} + f_{u\lambda})\varphi^2 dt \\ &\Gamma = \int_{t_1}^{t_2} f_{uu}\varphi^3 dt \end{aligned}$$

where all functions in the integrand are evaluated at $u_0(t,\lambda_0),t,\lambda_0$.

Consequently, if $p \neq 0$, $\Gamma \neq 0$, the implicit function theorem implies there is a solution of (5.4) different from $u_0(\cdot,\lambda)$ in a neighborhood of $\lambda = \lambda_0$.

An interesting paper on determining the stability of bifurcating solutions by using Leray-Schander degree has been written by Sattinger [1].

6. Analytic Systems, Regular and Irregular Singular Points.

In this section, we indicate the manner in which some of the classical results on the existence of analytic solutions of linear differential systems with or without singular points can be obtained using the above procedure. The presentation follows closely the paper of Harris, Sibuya and Weinberg [1]. The exploitation of the procedure for more complicated solutions has not been but should be undertaken.

Suppose $D = \mathrm{diag}(d_1,\ldots,d_n)$ where each d_j is a positive integer, $t^D = (t^{d_1},\ldots,t^{d_n})$, $B(t)$ is an $n \times n$ matrix analytic at $t = 0$, with an absolutely convergent power series for $|t| \leq \delta$.

Our problem is to find necessary and sufficient conditions for the existence of analytic solutions of the equation

$$t^D \dot{y} = B(t)y \tag{6.1}$$

To phrase this problem in the language of Section 2, let X be the set of all n-vector valued functions $x(t)$ whose components have absolutely convergent power series expansions for $|t| \leq \delta$, $\delta > 0$. For $x \in X$, $x(t) = \sum_{k=0}^{\infty} x_k t^k$, define

$$|x| = \sum_{k=0}^{\infty} |x_k| \delta^k.$$

With this norm, X is a Banach space.

For any $x \in X$, let $A: X \to X$ be defined by

$$(Ax)(t) = t^D \dot{x}(t), \quad |t| \leq \delta$$

Then, $\mathfrak{N}(A) = X_U$, $(Ux)(t) = x_0$, $|t| \leq \delta$, is the set of constant functions in X and $\mathscr{R}(A) = \{y \in X: y = t^D x,\ x \text{ in } X\}$.

If we define $Q_0: X \to X$ by

$$(Q_D x)(t) = \mathrm{col}\left(\sum_{k=0}^{d_1-1} x_{k_1} t^k, \ldots, \sum_{k=0}^{d_n-1} x_{k_n} t^k\right), \quad |t| \leq \delta,$$

where $x_k = \mathrm{col}(x_{k_1}, \ldots, x_{k_n})$ then Q_D is a projection on X and $\mathscr{R}(A) = X_{I-Q_0}$. Furthermore, a right inverse M of A on $\mathscr{R}(A)$ is given by

$$(Mx)(t) = \left(\sum_{k=0}^{\infty} x_{k_1+d_1} \frac{t^{k+1}}{k+1}, \ldots, \sum_{k=0}^{\infty} x_{k_n+d_n} \frac{t^k}{k+1}\right)$$

and $|Mx| \leq \max(\delta^{1-d_1}, \ldots, \delta^{1-\delta_n})|x|$. Therefore M is bounded.

If $N: X \to X$ is defined by $(Nx)(t) = B(t)x(t)$, $|t| \leq \delta$, then N is a bounded linear operator and finding a solution of (6.1) in X is equivalent to finding an $x \in X$ such that

$$Ax = Nx \tag{6.2}$$

From Lemma 2.1, (6.2) is equivalent to:

$$\text{(6.3)} \quad \begin{aligned} &\text{a)} \quad y = M(I-Q_D)N(y+z) \\ &\text{b)} \quad Q_D N(y+z) = 0, \quad y \in X_{I-U},\ z \in X_U \end{aligned}$$

If the linear operator $M(I-Q_D)N$ are a contraction operator, then (6.3a) would have a unique solution $y = Fz$ where $F: X_U \to X_{I-U}$ is a bounded linear operator. A solution of (6.1) would then be determined by solving the finite set of linear equations $Q_D N(Fz+z) = 0$. Unfortunately,

this operator may not be a contraction. To obtain a contraction we proceed as in Lemma 2.3. For $r \geq 0$ an integer, let $S_r: X \to X$ be the projection operator defined by $(S_r x)(t) = \sum_{k=1}^{r+1} x_k t^k$, $|t| \leq \delta$.

If $R_r = I-U-S_r$, then it is easy to see that

$$R_r M(I-Q_D) = M(I-Q_{D+rI})$$

and, furthermore, from Lemma 2.3 that (6.2) is equivalent to

$$(6.4)\quad \begin{aligned} &\text{a)}\quad v = M(I-Q_{D+rI})N(v+u) \\ &\text{b)}\quad Q_{D+rI}(A-N)(v+u) = 0,\ v \in X_{R_r},\ u \in X_{I-R_r}. \end{aligned}$$

It also follows immediately from the definition of M that there is a $\beta > 0$ such that

$$(6.5)\qquad |M(I-Q_{D+rI})| \leq \beta/(r+1)$$

Consequently, for r sufficiently large, the operator $M(I-Q_{D+rI})$ is a contraction. Therefore, for any fixed polynomial u of degree r, there is a unique solution Fu of (6.4a) with $F: X_{I-R_r} \to X_{R_r}$ a bounded linear operator. This implies that the equation (6.1) has an analytic solution if and only if the polynomial u can be chosen so that

$$(6.6)\qquad Q_{D+rI}(A-N)(Fu+u) = 0$$

Equation (6.6) is a polynomial vector equation with the jth component of degree d_j+r-1 for the $n(r+1)$ coefficients of the vector polynomial u of degree r; that is, these are at most $nr + \sum_{j=1}^{n} d_j$ linear homogeneous equations for $nr+n$ unknowns. Therefore, there are at least $n - \sum_{j=1}^{n} d_j$ linearly independent solutions. This proves the following theorem of Perron-Lettenmeyr:

<u>Theorem 6.1.</u> Equation (6.1) has at least $n - \sum_{j=1}^{n} d_j$ linearly independent analytic solutions for $|t| \leq \delta$.

Now let us consider the case of the regular singular point ($d_j = 1$ for all j). In this case, $Q_{D+rI} = Q_{(r+1)I} = U+S_r$; that is, $Q_{D+rI}x$ represents the vector polynomial of degree r consisting of the first $(r+1)$ terms of the expansion of x. Theorem 6.1 gives no information in this case, but we still may draw some conclusions. The function $(A-N)Fu$ has a power series expansion beginning with terms of degree $(r+1)$. Therefore, equation (6.6) is equivalent to

$$(U+S_r)(A-N)u = 0 \tag{6.7}$$

If $u = \sum_{k=0}^{r} u_k t^k$, $B(t) = \sum_{k=0}^{\infty} B_k t^k$, $|t| \leq \delta$, then equation (6.7) is equivalent to the system of linear equations

$$(kI-B_0)u_k = \sum_{j=1}^{k} B_j u_{k-j}, \quad k = 0,1,\ldots,r, \quad u_{-1} = 0. \tag{6.8}$$

Equations (6.8) are the equations for the first $r+1$ coefficients

of a formal solution $u(t) = \sum_{k=0}^{\infty} u_k t^k$ of

$$t\dot{u} = B(t)u \tag{6.9}$$

Consequently we have proved:

<u>Theorem 6.2.</u> Any formal solution of (6.9) is an analytic solution.

Let us now discuss the existence of solutions of (6.9) of the form $u(t) = t^{\lambda}v(t)$ where $v \in X$. Then

$$t\dot{v} = [B(t) - \lambda I]v(t)$$

Let n_λ be the dimension of $\mathfrak{N}(B_0-\lambda I)$ and let N_λ be the number of linearly independent solutions u of the above form. The determining equations corresponding to (6.8) are

$$[(k+\lambda)I-B_0]v_k = \sum_{j=1}^{k} B_j v_{k-j}, \quad k = 0,1,\ldots,r$$

if $v(t) = \sum_{k=0}^{\infty} v_k t^k$. Clearly $N_\lambda \leq n_\lambda + n_{\lambda+1} + \ldots$ and $N_\lambda \geq N_{\lambda+1}$.

We now show that $N_\lambda \geq n_\lambda$. Without loss in generality, we can assume $\lambda = 0$ and B_0 is in Jordan canonical form. Each vector in $\mathfrak{N}(B_0)$ corresponds to a zero row of B_0 and the corresponding d_j may be reduced to zero. Therefore, Theorem 6.1 implies at least $n - (n-n_0) = n_0$ solutions analytic at $t = 0$. This proves

<u>Theorem 6.3</u>. $\max(n_\lambda, n_{\lambda+1}, \ldots) \leq N_\lambda \leq n_\lambda + n_{\lambda+1} + \ldots$. In particular, if $B_0 = 0$, that is, system (6.9) has no singular point, then there is fundamental system of solutions analytic at $t = 0$. If $B_0 - kI$ is a nonsingular for each positive integer k, then $N_0 = n_0$ and, in fact, for any $u_0 \in \mathfrak{N}(B_0)$ there is a solution $u(t)$ of (6.9) with $u(0) = u_0$.

Similar results can be obtained by the same methods for nonlinear systems $t^D\dot{x}(t) = f(t,x(t))$ (see Harris [1]).

A more complete discussion of (6.9) is obtained by trying to find a fundamental matrix solution of (6.9) of the form $U(t) = P(t)t^G$, where G is a constant $n \times n$ matrix and $P(t)$ is an $n \times n$ matrix analytic at $t = 0$. We indicate how some information is easily obtained from Theorem 6.1. If $G = B_0$, then $U = Pt^{B_0}$ a matrix solution of (6.9) implies

$$t\dot{P} = B_0P - PB_0 + \bar{B}_1(t)P, \quad \bar{B}_1(0) = 0. \tag{6.10}$$

If we look at this equation as an equation for the n^2 components p of P, then

$$t\dot{p} = C_0p + C_1(t)p, \quad C_1(0) = 0. \tag{6.11}$$

If no two eigenvalues of B_0 differ by a positive integer, then no eigenvalue of C_0 is a positive integer and Theorem 6.3 implies there are as many linearly independent solutions of (6.11) as $\dim \mathfrak{N}(C_0)$.

Furthermore, any $p_0 \in \mathfrak{N}(C_0)$ yields a solution p of (6.11) analytic at $t = 0$ and $p(0) = p_0$. From the definition of p and C_0 the p corresponding to $P = I$ is in $\mathfrak{N}(C_0)$. Therefore, (6.10) has a solution P analytic at $t = 0$ and $P(0) = I$. This gives the classical

<u>Theorem 6.4.</u> If no two eigenvalues of B_0 differ by a positive integer, then there is a fundamental solution U of (6.9) of the form $U(t) = P(t)t^{B_0}$ where P is analytic at $t = 0$ and $P(0) = I$.

Golomb [1] has given some interesting results on the expansion of fundamental solutions of (6.1) even in the case of irregular singular points. The approach applies to even more general equations and the ideas are very closely related to those of Section 2.

7. Partial Differential Equations. In this section, we indicate how the above ideas have been used to discuss some particular problems in partial differential equations. We do not treat the most general situation, but try to bring out the main ideas and give appropriate references for more sophisticated results.

Consider the wave equation in one dimension:

$$u_{tt} - u_{xx} = \epsilon g(t,x,u,u_t,u_x), \quad \text{where}$$
$$g(t,x,u,p,q) = -g(t,-x,-u,-p,q) = g(t+2\pi,x,u,p,q) = g(t,x+2\pi,u,p,q) \tag{7.1}$$

for all $(t,x,u,p,q) \in \Omega(\rho) = \{(t,x) \in R^2 = (-\infty,\infty)x(-\infty,\infty);\ |u| + |p| + |q| \leq \rho\}$. The problem is to determine conditions on g which will ensure that equation (7.1) has a solution $u(t,x)$ satisfying

$$u(t+2\pi,x) = u(t,x+2\pi) = u(t,x) = -u(t,-x); \quad (t,x) \in R^2. \tag{7.2}$$

Due to the form of g any solution satisfying (7.2) will also satisfy

$$u(t,0) = u(t,\pi) = 0, \quad t \in (-\infty,\infty), \quad x \in [0,\pi]. \tag{7.3}$$

Therefore, we are solving for 2π-periodic solutions in t of the vibrating string with ends fixed at $x = 0$ and $x = \pi$.

Let $R = (-\infty,\infty)$ and define

$$T = \{\varphi: R^2 \to R;\ \varphi \text{ continuous, } \varphi(t+2\pi,x) = \varphi(t,x+2\pi) = \varphi(t,x) = -\varphi(t,-x)\};$$

$$W = \{\varphi: R^2 \to R;\ \varphi \text{ continuous, } \varphi(t,x) = p(x+t) - p(-x+t),\ p(t+2\pi) = p(t)\};$$

$$C_1^* = \{\varphi: R^2 \to R;\ \varphi \text{ and } \varphi_x \text{ continuous}\};$$

$$C_2 = \{\varphi: R^2 \to R;\ \varphi \text{ continuous together with derivatives up through order } 2\};$$

$$X = T \cap C_2;\quad Z = T \cap C_1^*.$$

All topologies in these spaces are those of uniform convergence, and norms will be desiginated by $|\cdot|_T$, $|\cdot|_W$, $|\cdot|_1^*$, $|\cdot|_2$, $|\cdot|_X$, $|\cdot|_Z$, respectively. Define operators A, N, and Q by

$$A: \mathscr{D}(A) \subset X \to Z;\quad (Au)(t,x) = u_{tt} - u_{xx};$$

$$N: X \to Z;\quad (N_u)(t,x) = \epsilon g(t,x,u(t,x),\ u_t(t,x),\ u_x(t,x));\qquad (7.4)$$

$$Q: T \to T;\quad (Q\varphi)(t,x) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(s,x+t-s)ds - \frac{1}{2\pi}\int_0^{2\pi} \varphi(s,-x+t-s)ds.$$

It is easy to see that Q is a projection of T onto $T_Q = W$. We can also consider Q to be a projection on X and Z, and as such, it is clear that $\mathfrak{N}(A) = X_Q$. Finding a solution of (7.1) is equivalent to finding a $u \in \mathscr{D}(A)$ such that

$$Au = Nu. \tag{7.5}$$

The following lemmas give more detailed properties of the operators Q and A.

<u>Lemma 7.1.</u> For any $\varphi \in T$, the following are equivalent:

i) $\varphi \in T_{I-Q}$,

ii) $\int_0^{2\pi} \varphi(s,y-s)ds = 0$ for all y;

iii) $\int_0^{2\pi}\int_0^{\pi} \varphi(t,x)\gamma(t,x)dxdt = 0$ for all γ in $T_Q = W$.

<u>Proof:</u>

(i) $\Longrightarrow$ (ii): If $\varphi \in T_{I-Q}$, then $(Q\varphi)(t,x) = 0$ implies $(Q\varphi)(t,-t) = 0$; i.e.

$$\int_0^{2\pi} \varphi(s,-s)ds = \int_0^{2\pi} \varphi(s,2t-s)ds \quad \text{for all } t.$$

Therefore

$$\begin{aligned}
0 = \int_0^{4\pi} (Q\varphi)(t,-t)dt &= 2\int_0^{2\pi} \varphi(s,-s)ds - \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{4\pi} \varphi(s,2t-s)dt\right)ds \\
&= 2\int_0^{2\pi} \varphi(s,-s)ds - \frac{1}{\pi}\int_0^{2\pi}\left(\int_0^{2\pi} \varphi(s,2t-s)dt\right)ds \\
&= 2\int_0^{2\pi} \varphi(s,-s)ds - \int_0^{2\pi}\left(\frac{1}{2\pi}\int_{-s}^{2\pi-s} \varphi(s,u)du\right)ds \\
&= 2\int_0^{2\pi} \varphi(s,-s)ds
\end{aligned}$$

since $\varphi(s,-u) = -\varphi(s,u)$. This proves (ii).

If (ii) is satisfied, then clearly (i) is satisfied. To prove the equivalence of (ii) and (iii), suppose $\gamma(t,x) = p(x+t) - p(-x+t)$ and observe that

$$\int_0^{2\pi}\int_0^{\pi} \varphi(t,x)[p(x+t) - p(-x+t)]dxdt = \int_0^{2\pi}\int_{t-\pi}^{t+\pi} \varphi(t,y-t)p(y)dydt$$

$$= \int_{-\pi}^{\pi}\left[\int_0^{2\pi} \varphi(t,y-t)dt\right]p(y)dy$$

<u>Lemma 7.2</u>. $\mathscr{R}(A) = Z_{I-Q}$ and A has a bounded right inverse $M: Z_{I-Q} \to X_{I-Q}$.

<u>Proof</u>: For any $\varphi \in Z$, consider the function

$$U(t,x) = \frac{1}{2}\int_0^t\int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta,$$

which belongs to C_2, and satisfies $U_{tt} - U_{xx} = \varphi$, $U(t,x+2\pi) = U(t,x)$, $U(t,-x) = -U(t,x)$. To prove that $\mathscr{R}(A) = Z_{I-Q}$, it is sufficient to show that $U(t,x) = U(t+2\pi,x)$ if and only if $\varphi \in Z_{I-Q}$. A straightforward calculation using the fact that $\varphi(t,-x) = -\varphi(t,x)$ gives

$$U(t+2\pi,x) = \frac{1}{2}\int_0^{t+2\pi}\left[\int_{x-2\pi-t+\theta}^{x+2\pi+t-\theta} \varphi(\theta,\xi)d\xi\right]d\theta$$

$$= \frac{1}{2}\int_0^{t+2\pi}\left[\int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi\right]d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta + \frac{1}{2}\int_0^{t}\int_{x-t+2\pi+\theta}^{x+t-2\pi-\theta} \varphi(\theta,\xi)d\xi d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta + U(t,x).$$

Therefore, $U(t+2\pi,x) = U(t,x)$ if and only if

$$\psi(t,x) \overset{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi}\int_{x-t+\theta}^{x+t-\theta} \varphi(\theta,\xi)d\xi d\theta = 0$$

for all t,x. This is equivalent to requiring that $\psi_t(t,x) = 0 = \psi_x(t,x)$, for all x,t, and that ψ is zero at one point. Using the fact that $\varphi(t,-x) = -\varphi(t,x)$, we obtain

$$\psi_t(t,x) = (Q\varphi)(t,x);$$

$$\psi_x(t,x) = (Q\varphi)(t,x) + \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta,-x+t-\theta)d\theta;$$

$$\psi(t,t) = -\psi(t,-t).$$

Since $\psi(0,0) = 0$, $\psi(t,x) \equiv 0$ is equivalent, by virtue of Lemma 7.1, to $(Q\varphi)(t,x) = 0$. This proves that $\mathscr{R}(A) = Z_{I-Q}$. Taking $M\varphi = (U-QU)\varphi$, the remainder of the lemma is obvious.

Lemma 7.2 implies that $Au = Nu$ is equivalent to

$$u = Qu + M(I-Q)Nu,$$

$$QNu = 0.$$

We can prove the following theorems by a straightforward application of the contraction principle.

<u>Theorem 7.1.</u> Suppose g has continuous first and second derivatives with respect to x,u,p,q in $\Omega(\rho)$ and let $0 < a < b < \rho$ be given positive constants. Then there is an $\epsilon_1 > 0$ with the following property: corresponding to each $\gamma \in X_Q = W \cap C_2$, $|\gamma|_X \leq a$, and to each ϵ, $|\epsilon| \leq \epsilon_1$, there is a unique function $\Gamma = \Gamma(\gamma,\epsilon)$ in X, $Q\Gamma = \gamma$, $|\Gamma|_X \leq b$, continuous together with its first derivatives in γ,ϵ, satisfying $\Gamma(\gamma,0) = \gamma$, $\Gamma'_\gamma(\gamma,0) = I$, and

$$\Gamma_{tt} - \Gamma_{xx} = \epsilon N\Gamma - \epsilon QN\Gamma.$$

<u>Theorem 7.2.</u> Under the hypotheses of Theorem 7.1, if there is an $\epsilon_2 \leq \epsilon_1$ and a function $\gamma(\epsilon)$ in X_Q, $|\gamma(\epsilon)|_X \leq a$, $|\epsilon| \leq \epsilon_2$, such that

$$QN\Gamma(\gamma(\epsilon),\epsilon) = 0, \tag{7.6}$$

then $\Gamma(\gamma(\epsilon),\epsilon)$ is a solution of (7.1) for $|\epsilon| \leq \epsilon_2$. Conversely, if (7.1) has a solution $u(\epsilon) \in X$, continuous in ϵ for $|\epsilon| \leq \epsilon_2$, $|u(\epsilon)|_X < b$, $|Qu(\epsilon)|_X \leq a$, $|\epsilon| \leq \epsilon_2$, then $u(\epsilon) = \Gamma(\gamma(\epsilon),\epsilon)$, where $\gamma(\epsilon) = Qu(\epsilon)$ and $\Gamma(\gamma(\epsilon),\epsilon)$ satisfies (7.6).

Equations (7.6) are the bifurcation equations and a solution $\gamma(\epsilon)$ of (7.6) is necessary and sufficient (as described in Theorem 7.2)

for the existence of a solution of (7.1) for ϵ small. A more convenient form for equations (7.6) is obtainable from Lemma 7.1. More specifically, equations (7.6) are equivalent to

$$H(\gamma,\epsilon) = 0, \quad \Gamma = \Gamma(\gamma,\epsilon); \quad \text{where}$$
$$2\pi\, H(\gamma,\epsilon)(y) \overset{\text{def}}{=} \int_0^{2\pi} g(s,y-s,\Gamma(s,y-s),\Gamma_t(s,y-s),\Gamma_x(s,y-s))ds \quad (0 \leq y \leq 2\pi) \tag{7.7}$$

The explicit formulae for $H(\gamma,0)$ and $H'_\gamma(\gamma,0)$ are

$$2\pi\, H(\gamma,0)(y) = \int_0^{2\pi} g(s,y-s,\gamma(s,y-s),\gamma_t(s,y-s),\gamma_x(s,y-s))ds; \tag{7.8}$$

$$2\pi[H'_\gamma(\gamma,0)\Delta](y) = \int_0^{2\pi} [g_u\Delta(s,y-s) + g_p\Delta_t(s,y-s) + g_q\Delta_x(s,y-s)]ds \tag{7.9}$$

where Δ is an arbitrary element of X and the arguments of g_u, g_p, g_q are the same as in (7.8).

The functions H, H'_γ are continuous mappings from X_Q into the space P_1 of continuously differentiable 2π-periodic functions. Thus we can apply the implicit function theorem to obtain a solution of (7.7) and thereby a solution of (7.1). More specifically, if there is a $\gamma_0 \in X_Q$, $|\gamma_0|_X < a$ such that

$$H(\gamma_0,0) = 0;$$
$$[H'_\gamma(\gamma_0,0)]^{-1}: \quad P_1 \to X_Q \tag{7.10}$$

is bounded then there will be a solution of (7.7) for ϵ small.

Remark 7.1. Since

$$(Q\mathcal{N}\Gamma)(t,x) = H(\gamma,\epsilon)(x+t) - H(\gamma,\epsilon)(-x+t),$$

we could apply the implicit function theorem directly to $Q\mathcal{N}\Gamma$, observing that the mean values of the functions in the domain of the inverse of the derivative operator are unimportant. This remark is convenient in the applications.

To see the nature of the bifurcation equations and especially the functions in (7.8), (7.9), we consider only one example. Let us examine the equation

$$u_{tt} - u_{xx} = \epsilon[u_t+bu+cu^3+f(t,x)] \tag{7.11}$$

where b,c are constants, $b \neq 0$, and $f(t,-x) = -f(t,x)$. The functions $H(\gamma,0)$, $H'_\gamma(\gamma,0)$, in (7.8), (7.9) are given by

$$\gamma(t,x) = p(x+t) - p(-x+t); \quad \Delta(t,x) = q(x+t) - q(-x+t);$$

$$H(\gamma,0)(y) = p'(y) + [b+3cm(p^2)]p(y) + cp^3(y) - cm(p^3) + h(y);$$

$$(H'_\gamma(\gamma,0)\Delta)(y) = q'(y) + [b+6cm(p^2)]q(y) - 3c[m(p^2q) - 2m(pq)];$$

$$h(y) = \frac{1}{2\pi}\int_0^{2\pi} f(s,y-s)ds, \qquad 0 \leq y \leq 2\pi;$$

where $' = d/dy$, $m(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(y)dy$ and we have assumed without loss

of generality that $m(p) = m(q) = 0$. For c sufficiently small one can show there is a p such that the conditions (7.10) are satisfied and, thus, there is a solution $u \in X$ of (7.11) (see Hale [3]).

Many more problems of the above type have been considered for equation (7.1) and for even more general equations. We have only tried to indicate the procedure and refer the interested reader to the papers of Cesari [6-9], Rabinowitz [1], Vejvoda [1-3], Hall [2], Naparstek [1], and Petrovanu [1,2] for even more complete results as well as additional references.

As another illustration, we briefly describe the ideas in the paper of Landesman and Lazer [1]. Let D be a bounded domain in R^n and let

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a^{ij} \frac{\partial}{\partial x_j}$$

be a second order, self-adjoint uniformly elliptic operator on D; that is, each $a^{ij} = a^{ji}$ is assumed to be real, bounded and measurable on D and there is a constant $c > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \xi_j \geq c \sum_{j=1}^{n} \xi_j^2$$

for all $x \in D$ and for all real ξ_j. Let $\overset{o}{H}_1$ be the closure through the (real) inner product

$$\langle u,v\rangle_1 = \int_D (u\cdot v+u_x\cdot v_x)dx$$

of C_1 functions with compact support in D; let $H_0 = L^2(D)$ where, for $f,g \in H_0$,

$$\langle f,g\rangle_0 = \int_D fgdx.$$

For $h \in H_0$, $\alpha > 0$ constant, and $g: R \to R$ bounded and continuous, the problem is to determine a weak solution of

$$\begin{aligned} Au + \alpha u + g(u) &= h(x), \quad x \in D; \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{7.12}$$

That is, if

$$B(u,v) = \sum_{i,j=1}^{n} \int_D a^{ij} u_{x_i} v_{x_j} dx,$$

we seek a $u \in \overset{o}{H}_1$ such that

$$(\varphi,Au) \overset{\text{def}}{=} B(\varphi,u) = \langle\varphi,h-\alpha u-g(u)\rangle_0. \tag{7.13}$$

for all $\varphi \in \overset{o}{H}_1$. For any $u \in H_0$, let $Nu = h - g(u)$. Equation (7.13) will be written symbolically as

$$(A+\alpha I)u = Nu. \tag{7.14}$$

With the representation (7.13) for the operator A, we can consider $A: \mathscr{D}(A) = \overset{\circ}{H}_1 \subset H_0 \to H_0$. A special case of the result of Landesman and Lazer [1] is the following:

Theorem 7.3. Suppose $\mathfrak{N}(A+\alpha I)$ is one-dimensional and w is a basis for $\mathfrak{N}(A+\alpha I)$, $w(x) > 0$ for $x \in D$, $|w|_0 = 1$. Assume also that $g(s) \to g(\infty)$, $g(-s) \to g(-\infty)$ as $s \to \infty$, $g(\infty)$, $g(-\infty)$ finite, and

$$g(-\infty) < g(s) < g(\infty), \quad s \in (-\infty,\infty). \tag{7.15}$$

A necessary and sufficient condition for the existence of a weak solution of (7.12) is

$$g(-\infty) < \langle h,w\rangle_0 < g(\infty). \tag{7.16}$$

Sketch of Proof: The Riesz representation theorem and Rellich's compactness theorem imply that $\mathscr{R}(A) = H_0$ and that A has a completely continuous right inverse $S: H_0 \to H_0$. Equation (7.14) is equivalent to the equation

$$u = -\alpha Su - S[g(u) - h].$$

Since S is symmetric, one easily obtains $\langle w,g(u) - h\rangle_0 = 0$, and, from the fact that $w(x) > 0$ in D, this implies

$$g(-\infty) < \langle h, w\rangle_0 < g(\infty)$$

is necessarily satisfied.

The proof of sufficiency is much more complicated and uses the ideas mentioned in Section 2. Let $\mathfrak{R}(A+\alpha I) = H_{0Q}$, where Q is the orthogonal projection through $\langle\cdot,\cdot\rangle_0$ onto the subspace spanned by w. The operator $A + \alpha I$ has a completely continuous right inverse $M: X_{0,I-Q} \to X_{0,I-Q}$. Equation (7.14) is, therefore, equivalent to the equations

$$\begin{aligned} u &= Qu + M(I-Q)Nu; \\ QNu &= 0. \end{aligned} \tag{7.17}$$

If we let $Qu = aw$, where a is a scalar, then these equations can be written as

$$\begin{aligned} u &= aw + M(I-Q)Nu \overset{\text{def}}{=} G_1(u,a) \\ a &= a - \langle Nu, w\rangle_0 \overset{\text{def}}{=} G_2(u,a). \end{aligned} \tag{7.18}$$

If we let $G = (G_1, G_2)$, then finding a fixed point of G in $H_0 \times R$ is equivalent to solving the problem. The difficult part of the proof remains. By using the hypotheses on g and some careful estimates, the authors show by means of Schauder's theorem that a fixed point of G exists.

Using techniques more closely related to those of Lazer and

Sanchez [1] discussed in Section 4, Landesman and Lazer have also proved the following interesting result: Suppose D is a domain whose boundary is regular enough for the existence of the Green's function for the boundary value problem

$$\Delta u = h(x),\ x \in D,\ u(x) = 0,\ x \in \partial D;$$

where Δ is the n-dimensional Laplacian and h is Holder continuous on $\bar{D}$. Suppose that for $\alpha_1 \leq \alpha \leq \alpha_2$ there exists no nontrivial solution of

$$\Delta u + \alpha u = 0,\ x \in D;\ u(x) = 0,\ x \in \partial D.$$

It is then concluded that for F a real-valued C_1 function on R satisfying $\alpha_1 \leq F'(u) \leq \alpha_2$ for all u, and for h Holder continuous on $D \cup \partial D$, the boundary value problem

$$\Delta u + F(u) = h(x),\ x \in D;\ \ u(x) = 0,\ x \in \partial D;$$

has a unique solution.

For other results on elliptic boundary value problems using the ideas of Section 2, see Cesari [10].

8. Admissibility. In this section, we indicate in a few remarks how the important concept of admissibility as defined by Antosiewicz [2] subsumes the ideas of Section 2. Suppose X, Y are Banach spaces, F is a vector space with at least the structure of a complete topological vector space, G is a subspace of F with at least the structure of a Banach space with the topology of G being finer than that induced by F. Let $u: X \to F$ be a homomorphism, $v: E \to F$ is a continuous, linear transformation, $w: G \to Y$ is a continuous transformation. The first problem is to find an $x \in X$, $g \in G$, such that

$$g = u(x) + v \circ w(g). \tag{8.1}$$

We say (Y,G) is admissible (relative to the pair of linear transformation (u,v) if, for any $y \in Y$, there is an $x \in X$ such that $u(x) + v(y) \in G$. An application of the closed graph theorem gives

Lemma 8.1. If (Y,G) is admissible, there exists a $\mu > 0$ and for each $y \in Y$ a point $x \in X$ such that $u(x) + v(y) \in G$ and

$$\mu\|u(x) + v(y)\| \leq \|y\|.$$

Lemma 8.1 is basic for the discussion of the existence of solutions of the nonlinear problem (8.1) when w is small in some sense. More specifically, for any given $g \in G$, one can find an

$x \in X$ such that $H(g) = u(x) + v \circ w(g) \in G$ and $H: G \to G$ will be continuous. If one can find a subset G_o of G such that $H: G_o \to G_o$ and H has the fixed point property, then there will be a solution of (8.1) in G_o. A result of this type is given by Antosiewicz [2]. Applications of this idea to the theory of asymptotic behavior of solutions of ordinary differential equations may be found in Hartman and Onuchic [1]. Implications in ordinary differential equations of an hypothesis of admissibility on the behavior of the solutions of a homogeneous equation may be found in Massera [1], Massera and Schäffer [1].

It is easily seen that the problem of Section 2 is a special case of the general problem considered here. Suppose X, Z are Banach spaces A, N as in Section 2, $\mathfrak{N}(A) = X_U$, $\mathscr{R}(A) = Z_E$. Then $Ax = Nx$ is equivalent to the system

$$\begin{aligned} &\text{a)} \quad x = Ux + MENx \\ &\text{b)} \quad (I-E)Nx = 0. \end{aligned} \tag{8.2}$$

If $Y = \mathscr{R}(A)$, $X = G = F$, $u = U$, $v = M$, $w = EN$, then the set (Y,X) is admissible by construction and the first equation (8.2a) is a special case of (8.1).

The concept of admissibility is a sweeping generalization of the Fredholm alternative and will continue to have significant applications to differential equations.

REFERENCES

ANTOSIEWICZ, H. A. [1] Boundary value problems for nonlinear ordinary differential equations. Pac. J. Math 17(1966), 191-197.

_____[2] Un analogue du principe du point fixé de Banach. Annali di Mat. Pura Appl. (4)74, 61-64.

_____[3] Newton's method and boundary value problem. J. Comput. System Sci. 2(1968), 177-202.

BANCROFT, S., HALE, J. K. and D. SWEET [1] Alternative problems for nonlinear functional equations. J. Differential Eqns. 4(1968), 40-56.

BARTLE, R. G. [1] Singular points of functional equations. Trans. Am. Math. Soc. 75(1953), 366-384.

BASS, R. W. [1] Equivalent linearization, nonlinear circuit analysis and the stabilization and optimization of control systems. Proc. Symp. Nonlinear Circuit Anal. 6(1956).

_____[2] Mathematical legitimacy of equivalent linearization by describing functions. IFAC Congress 1959.

CESARI, L. [1] Existence theorems for periodic solutions of nonlinear Lipschitzian differential systems and fixed point theorems. Contributions the theory of nonlinear oscillations, 5, 1960, 115-172.

_____[2] Existence theorems for periodic solutions of nonlinear differential systems. Boletin Soc. Mat. Mexicana 1960, 24-41.

_____[3] Asymptotic behavior and stability problems in ordinary differential equations, Springer Verlag, 2nd ed. 1963, viii + 271.

_____[4] Functional analysis and Galerkin's method, Michigan Math. J. 11, 1964, 385-418.

_____[5] Functional analysis and periodic solutions of nonlinear differential equations, Contributions to differential equations, Wiley 1, 1963, 149-187.

_____[6] Periodic solutions of hyperbolic partial differential equations. Nonlinear Differential Equations and Nonlinear Mechanics, Academic Press 1963, 33-57.

_____[7] A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations. Rend. Circolo Mat. Palermo (2) 14, 1965, 95-118.

_____[8] Existence in the large of periodic solutions of hyperbolic partial differential equations. Archive Rat. Mech. Anal. 20, 1965, 170-190.

_____[9] Smoothness properties of periodic solutions in the large of nonlinear hyperbolic differential systems. Funkcialaj Etvacioj 9, 1966, 325-338.

_____[10] A nonlinear problem in potential theory. Michigan Math. J., 16(1969), 3-20.

_____[11] Sulla stabilita delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici. Mem. Accad. Italia (6) 11, 1940, 633-692.

_____[12] Functional analysis and differential equations, p. 143-155. Advances in Differential and Integral Equations. SIAM Studies in Applied Math., No. 5(1969).

CRONIN, J. [1] Branch points of solutions of equations in a Banach space. Trans. Am. Math. Soc. 69(1950), 208-231.

FRIEDRICHS, K. [1] Special topics in analysis. Lecture Notes, New York University, 1953-1954.

GOLOMB, M. [1] On the reducibility of certain linear differential systems. J. Reine Angew. Math. 205(1960-1961), 171-185.

GRAVES, L. M. [1] Remarks on singular points of functional equations. Trans. Am. Math. Soc. 79(1955), 150-157.

HALE, J. K. [1] Periodic solutions of nonlinear systems of differential equations. Riv. Mat. Univ. Parma 5(1954), 281-311.

_____[2] Oscillations in Nonlinear Systems. McGraw-Hill, New York, 1963.

_____[3] Periodic solutions of a class of hyperbolic equations containing a small parameter. Arch. Rat. Mech. Anal. 23(1967),

_____[4] Ordinary Differential Equations. Wiley-Interscience, 1969.

HALL, W. S. [1] On the existence of periodic solutions for the equations $D_{tt}u + (-1)^p D_x^{2p} u = \epsilon f(\cdot,\cdot,u)$. J. Differential Eqns. 7(1970), 509-526.

_____[2] Periodic solutions of a class of weakly nonlinear evolution equations. Arch. Rat. Mech. Anal. 39(1970), 294-322.

HARRIS, W. A., JR. [1] Holomorphic solutions of nonlinear differential equations at singular points, pp. 184-187. Advances in Differential and Integral Equations. SIAM Studies in Applied Mathematics, No. 5, 1969.

HARRIS, W. A., JR., SIBUYA, Y. and L. WEINBERG [1] Holomorphic solutions of linear differential systems at singular points. Arch. Rat. Mech. Ana. 35(1969), 245-248.

HARTMAN, P. and N. ONUCHIC [1] On the asymptotic integration of ordinary differential equations. Pac. J. Math. 13(1963), 1193-1207.

KELLER, H. [1] Nonlinear bifurcation. Journal of Diff. Eqns. 3(1970), 417-435.

KELLER, J.B. [1] Bifurcation theory for ordinary differential equations, pp. 17-40; Bifurcation Theory and Nonlinear Eigenvalue Problems. Edited by Keller and Antman. Benjamin, 1967.

KNOBLOCH, H. W. [1] Comparison theorems for nonlinear second order differential equations, J. Differential Equations 1, 1965, 1-24.

KRASNOSELSKII, M. A. [1] Topological Methods in the Theory of Nonlinear Integral Equations. MacMillan, 1964.

LANDESMAN, E.M. and A. LAZER [1] Nonlinear perturbations of linear elliptic boundary value problems at resonance. J. Math. and Mec., 19(1970), 609-623.

LAZER, A. [1] On the computation of periodic solutions of weakly nonlinear differential equations. SIAM J. Appl. Math. 15(1967), 1158-1170.

LAZER, A. and D. Sanchez [1] On periodically perturbed conservative systems. Mich. Math. J., 16(1969), 193-200.

LAZER, A. and D.E. LEACH [1] Bounded perturbations of forced harmonic oscillators at resonance. Ann. Mat. Pura Appl.

LEWIS, D. C. [1] On the role of first integrals in the perturbationof periodic solutions. Ann. Math. 63(1956), 535-548.

_____ [2] Autosynartetic solutions of differential equations. Am. J. Math. 83(1961), 1-32.

LOCKER, J. [1] An existence analysis for nonlinear equations in Hilbert space. Trans. Amer. Math. Soc. 128, 1967, 403-413.

_____ [2] An existence analysis for nonlinear boundary value problems. To appear.

LOUD, W. [1] Periodic solutions of nonlinear differential equations of Duffing type, p. 199-224. Differential and Functional Equations. Benjamin, 1967.

MASSERA, J.L. [1] Sur l'existence de solutions bornees et periodiques des systemes quasi-lineaires d'equations differentielles. Ann. Mat. Pura Appl. 51(1960), 95-105.

MASSERA, J.L. and J.J. SCHAFFER [1] Linear Differential Equations and Functional Analysis. Academic Press,

MAWHIN, J. [1] Equations intégrales et solutions périodique des systèmes différentielles non linéaries. Bull. Acad. R. Belgique, Cl. Sci., (5) 55(1969), 934-947.

_____[2] Degré topologique et solutions périodiques des systèmes differentielles non linéaries. Bull. Soc. R. Sci. Liege 38(1969), 308-398.

_____[3] Periodic solutions of nonlinear functional differential equations. To appear in J. Diff. Eqns., 1971.

NASPARSTEK, A. [1] Periodic solutions of certain weakly nonlinear hyperbolic partial differential equations. A thesis at the University of Michigan, 1968.

NIRENBERG, L. [1] Functional Analysis, Lecture Notes, New York University, 1960-1961.

PERELLO, C. [1] Periodic solutions of differential equations with time lag containing a small parameter. J. of Diff. Eqns. 4(1968), 160-175.

PETROVANNU, D. [1] Solutions periodiques pour certaines equations hyperboliques, Analele Stintifice Iasi, 14(1968), 327-357.

RABINOWITZ, P. H. [1] Periodic solutions of a nonlinear wave equation, in Differential Equations and Dynamical Systems, pp. 69-74, Academic Press, New York, 1967.

SATTINGER, D. H. [1] Stability of bifurcating solutions by Leray-Schauder. To appear.

VAINBERG, M. M. and V. A. TRENOGIN [1] The method of Lyapunov and Schmidt in the theory of nonlinear differential equations and their further development. Mathematical Surveys 17(1962), 1-60.

VEJVODA, O. [1] Periodic solutions of a linear and weakly nonlinear wave equation in one-dimension, I. Czech. Mat. J. 14(89) 1964, 341-382.

_____[2] Periodic solutions of the first boundary value problem for a linear and weakly nonlinear heat equation. Svazek 13(1968), 466-477.

_____[3] Periodic solutions of nonlinear partial differential equations of evolution. Proc. of Conference in Bratislava, 1966, 293-300.

VEJVODA, O. [4] The mixed problem and periodic solutions of a linear and weakly nonlinear wave equation in one dimension. Rozpravy Cesk Acad. Ved. 80(1970), No. 3, 1078.

WILLIAMS, S. [1] A connection between the Cesari and Leray-Schauder methods. Michigan Math. J. 15(1968), 441-448.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*

Brown University

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

NA

3. REPORT TITLE

Applications of Alternative Problems

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(First name, middle initial, last name)*

Jack K. Hale

6. REPORT DATE

May 1971

7a. TOTAL NO. OF PAGES

69

7b. NO. OF REFS

8a. CONTRACT OR GRANT NO.

DA-ARO-D-31-124-71-G12

b. PROJECT NO.

20061102B14C

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

NA

9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

5145.68-M

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

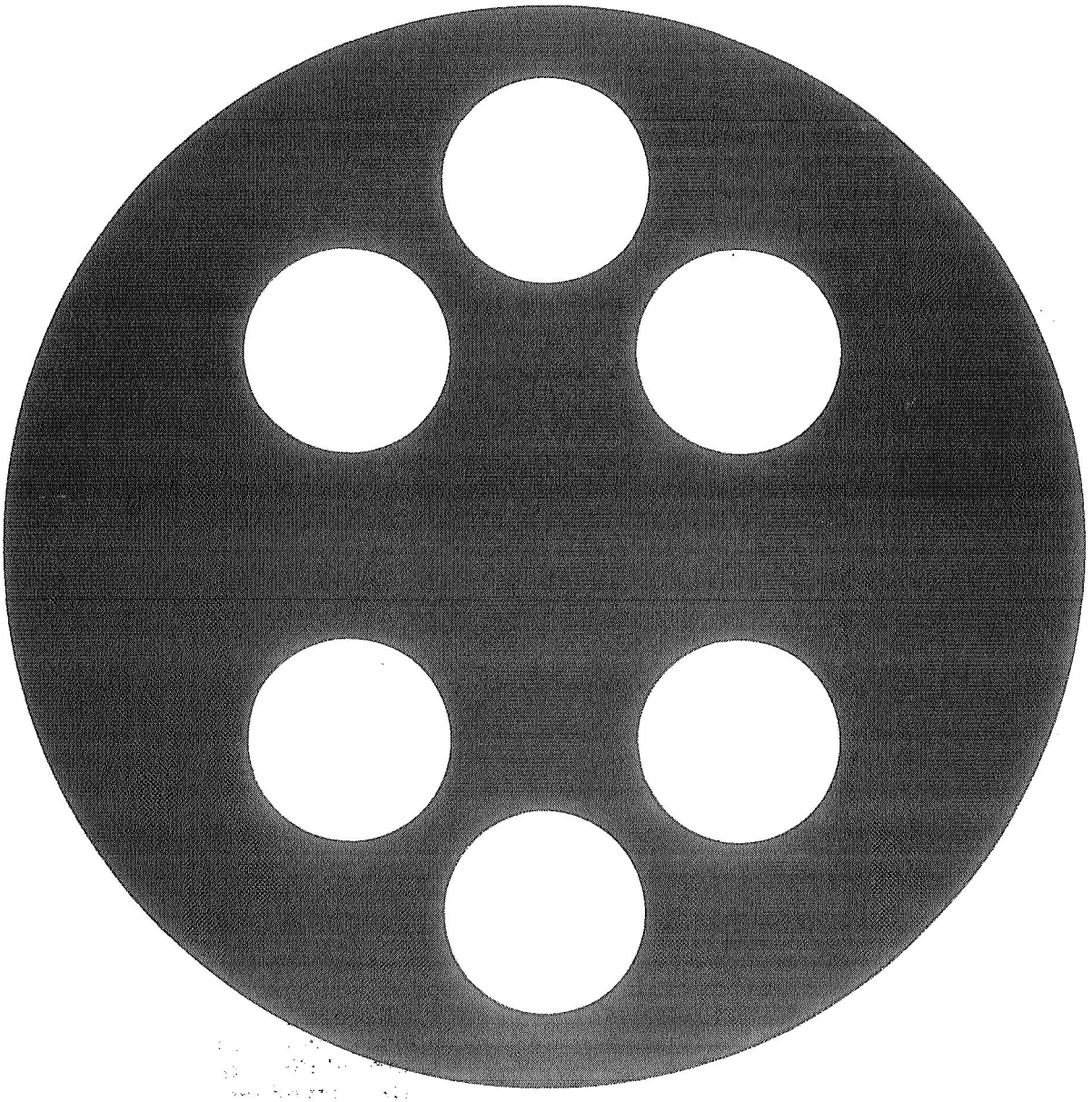
U. S. Army Research Office-Durham
Box CM, Duke Station
Durham, North Carolina 27706

13. ABSTRACT

This report presents a general approach to the solution of functional equations and indicates how to obtain the appropriate functional equations for a variety of applications.

14. KEY WORDS

Analysis (mathematics)
Applications of mathematics

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification