PREDICTIONS ON THE CHARACTERISTICS OF THE MINIMAL TWO-IMPULSE ORBITAL TRANSFER UNDER ARBITRARY TERMINAL CONDITIONS BY USING THE BOUNDING TRAJECTORIES

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The characteristics of the minimal total impulse solution of the two-terminal, two-impulse orbital transfer problems are predicted. The terminal conditions are assumed arbitrary. The concept of bounding trajectories is applied, from which qualitative and quantitative information on the minimal total impulse transfer are deduced without solving the octic equation, which governs the optimal transfer. To verify the predictions, numerical examples are presented.
ACKNOWLEDGEMENT

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NOMENCLATURE

a  semimajor axis, trajectory conic
A  semi-transversal axis, constraining hyperbola
B  semi-conjugate axis, constraining hyperbola
C  center-to-focus distance, constraining hyperbola
C_n  coefficient of the octic equation in $V_C^{**}$ (n = 0 to 8)
d  perpendicular distance, defined in Fig. 2
e  eccentricity, constraining hyperbola
f  total velocity impulse = $f_1 + f_2$
f_i  velocity impulse at terminal $i = |\Delta \dot{V}_i|$ (i = 1,2)
h  angular momentum per unit orbiting mass
k  orbital energy per unit orbiting mass
K  terminal parameter, defined by Eq. (10)
 l  distance between two terminal points
M_i, N_i  orthogonal projection of a terminal velocity vector on the $V_{Ri}$ and $V_C$ axes (see Fig. 3)
m_i, n_i  $M_i/\sqrt{L_i}$, $N_i/\sqrt{L_i}$
n  terminal distance ratio = $r_2/r_1$
n_c  terminal distance ratio, satisfying the coincidence condition, Eq. 15
P_i  terminal parameter, defined by Eq. (9)
R_n  coefficient of the octic equation in $V_{R**}$ (n = 0 to 8)
r  radial distance
r  semilatus rectum
r  position vector
center-to-vertex distance, evolute Lamé, defined in Fig. C-1

time

speed

velocity

velocity-increment vector at terminal i, $= |\mathbf{\dot{V}}_i - \mathbf{\dot{V}}_{01}|$

an unspecified trajectory parameter

direction angle of $\delta \mathbf{V}_i$ with reference to the normal of transfer plane, defined in Fig. 2

direction angle of $\delta \mathbf{V}_i$ with reference to local radial direction, defined in Fig. 2

direction angle of $\delta \mathbf{V}_i$ with reference to local transversal direction, defined in Fig. 2

directional eccentricity, trajectory conic

eccentricity vector, trajectory conic

true anomaly

gravitational strength of the Newtonian field

nondimensional speed $= V/\sqrt{\mu}$

included angle between $\hat{e}_C$ and $\hat{e}_{ri}$ (see Fig. C-1)

included angle between $\hat{e}_d$ and $\hat{e}_{\theta}$ (see Fig. C-1)

path angle, relative to local horizon

base angle, defined in Fig. 1

central angle, or angle of separation of two terminal position vectors, defined in Fig. 1
Subscripts
0 \quad \text{terminal orbit}
1, 2 \quad \text{terminal points}
i, j \quad \text{terminal index}
* \quad \text{stationary 1-impulse solution}
** \quad \text{minimal 2-impulse solution}
*j \quad \text{stationary 1-impulse (at terminal j) solution}
i*j \quad \text{quantity evaluated at terminal i, pertaining to stationary 1-impulse (at terminal j) solution}
i** \quad \text{quantity evaluated at terminal i, pertaining to minimal 2-impulse solution}
L, H \quad \text{low and high trajectories of a bounding pair}
C, R \quad \text{chordal and radial directions}
r, \theta \quad \text{radial and transversal directions}
P, N \quad \text{inplane and out-of-plane components}
\chi, \zeta \quad \text{directions along the interior and exterior base angle bisectors, defined in Fig. 4}

Superscripts
\quad \text{unrealistic}
* \quad \text{critical, or parabolic}
T \quad \text{transpose}

Unit Vectors
\hat{e}_c \quad \text{in chordal direction}
\hat{e}_d \quad \text{normal to } \hat{e}_c, \text{in transfer plane}
\hat{e}_N \quad \text{normal to transfer plane, defined in Fig. 2}
\hat{e}_{rl} \quad \tilde{r}_1/r_1
\[ \dot{e}_{r_2} = -\frac{r_2}{r_2} \]

\[ \dot{e}_\theta \quad \text{in transversal direction} \]

**Special Notations**

- **E** elliptic
- **H** hyperbolic
- **P** parabolic
- **S** simple region, hodograph plane (see Figs. 5 and D-1)
- **N** nonsimple region, hodograph plane (see Figs. 5 and D-1)
- **ST** short transfer
- **LT** long transfer
- **Q** tip of projection of velocity vector on the transfer plane
- **Q_{i*}** orthopoint, on Constraining hyperbola for terminal i
- **Q_{i*a}** orthopoint, corresponding to 1st (absolute) minimal 1-impulse (at terminal i) solution
- **Q_{i*b}** orthopoint, corresponding to maximal 1-impulse (at terminal i) solution
- **Q_{i*c}** orthopoint, corresponding to 2nd minimal 1-impulse (at terminal i) solution
- **Q_{i*d}** orthopoint, corresponding to 3rd minimal 1-impulse (at terminal i) solution
- **T_{*j}** transfer trajectory with stationary impulse at terminal j
- **T_{*ja}** transfer trajectory with absolute minimal impulse at terminal j
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T^*_{jb}$</td>
<td>transfer trajectory with maximal impulse at terminal $j$</td>
</tr>
<tr>
<td>$T^*_{jc}$</td>
<td>transfer trajectory with 2nd minimal impulse at terminal $j$</td>
</tr>
<tr>
<td>$T^*_{jd}$</td>
<td>transfer trajectory with 3rd minimal impulse at terminal $j$</td>
</tr>
<tr>
<td>$T^{**}$</td>
<td>minimal 2-impulse transfer trajectory</td>
</tr>
</tbody>
</table>
I. Introduction

The transfer between two space orbits by applying two terminal impulses under specified terminal conditions is a problem of both theoretical and practical interest in the fuel-optimal space maneuvers. The problem is to determine the optimal transfer trajectory so that the sum of the two terminal impulses is a minimum.

Investigations of the optimal two-impulse orbital transfer problem were first done by Hohmann,\(^1\) and analytical foundations of such investigations were mostly attributed to Lawden's work.\(^2\),\(^10\) The 2-terminal, 2-impulse transfer problem, a particular case of Lawden's more general problem, was first formulated and treated by Vargo,\(^3\) and later investigated by many contemporary authors. Among the previous work done on this problem, Altman and Pistiner\(^12\) established an eighth degree polynomial equation governing the optimization, which formed the basis for much of the current development, and a similar equation was also given by Lee.\(^16\) The octic equation was later reformulated in symmetric velocity coordinates and studied under broad terminal conditions by the author.\(^25\) As a result of such investigations, one beware of the following possible complications in the solution of the problem:

1. The presence of extraneous roots of the octic equation, which do not belong to the extremal impulse solution.
2. An extremal impulse solution of the octic may give a maximal total impulse instead of a minimal one.

3. There may exist more than one local minimal total impulse solution.

4. The arising of an unrealistic optimal solution, that is, a solution resulting in a transfer trajectory which leads to the final terminal via infinity.

In view of these possible complications, the determination of a realistic absolute minimal 2-impulse solution from the octic equation presents a formidable task, involving many pitfalls, in both theoretical analysis and numerical computations. In the author's previous work, \(^{25}\) instead of using an algebraic approach to the octic equation, a geometric approach in the velocity space is adopted, and some of the vital questions concerning the solutions were answered, and several necessary or sufficient conditions were derived. Based on this preliminary study, the present paper intends to give a systematic prediction on the characteristics of the minimal 2-impulse solution under various terminal conditions by using the bounding trajectories, a concept first introduced in Ref. 25. It will be seen that, by the proper choice of a bounding trajectory pair, a great deal of information on the minimal 2-impulse solution, qualitative and quantitative, may be obtained without solving the octic equation, and this information may in turn help to locate the optimal solution in numerical computation.
II. Formulation in Symmetric Velocity Coordinates

Let the terminal conditions be specified by the state coordinates \((\vec{r}_1, \vec{v}_{01})\) and \((\vec{r}_2, \vec{v}_{02})\) at the initial and final terminal points respectively, the problem is to minimize the total velocity impulse

\[ f = f_1 + f_2 \]  

where

\[ f_i = |\Delta \vec{V}_i| = |\vec{V}_i - \vec{V}_{0i}| \quad (i = 1, 2) \]  

and \(\vec{V}_i\) is the terminal velocity required for the transfer (Figures 1, 2). Resolving into the oblique velocity components along the terminal radial direction and the chordal direction (Figure 3), Godal's Compatibility conditions\(^4\) enable one to write

\[ \vec{V}_1 = V_c \hat{e}_c + V_r \hat{e}_{rl} \]

\[ \vec{V}_2 = V_c \hat{e}_c + V_r \hat{e}_{r2} \]  

where the velocity coordinates \(V_c\) and \(V_r\) are connected by

\[ V_c V_r = \frac{\mu}{d} \tan \frac{\psi}{2} \]  

The central angle \(\psi\) and the distance \(d\), as defined in Figure 3, are completely determined by the position vectors \(\vec{r}_1\) and \(\vec{r}_2\), which are assumed to be noncollinear, that is, \(0 < \psi < \pi\). The coordinate pair \((V_c, V_r)\) is known as the symmetric velocity coordinate pair in view of Equations (3).
FIG. 1 GEOMETRY OF 2-TERMINAL TRANSFER IN SPACE
FIG. 2 GEOMETRY OF VELOCITY VECTORS, 2-IMPULSE TRANSFER
FIG. 3 THE TRANSFER TRAJECTORY AND THE IN-PLANE VELOCITY COMPONENTS
and (4) which hold for all transfer trajectories between the two terminal points.

The analytic condition governing the optimal transfer is given by

\[ df_1 + df_2 = 0 \]  

(5)

which, after performing the differentiation together with Equations (2), (3) and (4), yields the two polynomial equations, known as the stationarity octics

\[ \sum_{n=0}^{8} C_n v_c^{n*} = 0 \]  

(6C)

\[ \sum_{n=0}^{8} R_n v_r^{n*} = 0 \]  

(6R)

where the coefficients \( C_n \) and \( R_n \) are functions of the following terminal parameters:

\[ M_{0i} = \vec{V}_{0i} \cdot \vec{e}_{ri} \]  

(7)

\[ N_{0i} = \vec{V}_{0i} \cdot \vec{e}_c \]  

(8)

\[ P_{0i} = V_{0i}^2 - 2K \cos \theta_i \]  

(9)

\[ K = \frac{\mu}{d} \tan \frac{\psi}{2} \]  

(10)

Thus we may write

\[ C_n = C_n (M_{01}, M_{02}, N_{01}, N_{02}, P_{01}, P_{02}, K) \]  

(11)

\[ R_n = R_n (M_{01}, M_{02}, N_{01}, N_{02}, P_{01}, P_{02}, K) \]

For fixed terminal conditions all these coefficients are constants, and Equations (6C) and (6R) define a pair of optimal...
values of $V_{C**}$ and $V_{R**}$ for an internal extremum of $f$. Derivations of the stationarity octics and the explicit forms of Equations (11) are given in Appendix A, and formulas for the transfer trajectory parameters in terms of the symmetric velocity coordinates $V_C$ and $V_R$ are summarized in Appendix B.

As shown in Reference 25, the minimal 2-impulse transfer trajectory $T_{**}$, defined by Equation (5), is bounded between the two transfer trajectories, $T_{*1}$ and $T_{*2}$, defined by

$$\text{df}_1 = 0 \quad \text{and} \quad \text{df}_2 = 0$$

(12)

respectively. In terms of the coordinate $V_C$, Equation (12), together with the constraint Equation (4), yields the two fourth degree equations

$$V_{C*1}^4 - N_{01}V_{C*1}^3 + K M_{01}V_{C*1} - K^2 = 0$$

(13C-1)

$$V_{C*2}^4 - N_{02}V_{C*2}^3 + K M_{02}V_{C*2} - K^2 = 0$$

(13C-2)

known as the stationarity quartics, one for each terminal. Similar equations may be written in terms of the variables $V_{R*1}$ and $V_{R*2}$. The trajectories $T_{*1}$ and $T_{*2}$, defined by Equations (13C-1, 2), have the physical significance of being the transfer trajectories between the same two terminal points $Q_1$ and $Q_2$, with stationary velocity impulses at $Q_1$ and $Q_2$ respectively, hence they will be referred to as the stationary 1-impulse transfer trajectories. Analytic studies of Equations (13C-1, 2) show that each quartic has at least two and at most four
real roots, depending on the terminal conditions. In other words, each stationarity quartic may yield two to four distinct stationary 1-impulse transfer trajectories. The choice of such trajectories for the bounding pair will be postponed until the transfer geometry in the velocity space is studied.
III. Preliminaries on the Two-Terminal Transfer

Based on the geometric studies of two-terminal transfers in the position and velocity spaces, some previously developed concepts and terminology which form the background of the present investigation will now be briefly given below.

A. On the Constraining Hyperbola

1. The tip of the transfer velocity vector at each terminal required for a 2-terminal transfer, is confined in the hodograph plane on a hyperbola, defined by Godal's compatibility condition, Eq. (4). Such a hyperbola is called the constraining hyperbola for the terminal velocity, and there is one for each terminal. The geometry of each constraining hyperbola is completely determined by the two position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Characteristics of the constraining hyperbola, and its principal geometric elements are summarized in Appendix C.

2. Each constraining hyperbola consists of two branches:

   the positive branch: $V_C > 0$, $V_R > 0$, associated with short transfers;

   the negative branch: $V_C < 0$, $V_R < 0$, associated with long transfers.
The positive branches of the two constraining hyperbolas constitute a short transfer pair, while the two negative branches, a long transfer pair (See Figure 4). The half-plane \( (V_r > 0) \) in which the positive branch lies will be designated as the positive half-plane, and that \( (V_r < 0) \) in which the negative branch lies, the negative half-plane.

3. All solution points in the hodograph plane for the two-terminal transfers, optimal or nonoptimal, are necessarily confined on the constraining hyperbolas. The solution point \( (Q_1) \) for the initial terminal velocity and its corresponding point \( (Q_2) \) for the final terminal velocity form a pair of transfer points. The line connecting a transfer point pair bisects the angle of separation \( \psi \).

4. The type of the transfer conic will be elliptic, hyperbolic, or parabolic according as the transfer point \( Q_1 \) lies inside, outside, or on the critical circle, \( V = V^* \), in the hodograph plane. Thus, each branch of the constraining hyperbola is divided by the critical circle into two portions: the elliptic portion and the hyperbolic portion as shown in Figure C-1, Appendix C. The points of intersection of the hyperbola and the critical circle are the critical points corresponding to parabolic transfers. The hyperbolic portion, including its end point, the critical point, in the half-plane, \( V_\chi > 0 \), is the unrealistic portion since it corresponds to unrealistic transfer trajectories.
FIG. 4 THE TRANSFER PAIR OF CONSTRAINING HYPERBOLAS
B. On the Stationary One-Impulse Transfer and the Orthopoints

1. Geometrically, the stationarity quartic, based on Equations (12) expresses the condition of orthogonality\(^8,^{20}\)

\[ \Delta \vec{V}_i \cdot \vec{dV}_i = 0 \]  

(14)

It follows that, when a terminal velocity \( \vec{V}_{0i} \) is prescribed, each solution point for the stationary one-impulse transfer is given by the foot of the normal drawn from the point \( Q_{0i} \), the projection of the tip of \( V_{0i} \) in the hodograph plane, to the constraining hyperbola. Such a point is called the orthopoint with respect to the fixed point \( Q_{0i} \), and is designated as \( Q_{i*} \). Hence each real root of the stationarity quartic corresponds to one orthopoint on the constraining hyperbola, and vice versa.

2. As each stationarity quartic may have two to four real roots, the number of orthopoints for a given terminal velocity point \( Q_{0i} \) range from two to four. Previous studies\(^{20}\) show that these orthopoints follow a general pattern as follows:

<table>
<thead>
<tr>
<th>Orthopoint Designation</th>
<th>Nature of ( f_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_{i*a} )</td>
<td>1st minimum, absolute.</td>
</tr>
<tr>
<td>( Q_{i*b} )</td>
<td>maximum.</td>
</tr>
<tr>
<td>( Q_{i*c} )</td>
<td>2nd minimum, local.</td>
</tr>
<tr>
<td>( Q_{i*d} )</td>
<td>3rd minimum, local.</td>
</tr>
</tbody>
</table>
Here the points $Q_{i*b}$ and $Q_{i*c}$ may be coinciding or missing in the real plane, depending on the location of $Q_0i$. We may speak of the orthopoint as elliptic, parabolic, or hyperbolic, and realistic or unrealistic, according to the nature of the portion of the constraining hyperbola on which it locates.

3. The hodograph plane may be divided into different regions for the terminal velocity point $Q_0i$ according to the number and nature of the orthopoints associated with it (See Figure 5).

The simple and nonsimple regions are separated by the evolute of the constraining hyperbola, which is a form of Lamé as follows:

<table>
<thead>
<tr>
<th>Region</th>
<th>Designation</th>
<th>Orthopoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple</td>
<td>S</td>
<td>2, one on each branch.</td>
</tr>
<tr>
<td>Nonsimple</td>
<td>N</td>
<td>4, three on the nearer branch</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(Q_{i<em>a}, Q_{i</em>b}, Q_{i*c})$, and one</td>
</tr>
<tr>
<td></td>
<td></td>
<td>on the other $(Q_{i*d})$</td>
</tr>
</tbody>
</table>

On the boundary two of the three points on the same branch coincide, $Q_{i*b} = Q_{i*c}$, where $f_i$ is neither minimum nor maximum; and at each vertex of the boundary, all three points on the same branch, $Q_{i*a}$, $Q_{i*b}$ and $Q_{i*c}$ coincide, with absolute minimum $f_i$. Typical distributions of the orthopoints are shown in Figure D-1, Appendix D (where the terminal subscript $i$ has been omitted for simplicity).
FIG. 5 THE HODOGRAPH REGION DIAGRAM
The realistic and unrealistic regions are partitioned according as the first minimal point, \( (Q_i^*) \), hence its associated trajectory, is realistic or unrealistic, as shown in Figure 5. The bounding lines consist of the critical lines, which are the normal lines through the critical points in the half-plane \( V_x > 0 \), and portions of the \( V_\zeta \) - axis.

The realistic region may be further divided into a number of subregions for the point \( Q_{0i} \) according to the types of the trajectories associated with the orthopoints as follows:

<table>
<thead>
<tr>
<th>Subregion</th>
<th>Designation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Double Elliptic</td>
<td>EE</td>
</tr>
<tr>
<td>Hyperbolic-Elliptic</td>
<td>HE</td>
</tr>
<tr>
<td>Double Hyperbolic</td>
<td>HH</td>
</tr>
</tbody>
</table>

Here the first letter indicates the type of the trajectory associated with \( Q_i^*a \), and the second letter, that associated with \( Q_i^*d \). The points \( Q_i^*b \) and \( Q_i^*c' \), if they exist, and their associated trajectories will be of the same type as that of \( Q_i^*a' \) or \( Q_i^*d \). On the critical lines, at least one of the trajectories is parabolic.

Likewise, the unrealistic region may be further divided as follows:

<table>
<thead>
<tr>
<th>Subregion</th>
<th>Designation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single Unrealistic</td>
<td>H'E</td>
</tr>
<tr>
<td>Double Unrealistic</td>
<td>H'H'</td>
</tr>
</tbody>
</table>
Here the same convention of designations used for the realistic subregions is adopted, with the superscript ' indicates unrealistic transfer.

All the foregoing divisions of hodographic regions apply, of course, to either terminal point. For details, see Appendix D.
IV. The Bounding Trajectories for the Minimal Two-Impulse Transfer

A. The Optimal Transfer Arc Pair

Assume the terminal velocity point $Q_{0i}$ is fixed, and let the transfer point $Q_1$ move along the constraining hyperbola. For convenience we designate the hyperbolic arc as positive or negative according as the distance $Q_{0i}Q_1 (=f_i)$ is increasing or decreasing as $Q_1$ moves from left to right. Evidently, the arc will change sign only when $Q_1$ passes through an orthopoint. The stationarity condition expressed by Equation (5) indicates clearly that the two-impulse optimal solution must locate on a transfer pair of arcs of opposite signs. The essential types of such arc pairs are shown in Figure 6.

In type (A) the endpoints of the arc pair are the minimal orthopoints, one on each arc, together with their cotrajectory points. They may be either $Q_{i*a}, Q_{i*c},$ or $Q_{i*d}$. It is assumed that no other orthopoints exist on either arc between its endpoints. On such an arc pair there is one and only one local minimal solution.†

Type (B) is a variation of type (A). It contains a maximal orthopoint on one of the arcs between its endpoints. Analytic studies show that there is either one local minimal solution and one local maximal solution on the arc pair, or none.† If such solutions exist, they

† See Appendix E
ONE LOCAL MIN. AND ONE LOCAL MAX, OR NONE.

ONE LOCAL MIN, (REALISTIC OR UNREALISTIC.)

ONE LOCAL MAX.

FIG. 6 TYPICAL PAIRS OF THE OPTIMAL TRANSFER ARCS
will actually locate on the subarc pairs defined by the
two orthopoints, one minimal and one maximal, on the same
arc.

Type (C) is another version of type (A), wherein one of
the minimal orthopoints is unrealistic, and the arc pair is
defined on the righthand side (Figure 6) by the unrealistic
critical point pair. The two-impulse minimum then may be
either realistic or unrealistic. In the latter case the
realistic optimal solution will be indefinite, given by an
arbitrary point pair on the arc pair, close to the
unrealistic critical point pair.20

In type (D) the arc pair is defined by the maximal
orthopoints, one on each arc, together with their cotra-
jectory points. It contains one maximal solution only, but
no minimal solution.†

Thus, in order to locate the two-impulse minimal
solution it is only the arc pairs of type (A) and its two
variations (B) and (C) which need to be examined. The
exclusion of the arcs of the same sign automatically
prevents the entering of the extraneous roots of the
stationarity octic, if any; and the exclusion of the arc
pair of type (D) further prevents the entering of the
maximal solution. Consequently the problem narrows down
to searching the absolute minimal solution on the arc
pairs of types (A), (B), and (C), where the local minimal
solutions are located.

†See Appendix E
Since each optimal arc pair is essentially defined by two orthopoints, one at each end, together with their cotrajectory points, we may specify such an arc pair by giving the two orthopoints as its coordinates, e.g., 

\[(Q_1{\ast}a, Q_2{\ast}d)\] is a typical optimal arc pair, which may also be written more compactly as \((a, d)\). By ignoring the order of the terminal points, we may regard the arc pairs \((a, d)\) and \((d, a)\) as of the same combination \((ad)\). Evidently, optimal arc pairs of the basic types (A) may have the following six combinations:

\[(aa), (ad), (dd), (ca), (cc), (cd).\]

By associating \(b\) with one of the endpoints, \(a\) or \(c\), we obtain the combinations for the arc pairs of the type (B). There are also six such combinations; namely,

\[(ab-a), (ab-c), (ab-d), (cb-a), (cb-c), (cb-d).\]

Arc pairs of type (B) and the last three combinations of type (A) would not be possible unless one or both of the terminal velocity points, \(Q_{01}\) and \(Q_{02}\) are in the nonsimple regions, of course. By replacing any one of the orthopoints by an unrealistic critical points as one endpoint, we obtain the optimal arc pairs of type (C).

As regards to the selection of the optimal arc pair for the absolute minimum, no rigorous rules are available at present. However, the following observations may serve as a guide:
1. When an optimal arc pair of the combination (aa) appears, the two absolute 2-impulse minimum are most likely on that pair.

2. The local minimum provided by the arc pair (dd) is usually not absolute.

Thus, to locate the absolute 2-impulse minimum we first look for the arc pair (aa). The arc pair (dd), if it exists, may usually be ignored. In the absence of arc pairs of the combination (aa) and (dd), or there is any doubt, one may always resort to the computation of all the local minimal solutions and comparison, of course.

B. The Bounding Trajectory Pair

Associated with each optimal arc pair there are two transfer trajectories, one corresponding to each endpoint pair. The existence of an interior minimum for the two-impulse transfer on such an arc pair shows that the minimal two-impulse transfer trajectory, denoted by T**, is actually bounded between the two bounding trajectories, hence the term "bounding trajectory pair". It will be shown that T** is not only bounded by such a trajectory pair in the position space, but also in the velocity space and many other parameter spaces. Thus essential information on the characteristics of the two-impulse minimum may be obtained by examining its bounding trajectory pair.

Since an endpoint pair of the optimal arc pair consist of basically one orthopoint and its cotrajectory point, a
Bounding trajectory is in general a stationary trajectory with respect to the velocity impulse at one of the terminals. In the special case wherein one of the endpoint pair is critical and unrealistic, the corresponding trajectory is the unrealistic parabola, which itself is unbounded in the position space; nevertheless, it may serve as a bounding trajectory. Designations of the bounding trajectories are made in accordance with the endpoints they associate with as follows:

<table>
<thead>
<tr>
<th>Endpoint</th>
<th>Bounding Trajectory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q&lt;sub&gt;i*a&lt;/sub&gt;</td>
<td>T&lt;sub&gt;*ia&lt;/sub&gt;, 1st minimal (abs.) one-impulse transfer</td>
</tr>
<tr>
<td>Q&lt;sub&gt;i*b&lt;/sub&gt;</td>
<td>T&lt;sub&gt;*ib&lt;/sub&gt;, maximal one-impulse transfer</td>
</tr>
<tr>
<td>Q&lt;sub&gt;i*c&lt;/sub&gt;</td>
<td>T&lt;sub&gt;*ic&lt;/sub&gt;, 2nd minimal one-impulse transfer</td>
</tr>
<tr>
<td>Q&lt;sub&gt;i*d&lt;/sub&gt;</td>
<td>T&lt;sub&gt;*id&lt;/sub&gt;, 3rd minimal one-impulse transfer</td>
</tr>
<tr>
<td>Q&lt;sub&gt;i*'&lt;/sub&gt;</td>
<td>T&lt;sub&gt;*i'&lt;/sub&gt;, unrealistic parabolic transfer</td>
</tr>
</tbody>
</table>

With this designation convention the coordinates specifying an optimal arc pair may now be extended to a bounding trajectory pair. For example, corresponding to the arc pair (a,d), we have the bounding trajectory pair (T<sub>*1a</sub>' , T<sub>*2d</sub>). Consequently, the different combinations previously given for the optimal arc pairs also apply to the bounding trajectory pairs. Thus corresponding to the six possible combinations for the arc pairs of type (A), there are six possible combinations of the bounding trajectory pair. The same can be said about the bounding trajectory pairs associated with the arc pairs of types (B) and (C).
Directly from the previous analysis of the optimal transfer arc pairs, the following observations may now be made:

1. Basically, a bounding trajectory pair is formed by two transfer trajectories under the same terminal conditions, one with a minimal initial velocity impulse, and the other with a minimal final velocity impulse. (Such a trajectory pair will be generally denoted by \((T_1, T_2)\). Subscripts will be added in accordance with the endpoints of the associated transfer arc pair whenever necessary.)

2. A bounding trajectory pair associated with the optimal transfer arc pair of type (A) will bound one and only one local minimal two-impulse transfer trajectory between them; and, in particular,

   (a) A bounding pair \((T_{1a}, T_{2a})\) formed by the two first minimal (absolute) one-impulse transfer trajectories with respect to the initial and final velocity impulses separately usually bounds the absolute minimal two-impulse transfer trajectory;

   (b) A bounding trajectory pair \((T_{1d}, T_{2d})\) formed by the two third minimal one-impulse transfer trajectories with respect to the initial and final velocity impulses separately bounds only a local minimal two-impulse transfer trajectory which is usually not the absolute one.

2. When the optimal arc pair is of the type (B), the bounding pair made of the two minimal one-impulse transfer trajectories may bound one local minimal two-impulse transfer
trajectory, or none. If it does bound one, then there exists a closer bounding pair formed by the two transfer trajectories, one with a minimal velocity impulse, and the other with a maximal velocity impulse, both at the same terminal, e.g. \( (T_{*1a}, T_{*1d}) \).

3. When one of the bounding trajectories is unrealistic (optimal arc pairs of type (C)), the minimal two-impulse trajectory bounded may become indefinite.

Several typical bounding trajectory pairs are illustrated in Figure 7.

It is to be noted that although there appears to be a great variety of the optimal transfer arc pairs and their associated bounding trajectory pairs, they do not all occur frequently. For example, when both terminal velocity points, \( Q_{01} \) and \( Q_{02} \), are in the realistic simple regions, as is usually the case. The pattern of the optimal arc pairs can fall under the following two classes only:

<table>
<thead>
<tr>
<th>Class</th>
<th>One Kind of Transfer</th>
<th>Other Kind of Transfer</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( (a, a) )</td>
<td>( (d, d) )</td>
</tr>
<tr>
<td>II</td>
<td>( (a, d) )</td>
<td>( (d, a) )</td>
</tr>
</tbody>
</table>

In Class I the absolute minimal two-impulse transfer trajectory will likely be bounded by the trajectory pair \( (T_{*1a}, T_{*2a}) \), but unlikely by the pair \( (T_{*1d}, T_{*2d}) \). Hence in this case it is only to the former pair our attention is to be focused. In Class II each of the bounding pairs \( (T_{*1a}, T_{*2d}) \) and \( (T_{*1d}, T_{*2a}) \), one in each kind of transfer, bounds a local minimal two-impulse transfer trajectory in that kind, and in the search of an absolute minimum, the consideration of
FIG. 7  TYPICAL PAIRS OF BOUNDING TRAJECTORIES
of both kinds is then necessary. In either case the number of bounding pairs to be considered is no more than two. Real complications can arise only when one or both of $Q_{01}$ and $Q_{02}$ are in the nonsimple and/or unrealistic regions, wherein more types of the optimal arc pairs may appear, and more bounding trajectory pairs are to be considered. Further discussions will be found in the next section.
V. Qualitative Predictions on the Minimal Two-Impulse Transfer

A. The Kind and Sense of the Transfer

For the transfer between two terminal points separated by a central angle $0 < \gamma < \pi$, there is a definite sense of motion around the field center, associated with each kind of transfer. In the following we will arbitrarily assign a positive sense to the short transfer, and a negative sense to the long transfer. It is clear that the two trajectories of a bounding pair $(T_{*1}, T_{*2})$, as defined in the preceding section, are of the same kind and sense, and so is the minimal two-impulse trajectory $T_{**}$ bounded between them. Thus, whenever a bounding trajectory pair is given, the kind and, hence, the sense of the minimal two-impulse trajectory bounded is fixed. Obviously the kind and sense of a bounding trajectory pair depend only on those of the optimal arc pair, but not on the particular endpoints defining it.

As pointed out in Reference 25, it is interesting to note that, while the sense of the minimal two-impulse transfer always agree with those of the two bounding trajectories, it does not necessarily agree with those of the two terminal orbits even though they have the same sense.†

†When the two terminal orbits are noncoplanar, it is to be understood that the sense of motion of each orbit refer to that of the projection of the orbit on the transfer plane.
This peculiar phenomenon stems from the fact that the stationary one-impulse transfer trajectory does not always agree in sense with the corresponding terminal orbit. The particular case in which two terminal orbits of the same sense call for a minimal two-impulse transfer in the opposite sense is illustrated in Reference 25.

B. Type of the Transfer Conic

A study of the hodograph geometry enables one to establish the following rules for determining the type of the minimal two-impulse transfer conic in terms of the bounding trajectories:

1. \( T_{**} \) will be elliptic if at least one of \( T_{*1} \) and \( T_{*2} \) is elliptic, and none of them is hyperbolic;
2. \( T_{**} \) will be hyperbolic if at least one of \( T_{*1} \) and \( T_{*2} \) is hyperbolic, and none of them is elliptic;
3. \( T_{**} \) will be parabolic if both \( T_{*1} \) and \( T_{*2} \) are parabolic.

Thus, once the bounding trajectory pair is chosen, the type of the minimal two-impulse transfer conic is uniquely determined under the foregoing three conditions. The only ambiguous case is that the bounding trajectory pair consists of one ellipse and one hyperbola, wherein the type of \( T_{**} \) is indeterminate.

The type of each bounding trajectory, \( T_{*i} \), is of course, determined by the terminal conditions. Once the terminal point \( Q_{0i} \) is located in the hodograph plane, the region
diagrams illustrated in Figure 5 will enable one to tell immediately the type of \( T_{*1} \).

Finally, it is to be mentioned that, while the two-impulse minimum always agrees in type with its two bounding trajectories of the same type, it is not necessarily so with the two terminal orbits of the same type. Just like in the case of kind and sense, this stems from the fact that a one-impulse minimal transfer trajectory does not always agree in type with the corresponding terminal orbit, a situation found in Reference 20. Thus, for minimal total impulse, it is possible that two elliptic orbits call for an hyperbolic transfer; and that two hyperbolic orbits, an elliptic transfer.

C. The Realistic and the Unrealistic Transfers

Concerning the nature of the minimal two-impulse transfer, realistic or unrealistic, the following criteria are evident:

1. \( T_{**} \) will be realistic if both \( T_{*1} \) and \( T_{*2} \) are realistic;

2. \( T_{**} \) will be unrealistic if both \( T_{*1} \) and \( T_{*2} \) are unrealistic.

Thus once a bounding trajectory pair is found, the nature of \( T_{**} \) is determined, unless the bounding pair consists of one realistic and one unrealistic, wherein the nature of \( T_{**} \) is not ascertained. The optimal transfer arc pair under Condition 2 actually reduces to one point pair--the
unrealistic critical one; and the two bounding trajectores, \( T*_{1} \) and \( T*_{2} \), both coinciding with the unrealistic parabolic trajectory.

It is to be noted that, while the two-impulse minimum in one kind of transfer is unrealistic, there may exist a realistic minimum in the other kind. Thus, it is sometimes advisable to examine the bounding trajectory pairs in both kinds. This is necessary when the two first minimal one-impulse transfer trajectories, \( T*_{1a} \) and \( T*_{2a} \), are of unlike kinds, for example, the condition under Class II, Section IV-B (last paragraph). In such a case it is quite possible to have one realistic absolute minimum in one kind, and one unrealistic local minimum in the other. The foregoing criteria apply to either kind, of course.

Obviously, the nature of each bounding trajectory is determined by the terminal conditions. For two fixed terminal points, such a determination may be readily made by using the hodographic region diagram in Figure 5 once the terminal velocity point \( QO_{1} \) is located. It is clear from such diagrams that Condition 1 is satisfied for both kinds when \( QO_{1} \) and \( QO_{2} \) are both in their realistic regions; and Condition 2 is satisfied for both kinds when they are both in their double unrealistic regions. In the latter case, there exists no realistic absolute minimum solution of the problem, and the solutions in both kinds are indefinite.
D. The Multiplicity of the Minimal Solutions

By multi-minimum we mean distinct transfer trajectories giving the same local minimal total impulse \( f_{**} \) under the same terminal conditions. Evidently, no multi-minimum in the same kind of transfer can be expected unless there are multiple pairs of bounding trajectories in the same kind for choice, corresponding to the multiple optimal transfer arc pairs of that kind. Thus, a pre-requisite for the occurrence of a multi-minimum of one kind is that at least one of the terminal velocity points, \( Q_{01} \) and \( Q_{02} \), is in its nonsimple region. Although there are six combinations for the optimal arc pairs of the basic type, as given in Section IV-A, studies of the distributions of the orthopoints in the constraining hyperbola show that there can be no more than three different arc pairs of the same kind. Consequently, no multiplicity higher than three can be expected for the same kind of transfer. Details of such studies are given in Appendix F, from which the following assertions may be made:

Concerning Multi-Minimum of the Same Kind.

1. No multi-minimum may arise when both terminal velocity points are in their simple regions.
2. When one and only one of the terminal velocity points is in its nonsimple region, there exists at most a double minimum.
3. No triple minimum can be expected unless both terminal velocity points are in their nonsimple regions.
4. No multiplicity higher than three can be expected under any terminal conditions.

The actual existence of a double minimum of the same kind has been illustrated in Reference 25. However, whether a triple minimum of the same kind actually exist has not been ascertained. A proof of its existence or nonexistence would be of theoretical interest.

Now considering both kinds of transfers, it is evident that a double minimum is possible even when both terminal velocity points are in their simple regions, since there is one local minimum in each kind in this case. Maximum multiplicity will be higher when one or both of the terminal velocity points are in their nonsimple regions. However, as shown in Appendix F, the total number of optimal arc pairs of both kinds cannot exceed four under any fixed terminal conditions. Thus a quadruple minimum of mixed kinds can be expected at most. A study of Appendix F enables one to further assert the following:

**Concerning Multi-Minimum of Mixed Kinds**

1. When both terminal velocity points are in their simple regions, there exists at most a double minimum.

2. When one and only one of the terminal velocity points is in its nonsimple region, there exists at most a triple minimum.
3. When both terminal velocity points are in their nonsimple regions, there exists at most a quadruple minimum.

4. No multiplicity higher than 4 can be expected under any terminal conditions.

As example of a quadruple minimum, consisting of two double minimal, one in each kind, all with the same minimal $f_\star\star$, is shown in Reference 25.

E. The Identical Minimal Two-Impulse and Minimal One-Impulse Solutions

It is obvious that when the two trajectories of a bounding pair becomes coincident, the two-impulse minimal transfer trajectory bounded between will necessarily coincide with them, that is,

$$T_{\star\star} = T_{\star 1} = T_{\star 2}$$

Thus, the minimal two-impulse solution will be identical to the two minimal one-impulse solutions, one with respect to the initial terminal impulse, and the other, the final terminal impulse, when they themselves are identical. This can also be easily seen by referring to the basic governing equations (5) and (12). In fact, the simultaneous validity of any two of the three equations assures the validity of the third one. Thus we conclude:

The coincidence of any two of the three trajectories, $T_{\star 1}$, $T_{\star 2}$ and $T_{\star\star}$ implies the coincidence of all three.

The optimal transfer arc pair now actually reduces to merely
a transfer point pair. The unrealistic case mentioned under Headingle C offers a special example of this case.

An analytic condition for the occurrence of such identical solutions, as deduced in Reference 25, is

\[ K\left( (M_0_2 - M_0_1)^2 - (N_0_2 - N_0_1)^2 \right)^2 = (M_0_2 - M_0_1)(N_0_2 - N_0_1) \]

\[ (M_0_2 N_0_1 - M_0_1 N_0_2)^2 \]  (15)

which may be written symbolically

\[ F(\vec{r}_1, \vec{r}_2, \vec{V}_0_1, \vec{V}_0_2) = 0 \]  (16)

Thus there is a definite relation to be satisfied by the four terminal vectors, \( \vec{r}_1, \vec{r}_2, \vec{V}_0_1 \) and \( \vec{V}_0_2 \) in order that the two-impulse minimization and the one-impulse minimizations at the initial and the final terminals separately will yield the same trajectory. Such a relation will be referred to as the coincidence condition for the two-impulse minimization and the two one-impulse minimizations for the two-terminal transfer. It can be shown that the condition given by Equation (15) is not only necessary, but also sufficient. It is interesting to note that Equation (15) is, in particular, satisfied by

\[ M_0_1 = M_0_2 \] and \[ N_0_1 = N_0_2 \]  (17)

In the case of apside-to-apside transfer, \( M_0_1 = M_0_2 = 0 \), Equation (17) lead to

\[ \frac{V_{op}^2}{V_{op}^1} = \frac{r_2}{r_1} \]  (18)

Thus, a sufficient condition for the coincidence of \( T_{*1}, T_{*2} \) and \( T_{**} \) is that the base triangle determined by the two
position vectors, and the velocity triangle determined by the two in-plane terminal orbit velocities are similar and orthogonal.

Finally, it is to be noted that, in the previous assertion on the coincidence of the three transfer trajectories, $T_{*1}$, $T_{*2}$ and $T_{*\star}$, it has been tacitly assumed that the two terminal impulse functions, $f_1$ and $f_2$, are both differentiable. This assertion and the coincidence condition, Equation (15) all break down when $f_1'$ and $f_2'$ do not both exist. Such a case may be called singular. In a singular case, it is possible to have $T_{*\star}$ coincident with one of $T_{*1}$ and $T_{*2}$, which do not themselves coincide. The special case wherein one of the terminal orbits passes through both terminal points, is a singular one. For example, if the initial terminal orbit also passes through the final terminal point, then this orbit itself is $T_{*1}$, and we may have $T_{*\star} = T_{*1} \perp T_{*2}$. 
VI. Quantitative Predictions on the Minimal Two-Impulse Transfer

So far the predictions have been made on the qualitative basis. Quantitative predictions on the various trajectory variables and elements are now in order. In the following, the upper and lower bounds of these trajectory quantities will be established by using the bounding trajectory pair.

A. The Position Vector

Consider a pair of trajectories in a two-terminal Keplerian trajectory family. It is obvious that the one with a higher initial path angle (with reference to the local transverse direction) will remain higher in radial distance on any intermediate radius vector throughout the trajectory range; for, otherwise, the two trajectories will intersect at least at one intermediate point between the two common terminal points, a fact impossible for two distinct Keplerian conics. Such an observation enables one to classify a pair of bounding trajectories as high and low, and indicate them by the subscripts \( H \) and \( L \) respectively. Thus, instead of \( T_1 \) and \( T_2 \), we write \( T_L \) and \( T_H \). Quantities pertaining to the high, or the low trajectory may be indicated in the same way. Such notations will be employed in the following formulations whenever it is convenient.
The existence of an interior point pair on an optimal transfer arc pair for the minimal two-impulse solution (see Section IV and Appendix E) implies that such a minimal trajectory is bounded between the two trajectories of the bounding pair in the position space. This assertion follows directly from the preceding argument, and will become more clear when we come to the terminal path angles under the next heading. Mathematically, we may express this fact by

\[ r_{*L} < r_{**} < r_{*H} \quad (0 \leq \Delta \theta \leq \psi) \]  

where the three radial distances \( r_{*L}, r_{*H}, \) and \( r_{**} \) are taken along the same radius vector between the two terminal position vectors \( \hat{r}_1 \) and \( \hat{r}_2 \) as shown in Figure 8(a) (where equality signs in the foregoing formula hold only on the terminal radius vectors \( \Delta \theta = 0, \psi \)). However, if they do hold on some intermediate radius vector, they will hold on every such radius vector, and the three trajectories, \( T_{*L}, T_{*H}, \) and \( T_{**} \) will coincide, a case in which the minimal two-impulse solution and the two minimal one-impulse solutions are identical, as presented in Section V-E. This special case will be excluded in the following analysis.

B. The Terminal Quantities

Direction of Departure and Arrival
FIG. 8  THE MINIMAL TWO-IMPULSE TRANSFER TRAJECTORY AND ITS BOUNDING TRAJECTORIES.
Consider a typical optimal transfer arc pair as shown in Figure 8(b), the geometry shows clearly that the three path angles $\phi_{*L}$, $\phi_{*H}$, and $\phi_{**}$ at the initial point satisfy the inequality

$$\phi_{1*L} < \phi_{1**} < \phi_{1*H}$$

which is, in fact, the basis for the Inequality (19). Thus, the high trajectory of a bounding pair has also a high initial path angle, and vice versa. However, at the final terminal point the roles of the high and low trajectories are reversed, and we have

$$\phi_{2*H} < \phi_{2**} < \phi_{2*L}$$

which is also evident from Figure 8(b). It is to be noted that, although the reference here is made to Figure 8(b), in which a transfer arc pair of short kind is shown, Inequalities (20-1,2) hold equally well for the long kind of transfer, if we always measure the path angle $\phi_i$ from the transversal direction in the direction of motion, hence, limiting it to $-\frac{\pi}{2} < \phi_i < \frac{\pi}{2}$ in each kind. These inequalities show that a minimal two-impulse transfer trajectory is bounded by its bounding trajectory pair in the directions of departure as well as arrival.
In view of Godal's compatibility condition, Equation (4), the chordal and radial components of the terminal transfer velocities change monotonically along the constraining hyperbola. Thus, with the aid of Figure 8(b), we deduce

\[ V_{R* L} < V_{R* *} < V_{R* H} \]  \hspace{1cm} (21)

and

\[ V_{C* H} < V_{C* *} < V_{C* L} \]  \hspace{1cm} (22)

From Inequality (22) we further deduce for the transversal component,

\[ (V_{\theta i})* H < (V_{\theta i})* * < (V_{\theta i})* L \]  \hspace{1cm} (i = 1, 2) \hspace{1cm} (23)

since \( V_{\theta i} \) is proportional to \( V_C \). No such simple statement is available for the other component \( V_r \) of the coordinate pair \((V_r, V_\theta)\) as it is more involved. From Inequalities (21 to 23) we see that each of the three terminal transfer velocity components \( V_{R* *} \), \( V_{C* *} \), and \((V_{\theta i})* * \) is bounded between the corresponding components of the bounding trajectory pair. However, this is not always true for the resultant transfer velocities, as it will be seen below.

In dealing with the resultant velocity at either terminal, it is important to note that, for the transfer between two fixed terminal points, there exists an overall minimum velocity at each terminal, given by the minimum energy points, which is the vertex of the branch of the terminal constraining hyperbola (see Table C-2, Appendix C). Thus, it is essential
to distinguish whether the optimal transfer arc contains the minimum energy point or not. A study of the hodograph geometry enables one to deduce that, when the optimal arc contains no minimum energy point,
\[ V_{iL}^* < V_{iH}^* < V_{iH}^* \quad \text{if} \quad V_{iL}^* < V_{iH}^* \quad (i = 1, 2) \quad (24) \]
\[ V_{iL}^* < V_{iH}^* < V_{iL}^* \quad \text{if} \quad V_{iL}^* > V_{iH}^* \]
In case it does contain such a point, we replace the lower bound by \((V_i)_{\text{min}}\), which has the magnitude,
\[ (V_i)_{\text{min}} = A_i = \sqrt{\frac{2\mu}{r_i}} \tan \frac{\psi}{2} \tan \frac{\psi_i}{2} \quad (25) \]

**The Terminal Velocity Impulses**

With reference to Fig. 8(b), it is evident that
\[ (\Delta V_{P1}^*)_{1} < (\Delta V_{P1}^*)_{2} < (\Delta V_{P1}^*)_{2} \quad (26-1) \]
\[ (\Delta V_{P2}^*)_{1} < (\Delta V_{P2}^*)_{2} < (\Delta V_{P2}^*)_{2} \quad (26-2) \]
where the \(\Delta V_p\)'s are the in-plane velocity impulses. Going from these in-plane components to the resultants in the noncoplanar case, we note first that the out-of-plane terminal velocity components, \((V_{ONi})\) if present, do not alter the location of the minimal 2-impulse solution in the hodograph plane; and second, that under fixed terminal conditions, such a component is a constant at each terminal, hence, its effect on each velocity impulse at the same terminal is to increase it by a constant component in accordance with
\[ f_i = \sqrt{(\Delta v_{p_i})^2 + v_{ONi}^2} \]  

Consequently, the preceding inequalities hold also for the resultant velocity impulses at each terminal:

\[ f_{1*1} < f_{1**} < f_{1*2} \]  

\[ f_{2*2} < f_{2**} < f_{2*1} \]  

from which we obtain immediately by addition,

\[ f_{1*1} + f_{2*2} < f** < f_{1*2} + f_{2*1} \]  

Thus, each of the two terminal velocity impulses and their sum required for a minimal 2-impulse transfer are well bounded, with their upper and lower bounds given by the two minimal 1-impulse solutions. In fact, two smaller upper bounds for \( f** \) can be found to be

\[ f*1 = f_{1*1} + f_{2*1} \]  

\[ f*2 = f_{1*2} + f_{2*2} \]  

where \( f*1 \) is the sum of the two terminal impulses required on \( T_{*1} \), and \( f*2 \) that on \( T_{*2} \), since

\[ f** < f*1 \]  

\[ f** < f*2 \]  

by definition. That the quantities \( f*1 \) and \( f*2 \) are both less than the upper bounds in the Inequality (29) can be easily seen since, again by definition, we have

\[ f_{1*1} < f_{1*2} , \quad f_{2*2} < f_{2*1} \]
C. The Trajectory Elements

The Angular Momentum and the Semilatus Rectum

Noting that the angular momentum \( h \) is related to the chordal component \( V_C \) of a terminal transfer velocity by

\[
h = V_C d
\]  

and that the distance \( d \) is a constant for the transfer between two fixed terminal points, we obtain immediately from Inequality (22),

\[
h_{*H} < h_{**} < h_{*L}
\]  

which also implies that

\[
\tilde{r}_{*H} < \tilde{r}_{**} < \tilde{r}_{*L}
\]  

in view of the orbital relation,

\[
\tilde{r} = \frac{h^2}{\mu}
\]  

where \( \tilde{r} \) is the semilatus rectum of the trajectory conic.

The Orbital Energy and the Semimajor Axis

From the Vis Viva Integral,

\[
k = \frac{1}{2}V^2 - \frac{\mu}{r}
\]  

we see that, to compare the orbital energies of different trajectories through the same terminal point, we need only to compare the magnitudes of their velocities. Here again the presence or absence of a minimum energy point in the optimal arc under consideration is of importance, and inequalities similar to those for the transfer velocities may be written.
for the orbital energy as follows:

In the absence of the minimum energy point,

\begin{align}
\kappa_{L} < k_{**} < \kappa_{H} & \quad \text{if } \kappa_{L} < \kappa_{H} \\
\kappa_{H} < k_{**} < \kappa_{L} & \quad \text{if } \kappa_{L} > \kappa_{H}
\end{align}

(38)

In case such a point is present, we replace the lower bounds in the preceding inequalities by \(k_{\text{min.}}\), which, in terms of the terminal parameters, is given by

\[ k_{\text{min.}} = -\frac{2\mu}{r_{1} + r_{2} + \ell} \]

(39)

The semimajor axis \(a\) of a trajectory conic in a given Newtonian field depends only on the orbital energy through the relation,

\[ a = \frac{\mu}{2|k|} \]

(40)

However, while \(k\) changes continuously along a constraining hyperbola, "a" changes discontinuously at the critical point \(Q^{*}\); it also has a local minimum in the elliptic portion at the minimum energy point. Thus, to establish the upper and lower bounds for the semimajor axis of a minimal 2-impulse transfer trajectory, it is essential to examine whether the optimal arc contains a minimal energy point, or a critical point. When both points are absent, we have

\begin{align}
\kappa_{L} < \kappa_{**} < \kappa_{H} & \quad \text{if } \kappa_{L} < \kappa_{H} \\
\kappa_{H} < \kappa_{**} < \kappa_{L} & \quad \text{if } \kappa_{L} > \kappa_{H}
\end{align}

(41)

Whenever the optimal arc contains a critical point, we replace
the upper bounds in the preceding inequalities by $a$. When it contains the minimum energy point alone, we replace the lower bounds by the elliptic minimum $a$, given by

$$a_{\text{min}} = \frac{1}{\kappa} (r_1 + r_2 + \ell)$$

(42)

However, when it contains both points, while we still replace the upper bounds in the preceding inequalities by $a$, care must be taken concerning the lower bound, since an hyperbolic semimajor axis may be well smaller than the elliptic minimum. Thus, in this case, we replace the lower bounds in Inequalities (41) by $a_{\text{min}}$ only when these bounds are greater than $a_{\text{min}}$. The foregoing analysis shows that, while the semimajor axis of $T_{**}$ is bounded when $T_{*1}$ and $T_{*2}$ are both elliptic, or both hyperbolic, or one of them is parabolic, it is not necessary so when one of them is elliptic, and the other is hyperbolic.

The Eccentricity Vector

Like the transfer velocity and orbital energy, there exists in a 2-terminal trajectory family an overall minimum for the numerical eccentricity, given by

$$e_{\text{min}} = \left| \frac{r_1 - r_2}{\kappa} \right|$$

(43)

The point on the constraining hyperbola corresponding to this least eccentric transfer trajectory is called the least eccentricity point, and it can be shown that there is such a
point on each branch of a terminal constraining hyperbola, located as shown in Figure C-1. Thus, to establish the upper and lower bounds for the numerical eccentricity of the minimal 2-impulse transfer trajectory, it is essential to examine whether the optimal arc under consideration contains this least eccentricity point or not. Similar to the inequalities deduced for the terminal transfer velocity and the orbital energy, we have in the absence of the least eccentricity point,

$$
\begin{align*}
\varepsilon_{*L} < \varepsilon^{**} &< \varepsilon_{*H} & \text{if} & \varepsilon_{*L} < \varepsilon_{*H} \\
\varepsilon_{*H} < \varepsilon^{**} &< \varepsilon_{*L} & \text{if} & \varepsilon_{*L} > \varepsilon_{*H}
\end{align*}
$$

(44)

In case the arc contains this least eccentricity point, we replace the lower bounds in the preceding inequalities by $\varepsilon_{\text{min}}$.

Furthermore, a study of the hodograph geometry shows that not only the numerical eccentricity of $T^{**}$ is so bounded, but also the direction of its eccentricity vector which is in the direction of the apsidal axis. Denoting the angle between the eccentricity vector of a transfer trajectory and the terminal position vector $\vec{r}_i$ by $\theta_i$, we have

$$
\begin{align*}
\theta_{i*L} < \theta_{i**} &< \theta_{i*H} & \text{if} & \theta_{i*L} < \theta_{i*H} \\
\theta_{i*H} < \theta_{i**} &< \theta_{i*L} & \text{if} & \theta_{i*L} > \theta_{i*H}
\end{align*}
$$

(i = 1, 2) (45)

Here the $\theta$'s are the true anomalies of the terminal point $Q_i$ measured on the three trajectories, $T^{*L}$, $T^{*H}$, and $T^{**}$ (see Fig. 8). So far as the bounding directions of the eccentricity
vectors are concerned, no reference to the least eccentricity point is necessary.

D. **Time of Flight**

It can be shown that the time of flight for the transfer between two fixed terminal points is a single-valued increasing function of the initial path angle. Thus, directly from Inequality (20-1) we deduce that

$$\Delta t_{*L} < \Delta t_{**} < \Delta t_{*H}$$

(46)

In addition to the few items presented above, the upper and lower bounds of many other trajectory quantities may be deduced in a similar way. However, no such exhaustive analysis will be attempted here. As a final remark, the following situation is worth mentioning:

When the two quantities, say $X_{*1}$ and $X_{*2}$ pertaining to a bounding trajectory pair, $T_{*1}$ and $T_{*2}$, respectively, bound the corresponding quantity $X_{**}$ of the minimal 2-impulse trajectory $T_{**}$, then the condition $X_{*1} = X_{*2}$ implies that

$$X_{**} = X_{*1} = X_{*2}$$

and

$$T_{**} = T_{*1} = T_{*2}$$

a case in which the minimal 2-impulse solution and the two minimal 1-impulse solutions are identical. However, this is not necessarily true when an absolute bound $X_{abs}$, upper or lower, is present unless $X_{*1} = X_{*2} = X_{abs}$.
For example, it is quite possible that a pair of bounding trajectories of the same eccentricity bounds a $T_{**}$ of less eccentricity if the optimal arc contains the least eccentricity point. When this is the case, we observe that the two quantities $X_{**1}$ and $X_{**2}$, being equal but distinct from $X_{abs}$, form an upper bound if $X_{abs}$ is an absolute lower bound, and, a lower bound if $X_{abs}$ is an absolute upper bound, and that they form no bound in the presence of both absolute upper and lower bounds.
VII. The Case of 180° Transfer

So far the analysis has been based on the assumption of $0 < \psi < \pi$. In the limiting case of $\psi = \pi$, although the stationarity Eqs. (6) and (13) no longer apply, the geometric analyses in Sections III and IV are still valid, and all the preceding qualitative and quantitative predictions still hold.

In fact, the situation is much simpler than in the general case, as the velocity constraining hyperbola for each terminal now degenerates into two straight lines both parallel to the line of terminals $Q_1Q_2$, its evolute disappears, leaving the hodograph plane consisting of only the simple region, and the transfer arc pair now becomes a pair of two straight line segments. As consequences of such simpler hodograph geometry, and in line with the preceding general conclusions, the two-impulse 180° transfer presents some particular features as follows:

1. There is one and only one optimal transfer arc pair, hence, one and only one bounding trajectory pair, in each sense of transfer (the distinction between short and long transfers now ceases to exist).

2. No multi-minimum for the transfer in the same sense is possible; and there exists at most a double minimum of opposite senses (direct consequence of item 1).

3. The optimal condition for minimal two-impulse transfer, Eq. (5), reduces to
\[
\sin \gamma_1 = \sin \gamma_2 \tag{47}
\]
for the \(180^\circ\) case. Here \(\gamma_i\) \((i = 1, 2)\) is the path angle of
the velocity-increment vector \(\Delta \hat{V}_i\) with reference to the local
radial direction; hence, Eq. (47) expresses the Law of Equal Slope.

4. The coincidence condition reduces to simple
\[
\hat{V}_{0r1} = \hat{V}_{0r2} \tag{48}
\]

5. In contrast with the non-\(180^\circ\) transfer, the two
position vectors, \(\hat{r}_1\) and \(\hat{r}_2\), now being collinear, do not
determine the orientation of the transfer plane. Hence
this orientation is open to choice.

Finally, it should be noted that, whereas no analytic
solution in closed form is possible for the minimal two-
impulse transfer in the general case, such a solution does
exist in the \(180^\circ\) case. For such a solution and the further
minimization of the total velocity impulse by optimizing the
orientation of the transfer plane, the reader may consult
Reference 27.
VIII. Numerical Examples

To verify the preceding predictions two sets of numerical examples have been worked out. The terminal conditions assumed and the corresponding transfer geometry are shown in Table 1. Set A consists of the transfers from a circular orbit to a series of coplanar, coaxial, and similar elliptic orbits of the same eccentricity $3/4$ but varying size. The point of departure on the circular orbit is, in each case, at $60^\circ$ from the point of arrival, which is the apocenter of the target ellipse. Both the initial and final orbits are assumed to be in the same sense of motion. Examples of set B are the same as those of set A, except that the final orbits are a series of similar hyperbolas of the same eccentricity $5/4$, and that the point of arrival is the pericenter of the target hyperbola in each case. In each set of examples, the absolute minimal 2-impulse solution for $T_{**}$, and the two minimal 1-impulse solutions defining the bounding trajectory pair, $T_{*1}$ and $T_{*2}$, are calculated for fixed values of the distance ratio $n$, ranging from 0.20 to 2.0. The principal results are graphically depicted in the nondimensional form in Figs. 9 to 14.

Tabulated values are found in Appendix G, and some numerical results of particular interest are summarized in Table 1.

From these results, it is seen that each of the three principal trajectory parameters, $V_C$, $V_R$ and $h$, calculated for
### TABLE 1. NUMERICAL EXAMPLES OF THE MINIMAL TWO-IMPULSE ORBITAL TRANSFER

<table>
<thead>
<tr>
<th></th>
<th>(A) CIRCLE-TO-ELLIPSE</th>
<th>(B) CIRCLE-TO-HYPERBOLA</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>TRANSFER GEOMETRY</strong></td>
<td>Q₁ Q₂ Q₃ Q₄</td>
<td>Q₁ Q₂ Q₃ Q₄</td>
</tr>
<tr>
<td><strong>TERMINAL CONDITIONS</strong></td>
<td>INITIAL</td>
<td>FINAL</td>
</tr>
<tr>
<td>Orbital Eccentricities</td>
<td>ε₁ = 0</td>
<td>ε₂ = 0.75</td>
</tr>
<tr>
<td>Velocities</td>
<td>ν₀₁ = 1</td>
<td>ν₀₂ = 0.5</td>
</tr>
<tr>
<td></td>
<td>φ₀₁ = 0</td>
<td>φ₀₂ = 0</td>
</tr>
<tr>
<td>Radial Distances</td>
<td>r₁</td>
<td>r₂ = nr₁</td>
</tr>
<tr>
<td>Angle of Separation</td>
<td>ψ = 60°</td>
<td>ψ = 60°</td>
</tr>
<tr>
<td>Minimal Total Impulse,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f_{**}/r_1$: lowest value</td>
<td>= 0.5 @ n = 1.0</td>
<td>= 0.5 @ n = 1.0</td>
</tr>
<tr>
<td>Distance Ratio for</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{**} = T_{*1} = T_{*2}$</td>
<td>n_c = 0.630</td>
<td>n_c = 1.31</td>
</tr>
</tbody>
</table>

For detailed tabulated values, see Appendix G; for graphs, see Figs. 9 to 14.
$T^{**}$ is indeed bounded between the corresponding quantities for the two bounding trajectories, $T_{*1}$ and $T_{*2}$, as predicted by the Inequalities (21, 22, and 34). (See Figs. 9 to 11). Also, the minimal total velocity impulse required for the transfer is bounded between its upper and lower bounds as predicted by Inequality (29) (See Fig. 12).

To compare the total velocity impulses required for the transfers along the three trajectories, $T_{*1}$, $T_{*2}$, and $T^{**}$, the values of $f_{*1}$, $f_{*2}$, and $f^{**}$ are found as shown in Fig. 13; and the relative saving in the total velocity impulse by 2-impulse minimization over the minimization of each terminal impulse is calculated from

$$\frac{\Delta f_{*1}}{f_{*1}} = \frac{f_{*1} - f^{**}}{f_{*1}} \quad (49-1)$$

$$\frac{\Delta f_{*2}}{f_{*2}} = \frac{f_{*2} - f^{**}}{f_{*2}} \quad (49-2)$$

and graphically shown in Fig. 14. From these plottings it is seen that the $f^{**}$ graph indeed remains below those of $f_{*1}$ and $f_{*2}$, as predicted by Inequalities (31-1,2), and that the savings $\Delta f_{*1}$ and $\Delta f_{*2}$ are positive throughout, justifying the two-impulse minimization.

In addition to the foregoing preliminary observations, the following are worth noting:

1. For each of the trajectory parameters, $V_C$, $V_R$, and $h$, calculated here, the three curves for the trajectories $T^{**}$, $T_{*1}$ and $T_{*2}$, intersect at a common point, indicating the
coincidence of $T_{**}^1$, $T_{**}^2$ and $T_{**}$. The same situations are found in the graphs of $f_{**}$ and its upper and lower bounds as shown in Figs. 12, where the three curves $f_{**}^1$, $f_{**}^1$ and $f_{**}^2$ touch each other at their common point. The values of $n$ at the common points given by the various graphs of the same set are, of course, the same. They are designated as $n_c$, as shown in Table 1. These values check with Eq. (18), as they should, since they belong to the class of apside-to-apside transfers.

2. For the inner transfer ($n < 1$) from a fixed initial terminal point, under constant angle of separation, and constant terminal velocities vectors, the total velocity impulse required for the transfer along each of the three trajectories, decreases as the final terminal distance $r_2$ increases; while in the outer transfer ($n > 1$), each of these impulses tend to increase with the final terminal distance within the present range of computation (see Figs. 12 and 13).

3. The case $n = 1$ is singular in each set of examples, since the initial circular orbit now passes through the final terminal point. Figures 9 to 11, and 13 show that each $T_{**}$ - curve touches the $T_{**}^1$ curve at $n = 1$, indicating $T_{**} = T_{**}^1 + T_{**}^2$. It is to be noted that, in examples B, the case of $n = 0.722$ is also singular since the final hyperbolic orbit now passes through the initial terminal point, and itself is $T_{**}^2$. This is confirmed by the present computation as the value of $f_{2*2}$ is indeed zero at this particular value of $n$. However, the computation results indicate that the absolute minimal $T_{**}$ is
different from $T_{*2}$ in this case. It can be verified that
$\dot{n} = 2.5$ is another singular case in example A, though beyond the
present range of plotting, since the final elliptic orbit
now passes through the initial terminal point.
\[ \psi = 60^\circ \]
\[ \nu_01 = 1.0 \]
\[ \nu_02 = 0.5 \]

\[ \text{DISTANCE RATIO, } n = r_2/r_1 \]

Fig 9 THE CHORDAL COMPONENT OF THE TERMINAL VELOCITY FOR MINIMAL 2-IMPULSE TRANSFER AND ITS UPPER AND LOWER BOUNDS.
Fig. 10 THE RADIAL COMPONENT OF THE TERMINAL VELOCITY FOR MINIMAL 2-IMPULSE TRANSFER AND ITS UPPER AND LOWER BOUNDS

(A) $\psi = 60^\circ$
$\nu_0 = 1.0$
$\nu_0' = 0.5$

(B) $\psi = 60^\circ$
$\nu_0 = 1.0$
$\nu_0' = 1.5$
FIG. 11 THE ANGULAR MOMENTUM FOR MINIMAL 2-IMPULSE TRANSFER AND ITS UPPER AND LOWER BOUNDS

\( \psi = 60^\circ \)
\( \nu_{01} = 1.0 \)
\( \nu_{02} = 0.5 \)
FIG. 12 THE MINIMAL TOTAL VELOCITY IMPULSE FOR 2-IMPULSE TRANSFER AND ITS UPPER AND LOWER BOUNDS.
**FIG. 13** THE TOTAL VELOCITY IMPULSES; THE MINIMAL 2-IMPULSE SOLUTION VERSUS THE MINIMAL INITIAL IMPULSE AND MINIMAL FINAL IMPULSE SOLUTIONS
FIG. 14 PERCENTAGE OF SAVING IN TOTAL VELOCITY IMPULSES, 2-IMPULSE MINIMIZATION OVER THE INITIAL IMPULSE MINIMIZATION AND FINAL IMPULSE MINIMIZATION.
IX. Summary of Conclusions

1. A minimal 2-impulse transfer trajectory $T_{**}$ is bounded between a pair of bounding trajectories between the same terminal points in the same sense of motion, one $T_{*1}$ with a minimal initial impulse, and the other $(T_{*2})$, a minimal final impulse.

   It is not only bounded by the two bounding trajectories, in the position space, but also in spaces of other trajectory parameters, such as (a) the directions of departure and arrival, (b) the terminal transfer velocities and their components, $V_C$, $V_R$ and $V_\theta$, (c) the terminal velocity impulses, (d) the angular momentum and semilatus rectum, (e) the orbital energy and semimajor axis, (f) the eccentricity vector, (g) time of flight, etc.

   Under each item the trajectory quantities, $X_{*1}$ and $X_{*2}$, pertaining to the bounding trajectory pair, $T_{*1}$ and $T_{*2}$ respectively, form a pair of upper and lower bounds of the corresponding quantity $X_{**}$ pertaining to the trajectory $T_{**}$ if no absolute upper and lower bounds are present. In case there exists an absolute bound, upper or lower, then it furnishes an additional choice for the proper bound $X_{*1}$ and $X_{*2}$. (For details, see Section VI.)

2. A minimal 2-impulse transfer trajectory always agree with its bounding trajectory pair in kind (short or long transfer), sense (counterclockwise or clockwise), type (elliptic, parabolic, or hyperbolic), and nature (realistic
or unrealistic) if they agree themselves; but not necessarily so with the two terminal orbits.

3. Under any terminal conditions, there exists at least one pair of bounding trajectories of each kind and sense; hence at least a local minimal 2-impulse solution, realistic or unrealistic, of each kind and sense.

4. There exist at most three bounding trajectory pairs of the same kind and sense, and at most a total of four such pairs of both kinds and senses. Hence there can be no more than a triple minimum of the same kind and sense of transfer and no more than a quadruple minimum of both kinds and senses.

5. Whenever the two trajectories of a bounding pair are coincident, the minimal 2-impulse transfer trajectory bounded between will coincide with them. A definite relation exists among the four terminal vectors, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{v}_{01}$, and $\mathbf{v}_{02}$, for such coincidence (see Eq. (15)). When and only when this coincidence condition is met, the 2-impulse minimization and the 1-impulse minimizations at the initial and final terminals separately will yield the same transfer trajectory.

All the foregoing conclusions are valid for any arbitrary central angle $0 < \Psi < \pi$. For the particular conclusions pertaining to the case of $\Psi = \pi$, see Section VII.
X. Final Remarks

As shown in the preceding Sections, a great deal of information concerning the minimal 2-impulse transfer may be obtained once the two bounding trajectories are determined. In many cases, definite qualitative conclusions may be asserted directly from the bounding trajectory pair; and quantitatively, the upper and lower bounds of the principal parameters pertaining to the minimal 2-impulse transfer may be established. Since each bounding trajectory is governed by a quartic equation, while the minimal 2-impulse transfer trajectory is governed by an octic equation, the present treatment amounts to solving two fourth degree equations instead of one eighth degree equation. In view of the fact that a quartic equation is much more tractable than an octic, and that an analytic solution in closed form exists for the former, such a treatment is advisable. The present geometric approach in the hodograph plane by examining the optimal transfer arc pairs, rather than an algebraic approach to the solutions of the pertinent equations, has the further advantage of eliminating the extraneous roots of the governing octic, which do not belong to the stationarity solution, as well as the roots for the maximal total impulse solutions, so that the problem narrows down to locating all the local minimal solutions and choosing an absolute minimal and realistic one. As each bounding trajectory has the particular
significance of having a stationarity impulse at one terminal, the existing knowledge on the comparatively simpler problem of determining the optimal 1-impulse transfer trajectory\textsuperscript{8, 20} may be utilized to aid the solution of the 2-impulse problem. Thus, in summary, the advantage of using the bounding trajectories for treating the 2-impulse transfer problem are as follows:

1. Solution of two quartic equations instead of a single but cumbersome octic equation.
2. Utilization of the existing knowledge on the optimal 1-impulse transfer problem to aid the solution of the optimal 2-impulse transfer problem.
3. The choice of a proper bounding trajectory pair eliminates the extraneous solutions as well as the maximal total impulse solutions.

At first sight it seems that the choice of a bounding trajectory pair is generally not unique, since each stationary quartic may yield as many as four distinct stationary 1-impulse trajectories. However, the present study shows that the number of such trajectory pairs cannot exceed three in one kind of transfer, and the total number of such pairs, counting both kinds, cannot exceed four (see Section V-D). Thus the number of possible bounding trajectory pairs is highly limited. In fact, the presence of three bounding
trajectory pairs of the same kind can happen only under the condition that both terminal velocities enter the nonsimple regions in the hodograph plane. This condition requires that each terminal velocity be of sufficient magnitude, \( V_{0p_1} > S_i \) (see Fig. C-1 and Eq. C-8, Appendix C), and that in a direction with limited deviation from the minimal energy direction, \( |\phi_1| < \frac{v_1}{2} \), see Fig. C-1). Such a requirement puts rather stringent conditions on the terminal orbits. For example, in a coplanar 60°-transfer at \( n = 2 \), it requires an initial terminal velocity \( V_{01} > 1.52 \, V_1^* \) and a final terminal velocity of \( V_{02} > 4.13 \, V_2^* \). Such conditions can be met only between two hyperbolic orbits of the eccentricities \( e_1 > 3.62 \) and \( e_2 > 3.34 \), a combination not likely to be encountered in practical problems. Thus in the usual cases, such as the transfer between two moderately eccentric Keplerian orbits, the two terminal velocity vectors will remain in the simple regions, and consequently, there is a unique bounding trajectory pair in each kind. Even under some unusual terminal conditions, when one or both of the terminal velocities do enter the nonsimple regions, and there are more than two bounding trajectory pairs, the first choice will usually be the pair of two absolute minimal 1-impulse transfer trajectories if such a pair exists. Thus the proper choice of a bounding trajectory pair, ordinarily does not present a problem.

In addition to yielding essential information on the
minimal 2-impulse transfer, the use of bounding trajectories may also aid theoretical studies of such transfers. The derivation of the coincidence condition, geometric as well as analytic, for the identical 2-impulse minimization and the two 1-impulse minimizations at the initial and final terminals separately furnishes an example (see Section V-E). Many other aspects of 2-terminal transfers may be investigated in the light of the bounding trajectories; however, such investigations are not intended in this report.

So far the present treatment has been kept perfectly general without any restrictions on the terminal conditions except that the two terminal orbits are assumed Keplerian. Thus the predictions made are applicable to all particular cases. In the case of 180° transfer, such predictions may not be necessary, since an analytic solution exists \(^2\), and the computation is direct and simple. However, the application of the bounding trajectory pair may still help to bring out easily many salient features of such a transfer, as illustrated in Section VII. No attempt is made here to cover other particular cases. However, the application of the present treatment to various cases under specialized terminal conditions should be straightforward.

Finally, it should be mentioned that, when the number of impulses are open to choice, three or more impulses may prove to be more economical than two impulses under the same initial and final terminal conditions in certain cases.\(^7,22,24,26\) Nevertheless, the two-impulse optimum will continue to be a practical mode of optimal transfer in most cases even though
optimal solutions with additional impulse or impulses do exist as the penalties on the implementational complexity and the duration of transfer may well offset the additional saving in fuel economy. A full discussion of the general multi-impulse transfer problem, however, is beyond the scope of the present report.
REFERENCES


APPENDIX A

Derivation of the Stationarity Octic Equations in Symmetric Velocity Coordinates

In terms of the symmetric coordinates \((V_C, V_R)\), the terminal velocity impulse required for the transfer is given by

\[
\mathbf{f}_i^2 = V_C^2 + V_R^2 - 2N_{oi}V_C - 2M_{oi}V_R + P_{oi} \quad (i=1, 2) \quad (A-1)
\]

where \(M_{oi}, N_{oi},\) and \(P_{oi}\) are defined by Equations (7 to 10).

Carrying out the differentiation of Equation (A-1) as indicated in Equation (5), and noting from Equation (4) the differential relation

\[
\frac{dV_C}{V_C} + \frac{dV_R}{V_R} = 0 \quad (A-2)
\]

we obtain, after simplification, the stationarity equation in the symmetric form

\[
a_4V_C^4 + a_3V_C^3 + a_2V_C^2 + a_1V_C + a_0 + a_{-1}V_R^4 + a_{-2}V_R^3 + a_{-3}V_R^2 + a_{-4}V_R = 0 \quad (A-3)
\]

where

\[
a_4 = N_{oi}^2 - N_{o2}^2 - P_{oi} + P_{o2}
\]
\[ a_3 = 4K(M_{o1} - M_{o2}) + 2N_{o1}N_{o2} (N_{o2} - N_{o1}) \]
\[ + 2(N_{o2}P_{o1} - N_{o1}P_{o2}) \]
\[ a_2 = 2K(N_{o2}M_{o2} - N_{o1}M_{o1} + 4N_{o1}M_{o2} - 4N_{o2}M_{o1}) + N_{o1}^2P_{o2} - N_{o2}^2P_{o1} \]
\[ a_1 = 4K^2(N_{o2} - N_{o1}) + 4KN_{o1}N_{o2}(M_{o1} - M_{o2}) \]
\[ + 2K(M_{o1}P_{o2} - M_{o2}P_{o1} + M_{o1}N_{o2}^2 - M_{o2}N_{o1}^2) \]
\[ a_0 = K^2(M_{o1}^2 - M_{o2}^2 + N_{o1}^2 - N_{o2}^2) + 2K^2(P_{o1} - P_{o2}) \]
\[ + 2K(N_{o2}M_{o2}P_{o1} - N_{o1}M_{o1}P_{o2}) \]
\[ a_{-1} = 4K^2(M_{o2} - M_{o1}) + 4KM_{o1}M_{o2}(N_{o1} - N_{o2}) \]
\[ + 2K(N_{o1}P_{o2} - N_{o2}P_{o1} + N_{o1}M_{o2}^2 - N_{o2}M_{o1}^2) \]
\[ a_{-2} = 2K(M_{o2}N_{o2} - M_{o1}N_{o1} + 4M_{o1}N_{o2} - 4M_{o2}N_{o1}) \]
\[ + 2M_{o1}^2P_{o2} - 2M_{o2}^2P_{o1} \]
\[ a_{-3} = 4K(N_{o1} - N_{o2}) + 2M_{o1}M_{o2}(M_{o2} - M_{o1}) \]
\[ + 2(M_{o2}P_{o1} - M_{o1}P_{o2}) \]
\[ a_{-4} = M_{o1}^2 - M_{o2}^2 - P_{o1} + P_{o2} \]  

(A-4)
Eliminating \( V_C \) and then \( V_R \) alternately from Equations (A-3) and (4) results in the octics in \( V_C \) and \( V_R \) respectively:

\[
\sum_{n=0}^{8} C_n V_C^{n**} = 0
\]  

(6-c)

\[
\sum_{n=0}^{8} R_n V_R^{n**} = 0
\]  

(6-R)

where

\[
C_n = a_{n-4} , \quad R_n = a_{4-n} \quad n = 4 \text{ to } 8 
\]

(A-5)

\[
C_n = k^{4-n} a_{n-4} , \quad R_n = k^{4-n} a_{4-n} \quad n = 0 \text{ to } 4 
\]

(A-5)

Note here the reciprocal relations among the coefficients:

\[
a_m = a_{-m}^T 
\]  

(A-6)

\[
C_n = k^{4-n} C_{8-n}^T = R_n^T 
\]  

(A-7)

\[
R_n = k^{4-n} R_{8-n}^T = C_n^T 
\]  

(A-8)

where the transpose (T) indicates the interchange of \( M_{oi} \) and \( N_{oi} \).
## APPENDIX B

### TABLE B: PRINCIPAL TRAJECTORY PARAMETERS OF TWO-IMPULSE TRANSFER IN SYMMETRIC VELOCITY COORDINATES

<table>
<thead>
<tr>
<th>Section</th>
<th>Basic Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Transfer Velocity</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Magnitude</strong></td>
<td>( V_i = \sqrt{v_{c}^2 + v_{R}^2 - 2K \cos \phi_i} ) (B-1)</td>
</tr>
<tr>
<td>((r, \theta)) components</td>
<td></td>
</tr>
<tr>
<td>( V_{ri} = v_{R} - v_{c} \cos \phi_i ) (B-2)</td>
<td></td>
</tr>
<tr>
<td>( V_{\phi i} = v_{c} \sin \phi_i ) (B-3)</td>
<td></td>
</tr>
<tr>
<td><strong>Path Angle</strong></td>
<td>( \phi_i = \tan^{-1} \left( \frac{v_{R} \csc \phi_i - \cot \phi_i}{v_{c}} \right) ) (B-4)</td>
</tr>
<tr>
<td><strong>Velocity-Increment (Terminal Impulse)</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Magnitude</strong></td>
<td>( f_i = \sqrt{v_{c}^2 + v_{R}^2 - 2N_{oi}v_{c} - 2M_{oi}v_{R} + v_{oi}^2 - 2K \cos \phi_i} ) (B-5)</td>
</tr>
<tr>
<td><strong>Direction Cosines</strong></td>
<td></td>
</tr>
<tr>
<td>( \cos \gamma_{ri} = (V_{R} - v_{c} \cos \phi_i - v_{ori} - f_i) ) (B-6)</td>
<td></td>
</tr>
<tr>
<td>( \cos \gamma_{\phi i} = (v_{c} \sin \phi_i - v_{osi} + f_i) ) (B-7)</td>
<td></td>
</tr>
<tr>
<td>( \cos \gamma_{Ni} = -v_{oni} f_i ) (B-8)</td>
<td></td>
</tr>
<tr>
<td><strong>Total Velocity-Impulse</strong></td>
<td></td>
</tr>
<tr>
<td>( f = \sum_{i=1}^{2} \sqrt{v_{c}^2 + v_{R}^2 - 2N_{oi}v_{c} - 2M_{oi}v_{R} + v_{oi}^2 - 2K \cos \phi_i} ) (B-9)</td>
<td></td>
</tr>
<tr>
<td><strong>Angular Momentum</strong></td>
<td>( h = v_{c} r_{i} \sin \phi_i ) (B-10)</td>
</tr>
<tr>
<td><strong>Orbital Energy</strong></td>
<td>( k = \ell(v_{c}^2 + v_{R}^2) - K \cos \phi_i - \frac{\mu}{r_{i}} ) (B-11)</td>
</tr>
</tbody>
</table>
APPENDIX B

TABLE B. (Cont'd)

Nondimensional Formulas

\[
\nu_i = \sqrt{\nu_{ci}^2 + \nu_{Ri}^2} - 2\tan\frac{\psi}{2} \cot \varphi_i \tag{B-1'}
\]

\[
\nu_{ri} = \nu_{Ri} - \nu_{ci} \cos \varphi_i \tag{B-2'}
\]

\[
\nu_{0i} = \nu_{ci} \sin \varphi_i \tag{B-3'}
\]

\[
\phi_i = \tan^{-1} \left( \frac{\nu_R}{\nu_C} \csc \varphi_i - \cot \varphi_i \right) \tag{B-4'}
\]

\[
\frac{f_i}{\mu/r_i} = \sqrt{\nu_c^2 + \nu_R^2 - 2n_{oi} \nu_c - 2m_{oi} \nu_R + \nu_{oi}^2 - 2\tan\frac{\psi}{2} \cot \varphi_i} \tag{B-5'}
\]

\[
\cos \gamma_{ri} = \left( \nu_R - \nu_c \cos \varphi_i - \nu_{ori} \right) / \frac{f_i}{\mu/r_i} \tag{B-6'}
\]

\[
\cos \gamma_{0i} = \left( \nu_c \sin \varphi_i - \nu_{00i} \right) / \frac{f_i}{\mu/r_i} \tag{B-7'}
\]

\[
\cos \gamma_{Ni} = -\nu_{oni} / \frac{f_i}{\mu/r_i} \tag{B-8'}
\]

\[
\frac{f}{\sqrt{\mu/r_i}} = \sqrt{\nu_{cl}^2 + \nu_{Rl}^2 - 2n_{oi} \nu_{cl} - 2m_{oi} \nu_{Rl} + \nu_{ol}^2 - 2\tan\frac{\psi}{2} \cot \varphi_i}
+ \sqrt{\nu_{c2}^2 + \nu_{R2}^2 - 2n_{o2} \nu_{c2} - 2m_{o2} \nu_{R2} + \nu_{o2}^2 - 2\tan\frac{\psi}{2} \cot \varphi_2} / \mu \tag{B-9'}
\]

\[
\frac{h}{\sqrt{\mu/r_i}} = \nu_{ci} \sin \varphi_i \tag{B-10'}
\]

\[
\frac{k_i}{\mu/r_i} = \nu_R \left( \nu_{ci}^2 + \nu_{Ri}^2 \right) - \tan\frac{\psi}{2} \cot \varphi_i - 1 \tag{B-11'}
\]
APPENDIX C

GEOMETRY OF THE TERMINAL VELOCITY CONSTRAINING HYPERBOLA
AND THE PERTINENT FORMULAS

TABLE C-1

The Principal Geometric Elements of the Constraining Hyperbola

<table>
<thead>
<tr>
<th>The Constraining Hyperbola</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eq. in Rectangular Coordinates</td>
</tr>
<tr>
<td>Semitransversal Axis</td>
</tr>
<tr>
<td>Semiconjugate Axis</td>
</tr>
<tr>
<td>Center-to-Focus Distance</td>
</tr>
<tr>
<td>Eccentricity</td>
</tr>
<tr>
<td>Included Angle Between the Asymptotes</td>
</tr>
</tbody>
</table>

The Evolute - Lame

| Eq. in Rectangular Coordinates | \( (A_i V_{x_i}^2) - (B_i V_{y_i}^2) = C \) (C-7) |
| Center-to-Vertex Distance | \( S_i = \sqrt{\frac{2\mu}{r_i}} \tan \frac{\psi}{2} \tan \frac{\phi_i}{2}/ \sin^2 \frac{\phi_i}{2} \) (C-8) |
| Included Angle Between the Asymptotes | \( \sigma_i = \phi_i \) (C-9) |
Notes:

1. The constraining hyperbola is asymptotic to the terminal radial direction and the chordal direction, while its involute, a form of Lame', is asymptotic to the terminal transversal direction and the direction perpendicular to the chord. The two sets of asymptotic directions are thus orthogonal to each other.

2. The constraining hyperbola and its involute have the interior and exterior bisectors of the base angle at the terminal as their common transversal and common conjugate axes respectively.

(A typical terminal constraining hyperbola is shown in Fig. C-1. For the particular points of interest on the constraining hyperbola, See Table C-2. For the relative orientation of the two terminal constraining hyperbolas, see Fig. 4.)
FIG. C-1  THE TERMINAL VELOCITY CONSTRAINING HYPERBOLA AND ITS EVOLUTE

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### TABLE C-2: PARTICULAR POINTS ON THE TERMINAL CONSTRAINING HYPERBOLA AND THEIR ASSOCIATED TRAJECTORIES

<table>
<thead>
<tr>
<th>Transfer Trajectory</th>
<th>Points on the Constraining Hyperbola</th>
<th>Pertinent Formulas $i = 1.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum Energy</td>
<td>$\mathcal{E}<em>+ \quad \mathcal{E}</em>-$</td>
<td>$(v_i)_{\text{min.}} = a_i$</td>
</tr>
<tr>
<td></td>
<td>Vertex of each branch</td>
<td>$k_{\text{min.}} = -\frac{2u}{r_1 + r_2 + \ell}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$a_{\text{min.}} = \frac{1}{k}(r_1 + r_2 + \ell)$</td>
</tr>
<tr>
<td>Least Eccentricity</td>
<td>$\mathcal{F}<em>+ \quad \mathcal{F}</em>-$</td>
<td>$\text{min.} = \left</td>
</tr>
<tr>
<td>Critical (Parabolic)</td>
<td></td>
<td>$v_i^* = \sqrt{\frac{2u}{r_i}}$</td>
</tr>
<tr>
<td>Realistic</td>
<td>$Q_+^* \quad Q_-^*$</td>
<td>$\tan \gamma_1^* = \sqrt{\cot \frac{\nu_1}{2} \tan \frac{\nu_2}{2}}$</td>
</tr>
<tr>
<td>Unrealistic</td>
<td>$Q_+^{1*} \quad Q_-^{1*}$</td>
<td>$\tan \gamma_2^* = \sqrt{\tan \frac{\nu_1}{2} \cot \frac{\nu_2}{2}}$</td>
</tr>
</tbody>
</table>
# APPENDIX D

## TABLE D: TERMINAL CONDITIONS AND THE DISTRIBUTION OF ORTHOPOINTS AND THEIR ASSOCIATED STATIONARY ONE-IMPULSE TRANSFER TRAJECTORIES†

<table>
<thead>
<tr>
<th>Location of Terminal Velocity Point</th>
<th>Location of Orthopoints and Types of the Associated Transfer Trajectories</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$Q_{oi}$</td>
</tr>
<tr>
<td></td>
<td>$Q_{i^a}$ $Q_{i^b}$ $Q_{i^c}$ $Q_{i^d}$</td>
</tr>
<tr>
<td>$EE - S^\pm$</td>
<td>E $^+$</td>
</tr>
<tr>
<td>$EE - N^\pm$</td>
<td>E $^+$</td>
</tr>
<tr>
<td>$HE - S^\pm$</td>
<td>H $^+$</td>
</tr>
<tr>
<td>$HE - N^\pm$</td>
<td>H $^+$</td>
</tr>
<tr>
<td>$HH - S^\pm$</td>
<td>H $^+$</td>
</tr>
<tr>
<td>Boundary between EE &amp; HE: S $^\pm$</td>
<td>Q*$^+$</td>
</tr>
<tr>
<td>N $^\pm$</td>
<td>Q*$^+$</td>
</tr>
<tr>
<td>Boundary between HE &amp; HH: S $^\pm$</td>
<td>H $^+$</td>
</tr>
</tbody>
</table>

| $H'E - S^\pm$                       | H$'^+$   | ----    | ----    | E $^+$      |
| $H'E - N^\pm$                       | H$'^+$   | H $^+$ (E$^\pm$) | H $^+$ (E$^\pm$) | E $^+$      |
| $H'H' - S^\pm$                      | H$'^+$   | ----    | ----    | H$'^+$      |

Notations: ' unrealistic transfer, + short transfer, - long transfer; for others, see nomenclature.

† Symbols in parenthesis are for the hatched portion of the HE-N$^\pm$ or H'E-N$^\pm$ regions only, see Figs. 5 and D-1.
FIG. D-1 TYPICAL DISTRIBUTIONS OF THE ORTHOPORTS ON THE CONSTRAINING HYPERBOLA
APPENDIX E

Proof of the Existence of a Two-Impulse Extremum on the Optimal Transfer Arc Pair

Assume $f_1$ and $f_2$ are continuous and twice differentiable. Consider a typical transfer arc pair of type (A). With reference to Fig. E-1, the end points of the arc pair $Q_{1*}$ and $Q_{2*}$ define a closed interval $[p, q]$. Since $Q_{1*}$ and $Q_{2*}$ are the minimal points on this arc pair, we have:

At $p$: \[
\begin{align*}
&f_1' = 0 \\
&f_2' < 0
\end{align*}
\] \[f' = f_1' + f_2' < 0\]

At $q$: \[
\begin{align*}
&f_1' > 0 \\
&f_2' = 0
\end{align*}
\] \[f' = f_1' + f_2' > 0\]

Thus $f'$ has opposite signs at the endpoints, hence there is at least a local extremal $f$ on the interval. Furthermore, the absence of any stationary point and inflection point on $f_1$ and $f_2$ in the interior of the interval shows that $f_1'$ and $f_2'$ are monotonically increasing on the interval, and so is $f'$. Thus the $f'$ curve crosses the $V_C$-axis only once, and $f''$ is positive throughout the interval. Consequently we conclude that,

There is one and only one interior extremal $f$

on the interval $[p, q]$, along the transfer arc pair (A), and this extremum is a local minimum.

† This condition is actually met in any interval excluding the origin, and where none of $f_1$ and $f_2$ vanishes.
Next, consider a typical transfer arc pair of type (B).

With reference to Fig. E-1B, we have, on the closed interval [p, q] defined by the endpoints Q_{1*a} and Q_{1*b} of the arc pair:

At p:
\[ f_1' = 0 \]
\[ f_2' = 0 \]
\[ f' = f_1' + f_2' < 0 \]

At q:
\[ f_1' = 0 \]
\[ f_2' = 0 \]
\[ f' = f_1' + f_2' < 0 \]

Thus \( f' \) has the same sign at the endpoints, hence there is either an even number of internal extrema of \( f \) or none.

Since, as assumed here, \( f_1 \) goes from one minimum to one maximum on the interval, there exist one and only one point of inflection on \( f_1' \), that is, there is one and only one interior extremal \( f_1' \) on the interval. On the other hand, in the absence of any stationary point and inflection point other than the endpoint, \( f_2' \) is monotonically increasing, and is negative throughout the interval. Consequently, \( f' \) first increases and then decreases, with one and only one interior extremum on the interval. Hence there are three possibilities:

1) \( f' \) cuts the \( V_c \) - axis at two points. There exists a pair of extremal values of \( f \), one maximal and one minimal.
2) $f'$ touches the axis. Then at the point of tangency, $f' = 0$ and $f'' = 0$, $f$ is neither a maximum nor a minimum.

3) $f'$ cuts the axis at no point. There exists no extremal $f$.

Consequently, we conclude that,

There is either one interior minimal $f$ and one interior maximal $f$, or none on the interval $[a, b]$, along the transfer arc pair (B).

The proof for the existence of a local maximum on the arc pair of type (D) is analogous to the proof for type (A).
FIG. E-1 VARIATION OF THE IMPULSE FUNCTION AND ITS DERIVATIVE ALONG AN OPTIMAL TRANSFER ARC PAIR
### APPENDIX F

**TABLE F: TERMINAL CONDITIONS AND THE MULTIPLICITY OF MINIMAL 2-IMPULSE SOLUTIONS**

<table>
<thead>
<tr>
<th>Case</th>
<th>Regional Locations of Terminal Velocity Points</th>
<th>Optimal Transfer Arc Pairs</th>
<th>Maximum Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( Q_{01} ) ( Q_{02} )</td>
<td>In One Kind and Sense</td>
<td>In Other Kind and Sense</td>
</tr>
<tr>
<td>1</td>
<td>( \pm ) ( \pm )</td>
<td>(a,a)</td>
<td>(d,d)</td>
</tr>
<tr>
<td>2</td>
<td>( \pm ) ( \mp )</td>
<td>(a,d)</td>
<td>(d,a)</td>
</tr>
<tr>
<td>3</td>
<td>( \pm ) ( \mp )</td>
<td>(a,a), (a,c)</td>
<td>(d,d)</td>
</tr>
<tr>
<td>4</td>
<td>( \pm ) ( \mp )</td>
<td>(a,d)</td>
<td>(d,a), (d,c)</td>
</tr>
<tr>
<td>5</td>
<td>( \mp ) ( \mp )</td>
<td>(a,a), (a,c), (c,c)</td>
<td>(d,d)</td>
</tr>
<tr>
<td>6</td>
<td>( \mp ) ( \pm )</td>
<td>(a,d), (c,d)</td>
<td>(d,a), (d,c)</td>
</tr>
</tbody>
</table>

Note: For the double sign, all upper signs go together in each case, and so are all the lower signs.
APPENDIX G
NUMERICAL RESULTS

TABLE G-1A. TRAJECTORY PARAMETERS FOR MINIMAL IMPULSE TRANSFERS: CIRCLE-TO-ELLIPSE
($v_{01} = 1.0, v_{02} = 0.5, \psi = 60^\circ$)

<table>
<thead>
<tr>
<th>Distance Ratio $n = r_2/r_1$</th>
<th>Transfer Velocities</th>
<th>Angular Momenta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chordal Components</td>
<td>Radial Components</td>
</tr>
<tr>
<td></td>
<td>$V_{C*1}$</td>
<td>$V_{C*2}$</td>
</tr>
<tr>
<td>0.2</td>
<td>1.7971</td>
<td>2.0854</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4845</td>
<td>1.6931</td>
</tr>
<tr>
<td>0.4</td>
<td>1.3184</td>
<td>1.4615</td>
</tr>
<tr>
<td>0.5</td>
<td>1.2251</td>
<td>1.3058</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1747</td>
<td>1.1933</td>
</tr>
<tr>
<td>0.7</td>
<td>1.1514</td>
<td>1.1084</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1447</td>
<td>1.0427</td>
</tr>
<tr>
<td>0.9</td>
<td>1.1476</td>
<td>.9912</td>
</tr>
<tr>
<td>1.0</td>
<td>1.1547</td>
<td>.9505</td>
</tr>
<tr>
<td>1.1</td>
<td>1.1632</td>
<td>.9183</td>
</tr>
<tr>
<td>1.2</td>
<td>1.1713</td>
<td>.8927</td>
</tr>
<tr>
<td>1.3</td>
<td>1.1783</td>
<td>.8724</td>
</tr>
<tr>
<td>1.4</td>
<td>1.1839</td>
<td>.8563</td>
</tr>
<tr>
<td>1.5</td>
<td>1.1880</td>
<td>.8434</td>
</tr>
<tr>
<td>1.6</td>
<td>1.1911</td>
<td>.8330</td>
</tr>
<tr>
<td>1.7</td>
<td>1.1931</td>
<td>.8247</td>
</tr>
<tr>
<td>1.8</td>
<td>1.1943</td>
<td>.8180</td>
</tr>
<tr>
<td>1.9</td>
<td>1.1950</td>
<td>.8126</td>
</tr>
<tr>
<td>2.0</td>
<td>1.1952</td>
<td>.8082</td>
</tr>
</tbody>
</table>
TABLE G-1B. TRAJECTORY PARAMETERS FOR MINIMAL IMPULSE TRANSFERS: CIRCLE TO HYPERBOLA

\( v_{01} = 1.0, \ v_{02} = 1.5, \ \psi = 60^\circ \)

<table>
<thead>
<tr>
<th>Distance Ratio ( n = r_2/r_1 )</th>
<th>Transfer Velocities</th>
<th>Angular Momenta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chordal Components</td>
<td>Radial Components</td>
</tr>
<tr>
<td></td>
<td>( V_{c1} )</td>
<td>( V_{c2} )</td>
</tr>
<tr>
<td>0.2</td>
<td>1.7971</td>
<td>3.4056</td>
</tr>
<tr>
<td>0.3</td>
<td>1.4845</td>
<td>2.8388</td>
</tr>
<tr>
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<td>1.3184</td>
<td>2.4924</td>
</tr>
<tr>
<td>0.4999</td>
<td>1.2251</td>
<td>2.2399</td>
</tr>
<tr>
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</tr>
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</tr>
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</tr>
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<td>1.1713</td>
<td>1.2374</td>
</tr>
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<td>1.1856</td>
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<td>1.1880</td>
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<td>1.1911</td>
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<td>1.0050</td>
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<tr>
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<td>1.1943</td>
<td>.9778</td>
</tr>
<tr>
<td>1.8999</td>
<td>1.1950</td>
<td>.9552</td>
</tr>
<tr>
<td>1.9999</td>
<td>1.1952</td>
<td>.9366</td>
</tr>
<tr>
<td>Distance Ratio</td>
<td>$f_{1*1}$</td>
<td>$f_{1*2}$</td>
</tr>
<tr>
<td>----------------</td>
<td>----------</td>
<td>----------</td>
</tr>
<tr>
<td>n=r_2/r_1</td>
<td>Initial</td>
<td>Final</td>
</tr>
<tr>
<td>.2000</td>
<td>.6635</td>
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</tr>
<tr>
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<td>.4881</td>
<td>1.3565</td>
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<tr>
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<td>.4051</td>
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</tr>
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<td>.5956</td>
</tr>
<tr>
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<td>.5202</td>
<td>.6034</td>
</tr>
</tbody>
</table>

TABLE G-2A. TERMINAL IMPULSES REQUIRED FOR MINIMAL IMPULSE TRANSFERS: CIRCLE-TO-ELLIPSE

($v_{01} = 1.0, v_{02} = 0.5, \psi = 60^\circ$)
TABLE G2B. TERMINAL IMPELS REQUIRED FOR MINIMAL IMPULSE TRANSFERS: CIRCLE-TO-HYPERBOLA

\((v_{01} = 1.0, \; v_{02} = 1.5, \; \psi = 60^{\circ})\)

<table>
<thead>
<tr>
<th>Distance Ratio</th>
<th>(T_{11})</th>
<th>(T_{12})</th>
<th>(T_{22})</th>
<th>(T_{22})</th>
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