A PRACTICAL METHOD OF DETERMINING WATER CURRENT VELOCITIES AND DIFFUSION COEFFICIENTS IN COASTAL WATERS BY REMOTE SENSING TECHNIQUES

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INTRODUCTION

Population and industrial centers along the coastal areas are generating large volumes of waste materials which present a threat to the environmental quality of the area. The environmental characteristics of the coastal area can be expected to generate a large population growth in the near future which will compound the pollution problems of the area.

Restricted mixing and circulation characteristics of the estuaries and nearshore waters limit the natural capacity to handle concentrated waste loadings from population centers. The major factors that determine the mixing characteristics in the coastal waters are (1) wind, (2) tidal range, (3) topography, (4) fresh water inflow and (5) density gradients. The salinity or density gradients are functions of the fresh and salt
water inflows and the circulation within the estuary. Energy must be provided to break down these gradients to allow complete mixing.

Diffusion coefficients and current velocities in the receiving waters are essential to waste disposal studies. Mathematical models of the coastal area require this information to predict the fate of future waste loadings. Since the coastal waters cover large areas and the diffusion coefficients and current velocities vary with both space and time, remote sensing provides a practical method of gathering this information.

Field work begins by dropping dye markers from an aircraft at selected locations throughout the receiving waters. The movement and spread of the resulting dye patches are measured remotely by two flights over the area. The movement of the patch between flights gives the current velocity while the change in size of the patch is used to determine diffusion coefficients. While the procedures described herein utilize aerial photography, the method can be adopted to automatic identification of dye patches and automatic processing with multispectral scanning imagers (Ellison, 1971). By using the ratio of light returned in two bands, one band in the region of maximum absorption and the second
band in the region of maximum re-emission of the dye, the dye patch can be identified and computation completed.

DIFFUSION STUDIES

Numerous investigators have employed solutions to the diffusion equations for the estimation of waste concentrations in a waste field. If the scale of the current eddies is much smaller than the dimensions of the waste field, then the Fickian form of diffusion equation can be applied. The basic equation is:

\[
\frac{\partial W}{\partial T} = \frac{\partial}{\partial Y} \left( \frac{\partial W}{\partial Y} \right) + \frac{\partial}{\partial X} \left( D_x \frac{\partial W}{\partial X} \right) + \frac{\partial}{\partial Z} \left( D_z \frac{\partial W}{\partial Z} \right) - \left[ \frac{\partial}{\partial Y} (V_y W) \right] + \frac{\partial}{\partial X} (V_x W) + \frac{\partial}{\partial Z} (V_z W) \right] + S \tag{1}
\]

where \( V \) is velocity, \( W \) is waste concentration, \( D \) is eddy diffusivity, \( X, Y, \) and \( Z \) are space coordinates and \( T \) represents time. The first three terms on the right are the diffusion terms, the next three are convection terms and \( S \) represents the sources and sinks.

Solutions to the equations have required various assumptions such as steady state condition, no vertical
or longitudinal mixing and unidirectional transport velocity in the X direction. With these assumptions, the equation becomes:

$$V_x \frac{\partial W}{\partial x} = \frac{\partial}{\partial y} (D_y \frac{\partial W}{\partial y}) + aW$$  \hspace{1cm} (2)

where $a$ is a first order decay constant and $aW$ represents a sink or loss in the system.

Investigators such as Pearson (1955-1967) Brooks (1960) and others have reported solutions to the diffusion equation for various conditions. Pearson points out that for a source with 1) steady unidirectional current, 2) uniform mixing of the waste and 3) continuous uniform flow from the source, solution to the diffusion equation is as follows:

$$S_{om} = \frac{2.35 \frac{d}{\sqrt{D_y V_x X}}}{Q}$$  \hspace{1cm} (3)

where $S_{om}$ is the minimum dilution along axis of waste plume at distance $X$ from a point source; $D_y$ is the assumed diffusivity in ft$^2$/sec; $X$ is the distance from source in feet; $V_x$ is the average velocity of water mass in ft/sec; $Q$ is the waste discharge, MGD and $d$ is the assumed mixing depth in feet.
Including the decay function for bacterial dieaway or disappearance, and expressing the waste concentration in terms of coliform concentration, the above expression becomes:

\[ \text{MPN} = \frac{0.425 QC_0}{d\sqrt{D_y} \sqrt{X} \exp (zX)} \]  

where MPN is the most probable number of organisms per ml on plume centerline at X; \( C_0 \) is the concentration of organisms in waste, MPN/ml; and \( a \) is the bacterial dieaway (decay) constant, sec\(^{-1}\).

Pearson (1967) further points out that: "The above equations assume a constant eddy diffusivity; correspondingly, the value of \( D_y \) employed must be representative of the overall or average scale of the diffusion phenomenon".

Brooks has reported a solution to the diffusion equation with a variable coefficient of diffusivity. It is assumed that the diffusivity coefficient, \( D_y \), varies as the four-thirds power of the scale of the diffusion phenomenon. \( D_y \approx \alpha 1^{4/3} \), where \( \alpha \) is a constant. Brooks' equation for a line source is as follows:

\[ C_m = C_0 e^{-at} \text{erf} \left[ \frac{3/2}{(1 + 2BX/(3b))^{3/2}} \right]^{1/2} \]
where: $C_o = \text{initial coliform concentration}$

$C_m = \text{maximum coliform concentration at time, } t$

$t = \text{time of travel } X/V_x$

$B = 12D_y/V_x b$

$a = \text{decay constant}$

$D_y = \text{eddy diffusivity at source (} X = 0)$

$b = \text{initial width of sewage field}$

Considering the foregoing solutions to the diffusion equation, the four characteristics of the receiving waters that have a major effect on waste concentration are the following:

1. $V_x$, average current speed
2. $D_y$, eddy diffusivity
3. $d$, average mixing depth
4. $a$, decay or dieaway constant of pollutant

The Allen Hancock Foundation conducted an investigation on the dilution and dispersion of a waste field in the sea (1964). In their study Rhodamine B dye was introduced as 1) slugs from a point source, 2) a continuous plume from a point source and 3) a continuous plume from a volume source. Dye concentration was measured with a Turner fluorometer. The mathematical models used for analysis of data were statistical models based on Gaussian distribution. The basic three-dimensional model for the dye slug was
\[ W(x, y, z, t) = \frac{M}{\pi \sqrt{2 \pi} \left( \sigma_x^2 \sigma_y^2 \sigma_z^2 \right)^{1/2}} \exp \left[ -\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2} \right] \]  

where \( W \) is the average concentration of a point \( x, y, z \), at time \( t \), \( M \) is the amount of dye initially discharged from an instantaneous point source and \( \sigma_x^2, \sigma_y^2, \sigma_z^2 \) are the average values of the variances of the concentration distribution.

Several of conclusions of the Allen Hancock investigation are listed below.

1. The rate of vertical diffusion can contribute significantly to the overall diffusion process at wind speeds greater than eight knots and/or low water column stability.
2. The rate of longitudinal and lateral diffusion appeared to be influenced by wind speed but not by water column stability.
3. The "4/3 law" relating the lateral coefficient of eddy diffusion as a function of average eddy scale did not hold in the particular oceanic areas studied.
Vertical mixing does occur in the waste field as well as horizontal mixing. As indicated by Wiegel (1964), vertical mixing is difficult to study in the laboratory because of limitations of tank size. In these studies the wind drags the surface water to the down wind end of the tank producing a hydraulic head which causes a flow in the opposite direction.

Laboratory studies have indicated that wind drag on the water surface produces very little mixing. However, when wind generated waves appear, extremely rapid mixing occurs as wind waves are rotational in the generating area. Masch (1961) conducted a wave study in a wave tank and developed the following relationship for the coefficient of eddy diffusivity:

$$D_y = 0.0038 (V_s + Q_w)^{3.2}$$

(7)

where $V_s$ is the surface current and $Q_w$ is the water particle orbit speed ($Q_w = H/T$, $H$ = significant wave height and $T$ = average wave period).
DYE PATCH STUDIES

The three pictures shown in figure 1 were taken July 7, 1969 using panchromatic film type 8401 with a Wratten 25A filter. While this is not the best film-filter combination for observing the dye patch, the change in shape of the dye patch can be seen. The dye was dropped at 12:19 and the first photo was taken at 12:25 from 3000 ft at which time the size of the dye field was 160 ft by 40 ft. The photo in figure 1b was taken at 13:13 from 4000 ft. After 54 minutes from the time the dye was dropped, the dye patch had grown to approximately 70 ft wide and an overall curved length of 1000 ft. The photo in figure 1c was taken an hour and a half later at 14:43 from 5000 ft. The dye field at this time is 2100 ft long and 1300 ft wide.

The example given in figure 1 was an extreme example of elongation, curvature and striation of a dye patch. The wind was 5 to 12 knots with a swell height of 4 to 6 ft and a water current velocity of 0.4 ft/sec. Most dye patches observed have been elongated in a direction nearly parallel to that of the water flow. Striations and curvature of the dye patch are common; however, in most experiments the overall shape of the patch resembles an ellipse.
Figure 1. Aerial photographs of a dye patch on July 7, 1969.

- **a**
  At 12:25 from 3000 ft

- **b**
  At 13:13 from 4000 ft

- **c**
  At 14:43 from 5000 ft
The elongation of the dye patch in the direction of flow suggests that dispersion due to a vertical velocity gradient can be an important consideration. The upper layers of water are first influenced by a change in wind velocity or direction and vertical water current gradients are formed. Wind waves and swell both have specific orientation and may cause diffusion to occur at different rates in a horizontal plane.

The basic diffusion equation given as equation 1 is reduced to a two dimensional model.

\[ \frac{\partial W}{\partial t} = D_y \frac{\partial^2 W}{\partial y^2} + D_x \frac{\partial^2 W}{\partial x^2} + aW \]  

(8)

A solution is

\[ W(x, y, t) = W_{\text{max}} \exp \left[ -\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2} \right] \]  

(9)

where the coordinate axis is assumed to move with the dye patch. In the equation \( W \) represents the dye concentration, \( x \) and \( y \) are the coordinates from the centroid of the dye patch parallel and transverse to the direction of flow, \( D_x \) and \( D_y \) are the longitudinal and lateral diffusion coefficients, \( a \) is a first order decay coefficient (which includes the loss to the lower layers due to vertical diffusion) and \( \sigma_x^2 \) and \( \sigma_y^2 \) represent the variances in
the X and Y directions. The relationship between the change in variance and the diffusion coefficient is given by

$$D = \frac{1}{2} \frac{\Delta \sigma^2}{\Delta t}$$  \hspace{1cm} (10)

The diffusion coefficient is equal to one half the change in variance ($\Delta \sigma^2$) divided by the time interval ($\Delta t$).

By dividing equation 9 by the maximum concentration at the centroid ($W_{\text{max}}$), taking the log of each side and multiplying by 2, the equation becomes

$$\frac{X^2}{\sigma_x^2} + \frac{Y^2}{\sigma_y^2} = 2 \ln \left[ \frac{W_{\text{max}}}{W} \right].$$  \hspace{1cm} (11)

Letting

$$a^2 = 2\sigma_x^2 \ln \left[ \frac{W_{\text{max}}}{W} \right]$$  \hspace{1cm} (12)

$$b^2 = 2\sigma_y^2 \ln \left[ \frac{W_{\text{max}}}{W} \right]$$  \hspace{1cm} (13)

equation 11 reduces to that of an ellipse.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$  \hspace{1cm} (14)
where a and b are the major and minor semi axis of an ellipse fitted to a line of equal concentration about the dye patch. Since the edge of the dye patch generally formed an irregularly shaped boundary and is characterized by relatively steep concentration gradient, the visible boundary of the patch was assumed to have a concentration of \( \frac{W_{\text{max}}}{2} \). With this assumption equations 12 and 13 reduce to

\[
\sigma_x^2 = 0.72a_e^2 \quad (15)
\]
\[
\sigma_y^2 = 0.72b_e^2 \quad (16)
\]

By computing the variances in both the longitudinal and transverse direction from two flights over the area, the diffusion coefficients can be determined from equation 10.

Since the dye patches seldom form a perfect ellipse, the major and minor semi axes of the patch can not be measured directly. In the computations, the irregular-shaped dye patch is replaced with an equivalent ellipse that has the same horizontal area and the same ratio of transverse to longitudinal variances as the real patch. The photographic coordinates of both the control points and the outlines of the dye patches are measured with an x-y coordinatograph. The control
points are necessary for photographic orientation as described in the following section.

PHOTOGRAPH ORIENTATION

Photographic orientation is accomplished by a non-linear solution to the collinearity condition equations (Keller and Tewinkel, 1966). Corrections are generally not required for atmospheric refraction, earth curvature, film shrinkage or lens distortion. The relationship between photo coordinates and ground coordinates is

\[
\begin{bmatrix}
  x_p - x_o \\
  y_p - y_o \\
  -f
\end{bmatrix}
- K[RM]
\begin{bmatrix}
  X_p - Y_c \\
  Y_p - Y_c \\
  Z_p - Z_c
\end{bmatrix} = 0
\]

where \(x_p\) and \(y_p\) are photo coordinates of image point \(p\), \(f\) is the camera focal length, \(x_o\) and \(y_o\) are photo coordinates of the principal point, \(K\) is a scale factor, the \(X, Y,\) and \(Z\) subscripted \(p\) and \(c\) refer to ground coordinates of the object and camera station respectively, and \(RM\) is the rotational matrix. The matrix is defined as
where: $m_{11} = \cos(\phi) \cos(K_a)$

$m_{12} = \cos(W_0) \sin(K_a) + \sin(W_0) \sin(\phi) \cos(K_a)$

$m_{13} = \sin(W_0) \sin(K_a) - \cos(W_0) \sin(\phi) \cos(K_a)$

$m_{21} = -\cos(\phi) \sin(K_0)$

$m_{22} = \cos(W_0) \cos(K_a) - \sin(W_0) \sin(\phi) \sin(K_a)$

$m_{23} = \sin(W_0) \cos(K_a) + \cos(W_0) \sin(\phi) \sin(K_a)$

$m_{31} = \sin(\phi)$

$m_{32} = \sin(W_0) \cos(\phi)$

$m_{33} = \cos(W_0) \cos(\phi)$

The three parameters $W_0$, $\phi$ and $K_a$ are the photographic rotations about the $X$, $Y$ and $Z$ axis, respectively.

The collinearity equations are obtained by dividing the first and second rows of equation 17 by the third row hereby eliminating the scale factor.

$$\frac{x_p - x_0}{-f} = \frac{m_{11}(X_p - X_c) + m_{12}(Y_p - Y_c) + m_{13}(Z_p - Z_c)}{m_{31}(X_p - X_c) + m_{32}(Y_p - Y_c) + m_{33}(Z_p - Z_c)}$$

(19)

$$\frac{y_p - y_0}{-f} = \frac{m_{21}(X_p - X_c) + m_{22}(Y_p - Y_c) + m_{23}(Z_p - Z_c)}{m_{31}(X_p - X_c) + m_{32}(Y_p - Y_c) + m_{33}(Z_p - Z_c)}$$

(20)
These equations insure that the camera station, image and object lie on a straight line. For each point two collinearity equations can be written. As there are six unknowns \((X_c, Y_c, Z_c, \omega, \phi, \text{ and } K_a)\) a minimum of three noncollinear control points are required for their solution. However, a least-squares solution permits the use of an unlimited number of control points.

The solution to the equations is obtained based on a set of initial approximations which are adjusted iteratively until the adjustments become small. The collinearity equations are linearized by the Taylor series with the expansion terminated at the first derivative.

When the initial approximations of \(B_j\) are close to the actual parameters values \((B)\)

\[
 f(B) = F(B_j) + \sum_{i=1}^{6} \frac{\partial f(B)}{\partial B_i} \Delta B_i \quad (21)
\]

Letting

\[
 Y = f(B) - f(B_j) \quad (22)
\]

\[
 Z_i = \frac{\partial f(B)}{\partial B_i} \quad \text{for } B = B_j \quad (23)
\]
then

\[ Y = \sum_{i=1}^{6} Z_i \Delta B_i + \epsilon \]

which is a linear form of the collinearity equations. The least squares solution in matrix notation is

\[ B = [Z^T Z]^{-1} Z^T Y \] (24)

The initial approximations of the parameters \( B_j \) are replaced by

\[ B_{j+1} = B_j + \Delta B_j \] (25)

This iterative process is continued until the solution converges, that is, until all \( \Delta B \)'s are less than some prespecified amount. In this space resection problem no test is made on the linear adjustments but the solution is terminated when the angular adjustments are less than about two seconds of arc.

By knowing the photo orientation, it is possible to determine the position vectors for any point on the photograph. The position vector \( X \) based on the state plane coordinate system axis is related to the photographic vector \( X_p \) by the equation
The ground coordinate of any point on the photograph can be computed from the unit position vector and the camera station coordinates.

**COMPUTATIONS**

The coordinates of the outline of the dye patches were measured with the x-y coordinatograph for both flights. Photo coordinates about the dye patch are converted to state plane coordinates and the centroid and moments of inertia computed from the following equations.

\[
A = \frac{1}{2} \sum_{i=1}^{n} X_i (Y_{i-1} - Y_{i+1}) \tag{27}
\]

\[
X = \frac{\sum X_i \Delta A_i}{A} \tag{28}
\]

\[
Y = \frac{\sum Y_i \Delta A_i}{A} \tag{29}
\]

\[
I_x = \sum X_i^2 \Delta A_i \tag{30}
\]

\[
X = [RM]^{-1}X_p \tag{26}
\]
\[
I_y = \sum Y_i^2 \Delta A_i \quad (31)
\]
\[
I_{xy} = \sum X_i Y_i \Delta A_i \quad (32)
\]

Care must be taken when programming these equations to avoid roundoff error. State plane coordinates are typically six to seven digit numbers.

The rotation angle (\(\alpha\)) of the axis to obtain the principal axis is given by

\[
\tan (2\alpha) = \frac{2I_{xy}}{I_x - I_y} \quad (33)
\]

The maximum and minimum moments of inertia about the principal axis are computed from

\[
I_{\text{max}} = \frac{I_x - I_y}{2} + \left[\frac{(I_x - I_y)}{2} + I_{xy}^2\right]^{1/2} \quad (34)
\]
\[
I_{\text{min}} = \frac{I_x - I_y}{2} - \left[\frac{(I_x - I_y)}{2} + I_{xy}^2\right]^{1/2} \quad (35)
\]

The irregular shaped dye patch is replaced with an equivalent ellipse that has the same horizontal area. In addition, the ratio of \(I_{\text{max}}\) to \(I_{\text{min}}\) are the same for the ellipse as for the dye patch. The principal moments of inertia for an ellipse are given by
where $a_e$ and $b_e$ are the major and minor semi axes of an equivalent ellipse. If the ratio (IR) of $I_{max}$ to $I_{min}$ remain the same for both the dye patch and the ellipse, the ratio of the variances are also the same (equations 15 and 16).

$$IR = \frac{I_{max}}{I_{min}} = \frac{a_e^2}{b_e^2} = \frac{\sigma_x^2}{\sigma_y^2}$$  \hspace{1cm} (38)

After computing the area (A) and the ratio of principal moments of inertia (IR) for the dye patch, the semi axes of the equivalent ellipse are determined from

$$a_e^2 = \frac{A}{\pi} (IR)^{-1/2}$$  \hspace{1cm} (39)

$$b_e^2 = \frac{A}{\pi} (IR)^{-1/2}$$  \hspace{1cm} (40)

Equations 15 and 16 provide the relationship between the semi axes and the variances of the equivalent ellipse. The real dye patch is replaced with an ellipse that has the same area and the same ratio of transverse
to longitudinal variances as the irregular-shaped, original patch. The diffusion coefficients are determined from the change in variances by equation 10 while the current velocity is determined from the change in position of the centroid of the dye patch between flights. The required time interval between photographic flights depends on the accuracy of orientation and positioning, but generally 30 to 60 minutes is adequate.

SUMMARY

This report presented a simplified procedure for determining water current velocities and diffusion coefficients. Dye drops which form dye patches in the receiving water are made from an aircraft. The changes in position and size of the patches are recorded from two flights over the area. The simplified data processing procedure requires only that the ground coordinates about the dye patches be determined at the time of each flight. With an automatic recording coordinatograph for measuring coordinates and a computer for processing the data, this technique provides a practical method of determining circulation patterns and mixing characteristics of large aquatic systems.
An effective waste management program for the preservation of water quality requires that the fate and effect of waste loadings to be predicted before they are allowed to be discharged. Mathematical models of the receiving waters can be used for making these predictions; however, they require as input the mixing and circulation characteristics of the aquatic system. Remote sensing offers a feasible method of measuring both the diffusion coefficients and the water currents of a receiving waters. This information is necessary to assess the environmental impact of waste water discharges and for industrial plant siting.
REFERENCES


The REMOTE SENSING CENTER was established by authority of the Board of Directors of the Texas A&M University System on February 27, 1968. The CENTER is a consortium of four colleges of the University: Agriculture, Engineering, Geosciences, and Science. This unique organization concentrates on the development and utilization of remote sensing techniques and technology for a broad range of applications to the betterment of mankind.