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CALCULATION OF TURBULENT BOUNDARY LAYERS WITH HEAT TRANSFER AND PRESSURE GRADIENT UTILIZING A COMPRESSIBILITY TRANSFORMATION

Part I - Summary Report

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16. Abstract This report presents the results of a theoretical investigation of compressible turbulent boundary layer flow with heat transfer and pressure gradient. The analysis utilizes a compressibility transformation but differs from previous work in that higher order closure rules are utilized to complete the transformation. Specifically, by requiring that the momentum equations in differential form be satisfied at the wall and at the sublayer edge, correspondence rules are obtained which relate the variable property (VP) flow of interest to a constant property (CP) flow in which mass transfer and pressure gradient occur simultaneously. To implement this approach a new CP formulation is developed which includes both of these effects. A computer program based on these formulations, together with a Crocco integral representation for the energy field, is developed. Details of the CP analysis and the computational procedures are presented in companion documents. Numerical results for a variety of cases are presented in this report. Comparisons with earlier forms of the transformation and with experiment are also included. For the zero pressure gradient case some differences between the various predictions are observed. However, for the several pressure gradient cases which are examined, the results obtained are found to be essentially identical to those given by first order closure rules; i.e., by a form of transformation which relates the VP flow to a CP flow with pressure gradient but zero mass transfer.

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FOREWORD

The present report is one of a series of three reports describing a new computer program which predicts turbulent boundary layer behavior under conditions involving both heat transfer and pressure gradient. Part I serves as a summary report and describes the general analysis which is utilized in the numerical calculation scheme. In Part II the requisite low speed formulation, consisting of a constant property flow with combined pressure gradient and mass transfer is described. Part III describes the numerical and computational procedures involved and serves as a computer program manual.

The titles in the series are:

Part I - Summary Report - "Calculation of Turbulent Boundary Layers with Heat Transfer and Pressure Gradient Utilizing a Compressibility Transformation," by C. Economos and J. Boccio.

Part II - "Constant Property Turbulent Boundary Layer Flow with Simultaneous Mass Transfer and Pressure Gradient," by J. Boccio and C. Economos.

Part III - "Computer Program Manual," by J. Schneider and J. Boccio

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SUMMARY

This report presents the results of a theoretical investigation of compressible turbulent boundary layer flow with heat transfer and pressure gradient. The analysis utilizes a compressibility transformation but differs from previous work in that higher order closure rules are utilized to complete the transformation. Specifically, by requiring that the momentum equations in differential form be satisfied at the wall and at the sublayer edge, correspondence rules are obtained which relate the variable property (VP) flow of interest to a constant property (CP) flow in which mass transfer and pressure gradient occur simultaneously. To implement this approach a new CP formulation is developed which includes both of these effects. A computer program based on these formulations, together with a Crocco integral representation for the energy field, is developed. Details of the CP analysis and the computational procedures are presented in companion documents. Numerical results for a variety of cases are presented in this report. Comparisons with earlier forms of the transformation and with experiment are also included. For the zero pressure gradient case some differences between the various predictions are observed. However, for the several pressure gradient cases which are examined, the results obtained are found to be essentially identical to those given by first order closure rules; i.e., by a form of transformation which relates the VP flow to a CP flow with pressure gradient but zero mass transfer.

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PART I: SUMMARY REPORT

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I. INTRODUCTION

In a superficial sense, the overall objective of the current investigation is identical to that of previous work reported in Reference 1. That is, both investigations address themselves to the same problem area (VP turbulent boundary layers) and both utilize a "compressibility transformation" approach to treat this problem. Nevertheless, the objectives differ in a very significant way since the present investigation is particularly concerned with development of a more self-consistent form of transformation which eliminates some of the anomalous behavior observed in the earlier work.

For a complete review of these deficiencies, the reader is referred to Reference 1. For the present purpose, it suffices to say that a major portion of the work to be described here involved development of a more general form of transformation which would allow introduction of an additional free parameter.

As outlined in Reference 2, such a generalization would imply a correspondence between the general VP flow case,and a CP flow with mass transfer and pressure gradient such that, even for the zero pressure gradient VP case, both mass transfer and pressure gradient would persist in the CP flow. Such a result appears to be attractive a priori since mechanisms are thereby introduced which might be expected to compensate for the two most striking deficiencies observed in the transformation; namely distortion of the wake portion of the velocity profile (Reference 3) and overestimation of skin friction coefficient at high heat transfer rates (Reference 4). This expectation is based, first of all, on the demonstrated improvement in skin friction prediction (Reference 2) which has been achieved for the high heat transfer case by use of a form of transformation which relates constant pressure VP flow with heat transfer to constant pressure CP flow with mass transfer. So far as the wake distortion is concerned, no corresponding development has been reported.* Nevertheless, since the wake distortion can qualitatively be associated with the existence of a pressure gradient (c.f., Reference 3), it may be anticipated that the form of transformation proposed here will simultaneously provide improvement in both areas of concern.

In order to exploit the proposed modification it is necessary, of course, to have available a suitable CP formulation which describes turbulent boundary layer development under the influence of both mass transfer and pressure gradient. At the inception of this program such a formulation did not exist and one has been developed during the current study. The details of this development, including presentation of numerical results and comparisons with experimental data, are given in Part II of this report. Subsequent to this development, a similar analysis was carried out by other investigators (Reference 5). Comparison of the two methods is also included in the aforementioned companion document.

The material presented in the subsequent sections is structured as follows. First, the modified closure rules which provide the desired correspondence are derived. We note here that two types are considered, one of which is rejected on the grounds that it does not exhibit uniformly valid behavior for limiting cases. Development of the working equations is then outlined with most of the algebraic detail either presented in the Appendices or omitted completely by citing appropriate references where such detail has previously been presented. Finally, a series of numerical results is presented and compared with appropriate experimental data and with predictions due to earlier forms of transformation. These results include zero and arbitrary pressure gradient cases both with and without heat transfer.

* In Reference 2, an empirical correction is developed to account for this effect. However, this does not represent a self-consistent modification of the compressibility transformation itself.

II. SYMBOLS

A1, ... A64	See Eq. (33) and Appendix B
A_{ij}	See Eq. (34)
A_i	See Eq. (46)
C2, C3, C4, C8	See Eq. (33) and Appendix B
C_i	See Eq. (34) and (46)
c_f, $\bar{c}_f$	$2\tau_w/\rho_e u_e^2$, $2\bar{\tau}_w/\bar{\rho}\bar{u}_e^2$
F_i	CP functional forms; see Part II
G_i	See Eq. (14), (21) and (26)
H_i	See Eq. (36) and (39)
H, $\bar{H}$	δ^*/θ, $\bar{\delta}^*/\bar{\theta}$
k_1,k_2	CP law of the wall constants; see Part II
ℓ	reference length
M	Mach number
$\tilde{m}_e$	See Eq. (31)
p, $\bar{p}$	pressure
P, $\bar{P}$	See Eq. (14)
R	gas constant
$\bar{R}$	$\bar{u}_{e_o}\ \bar{\delta}/\bar{\nu}$
R_{e_o}	$(u_e/\nu_e)_o$
R_y,$R_{\bar{y}}$	$u_e y/\nu_e$, $\bar{u}_e\bar{y}/\bar{\nu}$

R_δ, $R_{\bar{\delta}}$	$u_e\delta/\nu_e$, $\bar{u}_e\bar{\delta}/\bar{\nu}$
R_θ, $R_{\bar{\theta}}$	$u_e\theta/\nu_e$, $\bar{u}_e\bar{\theta}/\bar{\nu}$
T, T_t	temperature, total temperature
$\tilde{T}$	T/T_e
u, $\bar{u}$	streamwise velocity components
$\tilde{u}$	$u/u_e = \bar{u}/\bar{u}_e$
$\bar{u}^+$	$\bar{u} \cdot (\bar{\tau}_w/\bar{\rho})^{-\frac{1}{2}}$
U_e, $\bar{U}_e$	u_e/u_{e_o}, $\bar{u}_e/\bar{u}_{e_o}$
v, $\bar{v}$	normal velocity components
$\bar{v}^+_w$	$\bar{v}_w(\bar{\tau}_w/\bar{\rho})^{-\frac{1}{2}}$
$\bar{V}_w$	$\bar{v}_w/\bar{u}_{e_o}$
W	T_w/T_{t_e}
x, $\bar{x}$	streamwise coordinates
y, $\bar{y}$	normal coordinates
$\bar{y}^+_s$	$(\bar{y}_s/\bar{\nu})(\bar{\tau}_w/\bar{\rho})^{\frac{1}{2}}$
Z, $\bar{Z}$	See Eq. (14)
α	viscosity exponent
γ	isentropic exponent
Γ	See Appendix C
δ, $\bar{\delta}$	boundary layer thicknesses (.995 u_e, .995 $\bar{u}_e$)
δ^*, $\bar{\delta}^*$	displacement thicknesses
η, ξ, σ	the transformation parameters

$\tilde{\eta}$	$\rho_e \eta/\bar{\rho}$
$\bar{\eta}$	$\bar{y}/\bar{\delta}$
$\theta,\ \bar{\theta}$	momentum thicknesses
$\lambda_1,\ \lambda_2$	See Eq. (54) and Appendix D
$\Lambda_1,\ \Lambda_2$	CP functional forms - see Part II
$\mu,\ \bar{\mu}$	molecular viscosities
$\tilde{\mu}$	μ/μ_e
$\nu,\ \bar{\nu}$	$\mu/\rho,\ \bar{\mu}/\bar{\rho}$
π	CP wake parameter, see Eq. (67) and (68)
$\rho, \bar{\rho}$	densities
$\tilde{\rho}$	ρ/ρ_e
$\tilde{\sigma}$	$\mu_e \sigma/\bar{\mu}$
$\bar{\Sigma}$	$\bar{\theta}/\bar{\delta}$
$\tau,\ \bar{\tau}$	shear stress (including Reynolds stresses)
Υ	see Appendix C
$\bar{\varphi}$	$(\bar{c}_f/2)^{-\frac{1}{2}}$
$\chi,\ \bar{\chi}$	$(u_e/\nu_e)_o\ (x-x_o)\ ,\ \bar{u}_{e_o}\ (\bar{x}-\bar{x}_o)/\bar{\nu}$
$\psi,\ \bar{\psi}$	stream functions $\partial\psi/\partial y = \rho u\ ;\ \partial\psi/\partial x = \rho_w v_w - \rho v$ $\partial\bar{\psi}/\partial\bar{y} = \bar{\rho}\bar{u}\ ;\ \partial\bar{\psi}/\partial\bar{x} = \bar{\rho}\bar{v}_w - \bar{\rho}\ \bar{v}$

Subscripts

e	external conditions
o	initial values
s	sublayer edge conditions
w	wall conditions

Superscripts

$(\bar{\ })$	variables of the CP flow
$(\ )'$	differentiation w.r.t x
(1)	evaluation at $\bar{\eta} = 1$
$(\bar{\eta}^*)$, $*$	evaluation at $\bar{\eta}=\bar{\eta}^*=0.5$ (see Part II)

III. ANALYSIS

A. General Considerations

Most of the detailed development of the original Coles' compressibility transformation (Reference 6) has been worked out many times (c.f., References 1, 7 and 8) and need not be repeated here. For the present purpose, it suffices to note that by introduction of three x-dependent stretching parameters, ξ, η, σ, for the x and y coordinates and the stream function ψ, a VP flow with fluid dynamic behavior described by

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{1}$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y} - \frac{dp}{dx} \tag{2}$$

is transformed to a companion CP flow given by

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{3}$$

$$\bar{\rho}\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\rho}\bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial \bar{\tau}}{\partial \bar{y}} - \frac{d\bar{p}}{d\bar{x}} \tag{4}$$

To obtain closure (i.e., to complete the transformation) sufficient restraints must be imposed to define not only the transformation parameters, ξ, η and σ, but the pressure gradient term $d\bar{p}/d\bar{x}$ since this must be considered as an unknown in this formulation. If mass transfer is involved, the CP transpiration rate $\bar{\rho}\bar{v}_w$ is an additional unknown requiring still another restraint. Thus, for the most general case a total of five relations are required to achieve closure.

Up until the present time this more general case has not been worked out in detail although in Reference 2 a possible general procedure was outlined.*In Coles' original work, closure for the

* The general case has also been examined by Lewis (Ref. 9) but from a somewhat different point of view. A discussion of the differences and similarities of the two approaches may be found in Ref. 2.

special case of zero pressure gradient and zero transpiration in both planes was achieved by (a) satisfying Equations (2) and (4) at the outer edge of the boundary layer, (b) satisfying integrals of these equations from the wall to the edge, and (c) invoking the so-called sub-structure hypothesis which assumes the invarience, under the transformation, of a characteristic Reynolds number based on average thermodynamic properties within the viscous layer. Libby and Baronti (Reference 8) examined the pressure gradient case but with zero mass transfer. They also employed (a) and (b) but replaced (c) with the sub-layer hypothesis which evaluates the characteristic Reynolds number at the sublayer edge. The fourth restraint required to achieve closure in this case was obtained by satisfying Equations (2) and (4) at the wall. Economos (Reference 7) considered the mass transfer case with zero pressure gradient. Closure was achieved in a manner similar to that of Reference 8 but with the blowing rate $\overline{\rho v}_w$ replacing $d\bar{p}/d\bar{x}$ as an unknown.

With the exception of the sub-layer (or substructure) hypothesis, all of the restraints utilized by the various investigators can be thought of as "compatibility" conditions on the various unknown parameters which assure satisfaction of the describing equations at specified points in the flow regime. Any number of such conditions can be set down in a variety of ways so that, in principle, additional free parameters could be introduced with closure still being possible. For example, the approach used in Reference 2 to treat the case of simultaneous mass transfer and pressure and pressure gradient proposed utilizing the first derivatives of Equations (2) and (4) evaluated at the wall to achieve closure. Alternately, Equations (2) and (4) could be evaluated at points other than the wall; at the sublayer edge, say.

The point of view enumerated above implies non-uniqueness of solution to the problem of interest. Thus, to assess the usefulness of any particular choice, the entire formulation must be developed and implemented to the point where prediction of turbulent boundary layer behavior can be generated and compared with experiment. In the next section, two such possibilities are examined in detail. These include the use of the first derivatives of Equations (2) and (4) evaluated at the wall which will be referred to as "closure by wall compatibility." It will be shown that this particular choice is unsatisfactory by virtue of non-uniformly valid behavior for limiting cases of zero mass and heat transfer and zero pressure gradient. The second choice involves satisfying Equations (2) and (4) at the sublayer edge. This will be denoted by "closure by collocation." It is found that this second choice yields well behaved solutions for all ranges of conditions.

B. The Second Order Closure Relations

Closure by Wall Compatibility. - The starting point for deriving the closure relations is Equations (2) and (4) together with certain relations between the VP and CP flow parameters which follow from the basic transformation rules. The pertinent results, taken from Reference 1, are

$$\frac{\partial}{\partial y} = \eta \frac{\bar{\rho}}{\rho} \frac{\partial}{\partial \bar{y}} \tag{5}$$

$$u = \frac{\eta}{\sigma} \bar{u} \tag{6}$$

$$u/u_e = \bar{u}/\bar{u}_c \equiv \tilde{u} \tag{7}$$

$$\left(\frac{\partial u}{\partial y}\right)_w = \frac{\rho_w}{\bar{\rho}} \frac{\eta^2}{\sigma} \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_w \tag{8}$$

Consider now Equation (2) evaluated at y = 0. There results

$$\rho_w v_w \left(\frac{\partial u}{\partial y}\right)_w = \left(\frac{\partial}{\partial y} \mu \frac{\partial u}{\partial y}\right)_w - \frac{dp}{dx}$$

where mass transfer in the VP plane as reflected by the coefficient $\rho_w v_w$ has been included for generality and it has been recognized that in the vicinity of the wall the shear takes on a laminar form. Carrying out the indicated differentiation leads to

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_w = \left\{\frac{\rho_w v_w}{\mu_w} - \frac{1}{\mu_w}\left(\frac{\partial \mu}{\partial y}\right)_w\right\}\left(\frac{\partial u}{\partial y}\right)_w - \frac{1}{\mu_w} \rho_e u_e \frac{du_e}{dx} \tag{9}$$

where we have used the Euler relation $\rho_e u_e du_e/dx = - dp/dx$. If the same procedure is applied to Equation (4) there results

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_w = \frac{\bar{\rho}\bar{v}_w}{\bar{\mu}}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_w - \frac{1}{\bar{\mu}}\,\bar{\rho}\bar{u}_e\,\frac{d\bar{u}_e}{d\bar{x}} \tag{10}$$

which is identical in form to Equation (9) except that the viscosity derivative vanishes since, for CP flow, $\bar{\mu}$ is a constant. We now note that the relations (5), (6), and (8) imply

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_w = \frac{\sigma}{\eta^3}\left(\frac{\bar{\rho}}{\rho_w}\right)^2 \left\{\left(\frac{\partial^2 u}{\partial y^2}\right)_w - \frac{1}{\rho_w}\left(\frac{\partial \rho}{\partial y}\right)_w\left(\frac{\partial u}{\partial y}\right)_w\right\} \tag{11}$$

which permits elimination of the second derivatives appearing in Equations (9) and (10). There results

$$\frac{\bar{\rho}\bar{v}_w}{\bar{\tau}_w} - \frac{\bar{\rho}\bar{u}_e\bar{\mu}}{\bar{\tau}_w^2}\frac{d\bar{u}_e}{d\bar{x}} = \frac{\eta}{\sigma}\left\{\frac{\rho_w v_w}{\tau_w} - \frac{1}{(\partial u/\partial y)_w}\left[\frac{1}{\mu_w}\left(\frac{\partial \mu}{\partial y}\right)_w + \frac{1}{\rho_w}\left(\frac{\partial \rho}{\partial y}\right)_w\right]\right.$$
$$\left. - \frac{\rho_e u_e \mu_w}{\tau_w^2}\frac{du_e}{dx}\right\} \tag{12}$$

where we have recognized that the wall shear is related to the velocity field by

$$\bar{\tau}_w = \bar{\mu}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_w \quad ; \quad \tau_w = \mu_w\left(\frac{\partial u}{\partial y}\right)_w$$

In the further development it will be assumed that the thermodynamics can be related to the velocity field by means of a Crocco integral. Accordingly, we can write

$$\left(\frac{\partial \mu}{\partial y}\right)_w = \left(\frac{\partial \mu}{\partial u}\right)_w\left(\frac{\partial u}{\partial y}\right)_w$$
$$\left(\frac{\partial \rho}{\partial y}\right)_w = \left(\frac{\partial \rho}{\partial u}\right)_w\left(\frac{\partial u}{\partial y}\right)_w \tag{13}$$

Then Equation (12) can be written

$$\bar{Z} - \bar{P} = Z - P + G_{1_w} \tag{14}$$

where the following definitions have been introduced

$$\bar{Z} = \left(\frac{2}{\bar{c}_f}\right)\left(\frac{\overline{\rho v}_w}{\bar{\rho} \bar{u}_e}\right) \quad ; \quad Z = \left(\frac{2}{c_f}\right)\left(\frac{\rho_w v_w}{\rho_e u_e}\right)$$

$$\bar{P} = \left(\frac{2}{\bar{c}_f}\right)^2 \left(\frac{1}{\bar{U}_e} \frac{d\ln \bar{U}_e}{d\bar{\chi}}\right) \quad ; \quad P = \tilde{\mu}_w \left(\frac{2}{c_f}\right)^2 \frac{(U_e/\nu_e)_o}{(U_e/\nu_e)} \frac{d\ln U_e}{d\chi}$$

$$G_1 = -\frac{\partial \ln \tilde{\rho}\tilde{\mu}}{\partial \tilde{u}} \quad ; \quad G_{1_w} = -\left(\frac{\partial \ln \tilde{\rho}\tilde{\mu}}{\partial \tilde{u}}\right)_w$$

and the remaining parameters U_e, c_f, $\tilde{\rho}$, etc., have been defined in the List of Symbols. Note that the terms appearing in Equation (14) are of three types; terms proportional to an injection rate (Z, $\bar{Z}$), terms proportional to velocity gradients (P, $\bar{P}$), and a term G_{1_w} which depends solely upon the thermodynamics of the VP fluid.

It is useful at this point to show how special forms of Equation (14) have been utilized by other investigators to obtain closure. In addition, the effect of the fluid thermodynamics on the CP flow behavior will also be examined qualitatively. The latter is best accomplished by considering a specific thermodynamic system in order to evaluate explicitly the parameter G_{1_w}. For this purpose we assume that the CP fluid is a perfect gas with an exponential viscosity-temperature relation and unity Prandtl number. It is further assumed, for the case of injection, that the injectant gas is identical to the external fluid. Then the Crocco integral for total enthalpy implies

$$G_{1_w} = \frac{(1-\alpha)(1-W)}{W} \tag{15}$$

where α denotes the viscosity exponent (for most gases $\alpha < 1$) and W is the temperature ratio T_w/T_{t_e}. Evidently, for the adiabatic wall case ($W = 1$), Equation (15) implies that $G_{1_w} = 0$ while for heat transfer to the surface ($W<1$), $G_{1_w} > 0$. Note however, for the special case $\alpha = 1$ (i.e., a linear viscosity temperature relation) $G_{1_w} = 0$.

Now consider the special case $\bar{Z} = Z = 0$, which is the flow configuration examined in References 1 and 8. The closure relation becomes, from Equation (14)

$$\bar{P} = P - G_{1_w} \tag{16}$$

which implies some interesting possibilities insofar as the CP pressure gradient is concerned when $\alpha \neq 1$. It is apparent, for example, that $\bar{P}$ does not vanish when $P = 0$ except for the adiabatic wall case. It is also possible, of course, to achieve the case $\bar{P} \equiv 0$ by considering a VP flow where $P - G_{1_w} \equiv 0$; whether this corresponds to a physically realizable configuration is not known at this time and will not be considered further.

The most relevant possibility, for the present purpose, is the situation corresponding to VP flow with heat transfer and zero pressure gradient. This is of particular interest here in view of the fact that much of the anomalous behavior exhibited by earlier forms of the transformation has been associated with experimental data obtained under such conditions.

From Equation (16) with $P = 0$ we note that

$$\bar{P} = \gtrless 0 \text{ when } G_{1_w} \lessgtr 0.$$

That is, with heat transfer to the wetted surface ($G_{1_w} > 0$), this particular closure rule implies the existence of adverse pressure gradient ($\bar{P} < 0$) in the CP plane, while for the "hot wall" case ($G_{1_w} < 0$) a favorable pressure gradient occurs. The first of these results appears to be qualitatively inconsistent with observation, since, as pointed out in Reference 3, the distortion of the wake component is such as to imply a favorable pressure gradient. Note however that this distortion has been observed even for adiabatic flow for which this particular formulation yields no effect.

A somewhat parallel situation occurs for flow with mass transfer but zero pressure gradient as treated in Reference 7. The closure relation in this case was taken to be

$$\bar{Z} = Z + G_{1_w} \tag{17}$$

which follows from Equation (14) when $\bar{P} = P = 0$. It is now easy to see how the use of two different closure rules yields different solutions for the same physical problem. That is, the zero mass transfer, zero pressure gradient, VP case can be associated either to a CP flow with pressure gradient given by

$$\bar{P} = - G_{1_w} \tag{18}$$

or with constant pressure flow with transpiration (or suction) given by

$$\bar{Z} = G_{1_w} \tag{19}$$

The second of these formulations, of course, contributes nothing in the way of explaining the wake distortion. However, as demonstrated in Reference 2, it can improve skin friction prediction for the cold wall case $G_{1_w} > 0$. It might be anticipated that some intermediate combination of these effects would provide improvement in both areas, which is, of course, the motivation for the current investigation.

Returning now to the general problem, we require an additional relation involving the parameters $\bar{P}$ and $\bar{Z}$. Here we utilize the "second wall compatibility" relation obtained by differentiating Equations (2) and (4) with respect to y and $\bar{y}$, respectively, and evaluating these at y, $\bar{y}$ = 0.

This procedure yields

$$\left(\frac{\partial^3 u}{\partial y^3}\right)_w = \left\{ \frac{\rho_w v_w}{\mu_w} - \frac{2}{\mu_w}\left(\frac{\partial \mu}{\partial y}\right)_w \right\} \left(\frac{\partial^2 u}{\partial y^2}\right)_w - \frac{1}{\mu_w}\left(\frac{\partial^2 \mu}{\partial y^2}\right)_w \left(\frac{\partial u}{\partial y}\right)_w$$

$$\left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right)_w = \frac{\bar{\rho}\bar{v}_w}{\bar{\mu}} \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_w$$

But Equations (5) and (6) imply the following relation between the third derivatives:

$$\left(\frac{\partial^3 \bar{u}}{\partial \bar{y}^3}\right)_w = \left(\frac{\bar{\rho}}{\rho_w}\right)^3 \left(\frac{\sigma}{\eta^4}\right) \left\{ \left(\frac{\partial^3 u}{\partial y^3}\right)_w - \frac{3}{\rho_w} \left(\frac{\partial \rho}{\partial y}\right)_w \left(\frac{\rho_w}{\bar{\rho}}\right)^2 \left(\frac{\eta^3}{\sigma}\right) \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_w - \right.$$

$$\left. \frac{1}{\rho_w} \left(\frac{\partial^2 \rho}{\partial y^2}\right)_w \left(\frac{\rho_w}{\bar{\rho}}\right) \left(\frac{\eta^2}{\sigma}\right) \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_w \right\}$$

Now combine these last three equations to obtain

$$\bar{z}\left(\frac{\partial^2 \tilde{u}}{\partial \bar{y}^2}\right)_w = \left(\frac{\bar{\rho}}{\rho_w \eta}\right)^3 \left\{\left[\bar{z} - \frac{2}{\mu_w}\left(\frac{\partial \mu}{\partial \tilde{u}}\right)_w - \frac{3}{\rho_w}\left(\frac{\partial \rho}{\partial \tilde{u}}\right)_w\right]\left(\frac{\partial^2 \tilde{u}}{\partial y^2}\right)_w \right. +$$

$$\left. \frac{3}{\rho_w^2}\left(\frac{\partial \tilde{\rho}}{\partial u}\right)_w^2 - \left[\frac{1}{\rho_w}\left(\frac{\partial^2 \rho}{\partial y^2}\right)_w + \frac{1}{\mu_w}\left(\frac{\partial^2 \mu}{\partial y^2}\right)_w\right]\right\} \tag{20}$$

where we have also utilized (7), (8), and (13). To proceed, it is necessary to relate the last term on the right-hand-side of Equation (20) to the velocity field. Using Equation (13) it is easy to show that

$$\frac{1}{\rho_w}\left(\frac{\partial^2 \rho}{\partial y^2}\right)_w = \frac{1}{\rho_w}\left(\frac{\partial \rho}{\partial \tilde{u}}\right)_w\left(\frac{\partial^2 \tilde{u}}{\partial y^2}\right)_w + \frac{1}{\rho_w}\left(\frac{\partial^2 \rho}{\partial \tilde{u}^2}\right)_w\left(\frac{\partial \tilde{u}}{\partial y}\right)_w^2$$

and

$$\frac{1}{\mu_w}\left(\frac{\partial^2 \mu}{\partial y^2}\right)_w = \frac{1}{\mu_w}\left(\frac{\partial \mu}{\partial \tilde{u}}\right)_w\left(\frac{\partial^2 \tilde{u}}{\partial y^2}\right)_w + \frac{1}{\mu_w}\left(\frac{\partial^2 \mu}{\partial \tilde{u}^2}\right)_w\left(\frac{\partial \tilde{u}}{\partial y}\right)_w^2$$

The second derivatives can then be eliminated as before yielding the desired result which is

$$\bar{z}(\bar{z}-\bar{P}) = (Z+G_2)\ (Z+G_3) + G_4 - (Z+G_3)\ P \tag{21}$$

where

$$G_2 = -\left(\frac{\partial \ell n \tilde{\mu}}{\partial \tilde{u}}\right)_w$$

$$G_3 = 3G_{1_w} - \left(\frac{\partial \ell n \tilde{\rho}}{\partial \tilde{u}}\right)_w$$

$$G_4 = 3\left(\frac{\partial \ell n \tilde{\rho}}{\partial \tilde{u}}\right)_w^2 - \frac{1}{\tilde{\rho}_w}\left(\frac{\partial^2 \tilde{\rho}}{\partial \tilde{u}^2}\right)_w - \frac{1}{\tilde{\mu}_w}\left(\frac{\partial^2 \tilde{\mu}}{\partial \tilde{u}^2}\right)_w$$

Note that all of the G_i's depend only on fluid thermodynamics. As with the term G_{1_w}, they can also be expressed in terms of specified flow parameters according to

$$G_2 = -\alpha \frac{1-W}{W} \tag{22}$$

$$G_3 = (4-3\alpha) \frac{1-W}{W} \tag{23}$$

$$G_4 = (1+\alpha-\alpha^2) \left(\frac{1-W}{W}\right)^2 - \frac{2(1-\alpha)}{W} \frac{(\gamma-1)M_e^2/2}{1+(\gamma-1)M_e^2/2} \tag{24}$$

It is interesting to note the Mach number dependence of G_4 and the fact that it does not necessarily vanish for the adiabatic case.

Equations (14) and (21) represent the two additional restraints required to treat the general VP problem involving both mass transfer and pressure gradient. They should, however, be equally applicable and uniformly valid for limiting cases (i.e., $Z \to 0$, $P \to 0$, $W \to 1$). In order to examine this limiting behavior it is convenient to introduce some new notation. Let $Z = 0$; then Equation (21) reduces to

$$\bar{Z}(\bar{Z}-\bar{P}) = G_5 - G_3 P \tag{25}$$

where

$$G_5 \equiv G_2 G_3 + G_4 \tag{26}$$

Several limiting cases can now be considered as itemized in Table I. The first entry therein corresponds to the zero pressure gradient adiabatic wall case. From Equations (15) and (23) it follows that $G_{1_W} = G_3 = 0$ as has been indicated. Equations (24) and (26) imply that $G_5 \neq 0$ (assuming $M_e \neq 0$). In this circumstance Equations (14) and (25) imply

$$\bar{Z} - \bar{P} = 0$$

$$\bar{Z}(\bar{Z}-\bar{P}) \neq 0$$

The only way in which these equations can be satisfied simultaneously is for $\bar{Z}$ and $\bar{P}$ both to be infinitely large which is clearly unacceptable from both a physical and computational point of view. This irregular behavior could be suppressed by taking the viscosity exponent to be unity. The resulting situation is indicated by the second entry in Table I. In this case we find that $\bar{Z} = \bar{P}$ but that neither one is explicitly defined. Even if we consider this an acceptable situation, the irregular behavior has not really been eliminated. This is demonstrated by the third entry wherein the adiabatic case with non-zero pressure gradient is considered. Again, it is found that in order to satisfy the closure relations (14) and (25), both $\bar{Z}$ and $\bar{P}$ must take on infinitely large values.

In view of these results it is concluded that a consistent set of closure rules cannot be obtained in this manner, at least for the form of transformation utilized here (e.g., Eqs. (5), (6) and (8). Accordingly, an alternate approach is required and this is discussed in the next section.

Closure by Collocation. - In this approach Equation (14) is also utilized. However, Equation (21) is replaced by satisfying Equations (2) and (4) at the sublayer edge. This would yield

$$(\rho u)_s \left(\frac{\partial u}{\partial x}\right)_s + (\rho v)_s \left(\frac{\partial u}{\partial y}\right)_s = \left(\frac{\partial \tau}{\partial y}\right)_s - \frac{dp}{dx} \tag{27}$$

$$\bar{\rho}\bar{u}_s \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_s + \bar{\rho}\bar{v}_s \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_s = \left(\frac{\partial \bar{\tau}}{\partial \bar{y}}\right)_s - \frac{d\bar{p}}{d\bar{x}} \tag{28}$$

where subscript s indicates that the variables have been evaluated at their respective sublayer edge. In order to be consistent with the CP formulation, however, the following approximations are appropriate:

$$\frac{d\bar{p}}{d\bar{x}} + \bar{\rho}\bar{u}_s \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)_s << \left(\frac{\partial \bar{\tau}}{\partial \bar{y}}\right) - \bar{\rho}\bar{v}_s \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_s$$

$$\bar{v}_s \approx \bar{v}_w$$

By analogy we assume that similar approximations are appropriate for the VP flow. Thus if it is further assumed that $\rho_s \approx \rho_w$ and that at the sublayer edge the shear is due solely to laminar viscosity, Equations (27) and (28) reduce to

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_s = \left\{\frac{\rho_w v_w}{\mu_s} - \frac{1}{\mu_s}\left(\frac{\partial \mu}{\partial y}\right)_s\right\}\left(\frac{\partial u}{\partial y}\right)_s$$

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)_s = \frac{\bar{\rho}\bar{v}_w}{\bar{\mu}}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_s$$

By utilizing relations similar to Equation (11) and (13) and eliminating the second derivatives between these last two equations, there is obtained

$$\bar{Z}\,\frac{\bar{\tau}_w}{\bar{\tau}_s} = Z\,\frac{\tau_w}{\tau_s} + G_{1_s}$$

where

$$G_{1_s} = -\left(\frac{\partial \ln \widetilde{\rho\mu}}{\partial \tilde{u}}\right)_s$$

and

$$\bar{\tau}_s \equiv \bar{\mu}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)_s \quad ; \quad \tau_s \equiv \mu_s\left(\frac{\partial u}{\partial y}\right)_s$$

For the further development it is convenient to express the laminar shear $\bar{\tau}_s$ in terms of other variables of this formulation. For this purpose we can use Equation (10) of Part II of this report to show that

$$\bar{\tau}_s/\bar{\tau}_w = 1 + \tilde{u}_s\bar{Z}$$

where $\tilde{u}_s$ denotes the velocity ratio occurring at the edge of the sublayer. We will also restrict further consideration to the case of zero mass transfer in the VP plane. Accordingly, the desired closure relations become

$$\bar{Z} - \bar{P} = G_{1_w} - P \tag{29}$$

$$\frac{\bar{Z}}{1+\tilde{u}_s\bar{Z}} = G_{1_s} \tag{30}$$

where, to be consistent with Equation (15), G_{1_s} would be expressed as

$$G_{1_s} = (1-\alpha)\,\frac{(1-W)+2\tilde{m}_e\tilde{u}_s}{W+(1-W)\tilde{u}_s+\tilde{m}_e\tilde{u}_s^{2}} \tag{31}$$

where

$$\tilde{m}_e \equiv - \frac{(\gamma-1)M_e^2/2}{1+(\gamma-1)M_e^2/2}$$

Qualitatively, Equations (29) and (30) appear to be well behaved for all conditions of interest. We note especially that even for zero pressure gradient adiabatic flows neither $\bar{Z}$ nor $\bar{P}$ vanish which would certainly be the type of effect being sought here. Note also that as $M_e \to 0$ and $W \to 1$, there is obtained $\bar{Z} = 0$, $\bar{P} = P$ which implies that the system reduces uniformly to the CP problem as it should.

A more quantitative assessment of the general behavior of these relations is not possible without full implementation of the entire formulation. This is due to the fact that the velocity ratio $\tilde{u}_s$ is a function, not only of $\bar{Z}$, but of the CP skin friction coefficient as well. Nevertheless, some representative results have been generated for the zero pressure gradient case by considering $\tilde{u}_s$ as a parameter. That is, we let W, M_e, $\tilde{u}_s$ and α vary over reasonable range of interest and evaluate the corresponding values of $\bar{Z}$ from

$$\bar{Z} = \frac{G_{1_s}}{1-\tilde{u}_s G_{1_s}} \tag{32}$$

which follows directly from Equation (30). Some representative results of this procedure are shown in Figures 1 and 2. Figure 1 shows the effect of Mach number and wall temperature ratio at a fixed value of α with the velocity ratio $\tilde{u}_s$ as a parameter. In Figure 2, $\tilde{u}_s$ is maintained constant with α taken as the parameter. As may be noted, Equation (32) is well behaved throughout a wide range of all of the parameters even up to and including infinitely large Mach numbers and vanishingly small wall temperatures. By contrast, the first order closure rule represented by Equation (19), which incidentally does not include any Mach number dependence, exhibits singular behavior when $W \to 0$ as may be seen by examining Equation (15).

Superficially, it would appear that the second order closure rules derived here represent a means for resolving all of the difficulties previously encountered with the transformation. Closer examination of the data shown in Figures 1 and 2 implies however that this may not be the case. This is particularly evident by noting that for the adiabatic wall case W = 1, negative values of $\bar{Z}$ occur for all non-zero Mach numbers. But from Equation (29) with $G1_w = P = 0$, we have $\bar{Z} = \bar{P} < 0$ which implies the occurrence of adverse pressure gradient in the transformed plane. Since this is precisely opposite to the effect desired, the likelihood of improving the performance of the transformation approach by second order closure is substantially diminished. This will be borne out by the results presented in Section III.

C. Working Equations

General Remarks. - The final result of this section will be a system of ordinary differential equations whose matrix representation can be written

$$\begin{bmatrix} A1 & A9 & A17 & A25 & A33 & 0 & 0 & 0 \\ A2 & A10 & A18 & A26 & A34 & 0 & 0 & 0 \\ A3 & A11 & A19 & A27 & A35 & 0 & 0 & 0 \\ 0 & 0 & 0 & A28 & 0 & 0 & 0 & 0 \\ 0 & A13 & 0 & A29 & A37 & 0 & 0 & A61 \\ 0 & A14 & 0 & A30 & A38 & A46 & 0 & A62 \\ 0 & 0 & 0 & A31 & 0 & A47 & A55 & A63 \\ 0 & 0 & 0 & 0 & 0 & A48 & A56 & A64 \end{bmatrix} \begin{bmatrix} (\ln \bar{R})' \\ (\ln \bar{\varphi})' \\ (\pi)' \\ (\ln \bar{U}e)' \\ (\bar{V}_W)' \\ (\ln \tilde{\sigma})' \\ (\ln \tilde{\eta})' \\ (\chi)' \end{bmatrix} = \begin{bmatrix} 0 \\ C2 \\ C3 \\ C4 \\ 0 \\ 0 \\ 0 \\ C8 \end{bmatrix} \quad (33)$$

It is believed that by setting down Equation (33) initially, the detailed development which follows will be facilitated and clarified. This procedure will also provide an opportunity to compare the present formulation with that of Reference 1, at the outset. By virtue of this comparison, those similarities which occur in the two methods can be exploited to reduce much of the tedious algebra involved in the derivation of Equation (33).

In addition to this derivation, appropriate initial conditions will be developed in this section. For this purpose it will be necessary to outline a procedure for deducing skin friction coefficients from experimental velocity profile data according to the second order closure rules employed here. Similar procedures for the various other closure rules will also be set down. Although these so called "Clauser Plots" have been utilized by previous investigators (c.f. References 3, 7, 10) they have not, to the authors' knowledge, been set down in a systematic manner in any of the literature.

Finally, we note that the reason conventional notation has not been used in the coefficient matrix of Equation (33) is twofold. First of all it was desirable to avoid any possibility of confusing the various terms appearing here and in Reference 1. Secondly, the choice of notation anticipates the logical flow of certain "canned" subroutines used by the CDC 6600 computer for which this analysis has been programmed. A further discussion of this aspect may be found in Part III of this report.

<u>Comparison of First and Second Order Closure Formulations for the Pressure Gradient Case</u>. - In Section III.B, it has been pointed out that closure was achieved in Reference 1 by means of Equation (16). The resulting formulation has the matrix representation (c.f. Equation (56) of Reference 1):

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 & 0 \\ A_{51} & 0 & 0 & 0 & A_{55} & 0 & A_{57} \\ 0 & 0 & 0 & A_{64} & A_{65} & A_{66} & A_{67} \\ 0 & 0 & 0 & 0 & A_{75} & A_{76} & A_{77} \end{bmatrix} \begin{bmatrix} (\ln \bar{\omega})' \\ (\pi)' \\ (\ln \bar{R})' \\ (\ln \bar{U}e)' \\ (\ln \tilde{\sigma})' \\ (\ln \tilde{\eta})' \\ (\chi)' \end{bmatrix} = \begin{bmatrix} 0 \\ C_2 \\ C_3 \\ C_4 \\ 0 \\ 0 \\ C_7 \end{bmatrix} \qquad (34)$$

The most obvious difference between Equations (33) and (34) is that the former is an eight-dimensional system as compared to a 7 x 7 for the latter. Evidently this increase in dimension is due to the introduction of the additional free parameter corresponding to transpiration in the CP plane. In Eq. (33) this parameter is represented by the dependent variable $\bar{V}_w$ whose relation to the parameter $\bar{Z}$ previously introduced will be indicated in the subsequent development. Aside from this rather essential difference the two systems bear a considerable similarity in the sense that the same fundamental relations are utilized to derive the final working system of equations. For Equation (34) the specific relations, proceeding in order from the first row, were:

(a) the CP skin friction law (38)

(b) the CP momentum integral equation (34)

(c) the CP auxiliary equation (43)

(d) the first order closure relation (26)

(e) the sub-layer hypothesis (45)

(f) the edge compatibility condition (21)

(g) the VP momentum integral equation (30)

where the numbers in parenthesis refer to the appropriate equations as given in Reference 1. Note that item (d) is identical to Eq. (16) of this report.

The first point to be made regarding Equation (34) is that the first three relations (a) through (c) constitute the describing equations for the CP flow. If the CP pressure gradient parameter* $(\ln \bar{U}_e)'$ were a known function, these three equations could be solved independently of the remaining four equations. Here, the two groups of relations are coupled by virtue of the fact that $(\ln \bar{U}_e')$ must be considered as an unknown. As for this second group of relations, they represent the four restraints which are required to define the three stretching parameters of the transformation and the unknown CP pressure distribution.

The analogy with the present formulation should now be apparent. Again, a group of relations is required to describe the CP flow, but, of course, in this case they would involve both a pressure gradient and a mass transfer parameter. In Part II of this report such a system is developed and it is found that, once again, three equations are involved which are completely equivalent to items (a), (b) and (c) above. The second group of equations must now involve five rather than four, fundamental relations to account for the additional unknown parameter $\bar{V}_w$. Of these, three are exactly equivalent to items (e), (f) and (g). The remaining two are provided by the closure rules developed in the previous sections; i.e., Eqs. (29) and (30).

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* $\bar{U}_e$ bears the same relation to the parameter $\bar{P}$ that $\bar{V}_w$ does to $\bar{Z}$. The explicit correspondence between these variables will be indicated later.

With this as a background we are now in a position to proceed with the derivation of Equation (33) in a straightforward manner. This will be accomplished in the next three subsections.

In the first of these the working form of the two closure relations, which are unique to this formulation, will be derived. Then the relations corresponding to items (e), (f) and (g) will be considered. Finally, the CP formulation of Part II of this report will be arranged in a manner suitable for numerical integration and incorporation into the matrix representation of Equation (33).

<u>Working Form of the Closure Relations</u>. - The working dependent variables which are employed in the numerical integration scheme have been indicated in the solution vector appearing in Equation (33). The first five members of this vector all represent CP variables and have been defined in Part II of this report. Furthermore, with the exception of the mass transfer parameter $\bar{V}_w$, all of the variables are identical in meaning to those utilized in Reference 1. Accordingly, to proceed with this derivation, it is only necessary to show the relation between $\bar{Z}$ and $\bar{P}$ and these working variables. From the definition of these latter and those appearing on page 11 of this report, it follows immediately that

$$\bar{Z} = \frac{\bar{\varphi}^2 V_w}{\bar{U}_e} \qquad ; \qquad \bar{P} = \frac{\bar{\varphi}^4}{\bar{U}_e} (\ell n\bar{U}_e)'$$

Accordingly, Equation (29) can be written

$$\frac{\bar{\varphi}^4}{\bar{U}_e} \left[\frac{\bar{V}_w}{\bar{\varphi}^2} - (\ell n\bar{U}_e)' \right] = G_{1_w} - P$$

or, taking into account the definitions of P and the correspondence* between the CP and VP skin friction coefficients

$$(\ell n\bar{U}_e)' = \frac{\bar{V}_w}{\bar{\varphi}^2} - \frac{\bar{U}_e G_{1_w}}{\bar{\varphi}^4} + \left(\frac{1}{\tilde{\rho}_w^2 \tilde{\mu}_w \tilde{\sigma} \tilde{\eta}} \right) \frac{\tilde{\eta}_o}{\tilde{\sigma}_o} \frac{d\ell nU_e}{d\chi} \qquad (35)$$

---

* Correspondence between the CP and VP variables as implied by the transformation are listed in Appendix A.

This equation corresponds to the fourth row in Eq. (33) with A28 equal to unity and C4 taken equal to the right hand side of Eq. (35).* Complete evaluation of C4, of course, requires specification of both the external boundary conditions ($d\ell n\, U_e/d\chi$) as well as the wall boundary conditions ($G_{1w}, \tilde{\rho}_w, \tilde{\mu}_w$) the latter also involving the thermodynamic behavior of the VP fluid. Detailed evaluation of these parameters is discussed later in this section.

Consider now the second closure rule, Eq. (30). Proceeding as before, we can write

$$H_1 \equiv \frac{\bar{\varphi}^2 \bar{V}_w}{\bar{U}_e} \left[1 - (1 + \tilde{u}_s)\, G_{1s} \right] = 0 \qquad (36)$$

Since this is an algebraic equation it is necessary to differentiate it for the purpose of incorporating it in Eq.(33). In carrying out this differentiation the implicit dependence of $\tilde{u}_s$ and G_{1s} on the dependent variables must be recognized. From Eq. (20a) of Part II of this report it can be shown that

$$\tilde{u}_s = \frac{1}{\bar{\varphi}\bar{v}_w^+} \left[\exp\,(\bar{v}_w^+ \bar{y}_s^+) - 1 \right] \qquad (37)$$

where $\bar{v}_w^+ \equiv \frac{\bar{V}_w \bar{\varphi}}{\bar{U}_e}$

and $\bar{y}_s^+ = \mathrm{fcn}(\bar{v}_w^+)$ as determined from the transcendental equation (20c) of Part II. Accordingly,

$$\tilde{u}_s = \mathrm{fcn}(\bar{\varphi}, \bar{U}_e, \bar{V}_w)$$

In view of Eq. (31) we may also anticipate that in the general case, G_{1s} will exhibit a dependence on both $\tilde{u}_s$ and the

* All of the coefficients appearing in Eq. (33) are defined explicitly in Appendix B.

streamwise coordinate χ through a corresponding Mach number variation. Eq. (36) is then of the form

$$H_1(\bar{\varphi}, \bar{U}_e, \bar{V}_w, \chi) = 0$$

Thus, differentiation of Eq. (36) yields

$$\left(\frac{\partial H_1}{\partial \ell n \bar{\varphi}}\right) (\ell n \bar{\varphi})' + \left(\frac{\partial H_1}{\partial \ell n \bar{U}_e}\right) (\ell n \bar{U}_e)' + \left(\frac{\partial H_1}{\partial \bar{V}_w}\right)(\bar{V}_w)' + \left(\frac{\partial H_1}{\partial \chi}\right)(\chi)' = 0 \tag{36a}$$

which corresponds to the fifth row of Eq. (33) where the coefficients A13, A29, A37 and A61 are related to the partial derivatives in an obvious way. The latter are evaluated for a particular choice of thermodynamic behavior in Appendix C.

Working Form of the Remaining Transformation Restraints: -

(a) Sublayer Hypothesis - The basic form of this restraint is given by

$$\frac{\rho_s u_s y_s}{\mu_s} = \frac{\bar{\rho} \bar{u}_s \bar{y}_s}{\bar{\mu}} \tag{38}$$

By utilizing the various correspondences between the CP and VP variables given in Appendix A and the definitions this can be written

$$H_2 \equiv \tilde{\sigma} - \frac{\tilde{\rho}_s}{\tilde{\mu}_s} \frac{1}{\bar{y}_s^+} \int_0^{\bar{y}_s^+} \frac{d\bar{y}^+}{\tilde{\rho}} = 0 \tag{39}$$

The dependence of this relation on the thermodynamic behavior of the VP fluid is apparent. Proceeding as in the case of Eq. (36) we recognize that Eq. (39) is of the form

$$H_2 (\bar{\varphi}, \bar{U}_e, \bar{V}_w, \tilde{\sigma}, \chi) = 0$$

Differentiations yields

$$\frac{\partial \ell n H_2}{\partial \ell n \bar{\varphi}} (\ell n \bar{\varphi})' + \frac{\partial \ell n H_2}{\partial \ell n \bar{U}_e} (\ell n \bar{U}_e)' + \frac{\partial \ell n H_2}{\partial \bar{V}_w} (\bar{V}_w)' + (\ell n \tilde{\sigma})' + \frac{\partial \ell n H_2}{\partial \chi} \chi' = 0 \tag{39a}$$

which corresponds to the sixth row of Eq. (33). Explicit evaluation of these partial derivatives is presented in Appendix C.

(b) Edge Compatibility - This restraint follows directly from Eq. (6) evaluated at y, $\bar{y} \rightarrow \infty$. There results

$$u_e = \frac{\eta}{\sigma} \bar{u}_e$$

or, in terms of the working variables

$$U_e = \frac{\tilde{\eta}}{\tilde{\sigma}} \left(\frac{\tilde{\sigma}}{\tilde{\eta}}\right)_o \left(\frac{\nu_e}{\nu_{e_o}}\right) \bar{U}_e \tag{40}$$

The working form of this relation, corresponding to the seventh row in Eq. (33) follows by differentiation. There results

$$(\ell n \bar{U}_e)' - (\ell n \tilde{\sigma})' + (\ell n \tilde{\eta})' \frac{d \ell n U_e/\nu_e}{d\chi} \chi' = 0 \tag{40a}$$

where we have recognized that ν_{e_o} and $(\tilde{\sigma}/\tilde{\eta})_o$ are constants.

(c) VP Momentum Integral - Since this formulation will correspond to the case of zero mass transfer in the VP plane, the momentum integral equation remains identical to that used in Reference 1 and can be written

$$\frac{d\theta}{dx} + \frac{\theta}{u_e} \frac{du_e}{dx} (H+2) + \frac{\theta}{\rho_e} \frac{d\rho_e}{dx} = \frac{c_f}{2} \tag{41}$$

The CP counterpart of this equation must, of course, reflect the existence of mass transfer. Accordingly, it takes the form

$$\frac{d\bar{\theta}}{dx} + \frac{\bar{\theta}}{\bar{u}_e}\frac{d\bar{u}_e}{dx}(\bar{H}+2) = \frac{\bar{c}_f}{2} + \frac{\bar{v}_w}{\bar{U}_e} \tag{42}$$

After introduction of the various definitions these can be written

$$\frac{d\ell n\theta}{d\chi} + (H+2)\frac{d\ell nU_e}{d\chi} + \frac{d\ell n\rho_e}{d\chi} = \frac{c_f}{2\theta}\left(\frac{u_e}{\nu_e}\right)_o^{-1} \tag{41a}$$

$$(\ell n\bar{\theta})' + (\bar{H}+2)(\ell n\bar{U}_e)' = \frac{1}{\bar{\theta}}\left(\frac{1}{\bar{\varphi}^2} + \frac{\bar{V}_w}{\bar{U}_e}\right)\left(\frac{\bar{u}_{e_o}}{\bar{\nu}}\right)^{-1} \tag{42a}$$

Now the correspondence between θ and $\bar{\theta}$ according to Appendix A is given by $\theta = \bar{\theta}/\tilde{\eta}$. Accordingly

$$\frac{d\ell n\theta}{d\chi} = \frac{1}{\chi'}(\ell n\theta)' = \frac{1}{\chi'}\left[(\ell n\bar{\theta})' - (\ell n\tilde{\eta})'\right]$$

Hence, Eq. (41a) can be written after some rearrangement

$$(\bar{H}+2)(\ell n\bar{U}_e)' + (\ell n\tilde{\eta})' - \chi'\left\{(H+2)\frac{d\ell nU_e}{d\chi} + \frac{d\ell n\rho_e}{d\chi} - \frac{c_f}{2\theta}\left(\frac{u_e}{\nu_e}\right)_o^{-1}\right\} = \frac{1}{\bar{\theta}}\left(\frac{1}{\bar{\varphi}^2} + \frac{\bar{V}_w}{\bar{U}_e}\right)\left(\frac{\bar{u}_{e_o}}{\bar{\nu}}\right)^{-1}$$

It is also convenient to eliminate $(\ell n\bar{U}_e)'$ by means of Eq. (40a). There results

$$(\bar{H}+2)(\ell n\tilde{\sigma})' - (\bar{H}+1)(\ell n\tilde{\eta})' - \left\{(H-\bar{H})\frac{d\ell nU_e}{d\chi} - (\bar{H}+1)\frac{d\ell n\rho_e}{d\chi} + (\bar{H}+2)\frac{d\ell n\mu_e}{d\chi} - \frac{c_f}{2\theta}\left(\frac{u_e}{\nu_e}\right)_o^{-1}\right\}\chi' = \frac{1}{\bar{\theta}}\left(\frac{1}{\bar{\varphi}^2} + \frac{\bar{V}_w}{\bar{U}_e}\right)\left(\frac{\bar{u}_{e_o}}{\bar{\nu}}\right)^{-1} \tag{43}$$

In Part II of this report, a CP Reynolds number based on boundary layer thickness, $\bar{\delta}$, is defined according to $\bar{R} = \bar{u}_{e_o}\bar{\delta}/\bar{\nu}$ which is one of the working variables appearing in Eq. (33). Let the ratio $\bar{\theta}/\bar{\delta}$ be denoted by $\bar{\Sigma}$. Then

$$\frac{1}{\theta}\left(\frac{\bar{u}_{e_o}}{\bar{\nu}}\right)^{-1} = \frac{1}{\bar{\Sigma}\bar{R}}$$

and

$$\frac{c_f}{2\theta}\left(\frac{u_e}{\nu_e}\right)_o^{-1} = \frac{\tilde{\rho}_w\tilde{\mu}_w\tilde{\sigma}\,\tilde{\sigma}_o}{\bar{\varphi}^2\,\bar{\Sigma}\bar{R}}\left(\frac{\tilde{\eta}}{\eta_o}\right)$$

where we have utilized several of the correspondences given in Appendix A. Note that by introducing the parameter $\bar{R}$ the dependence of this system on the unit Reynolds number of the CP flow and, consequently, on its thermodynamic state, has totally been eliminated.

Equation (43) can now be written

$$(\bar{H}+2)(\ell n\tilde{\sigma})' - (\bar{H}+1)(\ell n\tilde{\eta})' - \left\{(H-\bar{H})\frac{d\ell nU_e}{d\chi} - (\bar{H}+1)\frac{d\ell n\rho_e}{d\chi} + (\bar{H}+2)\frac{d\ell n\mu_e}{d\chi} - \frac{\tilde{\rho}_w\tilde{\mu}_w\tilde{\sigma}\tilde{\sigma}_o}{\bar{\varphi}^2\,\bar{\Sigma}\bar{R}}\,\frac{\tilde{\eta}}{\eta_o}\right\}\chi' = \frac{1}{\bar{\Sigma}\bar{R}}\left\{\frac{1}{\bar{\varphi}^2} + \frac{\bar{V}_w}{\bar{U}_e}\right\} \tag{44}$$

In Eq. (44) the χ derivatives of U_e, ρ_e, and μ_e, are, of course, considered to be specified and hence are known. Furthermore, the CP formulation provides relations of the form

$$\bar{\Sigma} = \bar{\Sigma}\,(\bar{\varphi},\ \pi,\ \bar{U}_e,\ \bar{V}_w)\ ,\quad \bar{H} = \bar{H}\,(\bar{\varphi},\ \pi,\ \bar{U}_e,\ \bar{V}_w)$$

Consequently, provided only that the VP form factor H is expressed in terms of the working variables, Eq. (44) is in a form suitable for incorporation in Eq. (33). This representation must await specification of a particular thermodynamic model for the VP fluid since, in accordance with the correspondence (A8)

$$H = \bar{H} + \frac{1}{\theta}\int_o^{\bar{\delta}}\left(\frac{1}{\bar{\rho}} - 1\right)d\bar{y} \tag{45}$$

The integral appearing in this expression is evaluated later in this section. Equation (44) is represented by the last row ot Equation (33).

Working Form of the CP Formulation. - The describing equations for the CP formulation are given by Eq. (38) of Part II of this report. It takes the form

$$\begin{bmatrix} A_1 & A_4 & A_7 \\ A_2 & A_5 & A_8 \\ A_3 & A_6 & A_9 \end{bmatrix} \begin{bmatrix} (\ln\bar{\varphi})' \\ (\ln\bar{R})' \\ (\pi)' \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{46}$$

where derivatives of $\bar{V}_w$ and $\bar{U}_e$ are included in the forcing terms C_i. For the CP problem itself this is appropriate since these are considered to be given forcing functions. For the present application they represent unknowns so that this system has to be modified accordingly. The procedure is straightforward and involves only the transposition of the coefficients of $(\bar{V}_w)'$ and $(\ln\bar{U}_e)'$ from the right hand side of Eq. (46) to the left hand side. After some rearrangement there is obtained the first three rows of Eq. (33) where the coefficients in (33) and (46) are related by

$$A1 = A_4, \quad A2 = A_6, \quad A3 = A_5$$

$$A9 = A_1, \quad A10 = A_3, \quad A11 = A_2$$

$$A17 = A_7, \quad A18 = A_9, \quad A19 = A_8$$

$$A25\ (\ln\bar{U}_e)' + A33(\bar{V}_w)' = - C_1$$

$$A26\ (\ln\bar{U}_e)' + A34(\bar{V}_w)' - C2 = - C_3$$

$$A27\ (\ln\bar{U}_e)' + A35(\bar{V}_w)' - C3 = - C_2$$

For convenience in the further development it is useful to note here that the first of these equations arises by differentiation of the algebraic equation corresponding to the CP skin friction law; (Eq. (28b) of Part II). This equation can be written

$$\left(1+\bar{\varphi}^2 \frac{\bar{V}_w}{\bar{U}_e}\right)^{\frac{1}{2}} = 1 + \frac{\bar{\varphi}}{2}\frac{\bar{V}_w}{\bar{U}_e}\left\{k_2 + \frac{1}{k_1}\ln\left(\frac{\bar{R}\bar{U}_e}{\bar{\varphi}}\right) + \frac{2\pi}{k_1}\right\} \tag{47}$$

Thermodynamic Representation. - The influence of the VP fluid thermodynamics on Eq. (33) is reflected by several parameters appearing in Eqs. (35), (36), (39), and (45). These involve viscosity and density ratios at the wall and sublayer edge ($\tilde{\rho}_w$, $\tilde{\mu}_w$, $\tilde{\rho}_s$, $\tilde{\mu}_s$) and derivatives and integrals thereof, i.e.,

$$G_{1_w} = \left(\frac{\partial \ln \widetilde{\rho\mu}}{\partial \tilde{u}}\right)_w$$

$$G_{1_s} = \left(\frac{\partial \ln \widetilde{\rho\mu}}{\partial \tilde{u}}\right)_s$$

$$\frac{1}{\bar{y}_s} + \int_o^{\bar{y}_s^+} \frac{d\bar{y}^+}{\tilde{\rho}}$$

$$\int_o^{\bar{\delta}} \left(\frac{1}{\tilde{\rho}} - 1\right) d\bar{y}$$

Each of these must be related to the velocity field and subsequently to the working variables in order to complete the formulation. As indicated in Section III.B, a Crocco integral representation will be utilized for this purpose. For a perfect gas, taking into account that the pressure across the boundary layer is uniform, this implies that

$$\tilde{T} = \frac{1}{\tilde{\rho}} = \frac{W + (1-W)\,\tilde{u} + \tilde{m}_e \tilde{u}^2}{1 + \tilde{m}_e} \tag{48}$$

It will also be assumed here, for additional generality, that the VP fluid viscosity can be represented by a Sutherland type law

$$\mu \sim \frac{T^{3/2}}{T+198.6}$$

Then it can be shown that

$$\tilde{T}_w = \frac{1}{\tilde{\rho}_w} = \frac{W}{1+\tilde{m}_e} \tag{49}$$

$$\tilde{T}_s = \frac{1}{\tilde{\rho}_s} = \frac{W + (1-W)\ \tilde{u}_s + \tilde{m}_e \tilde{u}_s^2}{1 + \tilde{m}_e} \tag{50}$$

$$\tilde{\mu}_{w,s} = (\tilde{T}_{w,s})^{3/2} \left[\frac{1+198.6/T_e}{\tilde{T}_{w,s} + 198.6/T_e} \right] \tag{51}$$

$$G_{1_w} = \frac{1}{2}\ \frac{1-W}{W} \left[\frac{\tilde{T}_w - 198.6/T_e}{\tilde{T}_w + 198.6/T_e} \right] \tag{52}$$

$$G_{1_s} = \frac{1}{2} \left[\frac{(1-W) + 2\tilde{m}_e \tilde{u}_s}{\tilde{T}_s (1+\tilde{m}_e)} \right] \left[\frac{\tilde{T}_s - 198.6/T_e}{\tilde{T}_s + 198.6/T_e} \right] \tag{53}$$

Note that since $T_e = T_{te}(1+\tilde{m}_e)$, Eqs. (51), (52) and (53) require specification of the external stream stagnation temperature, which is, of course, taken as constant in this formulation.

To evaluate the first density integral we make use of the velocity distribution given by Eq. (20a) Part II. Taking into account Eq. (48) there results

$$\frac{1}{\bar{y}_s^+} \int_o^{\bar{y}_s^+} \frac{d\bar{y}^+}{\tilde{\rho}} = \frac{W + (1-W)\ (\lambda_1/\bar{\varphi}) + \tilde{m}_e (\lambda_2/\bar{\varphi}^2)}{1 + \tilde{m}_e} \tag{54}$$

where

$$\lambda_1 \equiv \frac{1}{\bar{y}_s^+} \int_o^{\bar{y}_s^+} \bar{u}^+ \, d\bar{y}^+ \quad , \quad \lambda_2 \equiv \frac{1}{\bar{y}_s^+} \int_o^{\bar{y}_s^+} \bar{u}^{+2} \, d\bar{y}^+$$

are functions of $\bar{v}_w^+$ only and are defined explicitly in Appendix D.

Evaluation of the second integral is somewhat more straightforward since the final result can be expressed in terms of previously defined parameters. That is, using Eq. (48) it can be shown that

$$\int_o^{\bar{\delta}} \left(\frac{1}{\tilde{\rho}} - 1\right) d\bar{y} = \frac{(W\ \bar{\delta}^* - \tilde{m}_e \bar{\theta})}{1+\tilde{m}_e} - \bar{\delta}^* \tag{55}$$

Thus, from Eq. (45) the correspondence for the form factor becomes

$$H = \frac{W\bar{H}-\tilde{m}_e}{1+\tilde{m}_e} \tag{56}$$

Boundary Conditions. - (a) Edge Conditions - Inviscid Flow: So far as the computer program is concerned, the edge conditions are imposed by inputting a table of Mach number versus streamwise length normalized with respect to a reference length ℓ. The internal logic of the program then smooths these data, if desired, and also numerically evaluates the first derivatives thereof. The details of this processing is given in Part III of this report. Here, we will indicate only how the various inviscid parameters appearing in Eq. (33) are related to this Mach number table.

For a calorically and thermally perfect gas the isentropic relations imply that

$$u_e = M_e(1+\tilde{m}_e)^{\frac{1}{2}} \ (\gamma R T_{t_e})^{\frac{1}{2}} \tag{57}$$

Accordingly, since the stagnation speed of sound is considered to be constant here

$$\frac{d\ell n U_e}{d\chi} = \frac{1+\tilde{m}_e}{M_e} \ \frac{dM_e}{d\chi}$$

or

$$\frac{d\ell n U_e}{d\chi} = \frac{1+\tilde{m}_e}{M_e} \ \frac{dM_e}{dx/\ell} \ (\frac{1}{Re_o \ell}) \tag{58}$$

where M_e and x/ℓ are the tabulated input values and Re_o denotes the initial unit Reynolds number $(u_e/\nu_e)_o$. Note that in deriving Eq. (58) we have made use of the relation

$$\frac{d\tilde{m}_e}{d\chi} = \frac{2\tilde{m}_e(1+\tilde{m}_e)}{M_e} \ \frac{dM_e}{dx/\ell} \ (\frac{1}{Re_o \ell}) \tag{59}$$

which follows directly from the definition of $\tilde{m}_e$.

Proceeding in a similar fashion it can be shown that

$$\frac{d\ell nT_e}{d\chi} = \frac{2\tilde{m}_e}{M_e} \left(\frac{dM_e}{dx/\ell}\right) \left(\frac{1}{Re_o \ell}\right) \tag{60}$$

$$\frac{d\ell n\rho_e}{d\chi} = -(1+\tilde{m}_e)\, M_e \left(\frac{dM_e}{dx/\ell}\right) \left(\frac{1}{Re_o \ell}\right) \tag{61}$$

In view of Eq. (60) a Sutherland-type viscosity law would then imply

$$\frac{d\ell n\mu_e}{d\chi} = \frac{2\tilde{m}_e}{M_e} \frac{dM_e}{dx/\ell} \left\{\frac{3}{2} - \frac{1+\tilde{m}_e}{(1+\tilde{m}_e)+198.6/T_{t_e}}\right\} \left(\frac{1}{Re_o \ell}\right) \tag{62}$$

The derivative of the kinematic viscosity ν_e follows directly from Eq. (61) and (62) since

$$\frac{d\ell n\nu_e}{d\chi} = \frac{d\ell n\mu_e}{d\chi} - \frac{d\ell n\rho_e}{d\chi} \tag{63}$$

Note that several of the foregoing relations are also required to evaluate the partial derivatives $\partial H_1/\partial\chi$ and $\partial \ell n H_2/\partial\chi$ appearing in Eqs. (36a) and (39a) respectively.

(b) Wall Conditions: A detailed examination of this system reveals that the state of flow at the wall is completely characterized by the parameter W which represents the ratio of wall temperature to external total temperature. This parameter is imposed in a manner identical to that used to input the external Mach number distribution. That is, a table of W vs x/ℓ is read in, smoothed and differentiated as discussed in Part III. Thus for the present purpose it is only necessary to recognize that

$$\frac{dW}{d\chi} = \frac{dW}{dx/\ell} \cdot \left(\frac{1}{Re_o \ell}\right) \tag{64}$$

Initial Conditions. - A detailed examination of the system represented by Eq. (33) reveals that

a) The initial value of the independent variable $\bar{\chi}$ is arbitrary.

b) The initial value of χ is arbitrary except that the choice must be consistent with the input Mach number and wall temperature tables.

c) The initial value of $\bar{U}_e$ and $\tilde{\eta}$ are arbitrary, as is $\tilde{\eta}_o$, but they must be selected in a consistent manner; i.e., they must satisfy the algebraic equation (40). This is most conveniently accomplished by taking $\bar{U}_{e_o} = \tilde{\eta}_o = 1$.

The five remaining dependent variables $\bar{R}$, $\bar{\omega}$, π, $\bar{V}_w$ and $\tilde{\sigma}$, are connected by three algebraic equations corresponding to the CP skin friction law, Eq. (47), and Eqs. (36) and (39). Accordingly, only two initial values need to be prescribed.* These will be taken to be $\bar{\varphi}$ and π. The reason for this choice will be made clear by virtue of the ensuing development.

As in Reference 1, two types of initial conditions will be considered; namely, conditions corresponding to a "leading edge" and conditions associated with some downstream station where boundary layer profiles are prescribed.

Since the formulation developed here corresponds to fully developed turbulent flow, initialization at a leading edge is necessarily an approximate concept. Nevertheless, if the concept is to be employed at all, it should at least exhibit qualitatively consistent behavior in the sense that the solutions sufficiently far downstream should be independent of the particular initial values selected.

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* As far as the computer program itself is concerned, it will accept arbitrary initial values for all of the variables in order to permit restart and continuation of any particular calculation. Such a procedure is admissible provided a self-consistent set of values is utilized. If this is not the case, the inconsistency will show up by means of certain output parameters which provide a running check on whether the algebraic equations are being satisfied as discussed in Part III of this report.

To establish appropriate initial values in line with this point of view, a series of numerical calculations were made. Some representative examples are shown in Figure 3. Here the variation of skin friction with streamwise distance corresponding to various choices of initial conditions are shown. It is apparent that the solutions exhibit virtually no dependence on the initial choice of π for values in the range associated with zero pressure gradient CP flows. On the other hand, although all solutions eventually merge, some effect of the initial choice of $\bar{\varphi}$ persists in the near region. Based on the criterion that the solutions be identical for $\chi \geq 10^5$, a suitable choice of $\bar{\varphi}_o$ appears to be $\bar{\varphi}_o = 15$. Thus, the "leading edge" option which has been provided in the computer program utilizes the values $\pi_o = 0.6$, $\bar{\varphi}_o = 15$.

We now wish to consider initialization at any downstream station where profile data have been prescribed. This requires the use of so called "Clauser Plots" which were mentioned briefly in a previous section. This procedure was first used by Clauser (Reference 11) in conjunction with analysis of CP flow profiles obtained on impermeable surfaces. For this case, the procedure is a relatively simple one and it is worthwhile to review this elementary case briefly here.

We suppose that the velocity distribution at some streamwise station has been determined experimentally in the form $\tilde{u}$ vs $R_{\bar{y}}$ where $R_{\bar{y}} \equiv \bar{u}_e \bar{y}/\bar{\nu}$. From this profile it is desired to infer the associated value of skin friction coefficient $\bar{c}_f/2 \equiv 1/\bar{\varphi}^2$. This is accomplished by comparing a semi-log plot of the experimental profile with a series of curves generated from the equation

$$\tilde{u} = \frac{1}{\bar{\varphi}} \left[k_2 + \frac{1}{k_1} \ln \left(R_{\bar{y}}/\bar{\varphi} \right) \right] \tag{65}$$

using several arbitrary values of the parameter $\bar{\varphi}$. Equation (65), of course, is the so-called "law of the wall" corresponding to CP flow without mass transfer and is obtained from Eq. (20b) of Part II by taking $\bar{v}_w^+ \equiv 0$. From this comparison the value of $\bar{\varphi}$ which best fits the given data can be determined by inspection. An example of such a "Clauser Plot" is shown in Figure 4. Note that the shear is quite unambiguously determined by this method and that it is in good agreement with the value obtained directly by means of a skin friction balance.

Generalization to the CP case with mass transfer is due to Stevenson (Reference 12) and has been used by others (References 5, 7). In this case the velocity distribution is compared with curves corresponding to Eq. (20b), Part II, which can be written:

$$\tilde{u} = \frac{1}{\bar{\varphi}} \Big\{ k_2 (1+k_2 \bar{V}_w \bar{\varphi}) + \frac{1}{k_1}(1+ \frac{k_1}{2} \bar{V}_w \bar{\varphi}) \ln (R_{\bar{y}}/\bar{\varphi}) + \frac{1}{4k_1^2} \bar{V}_w \bar{\varphi} \ln^2 (R_{\bar{y}}/\bar{\varphi}) \Big\} \tag{66}$$

Since $\bar{V}_w$ is known in the case of CP flow, Eq. (66) is exactly equivalent to Eq. (65) in the sense that a unique $\tilde{u}$ vs $R_{\bar{y}}$ variation results for any particular choice of $\bar{\varphi}$. A representative example of such a Clauser Plot has been given in Figure 1 of Part II and is repeated here as Figure 5, for convenience. Again, the determination of $\bar{\varphi}$ (and therefore of $\bar{c}_f$) is clear cut.

Once $\bar{\varphi}$ has been determined, it is also possible to assess the value of the profile parameter π for any particular profile. For this purpose the relations

$$\pi = \frac{k_1}{2} \Big\{ \bar{\varphi} - k_2 - \frac{1}{k_1} \ln (R_{\bar{\delta}}/\bar{\varphi}) \Big\} \tag{67}$$

$$\pi = \frac{k_1}{2} \Big\{ \frac{2}{\bar{V}_w \bar{\varphi}} (1+\bar{V}_w \bar{\varphi}^2)^{1/2} - 1 - k_2 - \frac{1}{k_1} \ln (R_{\bar{\delta}}/\bar{\varphi}) \Big\} \tag{68}$$

are utilized for the zero and non-zero mass transfer cases, respectively. These relations follow from the skin friction law, Eq. (47), where we have recognized that

$$\overline{RU}_e \equiv R_{\bar{\delta}} \equiv \bar{u}_e \bar{\delta}/\bar{\nu}$$

which can also be determined from the experimental profiles.

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* ~~Note that U_e has been taken equal to unity here which is~~ appropriate since the ultimate purpose of this procedure is to establish initial conditions.

The equivalent method for impermeable VP flow was developed by Baronti and Libby (Reference 3). In this case, however, the procedure is somewhat more involved since the measured profiles in the form $\tilde{u}$ vs $R_y \equiv u_e y/\nu_e$ must first be transformed to the corresponding CP form. Since $\tilde{u}$ can be interpreted as either the VP or CP velocity ratio (c.f. Eq. (7)) no stretching of the ordinate is required. On the other hand, points corresponding to various values of R_y must be transformed according to

$$R_{\bar{y}} = \tilde{\sigma} \int_0^{R_y} \tilde{\rho} \, d R_y$$
$$R_{\bar{\delta}} = \tilde{\sigma} \int_0^{R_\delta} \tilde{\rho} \, d R_y \tag{69}$$

as indicated in Appendix A. Thus, if the density variation has also been obtained experimentally, a semi-log plot of $\tilde{u}$ vs $R_{\bar{y}}/\tilde{\sigma}$ can be prepared from the measured data. This must then be compared with appropriate forms of the "law of the wall."

Depending upon the type of closure utilized, the $\tilde{u}$ vs $R_{\bar{y}}$ variation would be obtained from either Eq. (65) or (66). Note however that the abscissa must be divided by a corresponding value of $\tilde{\sigma}$ before the comparison can be made. Furthermore, if Eq. (66) is utilized, a consistent value of $\bar{V}_w$ must also be selected. The details of how this is accomplished for each of the various closure rules will now be outlined.

"Zero Order" Closure (ZC): This form of transformation is utilized in Reference 3 and presupposes that $\bar{V}_w = d\bar{p}/d\bar{x} = 0$. In this case, the appropriate value of $\tilde{\sigma}$ which is required would be obtained from Eqs. (39) and (54) specialized to the case $\bar{V}_w=0$. There results

$$\tilde{\sigma} = \frac{\tilde{\rho}_s}{\tilde{\mu}_s} \; \frac{W + (1-W)(5.3/\bar{\varphi}) + \tilde{m}_e \, (37.45/\bar{\varphi}^2)}{1 + \tilde{m}_e} \tag{70}$$

where $\tilde{\rho}_s$ and $\tilde{\mu}_s$ are evaluated from Eq. (50) and (51) with $\tilde{u}_s = 10.6/\bar{\varphi}$.* For prescribed values of $\tilde{m}_e$ and W, Eq. (70) is of the form $\tilde{\sigma} = \tilde{\sigma}(\bar{\varphi})$. Accordingly, for any choice of $\bar{\varphi}$, the

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* In Appendix E, it is shown that as $\bar{V}_w \to 0$, Eq. (37) implies $\tilde{u}_s = \bar{y}_s^+/\bar{\varphi} = 10.6/\bar{\varphi}$ while Eq. (54) takes on the form indicated.

variation of $\tilde{u}$ vs $R_{\bar{y}}/\bar{\delta}$ can be generated using Eqs. (65) and (70). An example of this procedure is shown in Figure 6. Once the value of $\bar{\varphi}$ which best fits the data has been selected, the corresponding value of π can be obtained from Eq. (67) by recognizing that $R_{\bar{\delta}}$ is determined by the second of the relations (69). Finally, the value of the VP skin friction coefficient associated with this profile is obtained by using the correspondence (A-9).

First Order Closure with Pressure Gradient (FCP): Since $\bar{V}_w$ is also assumed to be zero here, the procedure is identical to that for ZC in view of the fact that Eqs. (65), (67) and (70) retain identical forms despite non-zero values of $d\bar{p}/d\bar{x}$.

First Order Closure with Transpiration (FCT): - As has been noted above, the variation of $\tilde{u}$ with $R_{\bar{y}}/\tilde{\sigma}$ depends here not only on a choice of $\bar{\varphi}$ but on a value of $\bar{V}_w$ as well. This is true both for Eq. (66) and Eq. (39). The additional relation which is required is Eq. (19). That is, with W specified, G_{1_w} is known. Accordingly, with a value of $\bar{\varphi}$ selected, the appropriate value of $\bar{V}_w$ is obtained from

$$\bar{V}_w = \frac{G_{1_w}}{\bar{\varphi}^2} \tag{71}$$

which follows from Eq. (19) and the definition of $\bar{Z}$. With $\bar{\varphi}$ and $\bar{V}_w$ thus selected, the proper value of $\tilde{\sigma}$ is obtained from Eq. (39) using (50), (51) and (54). Equation (66) is then utilized to obtain the desired variation of $\tilde{u}$ vs $R_{\bar{y}}/\tilde{\sigma}$. The value of π follows from Eq. (68) while c_f is obtained as before. An example of an FCT Clauser Plot is shown in Figure 7.

Second Order Closure (SC): The procedure for the SC and FCT cases is similar except that the appropriate value of $\bar{V}_w$ must be determined by iteration using Eq. (36) which, for specified M_e and W is of the form $H_1(\bar{\varphi},\bar{V}_w) = 0$. Otherwise, calculation of $\tilde{u}$ vs $R_{\bar{y}}/\tilde{\sigma}$ and π for any value of $\bar{\varphi}$ is identical. Figure 8 shows the result of applying SC rules to particular experimental profiles. A comparison of the results obtained using the various methods is presented in the next section. A summary of the Clauser Plot procedures outlined above is presented in Table II.

IV. RESULTS AND DISCUSSION

A. Profile Analysis

As indicated in previous sections, one of the primary reasons for modifying the compressibility transformation was for the purpose of eliminating the observed distortion in the wake region of the transformed velocity profiles. This phenomenon, which was first pointed out in Reference 3 and also discussed in References 2 and 7, manifests itself in values of the wake parameter π which are substantially smaller than those which are expected to occur for constant pressure CP flows.

To determine whether the modified transformation provides any improvement in this respect, a series of profiles were analyzed according to the procedures set down in the previous sections. That is, Clauser plots for selected profiles were prepared and the values of c_f and π determined. A typical set of these Clauser plots is shown in Figures 6 and 8 corresponding to two choices of closure. The final estimated values of c_f, π, etc., are compared in Table III. As can be noted therein, the values of π inferred by using the SC rule are only slightly higher than the ones deduced by means of the ZC rules. Figure 9 shows how both of these values compare with the anticipated CP values. It appears that, although some improvement has been obtained, the distortion for the most part has not been eliminated. This situation prevailed for all of the profiles analyzed which included most of those listed in Table II of Reference 2.

B. Zero Pressure Gradient Results

To assess the usefulness of the form of transformation developed here in predicting boundary layer behavior for the VP zero pressure gradient case, a series of calculations were carried out corresponding to the range of conditions $0 \le M_e \le 6$, $0.25 \le W \le 1.0$. Similar calculations were also carried out using both the FCT and FCP forms of the transformation. To expedite comparison of these results with experiment, the predictions due to Spalding-Chi (Reference 13) were utilized to represent the experimental data since this methodology is generally recognized as a reasonably accurate correlation of most of the available data.

The comparison of all of these results is shown in Figure 10. From this comparison it would appear that for the range of parameters examined:

- The FCT form of transformation provides the best agreement with experiment.
- The FCP form is superior to the SC form for the adiabatic wall case W = 1.
- For moderate Mach number level ($M_e \leq 2$), the SC form of transformation provides improved agreement with experiment, relative to the FCP form, for flow configurations involving heat transfer.
- For Mach numbers above approximately $M_e = 4$, the FCP and SC forms give essentially the same results for the heat transfer case.

C. Pressure Gradient Results

The present methodology has been used to generate boundary layer predictions for two cases involving adverse pressure gradient. These include the adiabatic wall configuration of McLafferty and Barber (Reference 14) as well as the case examined by Kepler and O'Brien (Reference 15) which involved heat transfer as well. These results are compared with the experimental data in Figures 11 and 12 as well as with the predictions due to the FCP form of the transformation. As may be noted, the two predictions are essentially equivalent and neither agrees too well with the data in the region of maximum pressure gradient. Note, however, that this latter disagreement is most probably due to the normal pressure gradients which exist in the experimental setup and which are not accounted for in either of the theoretical calculations.

V. CONCLUDING REMARKS

On the basis of the results cited it would appear that the modified compressibility transformation developed here provides, at best, only marginal improvement over the methodology previously developed in Reference 1. This conclusion is valid only if the latter is utilized without the use of the so-called "Wake Parameter Correlation." In this case the earlier formulation is superior to the present method.

In principal, of course, the aforementioned correlation could also be employed in the present analysis. The resulting formulation, however, can be anticipated to be equivalent to that of Reference 1 with the correlation. Thus, it is concluded that such additional development would not be appropriate and it is recommended that, for the present, the method of Reference 1 be employed to treat VP flows with pressure gradient.

Although the present effort has not been entirely successful in achieving an improved form of transformation which is self-consistent, the basic objective remains valid and further effort in this area is recommended. In this connection, the approach outlined in Reference 16 appears promising and its further exploitation to achieve this objective is indicated.

APPENDIX A

CORRESPONDENCE BETWEEN CP AND VP VARIABLES

$$\frac{d\bar{x}}{dx} = \xi(x) \tag{A-1}$$

$$\frac{\bar{\rho}}{\rho}\frac{\partial \bar{y}}{\partial y} = \eta(x) \tag{A-2}$$

$$\frac{\bar{\psi}-\bar{\psi}_w}{\psi-\psi_w} = \sigma(x) \tag{A-3}$$

$$\bar{u} = \frac{\sigma}{\eta} u \tag{A-4}$$

$$\bar{\rho}\bar{v}-\bar{\rho}\bar{v}_w = \frac{\sigma}{\xi}\{(\rho v-\rho_w v_w) + \frac{\bar{\rho}u}{\eta}\frac{\partial \bar{y}}{\partial x} - \frac{\psi-\psi_w}{\sigma}\frac{d\sigma}{dx}\} \tag{A-5}$$

$$\theta = \frac{\bar{\theta}}{\tilde{\eta}} \tag{A-6}$$

$$\delta^* = \frac{1}{\tilde{\eta}}[\bar{\delta}^* + \int_0^{\bar{\delta}} (\frac{1}{\tilde{\rho}} -1)\, d\bar{y}\ ] \tag{A-7}$$

$$H = \bar{H} + \frac{1}{\bar{\theta}}\int_0^{\bar{\delta}} (\frac{1}{\tilde{\rho}} -1)\, d\bar{y} \tag{A-8}$$

$$c_f = \tilde{\rho}_w\tilde{\mu}_w\tilde{\sigma}\ \bar{c}_f \tag{A-9}$$

$$R_{\bar{y}} = \tilde{\sigma}\int_0^{R_y} \tilde{\rho}\ dRy \tag{A-10}$$

$$R_{\bar{\delta}} = \tilde{\sigma}\int_0^{R_\delta} \tilde{\rho}\ dRy \tag{A-11}$$

$$R_{\bar{\theta}} = \tilde{\sigma}\ R_\theta \tag{A-12}$$

APPENDIX B

ELEMENTS OF THE SYSTEM (33)*

$$A1 = 1/k_1$$

$$A2 = \Lambda_2^{(1)} + \Lambda_1^{(1)}$$

$$A3 = \Lambda_2^{(\bar{\eta}*)} + (2-\tilde{u}*)\ \Lambda_1^{(\bar{\eta}*)} + (1-\tilde{u}*)\ \bar{\eta}*$$

$$A9 = -\ 1/k_1 + \frac{2\bar{U}_e}{\bar{\varphi}\bar{V}_w}\left[(1+\frac{\bar{V}_w\bar{\varphi}^2}{\bar{U}_e})^{\frac{1}{2}} - 1\right] - 2\bar{\varphi}(1+\frac{\bar{V}_w\bar{\varphi}^2}{\bar{U}_e})^{-\frac{1}{2}}$$

$$A10 = F_1^{(1)}$$

$$A11 = F_1^{(\bar{\eta}*)}$$

$$A13 = \frac{\partial H_1}{\partial \ln \bar{\varphi}}$$

$$A14 = \frac{\partial \ln H_2}{\partial \ln \bar{\varphi}}$$

$$A17 = 2/k_1$$

$$A18 = F_3^{(1)}$$

$$A19 = F_3^{(\bar{\eta}*)}$$

$$A25 = -\ A9 - \bar{\varphi}(1+\frac{\bar{V}_w\bar{\varphi}^2}{\bar{U}_e})^{-\frac{1}{2}}$$

$$A26 = 2\Lambda_2^{(1)} + 3\Lambda_1^{(1)} - F_2^{(1)}$$

$$A27 = 2\Lambda_2^{(\bar{\eta}*)} + (4-\tilde{u}*)\ \Lambda_1^{(\bar{\eta}*)} + (1-\tilde{u}*)\ \bar{\eta}* - F_2^{(\bar{\eta}*)}$$

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* The parameters k_1, $\Lambda_1^{(1)}$, $\Lambda_2^{(1)}$, $\Lambda_2^{(\bar{\eta}*)}$, $\Lambda_1^{(\bar{\eta}*)}$, $\tilde{u}*$, $F_1^{(1)}$, $F_1^{(\bar{\eta}*)}$, $F_2^{(1)}$, $F_2^{(\bar{\eta}*)}$, $F_3^{(1)}$, $F_3^{(\bar{\eta}*)}$ are defined in Part II of this report.

$$A28 = 1$$

$$A29 = \frac{\partial H_1}{\partial \ell n \bar{U}_e}$$

$$A30 = \frac{\partial \ell n H_2}{\partial \ell n \bar{U}_e}$$

$$A31 = -1$$

$$A33 = (A1-A25)/\bar{V}_w$$

$$A34 = F_2^{(1)}/\bar{V}_w$$

$$A35 = F_2^{(\bar{\eta}^*)}/\bar{V}_w$$

$$A37 = \frac{\partial H_1}{\partial \bar{V}_w}$$

$$A38 = \frac{\partial \ell n H_2}{\partial \bar{V}_w}$$

$$A\ 46 = -1$$

$$A47 = 1$$

$$A48 = \bar{H} + 2$$

$$A55 = -1$$

$$A56 = -(\bar{H}+1)$$

$$A61 = \frac{\partial H_1}{\partial \chi}$$

$$A62 = \frac{\partial \ell n H_2}{\partial \chi}$$

$$A63 = \frac{d \ell n U_e/\nu_e}{d\chi}$$

$$A64 = (\bar{H}+1)\,\frac{d\ln\rho_e}{d\chi} - (H-\bar{H})\,\frac{d\ln U_e}{d\chi} - (\bar{H}+2)\,\frac{d\ln\mu_e}{d\chi}$$

$$+ \frac{\tilde{\rho}_w \tilde{\mu}_w\, \tilde{\sigma}\, \tilde{\sigma}_o}{\bar{\omega}^2\, \bar{\Sigma}\bar{R}}\left(\frac{\tilde{\eta}}{\tilde{\eta}_o}\right)$$

$$C2 = -\frac{1}{\bar{R}}\left(\frac{1}{\bar{\omega}^2} + \frac{\bar{V}_w}{\bar{U}_e}\right)$$

$$C3 = \frac{1}{\bar{R}}\left\{\bar{\tau}^* - \frac{1}{\bar{\omega}^2} - \frac{\bar{V}_w \tilde{u}^*}{\bar{U}_e}\right\}$$

$$C4 = \frac{\bar{V}_w}{\bar{\omega}^2} - \frac{G_{1_w}\bar{U}_e}{\bar{\omega}^4} + \left(\frac{\tilde{\eta}_o}{\tilde{\eta}}\right)\left(\tilde{\rho}_w^2 \tilde{\mu}_w\, \tilde{\sigma}\, \tilde{\sigma}_o\right)^{-1} \frac{d\ln\bar{U}_e}{d\chi}$$

APPENDIX C

THE PARTIAL DERIVATIVES APPEARING IN EQS. (36a) and (39a)

From Eq. (36)

$$\frac{\partial H_1}{\partial \ell n\bar{\varphi}} = \frac{2\bar{Z}}{(1+\tilde{u}_s\bar{Z})^2} - \left[\frac{\partial \tilde{u}_s}{\partial \ell n\bar{\varphi}} + \frac{\partial \tilde{u}_s}{\partial \ell n\bar{v}_w}\right]\left[\left(\frac{\bar{Z}}{1+\tilde{u}_s\bar{Z}}\right)^2 + \frac{\partial G_{1_s}}{\partial \tilde{u}_s}\right]$$

$$\frac{\partial H_1}{\partial \ell n\bar{U}_e} = -\frac{\bar{Z}}{(1+\tilde{u}_s\bar{Z})^2} + \frac{\partial \tilde{u}_s}{\partial \ell n\bar{v}_w} \left[\left(\frac{\bar{Z}}{1+\tilde{u}_s\bar{Z}}\right)^2 + \frac{\partial G_{1_s}}{\partial \tilde{u}_s}\right]$$

$$\frac{\partial H_1}{\partial \bar{V}_w} = -\frac{1}{\bar{V}_w}\frac{\partial H_1}{\partial \ell n\bar{U}_e}$$

$$\frac{\partial H_1}{\partial \chi} = -\left[\frac{\partial G_{1_s}}{\partial \tilde{m}_e}\frac{d\tilde{m}_e}{d\chi} + \frac{\partial G_{1_s}}{\partial W}\frac{dW}{d\chi} + \frac{\partial G_{1_s}}{\partial T_e}\frac{dT_e}{d\chi}\right],$$

where, from Eq. (53),

$$\frac{\partial G_{1_s}}{\partial \tilde{u}_s} = G_{1_s}\left[\Upsilon\frac{\partial \tilde{T}_s}{\partial \tilde{u}_s} + \frac{2\tilde{m}_e}{(1-W)+2\tilde{m}_e\tilde{u}_s}\right]$$

$$\frac{\partial G_{1_s}}{\partial \tilde{m}_e} = G_{1_s}\left[\Upsilon\frac{\partial \tilde{T}_s}{\partial \tilde{m}_e} + \frac{2\tilde{u}_s}{(1-W)+2\tilde{m}_e\tilde{u}_s} - \frac{1}{1+\tilde{m}_e}\right]$$

$$\frac{\partial G_{1_s}}{\partial W} = G_{1_s}\left[\Upsilon\frac{\partial \tilde{T}_s}{\partial W} - \frac{1}{(1-W)+2\tilde{m}_e\tilde{u}_s}\right]$$

$$\frac{\partial G_{1_s}}{\partial T_e} = 397.2\left(\frac{G_{1_s}}{T_e^2}\right)\left[\frac{\tilde{T}_s}{\tilde{T}_s^2-(198.6/T_e)^2}\right]$$

where, from Eq. (50)

$$\frac{\partial \tilde{T}_s}{\partial \tilde{u}_s} = \frac{(1-W)+2\tilde{m}_e\tilde{u}_s}{1+\tilde{m}_e}$$

$$\frac{\partial \tilde{T}_s}{\partial \tilde{m}_e} = \frac{\tilde{u}_s^2-\tilde{T}_s}{1+\tilde{m}_e}$$

$$\frac{\partial \tilde{T}_s}{\partial W} = \frac{1-\tilde{u}_s}{1+\tilde{m}_e}$$

with

$$\Upsilon \equiv \frac{397.2/T_e}{\tilde{T}_s{}^2-(198.6/T_e)^2} - \frac{1}{\tilde{T}_s}$$

and where $\frac{\partial \tilde{u}_s}{\partial \ln \bar{\varphi}}$ and $\frac{\partial \tilde{u}_s}{\partial \ln \bar{v}_w^+}$ are evaluated in Appendix D.

From Eqs. (39) and (54):

$$\frac{\partial \ln H_2}{\partial \ln \varphi} = \left(\frac{\partial \ln H_2}{\partial \tilde{u}_s}\right)\left(\frac{\partial \tilde{u}_s}{\partial \ln \bar{\varphi}}\right) + \left(\frac{\partial \tilde{u}_s}{\partial \ln \bar{v}_w^+}\right) + \frac{\left(\frac{1-W}{\bar{\varphi}}\right)\left(\frac{d\lambda_1}{d\ln \bar{v}_w^+} - \lambda_1\right) + \frac{\tilde{m}_e}{\bar{\varphi}^2}\left(\frac{d\lambda_2}{d\ln \bar{v}_w^+} - 2\lambda_2\right)}{W + (1-W)\ (\lambda_1/\bar{\varphi}) + \tilde{m}_e(\lambda_2/\bar{\varphi}^2)}$$

$$\frac{\partial \ln H_2}{\partial \ln \bar{U}_e} = \left(\frac{\partial \ln H_2}{\partial \tilde{u}_s}\right)\left(\frac{\partial \tilde{u}_s}{\partial \ln \bar{v}_w^+}\right) + \frac{\frac{1-W}{\bar{\varphi}}\ \frac{d\lambda_1}{d\ln \bar{v}_w^+} + \frac{\tilde{m}_e}{\bar{\varphi}^2}\ \frac{d\lambda_2}{d\ln \bar{v}_w^+}}{W + (1-W)\ (\lambda_1/\bar{\varphi}) + \tilde{m}_e(\lambda_2/\bar{\varphi}^2)}$$

$$\frac{\partial \ln H_2}{\partial \bar{V}_w} = -\frac{1}{\bar{V}_w}\ \frac{\partial \ln H_2}{\partial \ln \bar{U}_e}$$

$$\frac{\partial \ln H_2}{\partial \chi} = \frac{\partial \ln H_2}{\partial \tilde{m}_e}\ \frac{d\tilde{m}_e}{d\chi} + \frac{\partial \ln H_2}{\partial W}\ \frac{dW}{d\chi} + \frac{\partial \ln H_2}{\partial T_e}\ \frac{dT_e}{d\chi}$$

Where, taking into account also Eq. (51)

$$\frac{\partial \ln H_2}{\partial \tilde{u}_s} = \Gamma\ \frac{\partial \tilde{T}_s}{\partial \tilde{u}_s}$$

$$\frac{\partial \ell n H_2}{\partial \tilde{m}_e} = \Gamma \frac{\partial \tilde{T}_s}{\partial \tilde{m}_e} + \frac{\lambda_2/\bar{\varphi}^2}{W+(1-W)(\lambda_1/\bar{\varphi}) + \tilde{m}_e(\lambda_2/\bar{\varphi}^2)} - \frac{1}{1+\tilde{m}_e}$$

$$\frac{\partial \ell n H_2}{\partial W} = \Gamma \frac{\partial \tilde{T}_s}{\partial W} + \frac{1-(\lambda_1/\bar{\varphi})}{W+(1-W)(\lambda_1/\bar{\varphi}) + \tilde{m}_e(\lambda_2/\bar{\varphi}^2)}$$

$$\frac{\partial \ell n H_2}{\partial T_e} = \frac{198.6}{T_e^2} \quad \frac{(\tilde{T}_s-1)}{(1+198.6/T_e)(\tilde{T}_s+198.6/T_e)}$$

with $\Gamma \equiv -\left(\frac{1.5}{\tilde{T}_s}\right)\left(\frac{\tilde{T}_s-331/T_e}{\tilde{T}_s+198.6/T_e}\right)$

and where $\frac{d\lambda_1}{d\ell n\bar{v}_w^+}$ and $\frac{d\lambda_2}{d\ell n\bar{v}_w^+}$ are evaluated in Appendix D.

APPENDIX D

AUXILIARY CP PARAMETERS

Using Eq. (10) of Part II of this report

$$\lambda_1 \equiv \frac{1}{\bar{y}_s^+} \int_0^{\bar{y}_s^+} \bar{u}^+ d\bar{y}^+ = \frac{1}{\bar{v}_w^+} \left(\frac{\bar{u}_s^+}{\bar{y}_s^+} - 1 \right) \tag{D-1}$$

$$\lambda_2 \equiv \frac{1}{\bar{y}_s^+} \int_0^{\bar{y}_s^+} (\bar{u}^+)^2 \, d\bar{y}^+ = \frac{1}{\bar{v}_w^+} \left(\frac{\bar{u}_s^{+2}}{2\bar{y}_s^+} - \lambda_1 \right) \tag{D-2}$$

where

$$\bar{u}_s^+ \equiv \tilde{u}_s \bar{\varphi} = \frac{1}{\bar{v}_w^+} [\exp(\bar{v}_w^+ \bar{y}_s^+) - 1] \tag{D-3}$$

From (D-1) and (D-2) it follows that

$$\frac{d\lambda_1}{d\ln \bar{v}_w^+} = \left(1 + \frac{d\ln \bar{y}_s^+}{d\ln \bar{v}_w^+} \right) (\bar{u}_s^+ - \lambda_1) - \lambda_1 \tag{D-4}$$

$$\frac{d\lambda_2}{d\ln \bar{v}_w^+} = \left(1 + \frac{d\ln \bar{y}_s^+}{d\ln \bar{v}_w^+} \right) \left\{ (\bar{u}_s^{+2} - \lambda_2) + \frac{1}{\bar{v}_w^+} (\bar{u}_s^+ - \lambda_1) \right\}$$

$$- 2\lambda_2 - \frac{1}{\bar{v}_w^+} \left(\lambda_1 + \frac{d\lambda_1}{d\ln \bar{v}_w^+} \right) \tag{D-5}$$

while from (D-3)

$$\frac{\partial \tilde{u}_s}{\partial \ln \bar{\varphi}} = - \tilde{u}_s \tag{D-6}$$

$$\frac{\partial \bar{u}_s}{\partial \ln \bar{v}_w^+} = \frac{\bar{y}_s^+}{\bar{\varphi}} \left[1 + \frac{d\ln \bar{y}_s^+}{d\ln \bar{v}_w^+} \right] \left[\exp(\bar{v}_w^+ \bar{y}_s^+) \right] - \tilde{u}_s \tag{D-7}$$

where, from Eq. (15) of Part II of this report

$$\frac{d\ell n\bar{y}_s^+}{d\ell n\bar{v}_w^+} = \frac{k_1[\bar{y}_s^+ \exp(\bar{v}_w^+ \bar{y}_s^+/2)-k_2]-\ell n\bar{y}_s^+}{1-k_1\bar{y}_s^+ \exp(\bar{v}_w^+\bar{y}_s^+/2)} \tag{D-8}$$

APPENDIX E

LIMITING FORM OF THE ELEMENTS OF THE SYSTEM (33)

In view of the results shown in Figure 1, it can be anticipated that, in general, $\bar{V}_w$ can become vanishingly small during the course of any particular calculation. Since several of the coefficients appearing in Eq. (33) take on indeterminate forms when $\bar{V}_w = 0$, special treatment of these is required. Specifically, for $\bar{V}_w$ sufficiently small, special limiting forms of these coefficients are employed by the computer program.

These special forms are listed below:*

$$\lim_{\bar{V}_w \to 0} A9 = -\left(\frac{1}{k_1} + \bar{\varphi}\right)$$

$$\lim_{\bar{V}_w \to 0} A25 = \frac{1}{k_1}$$

$$\lim_{\bar{V}_w \to 0} A33 = \bar{\varphi}^3 / 4\bar{U}_e$$

$$\lim_{\bar{V}_w \to 0} A37 = \frac{\bar{\varphi}^2}{U_e} - \frac{\bar{\varphi}}{U_e} \frac{\partial \tilde{u}_s}{\partial \bar{v}_w^+} \frac{\partial G_{1_s}}{\partial \tilde{u}_s}$$

$$\lim_{\bar{V}_w \to 0} A38 = \frac{\bar{\varphi}}{U_e} \left\{ \frac{\partial \ell n H_2}{\partial \bar{v}_w^+} + \frac{\partial \ell n H_2}{\partial \tilde{u}_s} \frac{\partial \tilde{u}_s}{\partial \bar{v}_w^+} \right\}$$

where

$$\frac{\partial \tilde{u}_s}{\partial \bar{v}_w^+} = \frac{1}{\bar{\varphi}} \left\{ \frac{d\bar{y}_s^+}{d\bar{v}_w^+} + \frac{\bar{y}_s^{+2}}{2} \right\}$$

$$\frac{\partial \ell n H_3}{\partial \bar{v}_w^+} = \frac{\frac{1-W}{\bar{\varphi}} \frac{d\lambda_1}{d\bar{v}_w^+} + \frac{\tilde{m}_e}{\bar{\varphi}^2} \frac{d\lambda_2}{d\bar{v}_w^+}}{W + (1-W)(\lambda_1/\bar{\varphi}) + \tilde{m}_e(\lambda_2/\bar{\varphi}^2)}$$

---

* Limiting values of A34 and A35 are not listed here since the parameter $F_2^{(1)}$ and $F_2^{(\eta^*)}$ are themselves proportional to $\bar{V}_w$ which allows elimination of the indeterminancy in an obvious manner.

$$\frac{d\lambda_1}{d\bar{v}_w^+} = \frac{1}{2}\left\{ \frac{d\bar{y}_s^+}{d\bar{v}_w^+} + \frac{\bar{y}_s^{+2}}{3} \right\}$$

$$\frac{d\lambda_2}{d\bar{v}_w^+} = \frac{\bar{y}_s^+}{3}\left\{ 2\,\frac{d\bar{y}_s^+}{d\bar{v}_w^+} + \frac{5}{16}\,\bar{y}_s^{+2} \right\}$$

$$\frac{d\bar{y}_s^+}{d\bar{v}_w^+} = \frac{\bar{y}_s^{+3}}{4}\left\{ \frac{k_1}{1-k_1\bar{y}_s^+} \right\}$$

Note that these relations and others appearing in the various coefficients must be supplemented by limiting values of $\bar{y}_s^+$ and $\tilde{u}_s$. From Eq. (D-3) it is easy to show that

$$\lim_{\bar{v}_w \to 0} \tilde{u}_s = \lim_{\bar{v}_w^+ \to 0} \tilde{u}_s = \frac{\bar{y}_s^+}{\bar{\varphi}}$$

while from Eq. (15) of Part II of this report it can be shown that in the limit $\bar{v}_w \to 0$ $\bar{y}_s^+$ satisfies the transcendental equation

$$\bar{y}_s^+ = k_2 + \frac{1}{k_1}\,\bar{y}_s^+$$

which, for the choice of constants used here (i.e., $k_2 = 4.9$, $1/k_1 = 2.43$) leads to

$$\lim_{\bar{v}_w \to 0} \bar{y}_s^+ = 10.6$$

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TABLE I. BEHAVIOR OF CLOSURE RULES FOR LIMITING CASES

LIMITING CASES			VALUE OF COEFFICIENTS			CP VARIABLES	
W	P	α	G_{1_w}	G_3	G_5	$\bar{Z}-\bar{P}$	$\bar{Z}(\bar{Z}-\bar{P})$
1	0	$\neq 1$	0	0	$\neq 0$	0	$\neq 0$
1	0	1	0	0	0	0	0
1	$\neq 0$	1	0	0	0	$\neq 0$	0

TABLE II. SUMMARY OF EQUATIONS USED TO GENERATE CLAUSER PLOT REPRESENTATIONS

Flow Configuration	Closure Rule	$\bar{V}_w$ Calculated From	$\tilde{\sigma}$ Calculated From	Velocity Distribution Calculated From	π Calculated From	Comments
CP Impermeable Flow	-	$\bar{V}_w = 0$	$\tilde{\sigma} = 1$	Eq. (65)	Eq. (67)	
CP Transpired Flow	-	Given	$\tilde{\sigma} = 1$	Eq. (66)	Eq. (68)	
VP Impermeable Flow	ZC FCP	$\bar{V}_w = 0$	Eq. (70)	Eq. (65)	Eq. (67)	$\bar{y}_s^+ = 10.6; \tilde{u}_s = \frac{10.6}{\varphi}$
VP Impermeable Flow	FCT	Eq. (71)	Eq. (38)	Eq. (66)	Eq. (68)	
VP Impermeable Flow	SC	Eq. (36)	Eq. (38)	Eq. (66)	Eq. (68)	

TABLE III. FINAL CHOICE OF CP PARAMETERS FOR VELOCITY PROFILE DATA OF REFERENCE 19

TEST NO.	EXPERIMENTAL RESULTS		INFERRED FROM CLAUSER PLOTS				
	c_f	R_θ	Closure Rule	φ	c_f	$\tilde{\sigma}$	π
28	.00272	2980	ZC	23.03	.00251	.728	.374
			SC	22.80	.00256	.729	.421
29	.00218	6470	ZC	25.08	.00206	.709	.419
			SC	24.83	.00211	.710	.460
30	.00202	8570	ZC	25.99	.00190	.702	.456
			SC	26.77	.00195	.704	.495

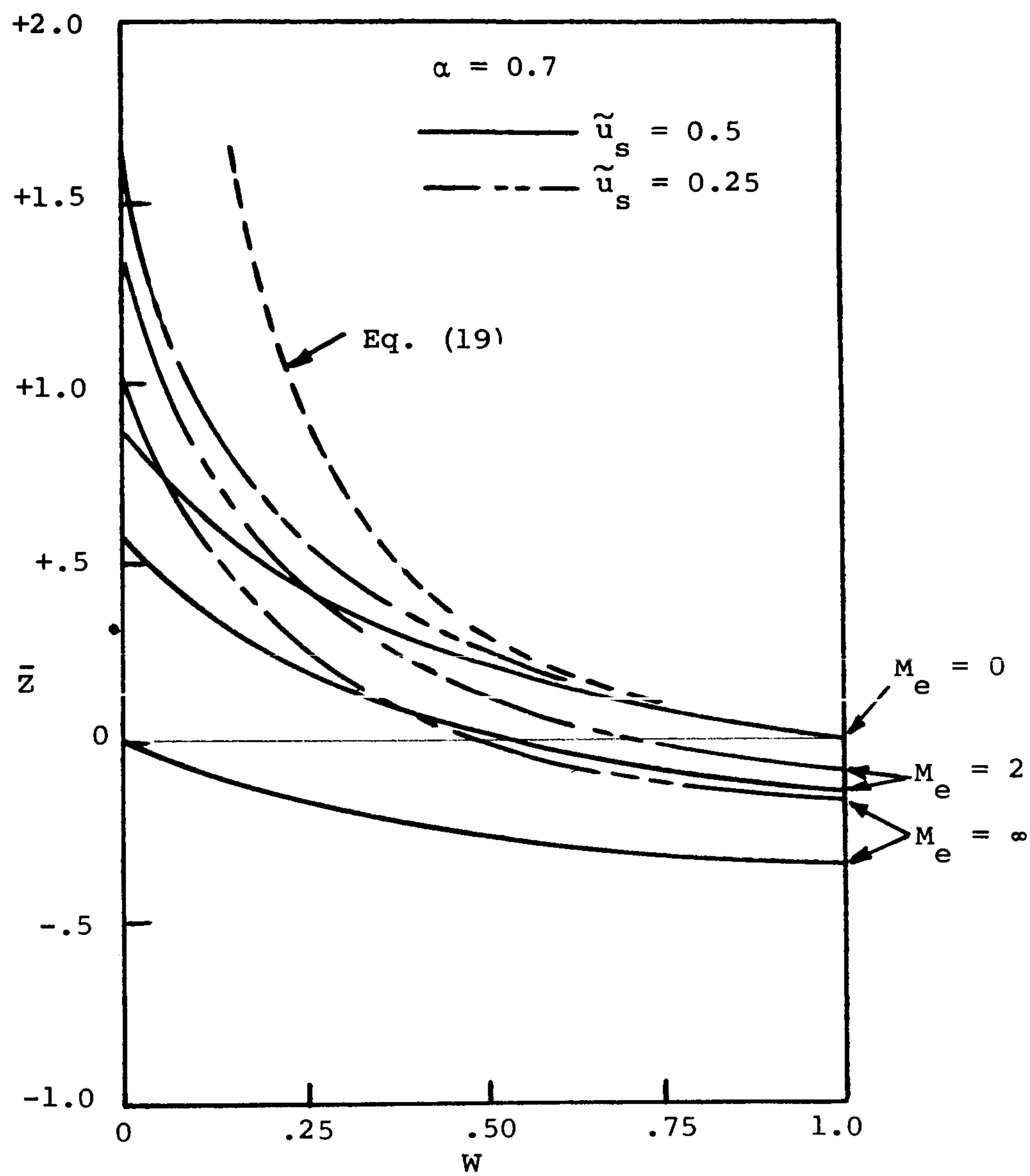


FIGURE 1: EFFECT OF MACH NUMBER, WALL TEMPERATURE AND SUBLAYER EDGE VELOCITY RATIO ON THE CP BLOWING PARAMETER ACCORDING TO SECOND ORDER CLOSURE RULES

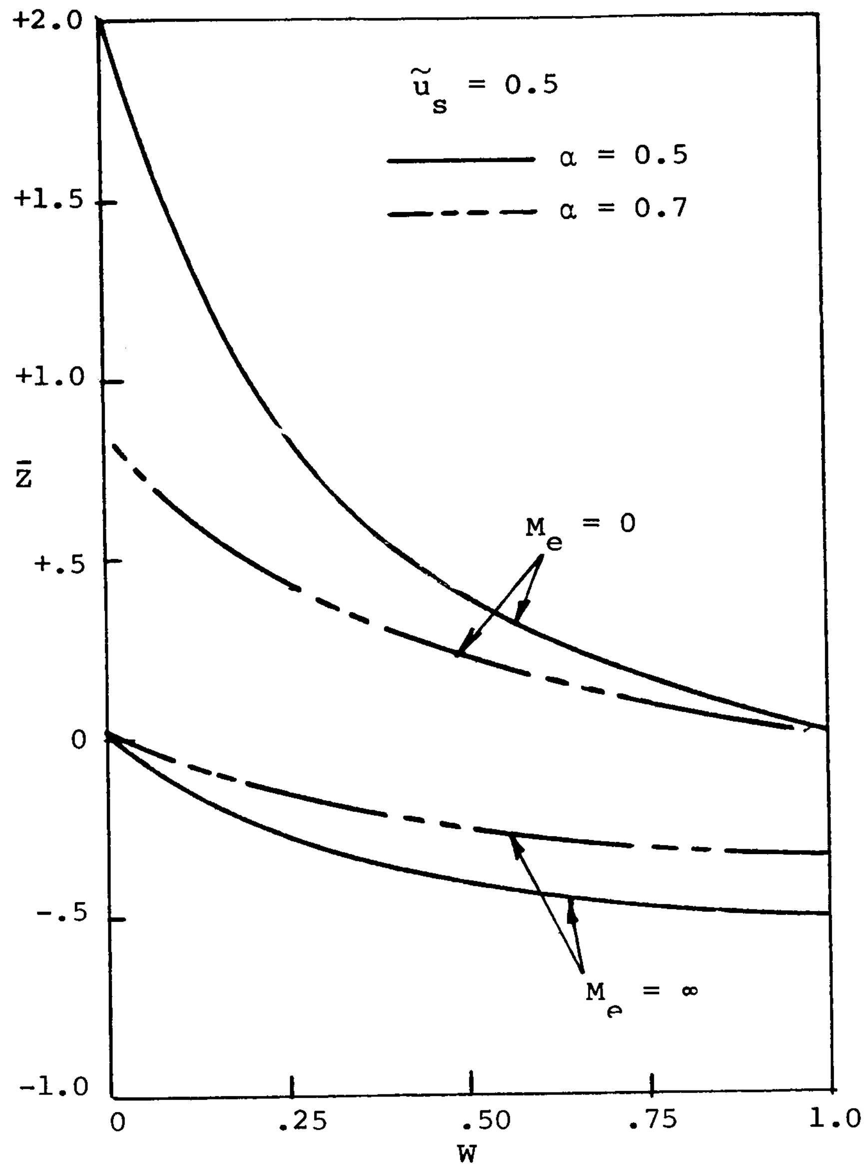


FIGURE 2: EFFECT OF MACH NUMBER, WALL TEMPERATURE RATIO AND VISCOSITY EXPONENT ON THE CP BLOWING PARAMETER ACCORDING TO SECOND ORDER CLOSURE RULES

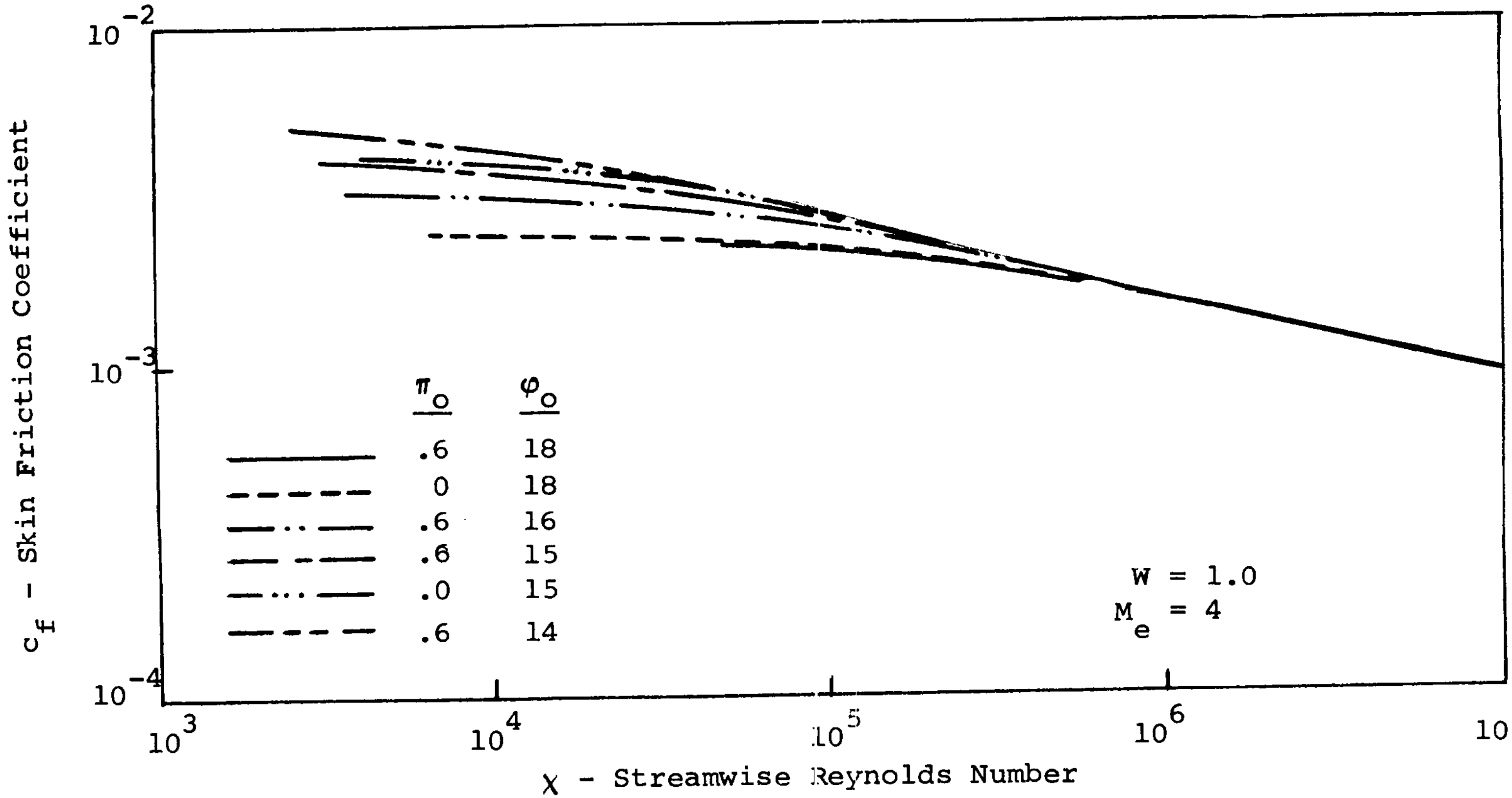


FIGURE 3: EFFECT OF INITIAL CONDITIONS ON BOUNDARY LAYER DEVELOPMENT FOR CONSTANT PRESSURE VP FLOW

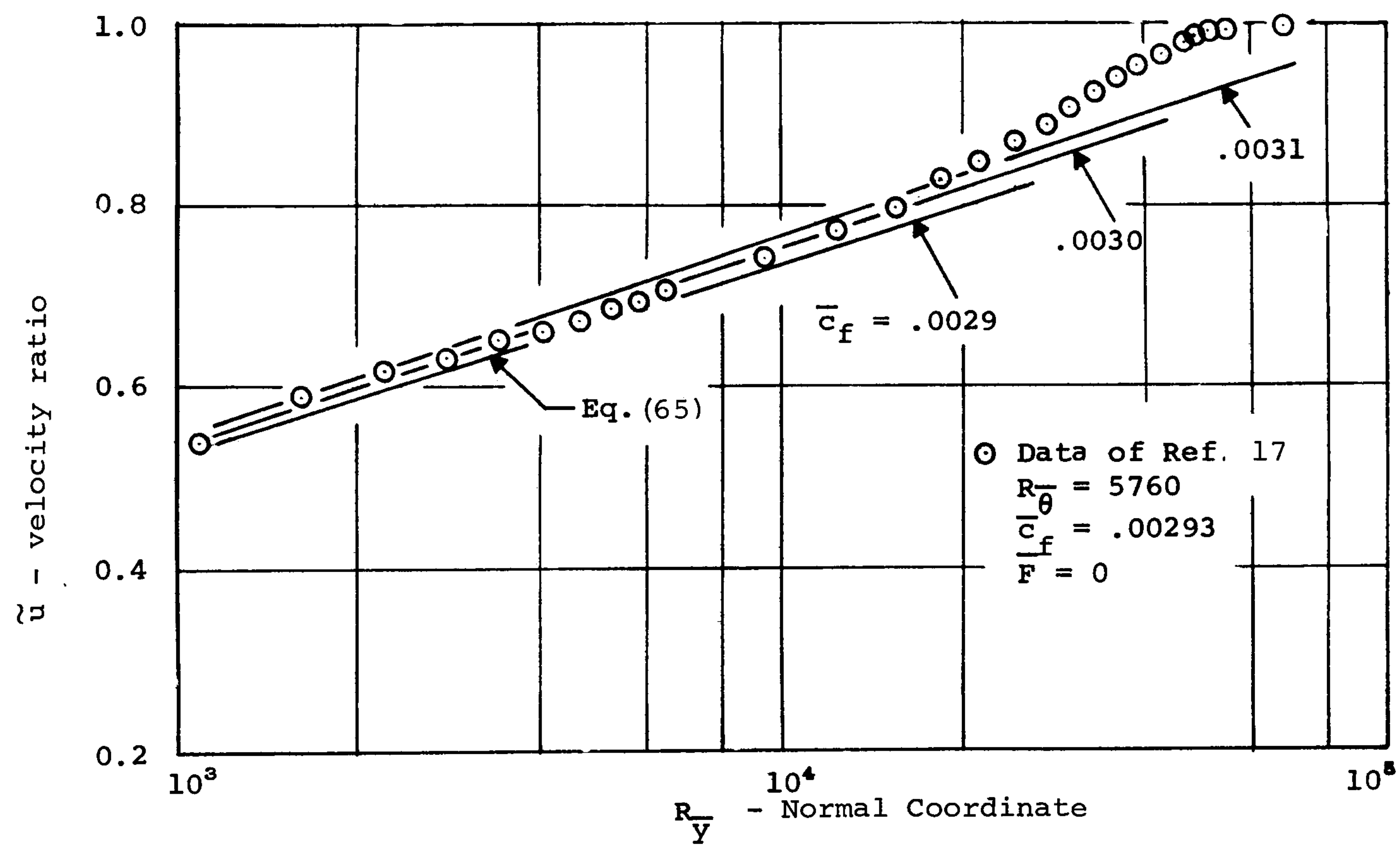


FIGURE 4: CLAUSER PLOT FOR CP IMPERMEABLE PROFILE DATA OF REFERENCE 17

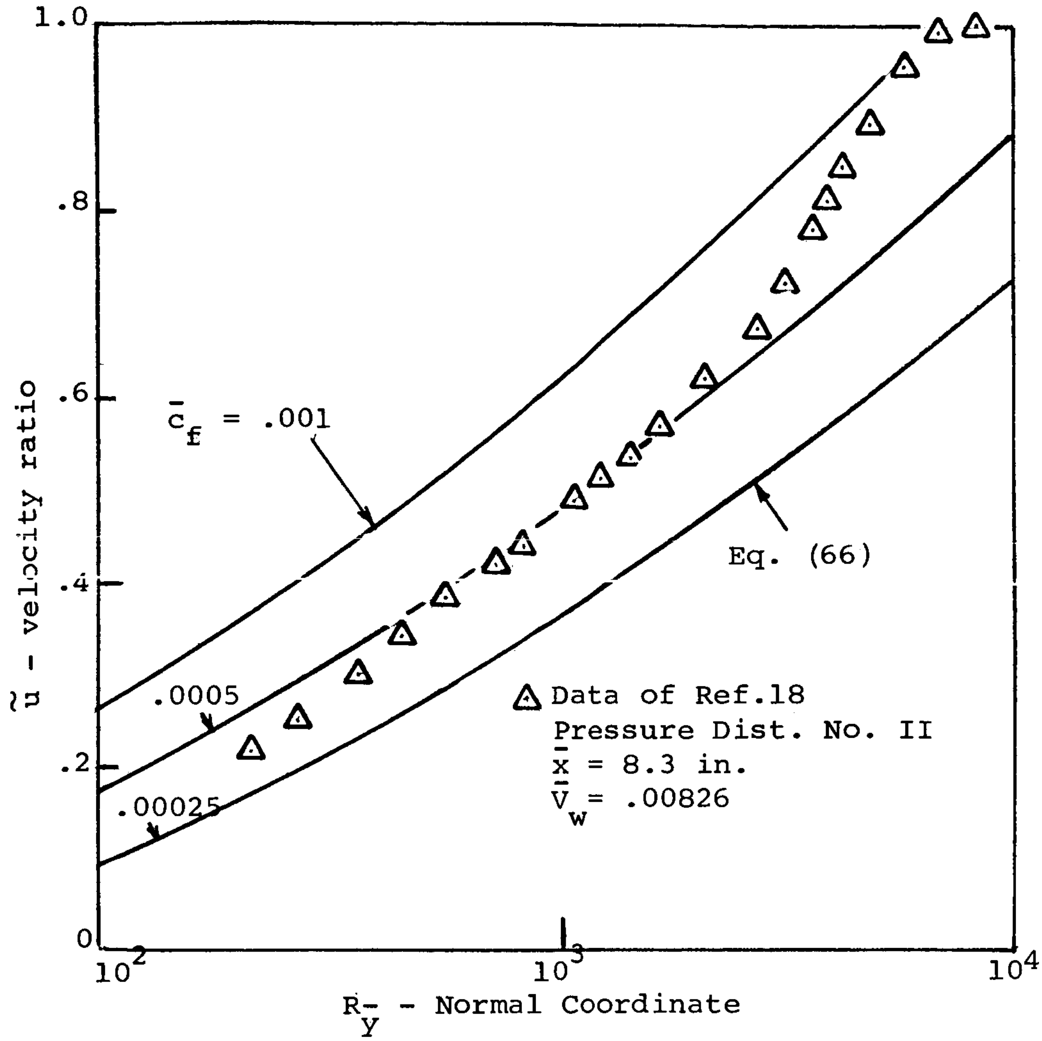


FIGURE 5: CLAUSER PLOT FOR CP TRANSPIRED PROFILE DATA OF REFERENCE 18

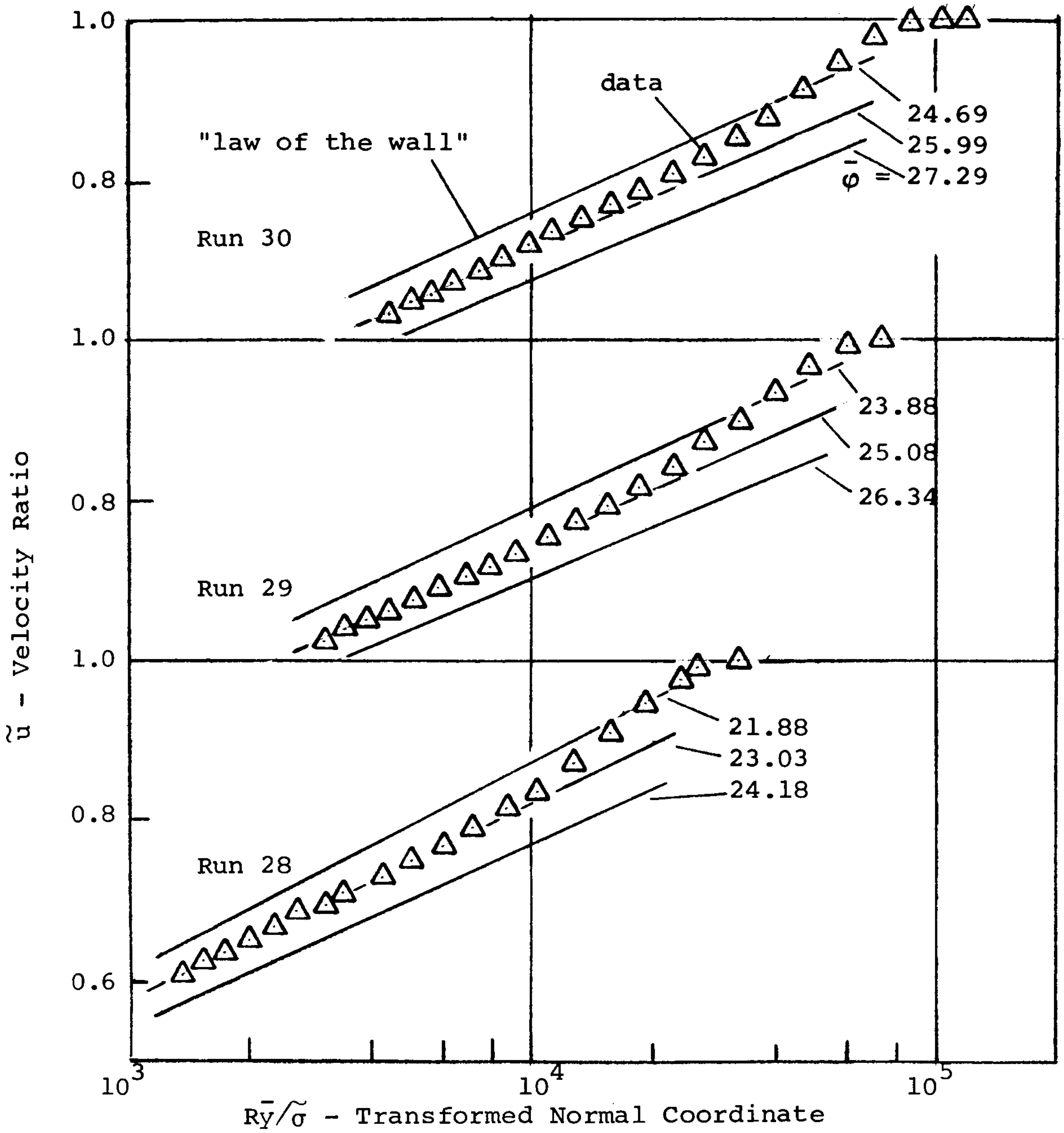
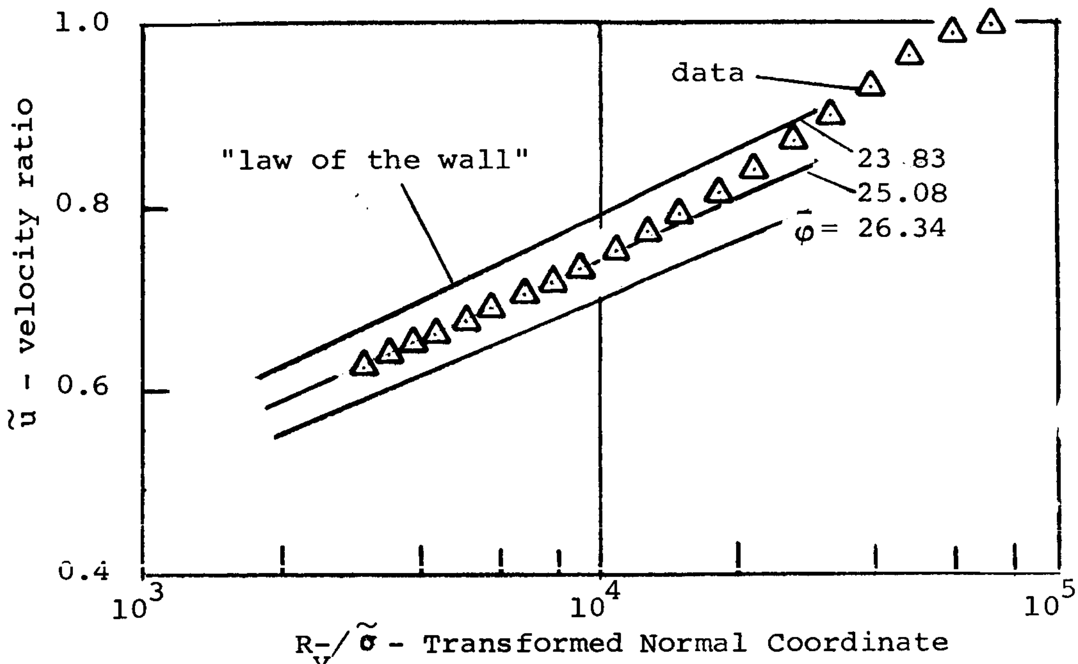


FIGURE 6: CLAUSER PLOTS ACCORDING TO ZERO ORDER CLOSURE FOR PROFILE DATA OF REFERENCE 19, $M_e \approx 2.0$, $W \approx 1$



$\bar{\varphi} = 25.08$
$c_f = .00206$
$\tilde{\sigma} = .709$
$\pi = .404$

Final Choice of CP Parameters inferred from Clauser Plot

FIGURE 7: CLAUSER PLOT ACCORDING TO FIRST ORDER CLOSURE WITH TRANSPIRATION FOR PROFILE DATA OF REFERENCE 19 - Run 29 (c_f = .00218, R_θ = 6470)

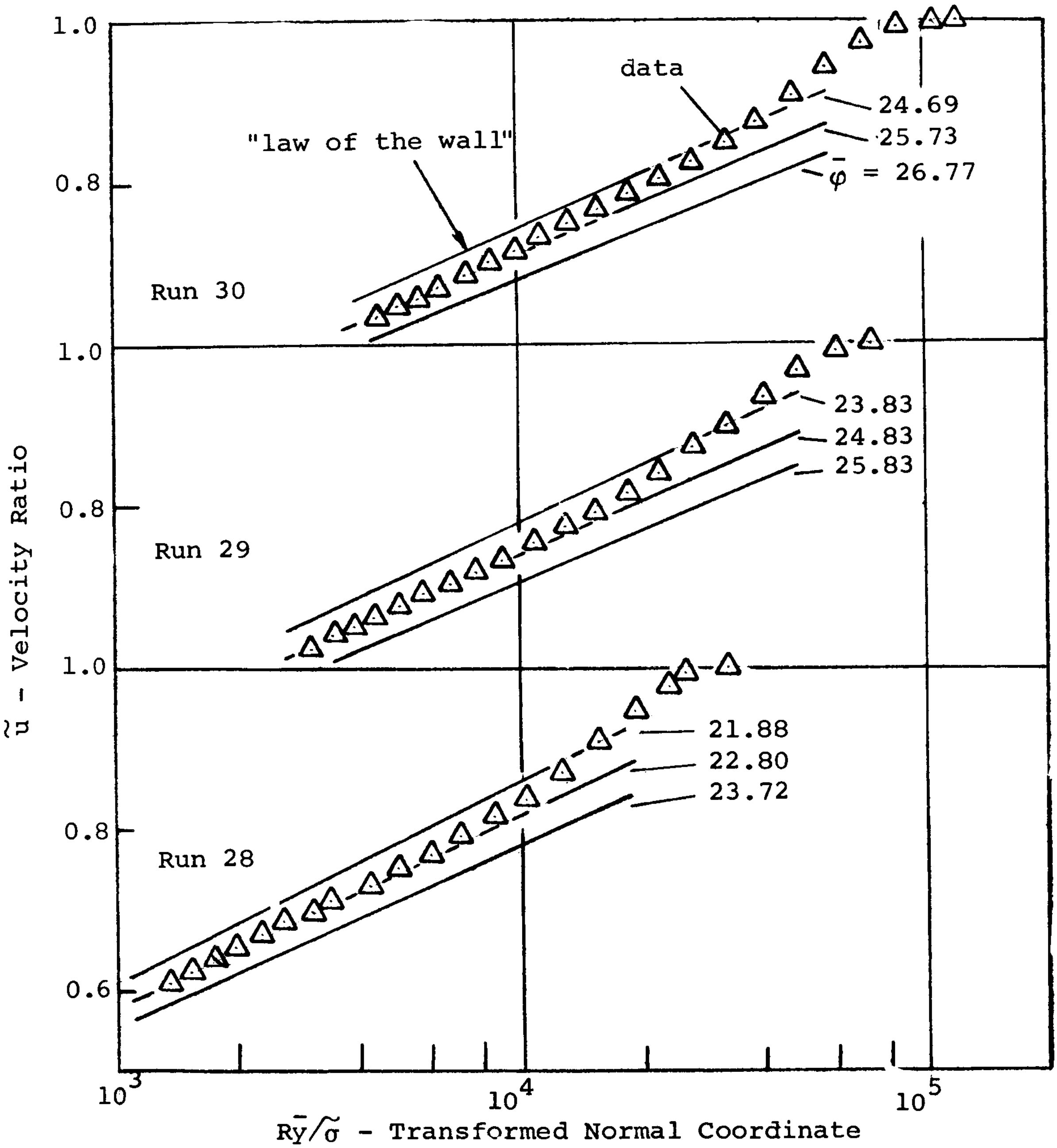


FIGURE 8: CLAUSER PLOTS ACCORDING TO SECOND ORDER CLOSURE FOR PROFILE DATA OF REFERENCE 19, $M_e \approx 2.0$, $W \approx 1$

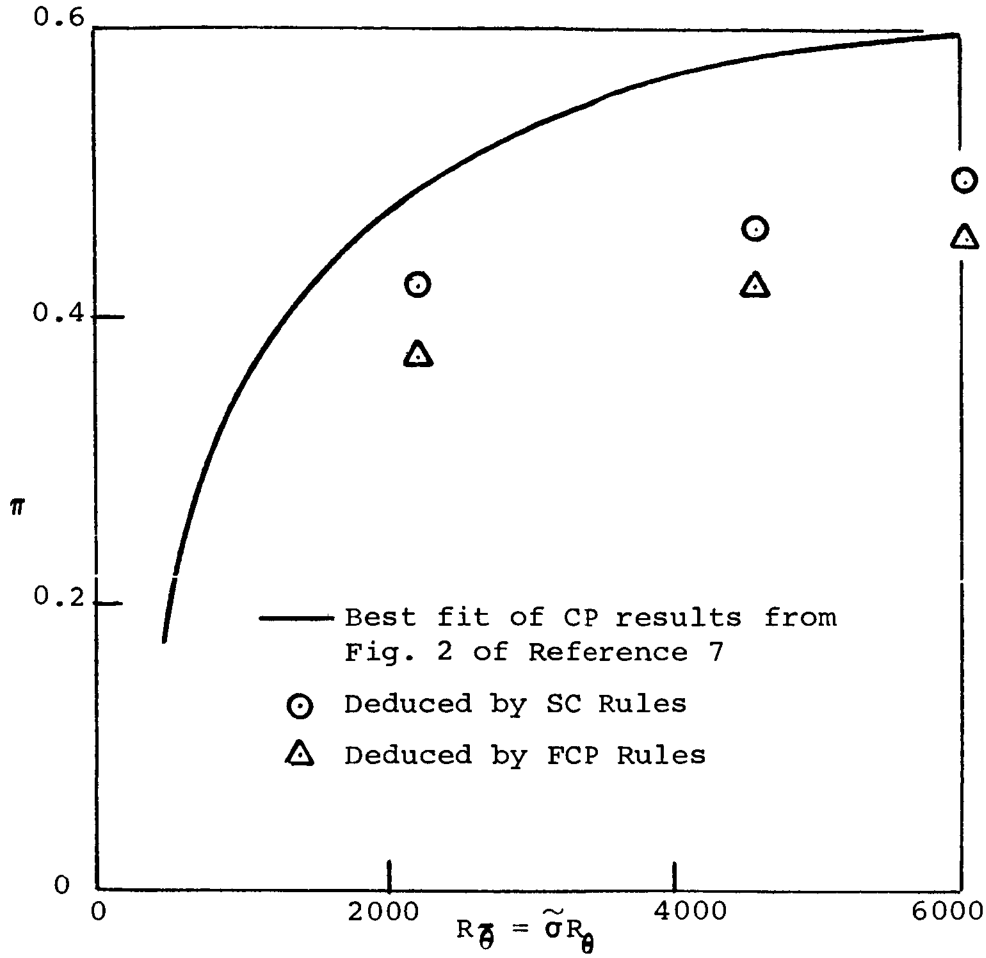


FIGURE 9: COMPARISON OF WAKE PARAMETER DEDUCED FROM PROFILES OF REFERENCE 19 WITH CP VALUES

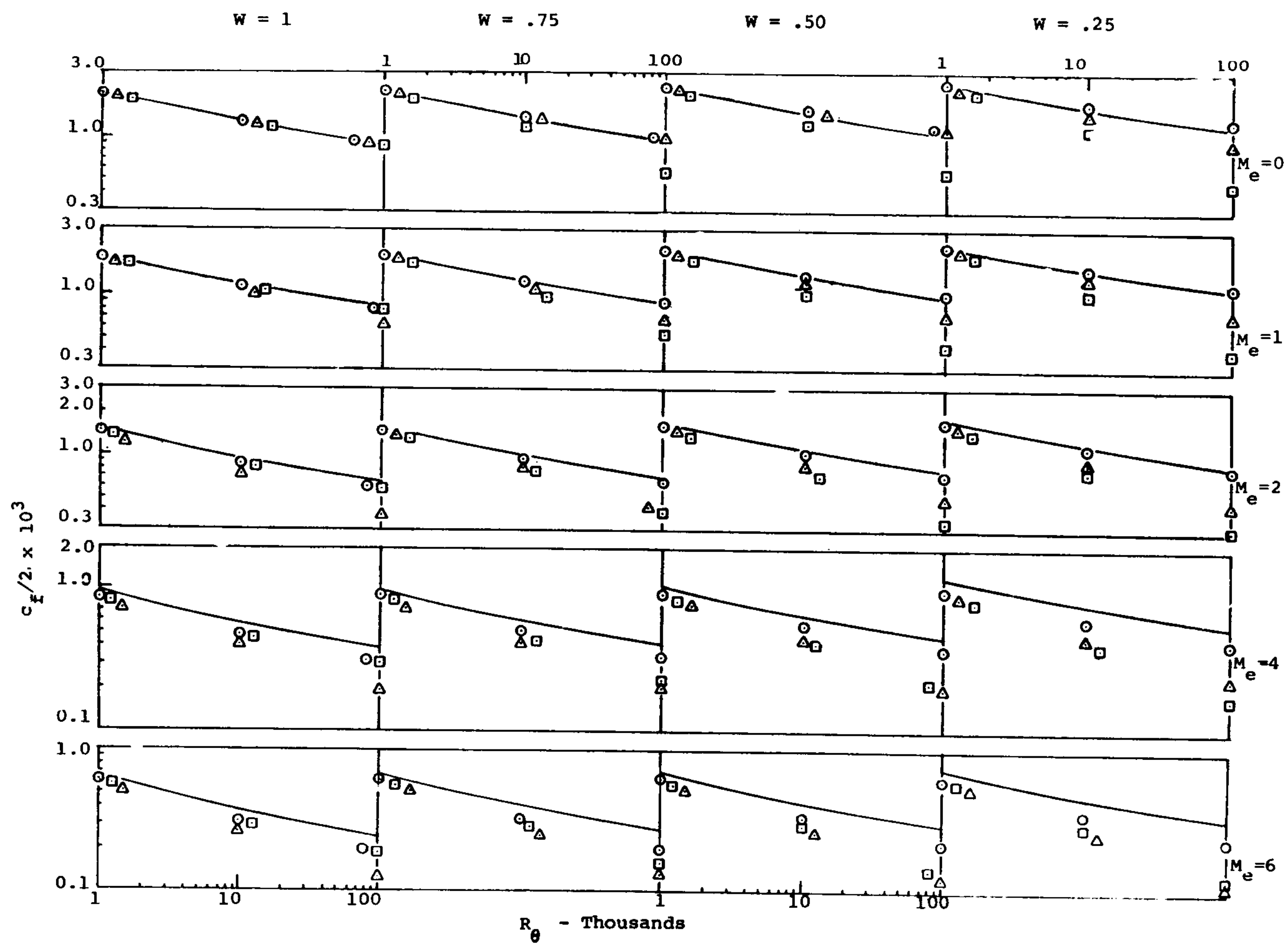


FIGURE 10: VARIATION OF SKIN FRICTION COEFFICIENT WITH MOMENTUM THICKNESS REYNOLDS NUMBER, MACH NUMBER AND WALL TEMPERATURE RATIO ACCORDING TO THE SEVERAL CLOSURE RULES AND COMPARISON WITH EXPERIMENT: SOLID LINE - SPALDING CHI (REFERENCE 13); CIRCLES - FCT PREDICTION: SQUARES - FCP PREDICTION: TRIANGLES - SC PREDICTION

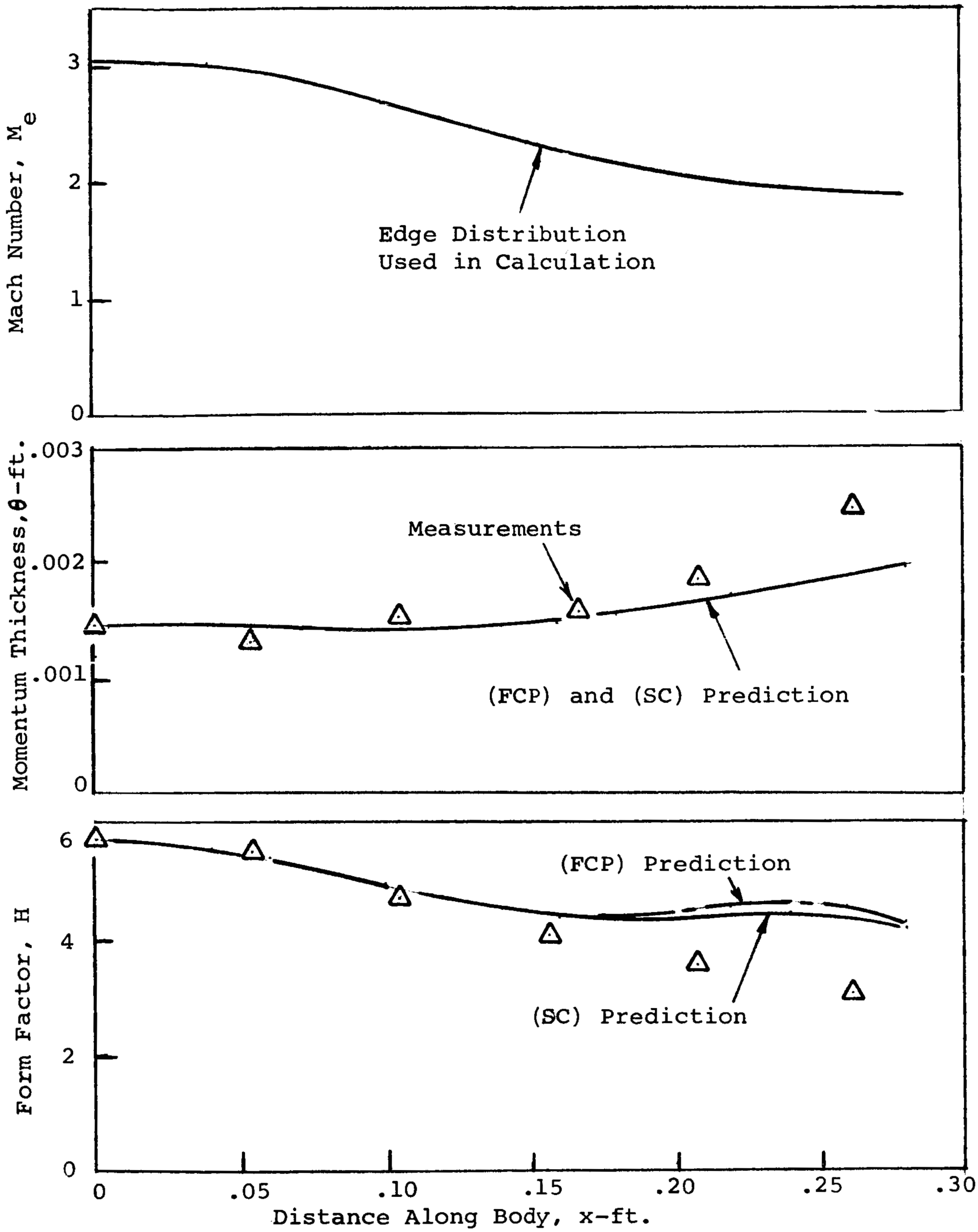


FIGURE 11: COMPARISON BETWEEN THEORETICAL PREDICTIONS AND MEASUREMENTS OF MC LAFFERTY AND BARBER (REFERENCE 14) FOR CURVED SURFACE - 1R(B)

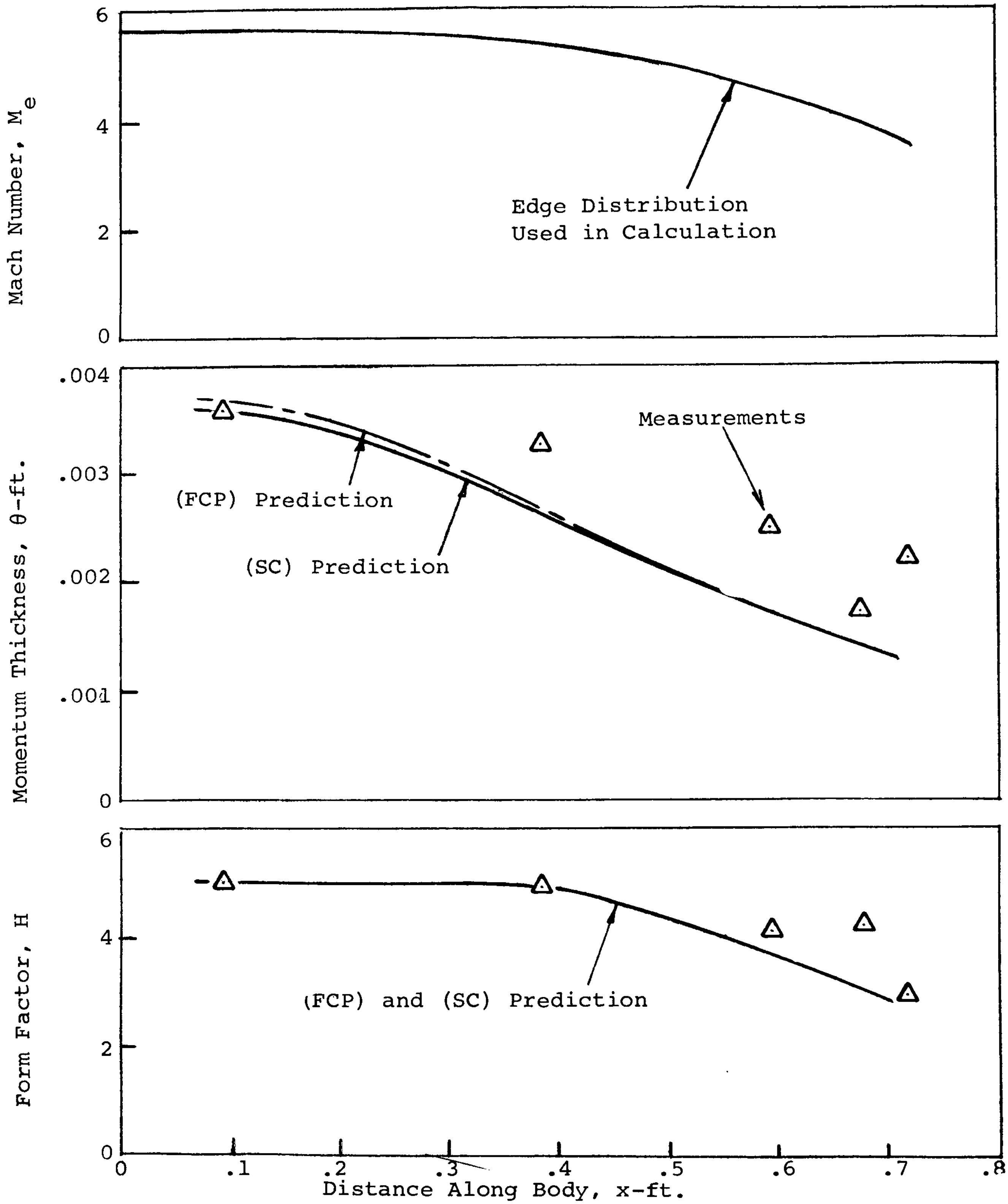


FIGURE 12: COMPARISON BETWEEN THEORETICAL PREDICTIONS AND MEASUREMENTS OF KEPLER AND O'BRIEN (REFERENCE 15) FOR CURVED SURFACE, HIGH HEAT TRANSFER, $M_\infty \approx 6.0$

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