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COMPARISON OF A DISCRETE STEEPEST ASCENT METHOD WITH THE CONTINUOUS STEEPEST ASCENT METHOD FOR OPTIMAL PROGRAMING

by A. Gary Childs

Langley Research Center
Hampton, Va. 23365

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16. Abstract

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COMPARISON OF A DISCRETE STEEPEST ASCENT METHOD WITH THE CONTINUOUS STEEPEST ASCENT METHOD FOR OPTIMAL PROGRAMING

By A. Gary Childs
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SUMMARY

A discrete steepest ascent method which allows controls which are not piecewise constant (for example, it allows all continuous piecewise linear controls) is derived for the solution of optimal programing problems. This method is based on the continuous steepest ascent method of Bryson and Denham and new concepts introduced by Kelley and Denham in their development of "compatible" adjoints for taking into account the effects of numerical integration. The method is a generalization of the algorithm suggested by Canon, Cullum, and Polak with the details of the gradient computation given. The discrete method is compared with the continuous method for an aerodynamics problem for which an analytic solution is given by Pontryagin's maximum principle, and numerical results are presented. The discrete method converges more rapidly than the continuous method at first, but then for some undetermined reason, loses its exponential convergence rate. A comparison is also made for the algorithm of Canon, Cullum, and Polak using piecewise constant controls. This algorithm is very competitive with the continuous algorithm.

INTRODUCTION

One of the simplest and most direct algorithms for the solution of optimal programing problems is the steepest ascent algorithm of Bryson and Denham. (See ref. 1.) This method, being a first-order method, suffers from poor terminal convergence. Kelley and Denham (ref. 2) claim convergence can be improved for a conjugate gradient algorithm by use of "compatible" adjoints. These adjoints satisfy adjoint difference equations for the numerical integration difference equations associated with the differential constraints and represent an attempt to take into account numerical calculations in the algorithm. In this paper, what is called "a discrete steepest ascent" algorithm is derived; this algorithm is based on the numerical integration difference equations and a reformulation of the optimal programing problem as a discrete problem. This algorithm is a generalization of the algorithm given by Canon, Cullum, and Polak in reference 3. The new algorithm is then compared with the Bryson-Denham algorithm for an aerodynamics problem for which an analytic solution is determined by Pontryagin's maximum principle. (See ref. 4.) Numerical results comparing the convergence rates of the two methods are presented in graphical form. Also, the algorithm of Canon, Cullum, and Polak is compared with the continuous algorithm for piecewise constant controls. Numerical results comparing convergence rates are presented.

SYMBOLS

Measurements and calculations were made in U.S. Customary Units. They are presented herein in the International System of Units (SI).

C_D drag coefficient

$C_{D_{C_L^2}}$ induced drag coefficient

$C_{D,o}$ zero-lift drag coefficient

C_L lift coefficient

C_{L_α} lift-curve slope

D drag, newtons

$d\beta = d\psi - \lambda^T_{\psi\Omega,o}\delta x_o$

$d\Omega, d\phi, d\psi$ defined by equations (7)

$(dP)^2 = \sum_{i=0}^{N^*} \delta u_i^T W_i \delta u_i$

$\mathcal{F}$ function of $x_i, u_i, u_{i+1}, t_i, t_{i+1}$ equal to Δx_i

$\mathcal{F}_i = \mathcal{F}(x_i, u_i, u_{i+1}, t_i, t_{i+1})$

$F_i = \frac{\partial \mathcal{F}_i}{\partial x_i}$

f function of $x(t)$, $u(t)$, and t equal to $\dot{x}(t)$

$f_x = \frac{\partial f}{\partial x}$

$\left.\begin{array}{l} f_x(1), f_x(2), \\ f_x(3), f_x(4) \end{array}\right\}$ defined by equations (C1) or (C2)

$G_i = \frac{\partial F_i}{\partial u_i}$

g gravity acceleration constant, 9.7759 meters/second^2

H Pontryagin pseudo-Hamiltonian

$H_i = \frac{\partial \mathcal{F}_i}{\partial u_{i+1}}$

h numerical integration step size

I identity matrix

$I_{\phi\phi}, I_{\psi\phi}, I_{\psi\psi}$ defined by equations (13)

i index used for difference equations, $\Delta x_i = \mathcal{F}_i$

J performance index for sample problem

$K_{D,1}, K_{D,2}, K_L$ defined by equations (B1)

L lift, newtons

m mass, kilograms

N maximum value of i

p_0, p_1, p_2 Pontryagin adjoints

S aerodynamic surface area, meters^2

T thrust, newtons

t time or independent variable, seconds

t_f final time, seconds

t_o	initial time, seconds
u	control vector or horizontal velocity, meters/second
u_0	initial control at initial increment
u_N	final control at final increment

$V = \left(u^2 + v^2\right)^{1/2}$, meters/second

v	vertical velocity, meters/second
W	weight, newtons
W_i	weighting matrix (symmetric)
x	state vector
x_o	initial state

$\alpha = \theta - \gamma$, radians

γ	flight-path angle, radians
θ	control angle, radians
λ_i	difference-equation adjoint vector
$\left.\begin{matrix}\lambda_{\Omega,i}, \lambda_{\psi,i}, \\ \lambda_{\phi,i}\end{matrix}\right\}$	difference-equation adjoints associated with Ω, ϕ, and ψ
$\lambda_{\phi\Omega,i}, \lambda_{\psi\Omega,i}$	defined by equations (9)
μ, ν	Lagrange multipliers
ρ	air density, kilograms/meter^3
ϕ	performance index for development of algorithm

ψ constraint vector function

Ω stopping condition function

Operations:

$\delta(\ )_i = (\ )_i - (\ )_i^*$

$\Delta(\ )_i = (\ )_{i+1} - (\ )_i$

$\triangleq$ definition

$\dot{(\ )}$ derivative of ( ) with respect to t

$d(\ )$ differential of ( ) with respect to $\begin{pmatrix} \delta u_0 \\ \cdot \\ \cdot \\ \cdot \\ \delta u_N \end{pmatrix}$

$(\ )^{-1}$ matrix inverse

$\frac{\partial[\ ]}{\partial(\ )}$ partial derivative of [ ] with respect to ( )

$(\ )^T$ matrix-vector transpose

A symbol with an asterisk denotes a nominal variable; a symbol without an asterisk denotes a general or arbitrary variable.

ANALYSIS

The Optimal Programing Problem

Let the following problem be considered. Maximize

$$\phi\left(x(t_f), t_f\right)$$

subject to

$$\dot{x} = f(x,u,t) \qquad \left(t_o \leqq t \leqq t_f\right)$$

and

$$\psi\left(x(t_f),t_f\right) = 0$$

with t_f determined by

$$\Omega\left(x(t_f),t_f\right) = 0$$

and t_o and x_o as given constants. For this problem, ϕ is a scalar function called the cost function, Ω is a scalar function called the stopping condition, x is a vector function called the state, u is a vector function called the control, and ψ is a vector function called the terminal constraint. The dimensions of the vector functions are, in general, different. The optimal programing problem in this form is discussed in reference 1 along with the continuous steepest ascent method for solving it.

The theory of reference 1 requires (strictly speaking) that all functions be known exactly. However, for the solution of the foregoing problem, one frequently uses numerical integrations for obtaining the state. Hence, an algorithm is required which recognizes the use of difference equations for the state, that is, a discrete algorithm. A discrete method is developed in an analogous fashion to the development of the continuous method. A reformulation of the problem taking into account some of the before-mentioned realities of computer calculations is made. Maximize

$$\phi\left(x_N,t_N\right)$$

subject to

$$\Delta x_i = x_{i+1} - x_i = \mathscr{F}\left(x_i,u_i,u_{i+1},t_i,t_{i+1}\right) \triangleq \mathscr{F}_i \qquad (i = 0, \ldots, N - 1) \tag{1}$$

and

$$\psi\left(x_N,t_N\right) = 0$$

with t_N determined by

$$\Omega\left(x_N,t_N\right) = 0$$

and t_o and x_o as given constants. For this problem, ϕ, Ω, x_i, u_i, and ψ are scalars or vectors as in the continuous formulation of the problem. In addition, it is specified that

$$t_{i+1} - t_i = h \qquad (i = 0, \ldots, N - 2)$$

where h is a constant. Also,

$$t_N - t_{N-1} \leqq h$$

Hence, the t_i are the time points of the numerical integration of the continuous state equations (represented by eq. (1)) with $t_N = t_f$.

Formulation of Discrete Method

Let u_i^* $(i = 0, \ldots, N^*)$ be a nominal control. Let x_i^* $(i = 0, \ldots, N^*)$ be the nominal state resulting from this control. Suppose a new control u_i $(i = 0, \ldots, N^*)$ is given which yields a new state x_i $(i = 0, \ldots, N^*)$. Then

$$\left.\begin{aligned} \delta u_i &\triangleq u_i - u_i^* \\ \delta x_i &\triangleq x_i - x_i^* \end{aligned}\right\} \qquad (i = 0, \ldots, N^*)$$

Expanding equations (1) about the nominal control and state and truncating to the linear terms yield

$$\Delta(\delta x_i) = \delta x_{i+1} - \delta x_i \approx F_i \delta x_i + G_i \delta u_i + H_i \delta u_{i+1} \qquad (i = 0, \ldots, N^* - 1) \tag{2}$$

where

$$\left.\begin{aligned} F_i &\triangleq \frac{\partial \mathcal{F}_i}{\partial x_i}\left(x_i^*, u_i^*, u_{i+1}^*, t_i, t_{i+1}\right) \\ G_i &\triangleq \frac{\partial \mathcal{F}_i}{\partial u_i}\left(x_i^*, u_i^*, u_{i+1}^*, t_i, t_{i+1}\right) \\ H_i &\triangleq \frac{\partial \mathcal{F}_i}{\partial u_{i+1}}\left(x_i^*, u_i^*, u_{i+1}^*, t_i, t_{i+1}\right) \end{aligned}\right\} \qquad (i = 0, \ldots, N^* - 1)$$

The adjoint equations for equations (2) are obtained from

$$\begin{aligned} \Delta\left(\lambda_i^T \delta x_i\right) &= \lambda_{i+1}^T \delta x_{i+1} - \lambda_i^T \delta x_i \\ &= \lambda_{i+1}^T \delta x_{i+1} - \lambda_{i+1}^T \delta x_i + \lambda_{i+1}^T \delta x_i - \lambda_i^T \delta x_i \\ &= \lambda_{i+1}^T \Delta\left(\delta x_i\right) + \Delta\left(\lambda_i^T\right) \delta x_i \\ &\approx \left[\lambda_{i+1}^T F_i + \Delta\left(\lambda_i^T\right)\right] \delta x_i + \lambda_{i+1}^T \left(G_i \delta u_i + H_i \delta u_{i+1}\right) \qquad (i = 0, \ldots, N^* - 1) \end{aligned} \tag{3}$$

Then defining the λ_i terms to be the compatible adjoints of reference 2 for the particular type of numerical integration used in equations (1),

$$\Delta(\lambda_i) = \lambda_{i+1} - \lambda_i = -F_i^T \lambda_{i+1} \qquad (i = 0, \ldots, N^* - 1) \tag{4}$$

and substituting equations (4) into equations (3) gives

$$\Delta\left(\lambda_i^T \delta x_i\right) \approx \lambda_{i+1}^T\left(G_i \delta u_i + H_i \delta u_{i+1}\right) \tag{5}$$

Summing equations (5) from $i = 0$ to N^* yields

$$\lambda_{N^*}^T \delta x_{N^*} \approx \lambda_0^T \delta x_0 + \sum_{i=0}^{N^*-1} \lambda_{i+1}^T\left(G_i \delta u_i + H_i \delta u_{i+1}\right) \tag{6}$$

Linear approximations for changes of ϕ, ψ, and Ω are

$$\left.\begin{aligned}
d\phi &\triangleq \frac{\partial \phi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) \delta x_{N^*} + \left[\frac{\partial \phi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) f\left(x_{N^*}^*, u_{N^*}^*, t_{N^*}\right) + \frac{\partial \phi}{\partial t_N}\left(x_{N^*}^*, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
&\approx \frac{\partial \phi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) \delta x_{N^*} + \left[\frac{\partial \phi}{\partial x_N}\left(x_{N^*}, t_{N^*}\right) f\left(x_{N^*}, u_{N^*}, t_{N^*}\right) + \frac{\partial \phi}{\partial t_N}\left(x_{N^*}, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
&\approx \frac{\partial \phi}{\partial x_N}\left(x_{N^*}, t_{N^*}\right)\left(x\left(t_f\right) - x_{N^*}\right) + \frac{\partial \phi}{\partial t_N}\left(x_{N^*}, t_{N^*}\right)\left(t_f - t_{N^*}\right) + \frac{\partial \phi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right)\left(x_{N^*} - x_{N^*}^*\right) \\
&\approx \phi\left(x\left(t_f\right), t_f\right) - \phi\left(x_{N^*}, t_{N^*}\right) + \phi\left(x_{N^*}, t_{N^*}\right) - \phi\left(x_{N^*}^*, t_{N^*}\right) \\
&= \phi\left(x\left(t_f\right), t_f\right) - \phi\left(x_{N^*}^*, t_{N^*}\right) \\[1em]
d\psi &\triangleq \frac{\partial \psi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) \delta x_{N^*} + \left[\frac{\partial \psi}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) f\left(x_{N^*}^*, u_{N^*}^*, t_{N^*}\right) + \frac{\partial \psi}{\partial t_N}\left(x_{N^*}^*, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
&\approx \psi\left(x\left(t_f\right), t_f\right) - \psi\left(x_{N^*}^*, t_{N^*}\right) \\[1em]
d\Omega &\triangleq \frac{\partial \Omega}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) \delta x_{N^*} + \left[\frac{\partial \Omega}{\partial x_N}\left(x_{N^*}^*, t_{N^*}\right) f\left(x_{N^*}^*, u_{N^*}^*, t_{N^*}\right) + \frac{\partial \psi}{\partial t_N}\left(x_{N^*}^*, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
&\approx \Omega\left(x\left(t_f\right), t_f\right) - \Omega\left(x_{N^*}^*, t_{N^*}\right)
\end{aligned}\right\} \tag{7}$$

where t_f is the final time associated with the control u_i $(i = 0, \ldots, N^*)$ and is determined by use of the stopping condition Ω. If Ω is not zero by the time t_{N^*}, the new control must be extrapolated in some way and the numerical integration of the state continued until a zero of Ω is found. Note that a reversion to the continuous system $(\dot{x} = f(x,u,t))$ is used to obtain an approximation for $x(t_f) - x_{N^*}$.

Let

$$\lambda^T_{\phi,N^*} \triangleq \frac{\partial\phi}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right)$$

$$\lambda^T_{\psi,N^*} \triangleq \frac{\partial\psi}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right)$$

$$\lambda^T_{\Omega,N^*} \triangleq \frac{\partial\Omega}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right)$$

and using equation (6), equations (7) become

$$\left.\begin{aligned}
d\phi &= \lambda^T_{\phi,0}\delta x_o + \sum_{i=0}^{N^*-1} \lambda^T_{\phi,i+1}\left(G_i\delta u_i + H_i\delta u_{i+1}\right) \\
&\quad + \left[\frac{\partial\phi}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right) f\left(x^*_{N^*}, u^*_{N^*}, t_{N^*}\right) + \frac{\partial\phi}{\partial t_N}\left(x^*_{N^*}, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
d\psi &= \lambda^T_{\psi,0}\delta x_o + \sum_{i=0}^{N^*-1} \lambda^T_{\psi,i+1}\left(G_i\delta u_i + H_i\delta u_{i+1}\right) \\
&\quad + \left[\frac{\partial\psi}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right) f\left(x^*_{N^*}, u^*_{N^*}, t_{N^*}\right) + \frac{\partial\psi}{\partial t_N}\left(x^*_{N^*}, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right) \\
d\Omega &= \lambda^T_{\Omega,0}\delta x_o + \sum_{i=0}^{N^*-1} \lambda^T_{\Omega,i+1}\left(G_i\delta u_i + H_i\delta u_{i+1}\right) \\
&\quad + \left[\frac{\partial\Omega}{\partial x_N}\left(x^*_{N^*}, t_{N^*}\right) f\left(x^*_{N^*}, u^*_{N^*}, t_{N^*}\right) + \frac{\partial\Omega}{\partial t_N}\left(x^*_{N^*}, t_{N^*}\right)\right]\left(t_f - t_{N^*}\right)
\end{aligned}\right\} \quad (8)$$

Setting $d\Omega = 0$ gives a formula for $t_f - t_{N^*}$ which modifies equations (8) to

$$\left.\begin{aligned} d\phi &= \lambda^T_{\phi\Omega,0}\delta x_0 + \sum_{i=0}^{N^*-1} \lambda^T_{\phi\Omega,i+1}\left(G_i\delta u_i + H_i\delta u_{i+1}\right) \\ d\psi &= \lambda^T_{\psi\Omega,0}\delta x_0 + \sum_{i=0}^{N^*-1} \lambda^T_{\psi\Omega,i+1}\left(G_i\delta u_i + H_i\delta u_{i+1}\right) \end{aligned}\right\} \tag{9}$$

where

$$\left.\begin{aligned} \lambda^T_{\phi\Omega,i} &\triangleq \lambda^T_{\phi,i} - \frac{\frac{\partial\phi}{\partial x_N}\left(x^*_{N^*},t_{N^*}\right) f\left(x^*_{N^*},u^*_{N^*},t_{N^*}\right) + \frac{\partial\phi}{\partial t_N}\left(x^*_{N^*},t_{N^*}\right)}{\frac{\partial\Omega}{\partial x_N}\left(x^*_{N^*},t_{N^*}\right) f\left(x^*_{N^*},u^*_{N^*},t_{N^*}\right) + \frac{\partial\Omega}{\partial t_N}\left(x^*_{N^*},t_{N^*}\right)}\lambda^T_{\Omega,i} \\ \lambda^T_{\psi\Omega,i} &\triangleq \lambda^T_{\psi,i} - \frac{\frac{\partial\psi}{\partial x_N}\left(x^*_{N^*},t_{N^*}\right) f\left(x^*_{N^*},u^*_{N^*},t_{N^*}\right) + \frac{\partial\psi}{\partial t_N}\left(x^*_{N^*},t_{N^*}\right)}{\frac{\partial\Omega}{\partial x_N}\left(x^*_{N^*},t_{N^*}\right) f\left(x^*_{N^*},u^*_{N^*},t_{N^*}\right) + \frac{\partial\Omega}{\partial t_N}\left(x^*_{N^*},t_{N^*}\right)}\lambda^T_{\Omega,i} \end{aligned}\right\} \quad (i = 0, \ldots, N^*)$$

It is desired to maximize

$$d\phi = \left(\lambda^T_{\phi\Omega,0} - \nu^T\lambda^T_{\psi\Omega,0}\right)\delta x_0 + \sum_{i=0}^{N^*}\left[\left(\lambda^T_{\phi\Omega,i+1} - \nu^T\lambda^T_{\psi\Omega,i+1}\right)G_i + \left(\lambda^T_{\phi\Omega,i} - \nu^T\lambda^T_{\psi\Omega,i}\right)H_{i-1} - \mu\delta u_i^T W_i\right]\delta u_i + \nu^T d\psi + \mu(dP)^2$$

with respect to δu_i $(i = 0, \ldots, N^*)$ where δx_0, $d\psi$, and the step size

$$(dP)^2 \triangleq \sum_{i=0}^{N^*} \delta u_i^T W_i \delta u_i \tag{10}$$

(where W_i is a symmetric $m \times m$ matrix) are assumed to be specified and ν and μ are Lagrange multipliers. Also, G_{N^*} and H_{-1} are defined to be zero matrices in constructing a form for $d\phi$ which will be easier to maximize. Taking the differential yields

$$d(d\phi) = \sum_{i=0}^{N^*} \left[\left(\lambda_{\phi\Omega,i+1}^T - \nu^T\lambda_{\psi\Omega,i+1}^T\right)G_i + \left(\lambda_{\phi\Omega,i}^T - \nu^T\lambda_{\psi\Omega,i}^T\right)H_{i-1} - 2\mu\,\delta u_i^T W_i\right]d(\delta u_i)$$

This relation is zero for all values of $d(\delta u_i)$ $(i = 0, \ldots, N^*)$ if

$$\delta u_i = \frac{1}{2\mu} W_i^{-1}\left[G_i^T\left(\lambda_{\phi\Omega,i+1} - \lambda_{\psi\Omega,i+1}\nu\right) + H_{i-1}^T\left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i}\nu\right)\right] \qquad (i = 0, \ldots, N^*) \tag{11}$$

Substituting these control increments into the equations for $d\psi$ and $(dP)^2$ gives values for ν and μ. By use of these values, the δu_i $(i = 0, \ldots, N^*)$ are completely specified. This process is carried out in appendix A. The result is

$$\delta u_i = \left[\frac{(dP)^2 - d\beta^T I_{\psi\psi}^{-1}\,d\beta}{I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi}^{-1} I_{\psi\phi}}\right]^{1/2} W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T\left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i} I_{\psi\psi}^{-1} I_{\psi\phi}\right)$$

$$+ W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T \lambda_{\psi\Omega,i} I_{\psi\psi}^{-1}\,d\beta \qquad (i = 0, \ldots, N^*) \tag{12}$$

where

$$\left.\begin{aligned}
I_{\psi\psi} &= \sum_{i=0}^{N^*} \lambda_{\psi\Omega,i}^T\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T \lambda_{\psi\Omega,i} \\
I_{\psi\phi} &= \sum_{i=0}^{N^*} \lambda_{\psi\Omega,i}^T\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T \lambda_{\phi\Omega,i} \\
I_{\phi\phi} &= \sum_{i=0}^{N^*} \lambda_{\phi\Omega,i}^T\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T \lambda_{\phi\Omega,i}
\end{aligned}\right\} \tag{13}$$

The change of ϕ caused by this control change is (by use of eq. (4))

$$d\phi = \lambda_{\phi\Omega,0}^T\delta x_o + \sum_{i=0}^{N^*} \lambda_{\phi\Omega,i}^T\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]\delta u_i$$

$$= \lambda_{\phi\Omega,0}^T\delta x_o + \left\{\left[(dP)^2 - d\beta^T I_{\psi\psi}^{-1}\,d\beta\right]\left(I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi}^{-1} I_{\psi\phi}\right)\right\}^{1/2} + I_{\psi\phi}^T I_{\psi\psi}^{-1}\,d\beta \tag{14}$$

The Sample Problem

A problem was chosen for the comparison of the usual steepest ascent method with the discrete steepest ascent method. The final vertical velocity of an airplane with aerodynamic parameters similar to those of the X-15 was to be maximized. The airplane was flying in a vertical plane, that is, it was constrained to two-dimensional motion. It had no initial horizontal velocity and some positive vertical velocity. The airplane was considered to be a point mass. The coordinate system and force diagram used for the problem are shown in figure 1. The equations of motion are

$$m\dot{u} = T \cos \theta - D \cos \gamma - L \sin \gamma$$

$$m\dot{v} = T \sin \theta - W - D \sin \gamma + L \cos \gamma$$

where T is the (constant) thrust, $W = mg$ is the weight, D is the drag, L is the lift, m is the mass, g is the earth's gravitational acceleration constant, u is the horizontal speed, v is the vertical speed, θ is the control angle, and γ is the flight-path angle. The formulas used for the lift and drag are

$$L = C_L \frac{\rho V^2 S}{2}$$

$$D = C_D \frac{\rho V^2 S}{2}$$

where

$$C_L = C_{L_\alpha} \alpha$$

$$C_D = C_{D,o} + C_{D_{C_L^2}} C_{L_\alpha}^2 \alpha^2$$

and ρ is the air density (constant), $V = \left(u^2 + v^2\right)^{1/2}$ is the speed, S is the wing surface area, and C_{L_α}, $C_{D,o}$, and $C_{D_{C_L^2}}$ are constant coefficients determined by the airplane configuration. The values of constants used in the computer study are as follows:

T, N	250 000
m, kg	15 000
g, m/sec^2	9.7759
C_{L_α}	3.61
$C_{D,o}$	0.055
$C_{D_{C_L^2}}$	0.398
ρ, kg/m^3	0.41351
S, m^2	20

The problem, then, was to maximize $v(t_f) - v(t_o)$ where t_f was given and $u(t_o)$ was zero, $v(t_o) = v_o > 0$. This problem was solved analytically by using Pontryagin's maximum principle. (See ref. 4.) The solution was to fly with θ constantly equal to 90^o. This solution is derived in appendix B. The problem was programed for solution by both the usual steepest ascent method and the discrete steepest ascent method. Fourth-order Runge-Kutta integration was used (ref. 5), and appendix C gives the relationship between the adjoints of the two methods. Linear interpolation was used to obtain intermediate values of the control. A comparison of the results from the two methods obtained by the computer is given in the next section.

RESULTS AND DISCUSSION

Computer solutions of the problem were programed for both steepest ascent algorithms. The discrete algorithm compared very unfavorably with the continuous algorithm. The discrete algorithm gave corrections for θ_o and θ_{N^*} which are only about half as much as the corrections given by the continuous algorithm. Recall that H_{-1} and G_{N^*} in equation (12) are zero matrices. Hence, there are only half as many terms in the computation for δu_o and δu_{N^*} as for the other δu_i $(i = 1, \ldots, N^* - 1)$. As can be seen in reference 1, for the continuous algorithm, the equation corresponding to equation (12) has an equal number of terms for the calculation of all $\delta u(t)$ $\left(t_o \leqq t \leqq t_f^*\right)$. If one traces back both from equation (12) and the corresponding equation in reference 1, one discovers that the difference just noted in these equations appears first in the state equations. In this paper, the equations are

$$\Delta x_i = F\left(x_i, u_i, u_{i+1}, t_i, t_{i+1}\right) \qquad (i = 0, \ldots, N^* - 1)$$

In reference 1, the equations are

$$\dot{x} = f(x,u,t) \qquad \left(t_o \leqq t \leqq t_f\right)$$

The difference arises because both u_i and u_{i+1} appear in the first set of equations. Thus, in summing to get x_{N^*}, the u_i $(i = 1, \ldots, N^* - 1)$ appear twice as often as u_o and u_{N^*}. For the continuous equations of motion, every $u(t)$ $\left(t_o \leqq t \leqq t_f\right)$ has equal influence in determining $x(t_f)$. Since the first set of equations represents a numerical integration process for the second set, the difficulty appears in the attempted discrete modeling of a continuous model for the dynamics of the problem. The most obvious solution for such a difficulty is to find directly a discrete model for the dynamics of the problem. This model would be a set of difference equations of the form

$$\Delta x_i = \mathscr{F}(x_i, u_i, t_i)$$

that is, equations (1) where it is assumed that u_{i+1} and t_{i+1} do not appear.

This form of the difference equations for the state is essentially the same as that used in reference 3. It can be achieved by requiring the control of the sample problem to be constant over the numerical integration intervals. In general, however, there might exist "wild" piecewise constant controls (that is, where the control would have, at some interval end points, large jump discontinuities) which, although optimal, would be impractical to implement. Also, even though such controls, as optimal controls, might not occur, the possibility exists that such controls would be encountered in the iterations of the algorithm and could precipitate divergence. Putting u_{i+1} and t_{i+1} in the difference equations allows continuity restrictions to be used to keep the control reasonable. The foregoing should be considered an argument justifying the study of the algorithm of this paper rather than as a criticism of the effectiveness of the algorithm of reference 3. Indeed, results presented for this algorithm in figure 2 reveal that it is very competitive with the continuous algorithm for this problem where the restriction has been made for both methods that θ be piecewise constant. Appendix D gives the derivation of the equations used in this study which was suggested by Terry A. Straeter of the Langley Research Center.

To attempt an alleviation of the before-mentioned difficulty, an empirical device was employed. The corrections δu_0 and δu_{N^*} were doubled. It was hoped that this doubling would give δu_0 and δu_{N^*} an almost equal influence with the other δu_i terms. For the sample problem, $\delta\theta_0$ and $\delta\theta_{N^*}$ would be doubled and would be about right.

This procedure helped, but performance of the discrete algorithm still was not as good as that of the continuous algorithm. Further computer experience revealed that the $\delta\theta_i$ terms ($i = 0, \ldots, N^*$) were larger for the discrete algorithm than for the continuous algorithm. The formula for these corrections for the sample problem is

$$\delta\theta_i = \left[\frac{(dP)^2}{I_{\phi\phi}}\right]^{1/2} \left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right]^T \lambda_{\phi\Omega,i}$$

Observe that an increase in $I_{\phi\phi}$ would make these quantities smaller. On the basis of the problems with $\delta\theta_0$ and $\delta\theta_{N^*}$, examination of the equation for $I_{\phi\phi}$ of equations (13) revealed that there are only half as many terms for $i = 0$ and $i = N^*$ as there are for the other values of i. Therefore, the $i = 0$ and $i = N^*$ terms were doubled. After these two changes, the performance of the discrete algorithm modified in this fashion during the first few iterations, was better than that of the continuous algorithm.

Figures 3 to 9 give the results of a comparison of the two algorithms as they are presently formulated. Each case was run for 25 iterations. Points are omitted for iterations where the automatic convergence scheme rejected a forward trajectory which did not give as good a performance index as the trajectory of the previous iteration.

For figure 3, the final time was 10 seconds, the computation interval was 0.5 second, and the nominal control was $\theta \equiv 70^\circ$. The computation interval of 0.5 second produces terms twice as large for the discrete $(dP)^2$ as would appear in an approximation for the continuous $(dP)^2$; that is,

$$\text{Discrete:} \quad (dP)^2 \triangleq \sum_{i=0}^{N^*} \delta\theta_i^2$$

$$\text{Continuous:} \quad (dP)^2 \triangleq \int_{t_o}^{t_f} \delta\theta^2 dt \approx \sum_{i=0}^{N^*-1} \frac{1}{2}\delta\theta_i^2$$

Hence, the initial $(dP)^2$ for the discrete algorithm was chosen to be twice the value of the initial $(dP)^2$ for the continuous algorithm to allow an equal amount of control change for both algorithms.

For a fixed-final-time fixed-initial-condition problem,

$$d\phi = \left[I_{\phi\phi}(dP)^2\right]^{1/2}$$

Therefore, an $I_{\phi\phi}$ for the discrete algorithm equal to half the value of the $I_{\phi\phi}$ for the continuous algorithm will give just as much change for ϕ. Hence, the ordinates for figure 3 are $I_{\phi\phi}$ for the continuous method and $2I_{\phi\phi}$ for the discrete method. Similar adjustments were made for the results shown in figures 4 to 9.

The case of figure 3 was considered as the base problem. Figures 4 to 9 show results of perturbations from this case. These cases were run to determine whether any changes of certain parameters would improve the performance of the discrete algorithm relative to the continuous algorithm. In figures 4 and 5 results are presented for perturbations in the computation interval to 1 second and 5 seconds, respectively. In figures 6 and 7 results are given for perturbations from the base problem nominal control to $\theta \equiv 80^\circ$ and $\theta \equiv 60^\circ$, respectively. The values of initial $(dP)^2$ for these two cases were chosen to give less and more control effort to effect convergence in about 25 iterations. Figures 8 and 9 give results for final times of 20 seconds and 30 seconds, respectively. For both cases the computation interval was 5 seconds. This calculation was an attempt to determine the effect of large errors in the numerical integration.

Both methods converge for the base problem. As shown in figure 3, the discrete method converges faster than the continuous method until it loses its exponential convergence rate for some unknown reason. The loss of convergence rate is seen as a "flattening out" on the semi-log plot. This behavior is typical of the discrete method and is seen in the other figures where the discrete method converges. No good hypothesis has been formulated to explain the behavior. It is noted, however, that the discrete method stops working in each case after about the same number of iterations. An examination of the computer output reveals that the control after this number of iterations is "rough;" that is, the values of the θ_i are scattered on both sides of the theoretically optimal 90°. For the continuous method, the control is kept relatively "smooth" in the same sense. Hence, the reason for the loss of convergence of the discrete algorithm may be involved with some implicit constraints which the continuous algorithm exerts on the control to keep it smooth. This reasoning leads one to think about the possibility of some uniqueness problem for the discrete algorithm (caused by the lack of the implicit constraints), especially since such problems are commonly encountered in dealing with discrete models for continuous systems. Another possible explanation for the flattening out is that it is due to the modifications of the discrete algorithm described in the first part of the section "Results and Discussion;" that is, a switch back to the original algorithm when the flattening out begins may improve convergence. Unfortunately, time limitations prevented a study of this possibility.

Comparison of figures 3 to 5 reveals the effect of a change in the computation interval. Both methods converge more slowly for a larger computation interval. The discrete method is notably more sensitive to a degrading of numerical accuracy in the integration, particularly in the flat section. A possible reason for this greater sensitivity is that as the final time remains the same and the computation interval increases, the number of integration points becomes less. Hence, the problems mentioned earlier with respect to $\delta\theta_0$ and $\delta\theta_N{}^*$ have a greater influence on results. The computation interval of 5 seconds, yielding only three integration points, affected the discrete algorithm so adversely that it diverged.

Comparison of figures 3, 6, and 7 demonstrates an interesting phenomenon. The flat part of the discrete algorithm plot occurs at different levels for different choices of the nominal control. It is especially surprising that the convergence "settles out" for a 60° nominal at a lower value of the gradient than for a 70° nominal. The convergence rates with this exception are about the same for different nominal controls; that is, the average slope of a curve connecting the points would be about the same for the three plots.

The discrete method diverged for figure 8. The continuous method was converging very slowly for the case of figure 8, as is indicated by the presence of points for up to 24 iterations. Both methods diverged in figure 9. The erratic behavior of the points represents an inaccurate calculation of the gradient and explains the poor convergence rate. The state integration up to 10 seconds is accurate to three significant figures for the case of figure 8. The same would, of course, be true for the case of figure 9 since the computation interval is the same as that for figure 8. However, the error over the 30 seconds of figure 9 would be greater than the error over the 20 seconds of figure 8. Therefore, the state integration error probably contributed substantially to the divergence in these cases.

CONCLUDING REMARKS

A discrete steepest ascent algorithm which takes into account numerical integration of the differential constraints and allows controls which are not piecewise constant has been formulated for the solution of optimal programing problems. The algorithm was compared with the continuous steepest ascent algorithm of Bryson and Denham for an aerodynamic problem for which an analytic solution had been obtained by using Pontryagin's maximum principle. For this problem the discrete algorithm converged somewhat faster initially but eventually slowed its convergence rate greatly. Basic difficulties with the discrete method which caused this condition are described. The prime and seemingly unavoidable difficulty is that the method of this paper uses a discrete model for a problem stated in terms of a continuous model. For physical problems formulated by use of such a continuous model, the method would apparently be of limited usefulness. However, if the problem is stated in terms of a discrete model (that is, using piecewise constant controls), the algorithm of Canon, Cullum, and Polak, of which the algorithm of this paper is a generalization, is very competitive with the continuous algorithm.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., October 13, 1971.

APPENDIX A

DERIVATION OF EQUATIONS FOR CONSTANT LAGRANGE MULTIPLIERS

In this appendix formulas are determined for the Lagrange multipliers, ν and μ introduced in the section "Analysis." The control corrections δu_i are then given by using these formulas. Using equations (9) and (11) for $d\psi$ and δu_i $(i = 0,\ldots,N^*)$ yields

$$d\psi = \lambda^T_{\psi\Omega,0}\delta x_o + \sum_{i=0}^{N^*}\left(\lambda^T_{\psi\Omega,i+1}G_i + \lambda^T_{\psi\Omega,i}H_{i-1}\right)\frac{1}{2\mu}W_i^{-1}\left[G_i^T\left(\lambda_{\phi\Omega,i+1} - \lambda_{\psi\Omega,i+1}\nu\right)\right.$$

$$\left. + H^T_{i-1}\left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i}\nu\right)\right] = \lambda^T_{\psi\Omega,0}\delta x_o + \frac{1}{2\mu}\left(I_{\psi\phi} - I_{\psi\psi}\nu\right) \tag{A1}$$

where (by using eqs. (4))

$$I_{\psi\phi} = \sum_{i=0}^{N^*}\left(\lambda^T_{\psi\Omega,i+1}G_i + \lambda^T_{\psi\Omega,i}H_{i-1}\right)W_i^{-1}\left(G_i^T\lambda_{\phi\Omega,i+1} + H^T_{i-1}\lambda_{\phi\Omega,i}\right)$$

$$= \sum_{i=0}^{N^*}\lambda^T_{\psi\Omega,i}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T\lambda_{\phi\Omega,i}$$

$$I_{\psi\psi} = \sum_{i=0}^{N^*}\left(\lambda^T_{\psi\Omega,i+1}G_i + \lambda^T_{\psi\Omega,i}H_{i-1}\right)W_i^{-1}\left(G_i^T\lambda_{\psi\Omega,i+1} + H^T_{i-1}\lambda_{\psi\Omega,i}\right)$$

$$= \sum_{i=0}^{N^*}\lambda^T_{\psi\Omega,i}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]W_i^{-1}\left[\left(I + F_i\right)^{-1}G_i + H_{i-1}\right]^T\lambda_{\psi\Omega,i}$$

Solving for ν from equation (A1) yields

$$\nu = I^{-1}_{\psi\psi}\left(I_{\psi\phi} - 2\mu\ d\beta\right) \tag{A2}$$

where

$$d\beta = d\psi - \lambda^T_{\psi\Omega,0}\delta x_o$$

Using equations (10) and (11) for $(dP)^2$ and δu_i $(i = 0, \ldots, N^*)$ and equation (A2)

$$(dP)^2 = \frac{1}{4\mu^2} \sum_{i=0}^{N^*} \left[\left(\lambda_{\phi\Omega,i+1}^T - \nu^T \lambda_{\psi\Omega,i+1}^T\right) G_i + \left(\lambda_{\phi\Omega,i}^T - \nu^T \lambda_{\psi\Omega,i}^T\right) H_{i-1}\right] W_i^{-1} \left[G_i^T \left(\lambda_{\phi\Omega,i+1} - \lambda_{\psi\Omega,i+1}\nu\right) + H_{i-1}^T \left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i}\nu\right)\right]$$

$$= \frac{1}{4\mu^2}\left(I_{\phi\phi} - 2\nu^T I_{\psi\phi} + \nu^T I_{\psi\psi}\nu\right)$$

$$= \frac{1}{4\mu^2}\left[I_{\phi\phi} - 2\left(I_{\psi\phi}^T - 2\mu\ d\beta^T\right) I_{\psi\psi}^{-1} I_{\psi\phi} + \left(I_{\psi\phi}^T - 2\mu\ d\beta^T\right) I_{\psi\psi}^{-1} I_{\psi\psi} I_{\psi\psi}^{-1} \left(I_{\psi\phi} - 2\mu\ d\beta\right)\right]$$

$$= \frac{1}{4\mu^2}\left(I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi}^{-1} I_{\psi\phi} + 4\mu^2 d\beta^T I_{\psi\psi}^{-1}\ d\beta\right) \tag{A3}$$

where (by using eqs. (4))

$$I_{\phi\phi} = \sum_{i=0}^{N^*} \left(\lambda_{\phi\Omega,i+1}^T G_i + \lambda_{\phi\Omega,i}^T H_{i-1}\right) W_i^{-1} \left(G_i^T \lambda_{\phi\Omega,i+1} + H_{i-1}^T \lambda_{\phi\Omega,i}\right)$$

$$= \sum_{i=0}^{N^*} \lambda_{\phi\Omega,i}^T \left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right] W_i^{-1} \left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right]^T \lambda_{\phi\Omega,i}$$

Solving for 2μ from equation (A3) yields

$$2\mu = \left[\frac{I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi}^{-1} I_{\psi\phi}}{(dP)^2 - d\beta^T I_{\psi\psi}^{-1}\ d\beta}\right]^{1/2} \tag{A4}$$

where the positive sign has been chosen to make $d\phi$ positive.

By using equations (11), (A2), and (A4), the control change equations are given by

$$\delta u_i = \frac{1}{2\mu} W_i^{-1}\left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right]^T \left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i}\nu\right)$$

$$= \left[\frac{(dP)^2 - d\beta^T I_{\psi\psi}^{-1}\, d\beta}{I_{\phi\phi} - I_{\psi\phi}^T I_{\psi\psi}^{-1} I_{\psi\phi}}\right]^{1/2} W_i^{-1}\left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right]^T \left(\lambda_{\phi\Omega,i} - \lambda_{\psi\Omega,i} I_{\psi\psi}^{-1} I_{\psi\phi}\right)$$

$$+ W_i^{-1}\left[\left(I + F_i\right)^{-1} G_i + H_{i-1}\right]^T \lambda_{\psi\Omega,i} I_{\psi\psi}^{-1}\, d\beta \qquad (i = 0, \ldots, N^*)$$

APPENDIX B

ANALYTICAL SOLUTION DERIVED BY USE OF PONTRYAGIN'S PRINCIPLE

Pontryagin's maximum principle (ref. 4) is applied to the same problem introduced earlier. It is established that the control $\theta \equiv 90^{o}$ satisfies a necessary condition for optimality. Let

$$\left.\begin{aligned} K_{D,1} &\triangleq \left(\frac{\rho S}{2}\right) C_{D,o} \\ K_{D,2} &\triangleq \left(\frac{\rho S}{2}\right) C_{D_{C_L{}^2}} C_{L_\alpha}^2 \\ K_L &\triangleq \left(\frac{\rho S}{2}\right) C_{L_\alpha} \end{aligned}\right\} \tag{B1}$$

From figure 1, it is clear that

$$\left.\begin{aligned} \cos \gamma &= \frac{u}{V} \\ \sin \gamma &= \frac{v}{V} \\ \alpha &= \theta - \gamma = \theta - \cos^{-1} \frac{u}{V} \end{aligned}\right\} \quad (0 \leqq \gamma \leqq \pi) \tag{B2}$$

where α is the angle of attack of the airplane. By using equations (B1) and (B2), the equations of motion can be rewritten as

$$\left.\begin{aligned} m\dot{u} &= T \cos \theta - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)u + K_L \alpha v\right]\left(u^2 + v^2\right)^{1/2} \\ m\dot{v} &= T \sin \theta - mg - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)v - K_L \alpha u\right]\left(u^2 + v^2\right)^{1/2} \end{aligned}\right\} \tag{B3}$$

It is desired to minimize

$$J = \int_{t_o}^{t_f} - \dot{v}\, dt = -\left[v(t_f) - v(t_o)\right]$$

The Pontryagin pseudo-Hamiltonian is

$$H \triangleq p_0(-\dot{v}) + p_1\dot{u} + p_2\dot{v} = \frac{p_2 - p_0}{m}\left\{T \sin\theta - mg - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)v - K_L\alpha u\right]\left(u^2 + v^2\right)^{1/2}\right\}$$

$$+ \frac{p_1}{m}\left\{T \cos\theta - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)u + K_L\alpha v\right]\left(u^2 + v^2\right)^{1/2}\right\}$$

where

$$\dot{p}_1 \triangleq -\frac{\partial H}{\partial u} = -\frac{p_1}{m}\left\{-\left[\left(2K_{D,2}\alpha\frac{\partial\alpha}{\partial u}\right)u + \left(K_{D,1} + K_{D,2}\alpha^2\right) + K_L\frac{\partial\alpha}{\partial u}v\right]\left(u^2 + v^2\right)^{1/2}\right.$$

$$\left. - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)u + K_L\alpha v\right]u\left(u^2 + v^2\right)^{-1/2}\right\} - \frac{p_2 - p_0}{m}\left\{-\left[\left(2K_{D,2}\alpha\frac{\partial\alpha}{\partial u}\right)v\right.\right.$$

$$\left.\left. - K_L\frac{\partial\alpha}{\partial u}u - K_L\alpha\right]\left(u^2 + v^2\right)^{1/2} - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)v - K_L\alpha u\right]u\left(u^2 + v^2\right)^{-1/2}\right\}$$

$$\dot{p}_2 \triangleq -\frac{\partial H}{\partial v} = -\frac{p_1}{m}\left\{-\left[\left(2K_{D,2}\alpha\frac{\partial\alpha}{\partial v}\right)u + K_L\frac{\partial\alpha}{\partial v}v + K_L\alpha\right]\left(u^2 + v^2\right)^{1/2}\right.$$

$$\left. - \left[\left(K_{D,1} + K_{D,2}\alpha^2\right)u + K_L\alpha v\right]v\left(u^2 + v^2\right)^{-1/2}\right\} - \frac{p_2 - p_0}{m}\left\{-\left[\left(2K_{D,2}\alpha\frac{\partial\alpha}{\partial v}\right)v\right.\right.$$

$$+ \left(K_{D,1} + K_{D,2}\alpha^2\right) - K_L\frac{\partial\alpha}{\partial v}u\Big]\left(u^2 + v^2\right)^{1/2} - \Big[\left(K_{D,1} + K_{D,2}\alpha^2\right)v$$

$$\left.\left. - K_L\alpha u\right]v\left(u^2 + v^2\right)^{-1/2}\right\} \qquad \text{(B4)}$$

By using the equation for α from equations (B2),

$$\frac{\partial\alpha}{\partial u} = \frac{\sqrt{v^2}}{V^2}$$

$$\frac{\partial\alpha}{\partial v} = -\frac{uv}{\sqrt{v^2}V^2}$$

One has also

$$p_0 = -1$$

$$p_1(t_f) = 0$$

$$p_2(t_f) = 0$$

From reference 4, in order to minimize J with respect to θ, H must be maximized with respect to θ. Therefore,

$$\frac{\partial H}{\partial \theta} = \frac{p_1}{m}\left\{-T \sin \theta - \left[\left(2K_{D,2}\alpha\right)u + K_L v\right]\left(u^2 + v^2\right)^{1/2}\right\} + \frac{p_2 - p_0}{m}\left\{T \cos \theta\right.$$

$$\left. - \left[\left(2K_{D,2}\alpha\right)v - K_L u\right]\left(u^2 + v^2\right)^{1/2}\right\} = 0 \qquad \left(t_o \leqq t \leqq t_f\right)$$

The claim is made that $\theta = \frac{\pi}{2}\left(t_o \leqq t \leqq t_f\right)$ will be a solution. By putting $\theta = \frac{\pi}{2}$ into the equation for $\dot{u}$ of equations (B3), it is seen that $u = 0\left(t_o \leqq t \leqq t_f\right)$ satisfies the equation. This result is in agreement with the boundary condition $u(t_o) = 0$. From the equation for α from equations (B2), $\alpha = 0\left(t_o \leqq t \leqq t_f\right)$. By putting $\theta = \frac{\pi}{2}$ and $u = 0$ into the equation for $\dot{p}_1$ of equations (B4), $p_1 = 0\left(t_o \leqq t \leqq t_f\right)$ is seen to be a solution which also satisfies $p_1(t_f) = 0$. With $\theta = \frac{\pi}{2}$, $u = 0$, and $p_1 = 0$, it is seen that $\frac{\partial H}{\partial \theta} = 0\left(t_o \leqq t \leqq t_f\right)$.

The second partial is

$$\frac{\partial^2 H}{\partial \theta^2} = \frac{p_1}{m}\left\{-T \cos \theta - \left[\left(2K_{D,2}\right)u\right]\left(u^2 + v^2\right)^{1/2}\right\} + \frac{p_2 - p_0}{m}\left\{-T \sin \theta - \left[\left(2K_{D,2}\right)v\right]\left(u^2 + v^2\right)^{1/2}\right\}$$

$$= \frac{p_2 + 1}{m}\left(-T - 2K_{D,2}v\sqrt{v^2}\right)$$

where $\theta = \frac{\pi}{2}$, $u = 0$, $p_0 = -1$, and $p_1 = 0$ have been used. The value $\frac{\partial^2 H}{\partial \theta^2}$ will be less than zero for $t_o \leqq t \leqq t_f$ if $p_2 + 1 > 0$, and $-T - 2K_{D,2}v\sqrt{v^2}$ will be less than zero for $t_o \leqq t \leqq t_f$. These two conditions can be shown to hold if it is further assumed that $T > mg + K_{D,1}v_0^2$ and that v is continuous. Thus, under these conditions, $\theta = \frac{\pi}{2}\left(t_o \leqq t \leqq t_f\right)$ satisfies a sufficient condition for a local maximum of H and thereby satisfies a necessary condition for optimality.

APPENDIX C

DERIVATION OF COMPATIBLE ADJOINTS FOR FOURTH-ORDER RUNGE-KUTTA INTEGRATION

In this appendix the adjoints for both the continuous method and the discrete method for fourth-order Runge-Kutta integration are developed. Relationships between the two methods are discussed. For both methods x^* is given by Runge-Kutta integration of the equation

$$\dot{x}^* = f(x^*,u^*,t)$$

with initial condition $x^*(t_o) = x_o$ and the interval $t_o \leqq t \leqq t_f^*$ is partitioned as in the section "Analysis."

The adjoints for the usual steepest ascent method are generated by

$$\dot{\lambda} = -f_x^T(x^*,u^*,t)\lambda$$

with the terminal condition $\lambda(t_f)$ for ϕ and ψ the same as for the discrete method. For any $i = 0, 1, \ldots, N^* - 2$, by using backwards Runge-Kutta fourth-order integration,

$$\begin{aligned}
\lambda_i = \lambda_{i+1} + \frac{h}{6}\Bigg(& f_x^T(4)\lambda_{i+1} + 2f_x^T(3)\left[\lambda_{i+1} + \frac{h}{2} f_x^T(4)\lambda_{i+1}\right] + 2f_x^T(2)\left[\lambda_{i+1} + \frac{h}{2} f_x^T(3)\left(\lambda_{i+1}\right.\right. \\
&\left.\left. + \frac{h}{2} f_x^T(4)\lambda_{i+1}\right)\right] + f_x^T(1)\left\{\lambda_{i+1} + hf_x^T(2)\left[\lambda_{i+1} + \frac{h}{2} f_x^T(3)\left(\lambda_{i+1} + \frac{h}{2} f_x^T(4)\lambda_{i+1}\right)\right]\right\}\Bigg) \\
= \Bigg\{ I &+ h\left[\frac{1}{6} f_x^T(4) + \frac{1}{3} f_x^T(3) + \frac{1}{3} f_x^T(2) + \frac{1}{6} f_x^T(1)\right] + h^2\left[\frac{1}{6} f_x^T(3)\, f_x^T(4) + \frac{1}{6} f_x^T(2)\, f_x^T(3)\right. \\
&\left. + \frac{1}{6} f_x^T(1)\, f_x^T(2)\right] + h^3\left[\frac{1}{12} f_x^T(2)\, f_x^T(3)\, f_x^T(4) + \frac{1}{12} f_x^T(1)\, f_x^T(2)\, f_x^T(3)\right] \\
& + h^4\left[\frac{1}{24} f_x^T(1)\, f_x^T(2)\, f_x^T(3)\, f_x^T(4)\right]\Bigg\}\lambda_{i+1}
\end{aligned}$$

where

$$\left.\begin{aligned} f_x(1) &\triangleq f_x\left(x_i^*,u_i^*,t_i\right) \\ f_x(2) = f_x(3) &\triangleq f_x\left(x^*\left(t_{i+1} - \frac{h}{2}\right),u^*\left(t_{i+1} - \frac{h}{2}\right),t_{i+1} - \frac{h}{2}\right) \\ f_x(4) &\triangleq f_x\left(x_{i+1}^*,u_{i+1}^*,t_{i+1}\right) \end{aligned}\right\} \tag{C1}$$

The same equation holds for $i = N^* - 1$, except h is replaced by $t_f^* - \left[t_o + (N^* - 1)h\right]$.

The adjoints for the discrete steepest ascent method are generated by

$$\Delta\lambda_i = -F_i^T\lambda_{i+1} \qquad (i = 0, \ldots, N^* - 1)$$

or

$$\lambda_i = \left(I + F_i^T\right)\lambda_{i+1}$$

where

$$F_i \triangleq \frac{\partial \mathcal{F}_i}{\partial x_i}\left(x_i^*,u_i^*,u_{i+1}^*,t_i,t_{i+1}\right)$$

and

$$\mathcal{F}_i = x_{i+1} - x_i$$

as given by the forward Runge-Kutta fourth-order integration. By using the formulas for the numerical integration with $i = 0, 1, \ldots, N^* - 2$,

$$\begin{aligned} \mathcal{F}_i = \frac{h}{6}\Bigg[&f\left(x_i,u_i,t_i\right) + 2f\left(x_i + \frac{h}{2} f\left(x_i,u_i,t_i\right),u\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\right) + 2f\Bigg(x_i + \frac{h}{2} f\Bigg(x_i \\ &+ \frac{h}{2} f\left(x_i,u_i,t_i\right),u\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\Bigg),u\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\Bigg) + f\Bigg(x_i + hf\Bigg(x_i + \frac{h}{2} f\Bigg(x_i \\ &+ \frac{h}{2} f\left(x_i,u_i,t_i\right),u\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\Bigg),u\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\Bigg),u_{i+1},t_{i+1}\Bigg)\Bigg] \end{aligned}$$

Using this equation yields

$$F_i = \frac{h}{6}\left(f_x(1) + 2f_x(2)\left(I + \frac{h}{2} f_x(1)\right) + 2f_x(3)\left[I + \frac{h}{2} f_x(2)\left(I + \frac{h}{2} f_x(1)\right)\right]\right.$$
$$\left. + f_x(4)\left\{I + hf_x(3)\left[I + \frac{h}{2} f_x(2)\left(I + \frac{h}{2} f_x(1)\right)\right]\right\}\right)$$

where

$$\left.\begin{aligned}
&f_x(1) \triangleq f_x\left(x_i^*,u_i^*,t_i\right)\\
&f_x(2) \triangleq f_x\left(x_i^* + \frac{h}{2} f\left(x_i^*,u_i^*,t_i\right),u^*\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\right)\\
&f_x(3) \triangleq f_x\left(x_i^* + \frac{h}{2} f\left(x_i^* + \frac{h}{2} f\left(x_i^*,u_i^*,t_i\right),u^*\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\right),u^*\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\right)\\
&f_x(4) \triangleq f_x\left(x_i^* + hf\left(x_i^* + \frac{h}{2} f\left(x_i^* + \frac{h}{2} f\left(x_i^*,u_i^*,t_i\right),u^*\left(t_i + \frac{h}{2}\right),t_i + \frac{h}{2}\right),u^*\left(t_i\right.\right.\right.\\
&\qquad \left.\left.\left. + \frac{h}{2}\right),t_i + \frac{h}{2}\right);u_{i+1}^*,t_{i+1}\right)
\end{aligned}\right\} \tag{C2}$$

This procedure gives

$$\lambda_i = \left\{I + h\left[\frac{1}{6} f_x^T(1) + \frac{1}{3} f_x^T(2) + \frac{1}{3} f_x^T(3) + \frac{1}{6} f_x^T(4)\right]\right.$$
$$+ h^2\left[\frac{1}{6} f_x^T(1)\, f_x^T(2) + \frac{1}{6} f_x^T(2)\, f_x^T(3) + \frac{1}{6} f_x^T(3)\, f_x^T(4)\right]$$
$$+ h^3\left[\frac{1}{12} f_x^T(1)\, f_x^T(2)\, f_x^T(3) + \frac{1}{12} f_x^T(2)\, f_x^T(3)\, f_x^T(4)\right]$$
$$\left. + h^4\left[\frac{1}{24} f_x^T(1)\, f_x^T(2)\, f_x^T(3)\, f_x^T(4)\right]\right\}\lambda_{i+1}$$

The same equation holds for $i = N^* - 1$, except h is replaced by $t_f^* - \left[t_o + (N^* - 1)h\right]$.

Therefore, the only difference between the two methods with respect to the adjoints is that x^*_{i+1} is replaced by

$$x^*_i + hf\left(x^*_i + \frac{h}{2} f\left(x^*_i + \frac{h}{2} f\left(x^*_i, u^*_i, t_i\right), u^*\left(t_i + \frac{h}{2}\right), t_i + \frac{h}{2}\right), u^*\left(t_i + \frac{h}{2}\right), t_i + \frac{h}{2}\right)$$

Since $x^*\left(t_{i+1} - \frac{h}{2}\right)$ must be approximated from x^*_i and x^*_{i+1}, the approximations might as well be chosen to make $f_x(2)$ and $f_x(3)$ the same for both methods.

It has been noted that the replacement for x^*_{i+1} is an approximation for x^*_{i+1} used in the forward Runge-Kutta integration. Likewise, the two approximations (used in $f_x(2)$ and $f_x(3)$) for $x^*\left(t_{i+1} - \frac{h}{2}\right)$ are used in the forward integration. Thus, one sees how the backward integration is being made compatible with the forward integration.

APPENDIX D

COMPATIBLE ADJOINTS FOR PIECEWISE CONSTANT CONTROLS

In this appendix, the discrete algorithm is simplified for the sample problem by the assumption of piecewise constant controls. The form for the difference equations representing the state can, in this case, be reduced to

$$x_{i+1} - x_i = f_i(x_i,u_i) \qquad (i = 0, \ldots, N^* - 1) \tag{D1}$$

which is the form discussed at length in reference 3. All changes in the discrete method arise from the fact that H_i is now zero.

Hence, equation (2) becomes

$$\Delta(\delta x_i) \approx F_i \delta x_i + G_i \delta u_i \qquad (i = 0, \ldots, N^* - 1)$$

where

$$\left.\begin{aligned} F_i &= \frac{\partial f_i}{\partial x_i}\left(x_i^*,u_i^*\right) \\ G_i &= \frac{\partial f_i}{\partial u_i}\left(x_i^*,u_i^*\right) \end{aligned}\right\} \qquad (i = 0, \ldots, N^* - 1)$$

Equation (3) becomes

$$\Delta\left(\lambda_i^T \delta x_i\right) \approx \left[\lambda_{i+1}^T F_i + \Delta\left(\lambda_i^T\right)\right]\delta x_i + \lambda_{i+1}^T G_i \delta u_i \qquad (i = 0, \ldots, N^* - 1)$$

Equation (4) remains the same; that is, the compatible adjoints of reference 2 are still used. Equation (5) becomes

$$\Delta\left(\lambda_i^T \delta x_i\right) \approx \lambda_{i+1}^T G_i \delta u_i$$

For the sample problem δx_o is zero; therefore, equation (6) is

$$\lambda_{N^*}^T \delta x_{N^*} \approx \sum_{i=0}^{N^*-1} \lambda_{i+1}^T G_i \delta u_i$$

For the sample problem t_f is fixed; thus, equations (7) are

$$d\phi \triangleq \frac{\partial \phi}{\partial x_N}\left(x_{N^*}^*,t_{N^*}\right)\delta x_{N^*} \approx \phi\left(x_{N^*},t_{N^*}\right) - \phi\left(x_{N^*}^*,t_{N^*}\right)$$

where $d\psi$ and $d\Omega$ are omitted since there are no constraints and no stopping condition. Equation (8) becomes

$$d\phi = \sum_{i=0}^{N^*-1} \lambda_{\phi,i+1}^{T} G_i \delta u_i \tag{D2}$$

It is desired to maximize $d\phi$ subject to

$$(dP)^2 = \sum_{i=0}^{N^*-1} \delta u_i^T \delta u_i \tag{D3}$$

This procedure is the same as maximizing

$$d\phi = \sum_{i=0}^{N^*-1} \left(\lambda_{\phi,i+1}^{T} G_i - \mu \delta u_i^T\right)\delta u_i + \mu (dP)^2$$

and then solving for the μ which will satisfy equation (D3). Taking the differential yields

$$d(d\phi) = \sum_{i=0}^{N^*-1} \left(\lambda_{\phi,i+1}^{T} G_i - 2\mu \delta u_i^T\right) d\left(\delta u_i\right)$$

This differential will be zero for all $d\left(\delta u_i\right)$ if and only if

$$\delta u_i = \frac{1}{2\mu} G_i^T \lambda_{\phi,i+1} \qquad (i = 0, \ldots, N^* - 1)$$

Substituting the differential into equation (D3) yields

$$(dP)^2 = \frac{1}{4\mu^2} \sum_{i=0}^{N^*-1} \lambda_{\phi,i+1}^{T} G_i G_i^T \lambda_{\phi,i+1} \triangleq \frac{1}{4\mu^2} I_{\phi\phi}$$

which will be satisfied if

$$\frac{1}{2\mu} = \left[\frac{(dP)^2}{I_{\phi\phi}}\right]^{1/2}$$

which gives for the control corrections

$$\delta u_i = \left[\frac{(dP)^2}{I_{\phi\phi}}\right]^{1/2} G_i^T \lambda_{\phi,i+1} \qquad (i = 0, \ldots, N^* - 1)$$

Substituting δu_i into equation (D2) gives

$$d\phi = \sum_{i=0}^{N^*-1} \lambda_{\phi,i+1}^T G_i \left[\frac{(dP)^2}{I_{\phi\phi}}\right]^{1/2} G_i^T \lambda_{\phi,i+1} = \left[I_{\phi\phi}(dP)^2\right]^{1/2}$$

and, as a result, the gradient squared is

$$\left(\frac{d\phi}{dP}\right)^2 = I_{\phi\phi}$$

These equations were used in the computer study.

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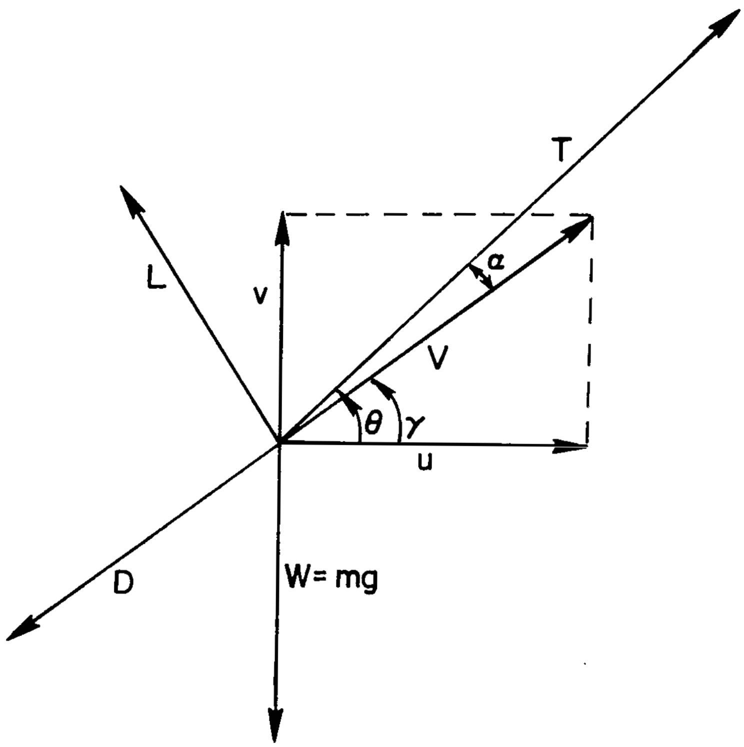


Figure 1.- Coordinate system and force diagram for sample problem.

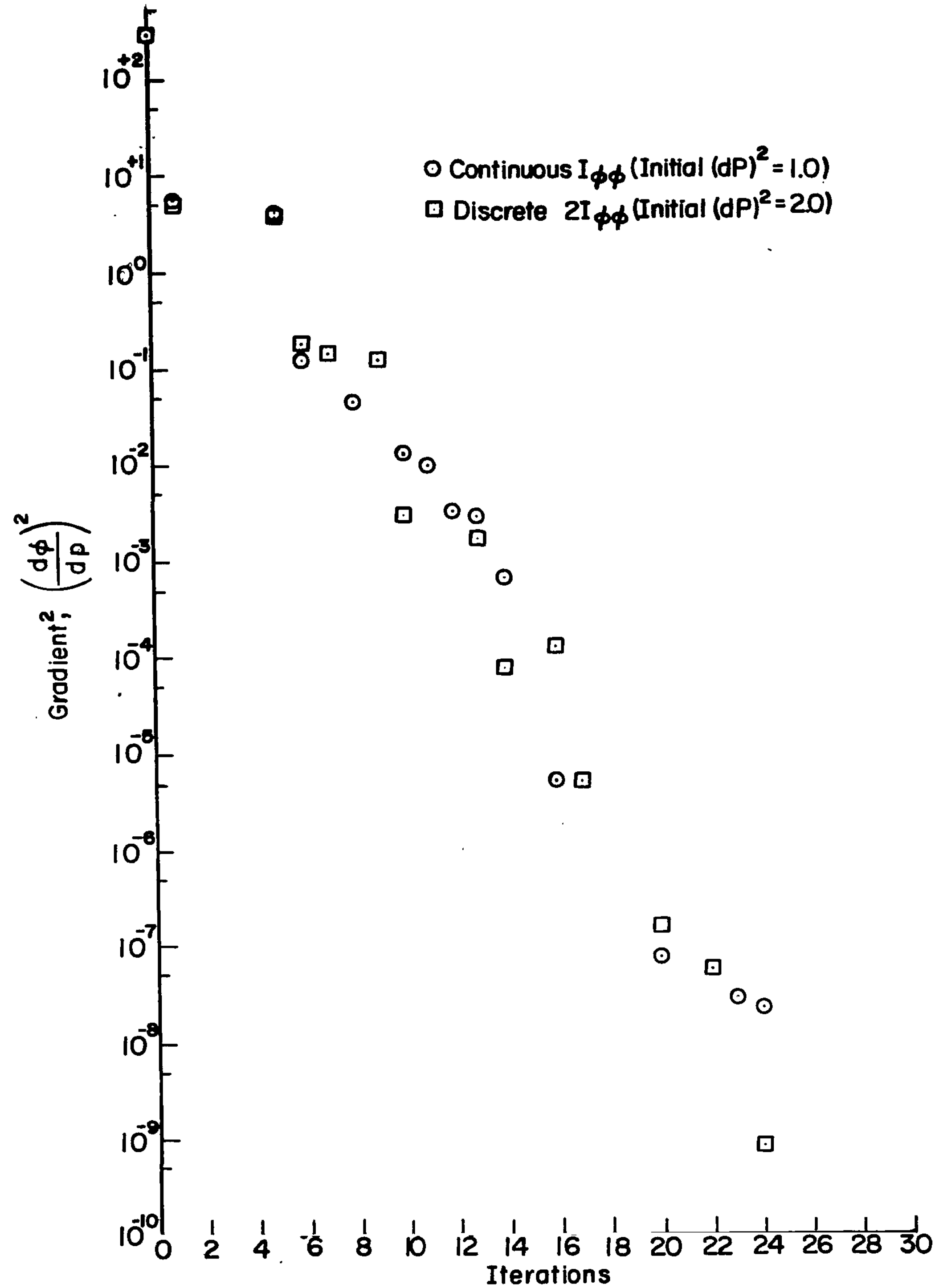


Figure 2.- Square of gradient plotted against iteration count (Canon-Cullum-Polak algorithm for discrete). Input control, 70°; final time, 10 seconds; computing interval, 0.5 second.

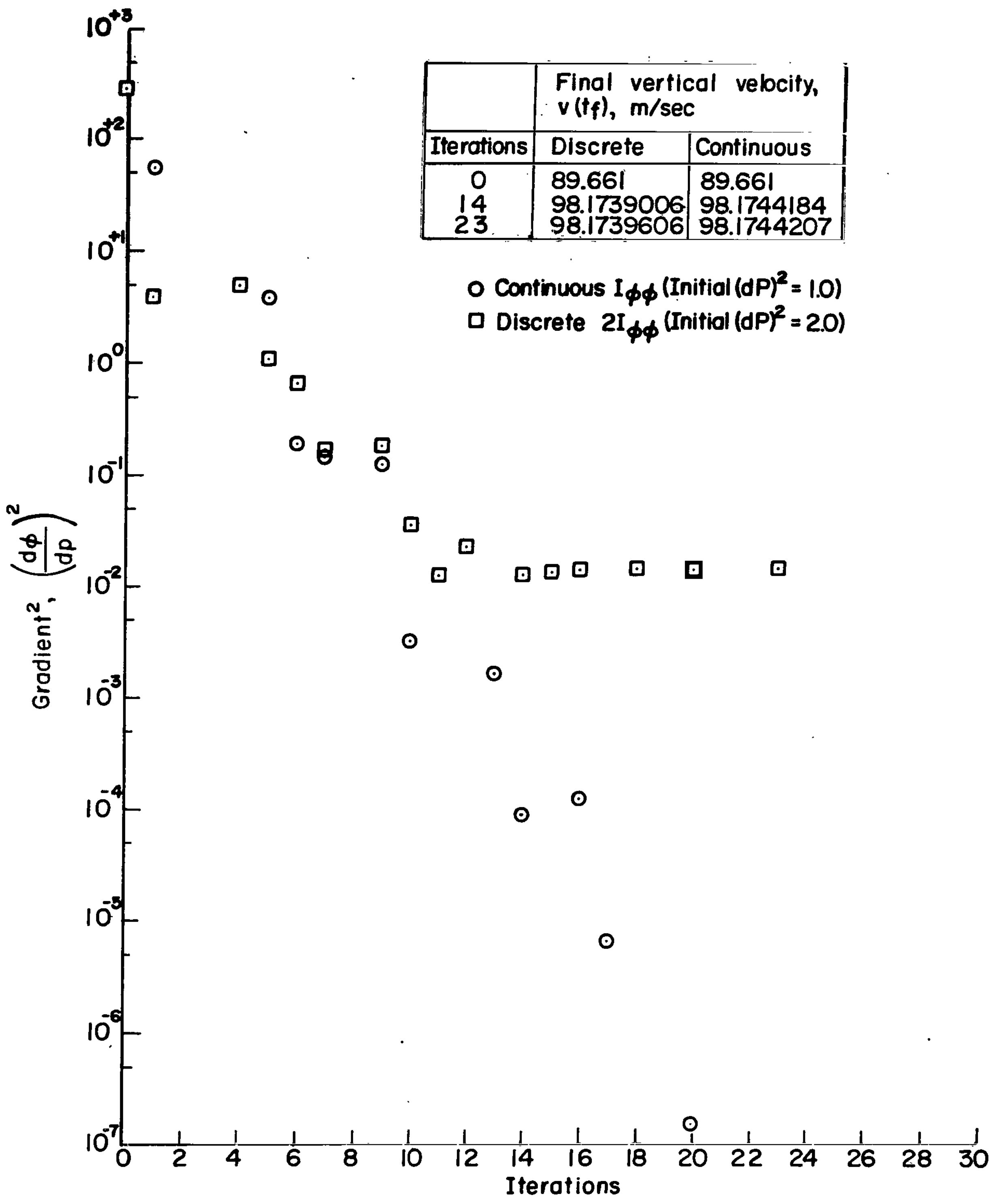
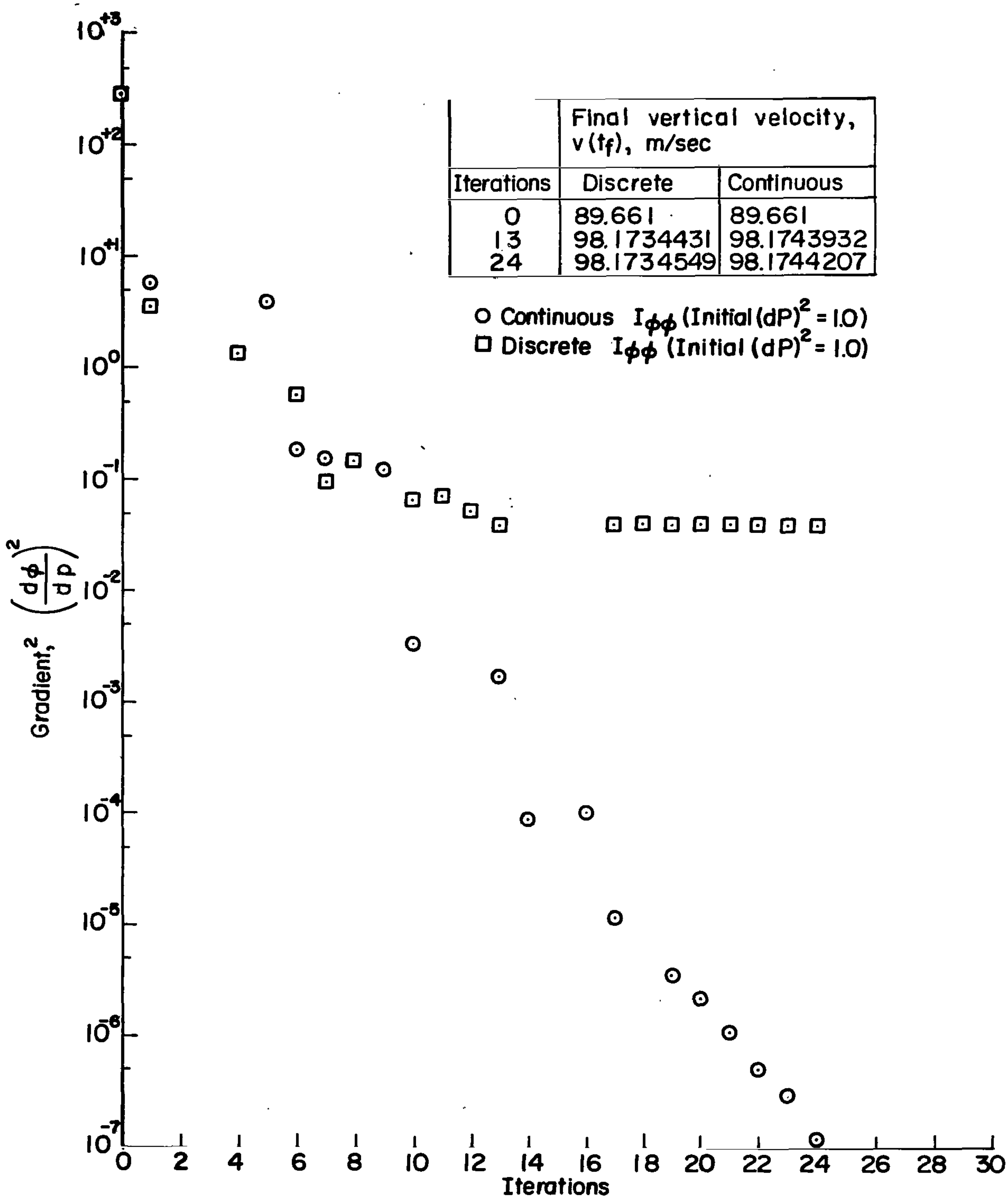


Figure 3.- Square of gradient plotted against iteration count. Input control, 70°; final time, 10 seconds; computing interval, 0.5 second.



	Final vertical velocity, $v(t_f)$, m/sec	
Iterations	Discrete	Continuous
0	89.661	89.661
13	98.1734431	98.1743932
24	98.1734549	98.1744207

Figure 4.- Square of gradient plotted against iteration count. Input control, 70°; final time, 10 seconds; computing interval, 1 second.

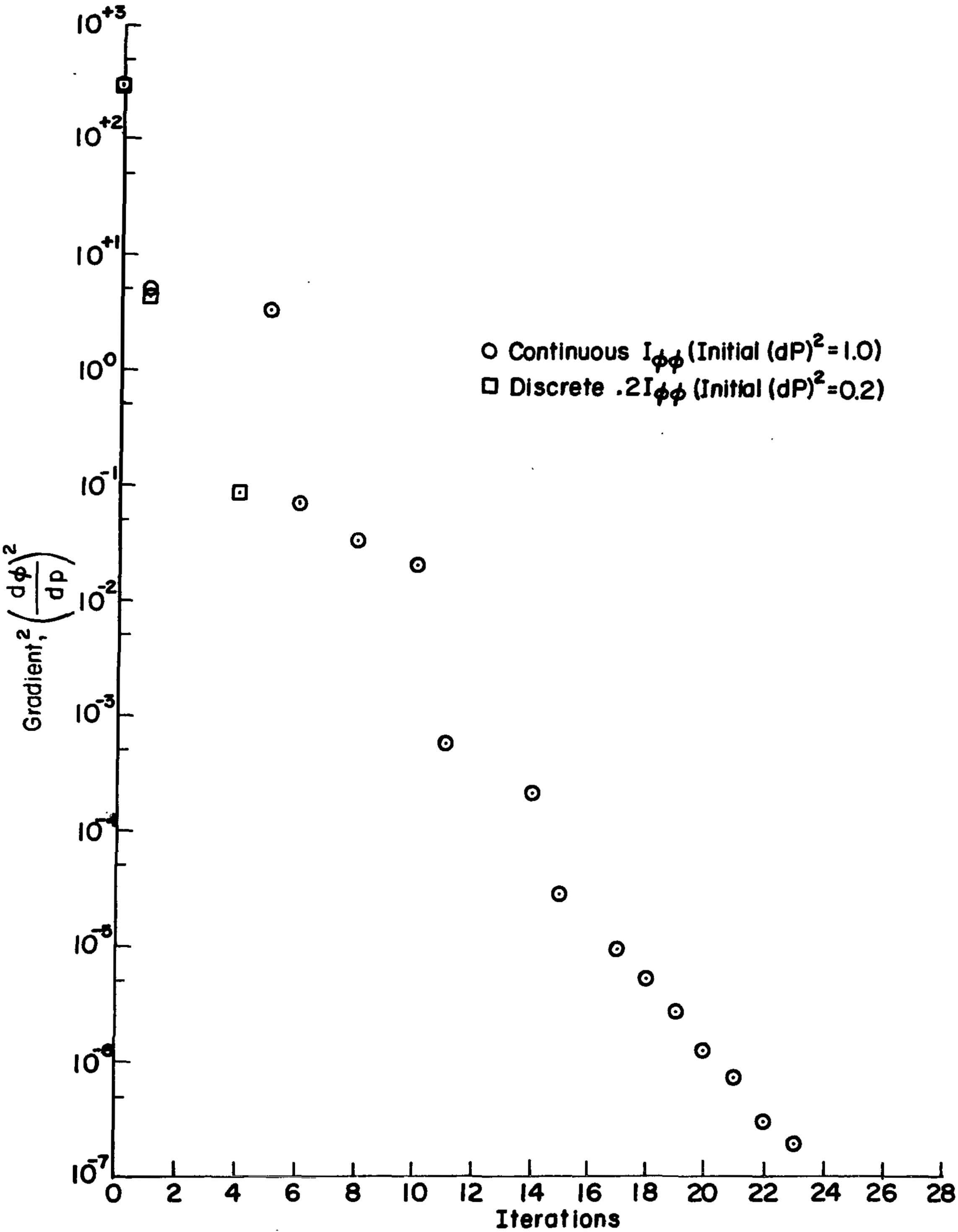
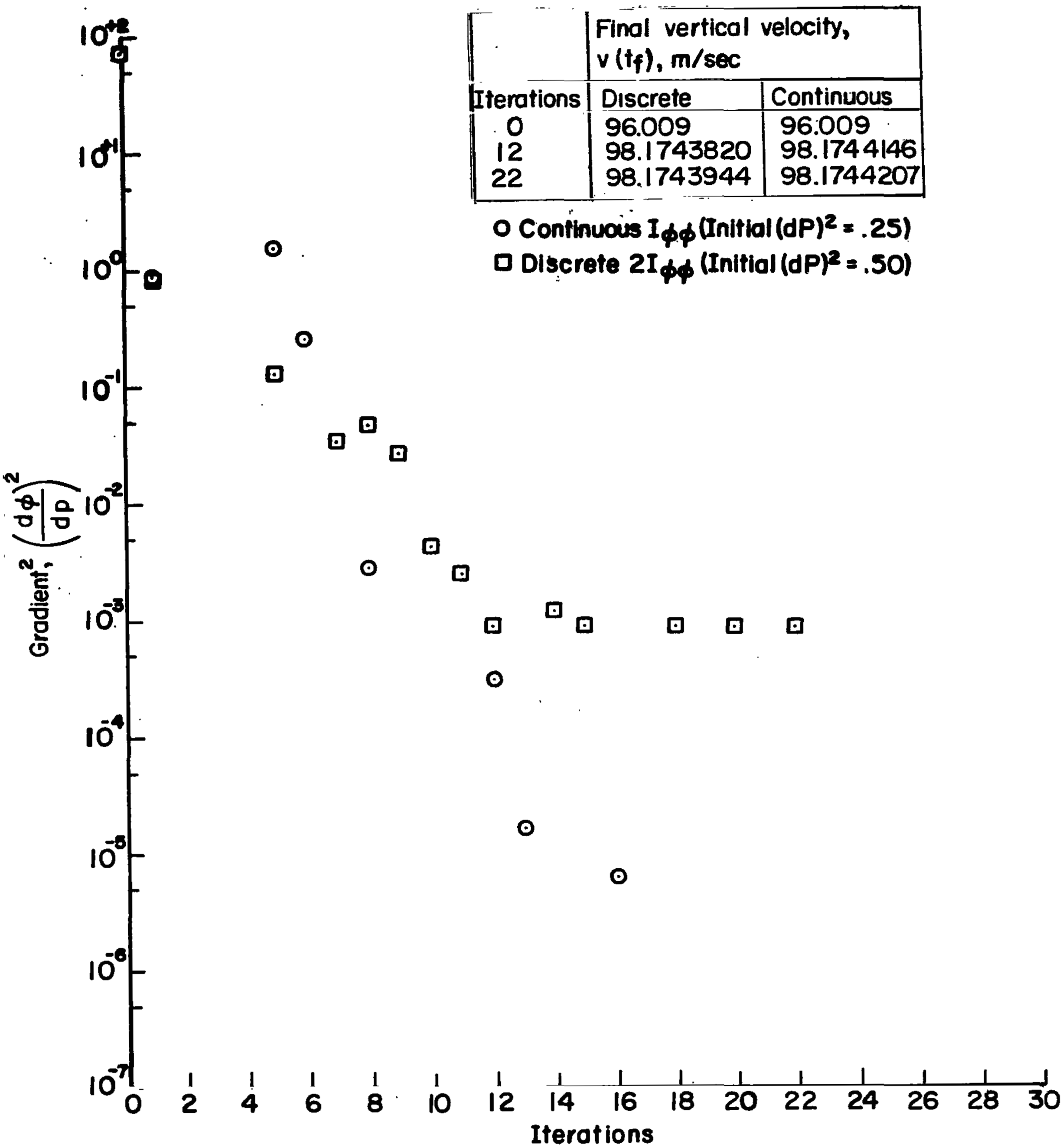


Figure 5.- Square of gradient plotted against iteration count. Input control, 70°; final time, 10 seconds; computing interval, 5 seconds.



	Final vertical velocity, $v(t_f)$, m/sec	
Iterations	Discrete	Continuous
0	96.009	96.009
12	98.1743820	98.1744146
22	98.1743944	98.1744207

Figure 6.- Square of gradient plotted against iteration count. Input control, 80°; final time, 10 seconds; computing interval, 0.5 second.

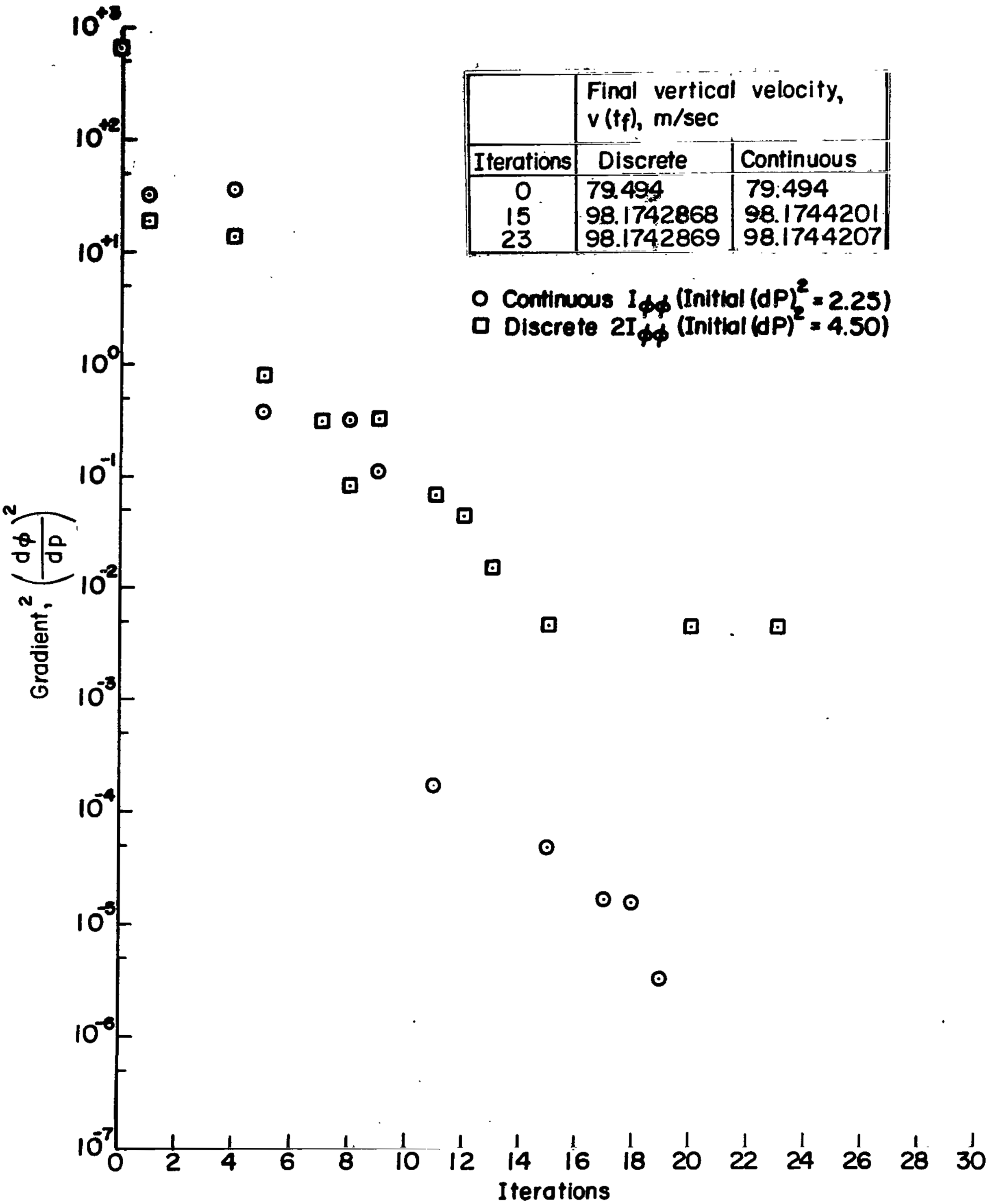


Figure 7.- Square of gradient plotted against iteration count. Input control, 60°; final time, 10 seconds; computing interval, 0.5 second.

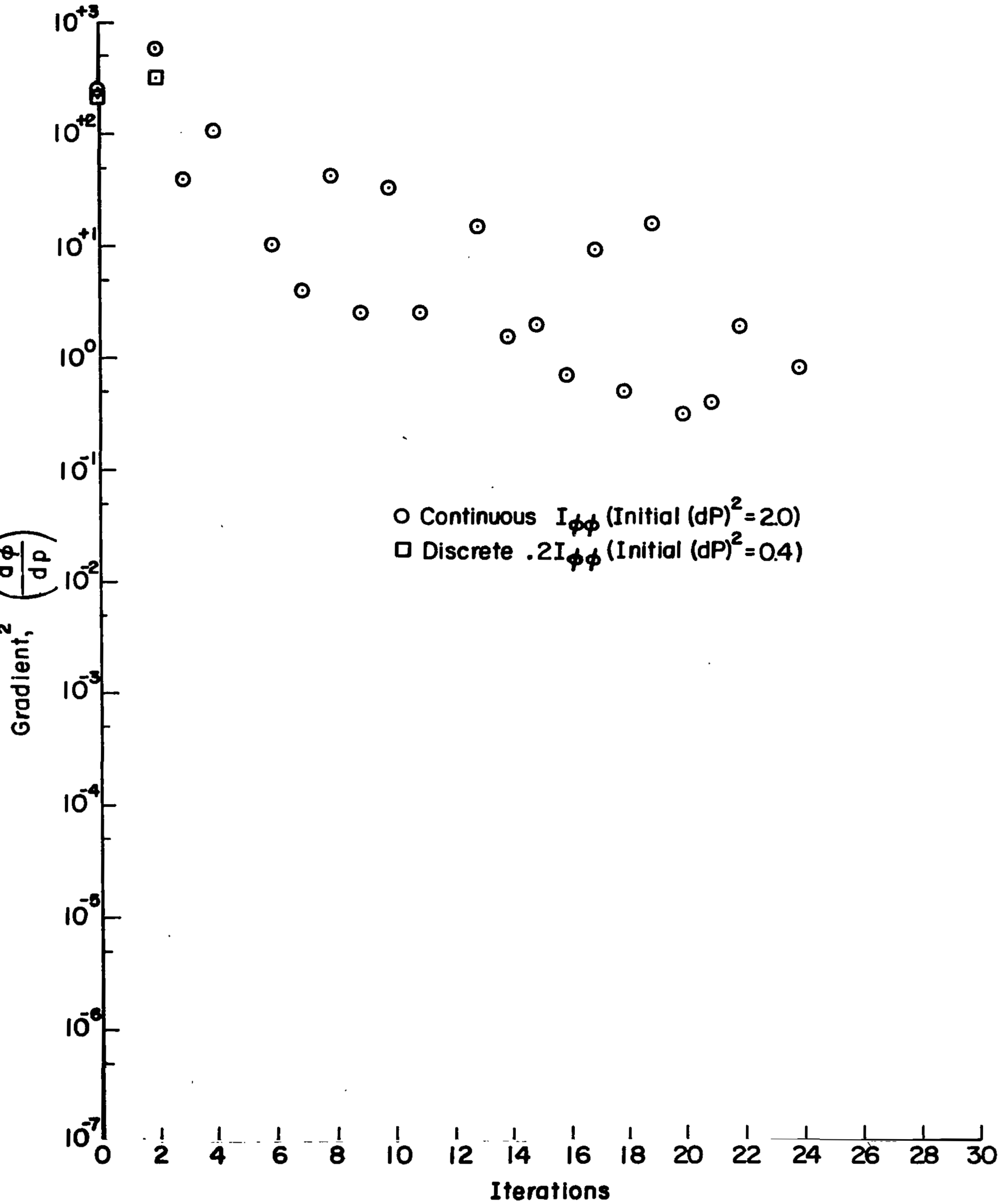


Figure 8.- Square of gradient plotted against iteration count. Input control, 70°; final time, 20 seconds; computing interval, 5 seconds.

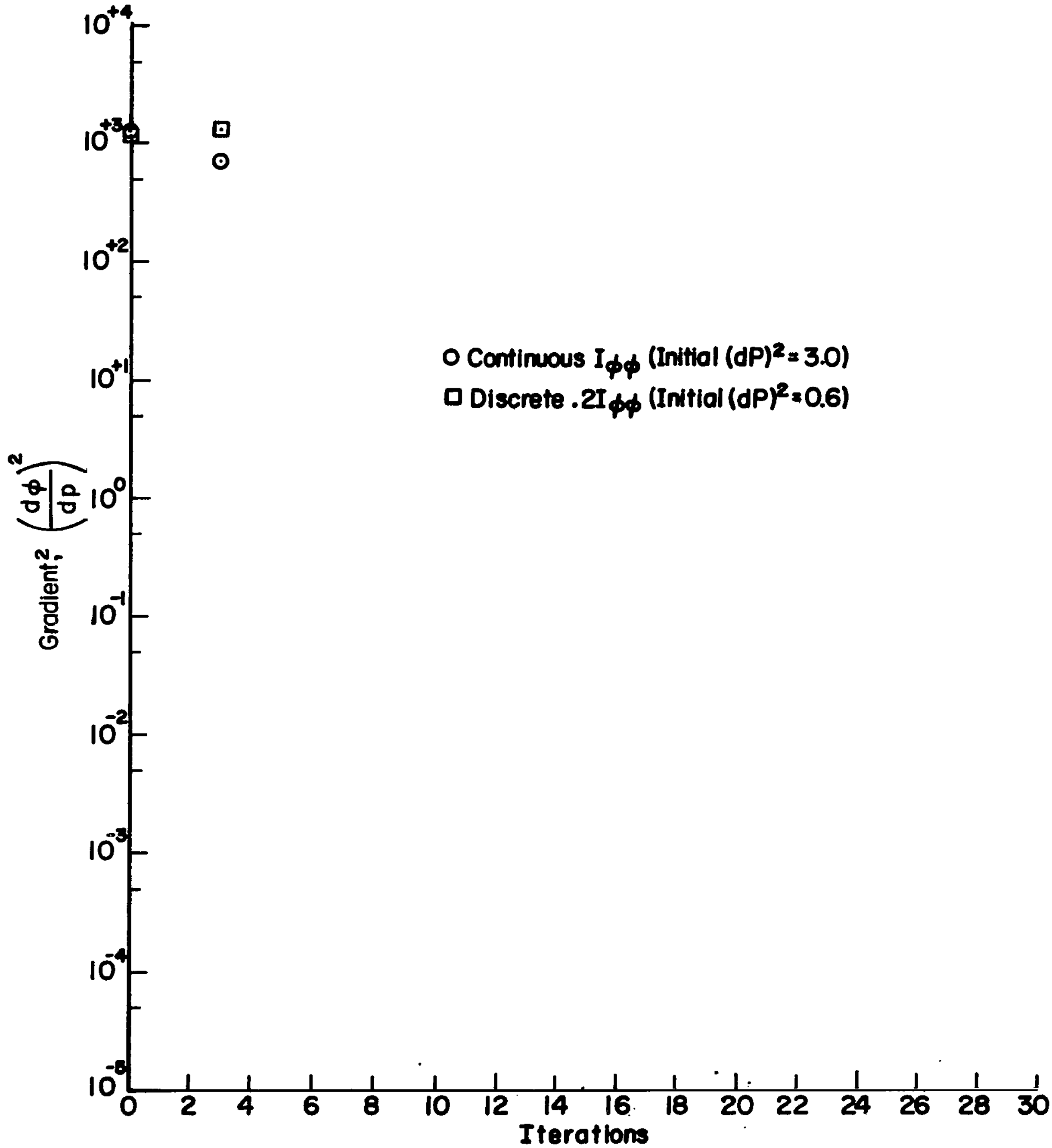


Figure 9.- Square of gradient plotted against iteration count. Input control, 70°; final time, 30 seconds; computing interval, 5 seconds.

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