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AIRCRAFT ALTITUDE DETERMINATION USING MULTIPATH INFORMATION IN AN ANGLE-MEASURING NAVIGATION SATELLITE SYSTEM

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ABSTRACT

In an angle-measuring navigation satellite system using a pair of crossed interferometers located on a satellite in synchronous orbit, three parameters are needed to determine a user's position unambiguously: the phase difference between received signals which had been transmitted by the two antennas on each of the two interferometers, and the user's altitude.

The two phase difference measurements yield a line of possible user locations, and the addition of the altitude measurement reduces this line to a single point. Instead of measuring altitude via a barometric altimeter, a method is proposed here which makes use of the navigation signals received after reflection off the earth's surface. The iterative procedure used here employs the arrival time difference between direct and reflected signals.

Based on previous calculations of errors in measuring the electrical parameters it is concluded that, for North Atlantic coverage and specular reflection, altitude measurements can be made with a 1-σ error of 65 meters.

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1.0 INTRODUCTION

In an angle-measuring navigation satellite system using a pair of crossed interferometers located on a satellite in synchronous orbit, three parameters are needed to determine a user's location unambiguously: the phase difference between received signals which had been transmitted by the two antennas on each of the two interferometers, and the user's altitude. The satellite is assumed in this study to be transmitting at L-band and to be nominally located over the equator at 30°W longitude, thereby providing complete North Atlantic coverage.

Each of the two phase difference measurements places the user on the surface of a cone whose vertex is the satellite (an approximation good for great distances from the satellite). The intersection of these two cones is a straight line extending from the satellite towards the user.

A knowledge of the user's height above the earth reduces this line of possible positions to a single point. Normally, the altitude would be determined by a barometric altimeter. An alternate altitude determination method, which is proposed here, makes use of the satellite signals received after reflection off the earth's surface. This procedure, which involves measuring the arrival time difference between direct and reflected signals, could be used as a backup for the barometric altimeter, or even in place of it.

Several types of signaling (AM, FM, RF pulses) have been discussed in the literature (Ref. 1, 2, 3) and will not be repeated here. It will be mentioned, however, that regardless of the signaling employed, separation of the direct and reflected signals is necessary for accurate measurement of the arrival time difference.

In this study it is merely assumed that the user is able to make the proper phase difference measurements and time delay measurements, regardless of the type of signals used in the system.

A brief description of the geometry of position-finding in an angle-measuring system is found in Sec. 2.0. Section 3.0 is concerned with an analysis of the height determining procedure which makes use of multipath information, while Section 4.0 deals with an analysis of the errors involved. The results of a system evaluation and some conclusions are found in Section 5.0.

2.0 MATHEMATICS OF ANGLE MEASURING NAVIGATION SATELLITE SYSTEM

The synchronous satellite will be considered located nominally over the equator at 30°W longitude. One interferometer boom is located in the equatorial plane in an east-west position, while the other interferometer boom is in a north-south orientation. A nominal coordinate system, x_o - y_o - z_o, is considered with the origin at the nominal satellite location. This nominal coordinate system is shown in Fig. 1.

The satellite will usually be at some location other than its nominal one, and for this reason it is given a local coordinate system, x-y-z, also shown in Fig. 1.

The satellite altitude is constantly monitored by ground-based tracking stations which relay the pertinent information to the field of users at regular intervals.

The matrices used in transforming the nominal coordinate system into the local coordinate system, and vice versa, are found in the Appendix.

Consider the pair of antennas along the x-axis, as shown in Fig. 2. The measured phase difference at the aircraft between the signals transmitted by each antenna is given by

$$\Delta\varphi_x = 2\pi \frac{D}{\lambda} \cos \psi_x - n2\pi \qquad (2\text{-}1)$$

where λ is the wavelength of the L-band signal (1.6 GHz), and n is the largest integer in $D/\lambda \cos \psi_x$, since measurements are made modulo 2π. Clearly, there are several values of ψ_x which would yield a particular measured value of $\Delta\varphi_x$:

$$\psi_x = \cos^{-1} \left[\frac{\lambda}{2\pi D} (\Delta\varphi_x + n2\pi) \right] . \qquad (2\text{-}2)$$

A user, then, would be located on the surface of a cone whose vertex is the x-y-z origin, whose axis is the x-axis, and whose half-angle is ψ_x. Each cone, corresponding to each possible value of ψ_x, intersects the earth forming a curve. It can be shown (ref. 1) that if the boom length, D, is 20 wavelengths, the minimum distance between these curves is 1000 nautical miles. Thus, if a user knows his position to within 1000 nmi, he then knows the proper cone and proper value of ψ_x.

The equation of the cone is given by

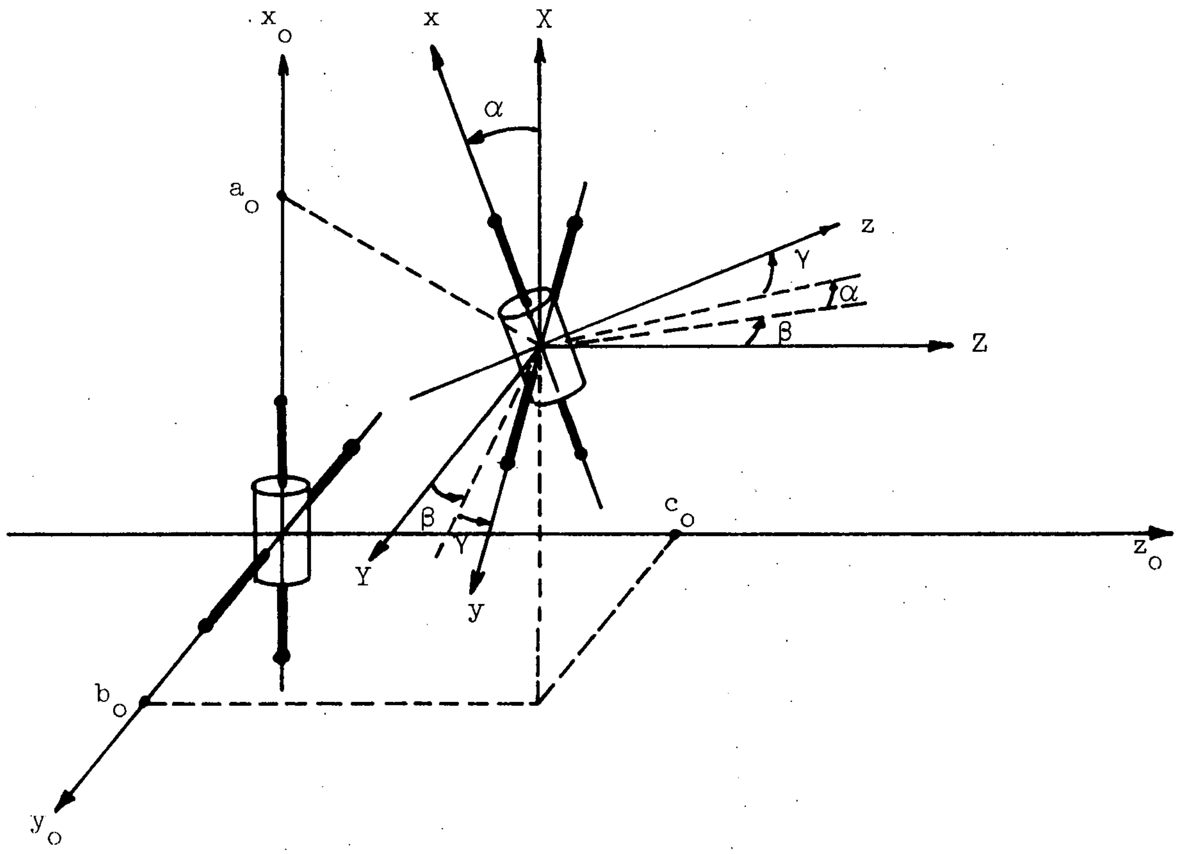

Fig. 1 General Satellite Coordinate System

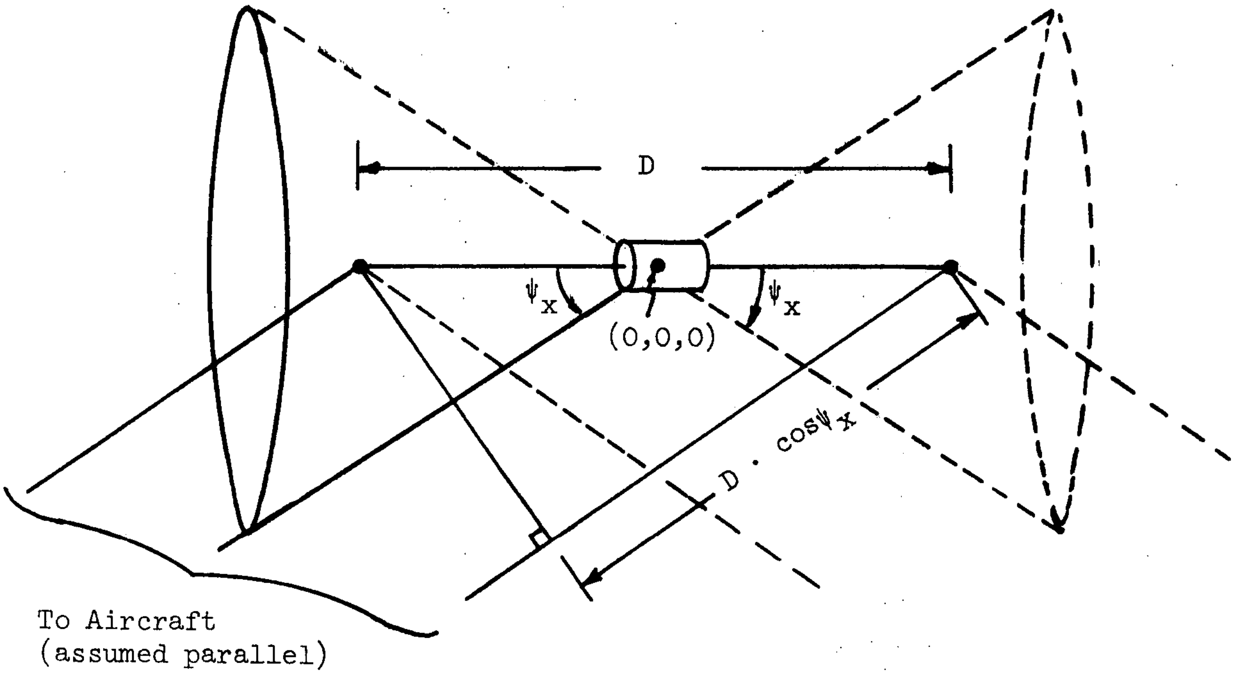

Fig. 2 x-Axis Interferometer Geometry

$$\left[\tan^2 \psi_x\right] x^2 - y^2 - z^2 = 0. \qquad (2\text{-}3)$$

Actually, eq. (2-3) describes a pair of cones, the second one of which is shown in Fig. 2 by the dotted lines. The presence of two cones is of importance only when their intersections with the earth are close. However, since a user will be making new measurements every minute or so, for instance, he will be able to determine the proper choice of cones.

Similarly, by measuring the phase difference between the signals from each of the two antennas on the y-axis, the user finds himself situated on the surface of a second cone with vertex at the x-y-z origin, with axis along the y-axis, and with a half-angle of ψ_y. This cone's equation--again, there is really a pair of cones--is given by

$$\left[\tan^2 \psi_y\right] y^2 - x^2 - z^2 = 0. \qquad (2\text{-}4)$$

The intersection of the cones is a straight line which passes through both the satellite center and the aircraft, and which is described by the following equations:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}\,, \qquad (2\text{-}5)$$

where

$$\left.\begin{aligned} a &= \pm \cos \psi_x \\ b &= \cos \psi_y \\ c &= \pm \sqrt{1 - \cos^2 \psi_x - \cos^2 \psi_y} \end{aligned}\right\} \qquad (2\text{-}6)$$

The reason for the choice of signs in eq. (2-6) is easily seen in Fig. 3. Since there are two cones centered on the x-axis and two centered on the y-axis, their intersections result in four straight lines. The field of users will be sent data regarding satellite attitude as determined by the ground stations, so that determination of the proper signs for a and c is easily obtained.

The center of the earth (assumed to be a sphere of radius R_e) in the x-y-z system is at the point (x_e, y_e, z_e), and the equations of the line from the satellite to earth center are given by

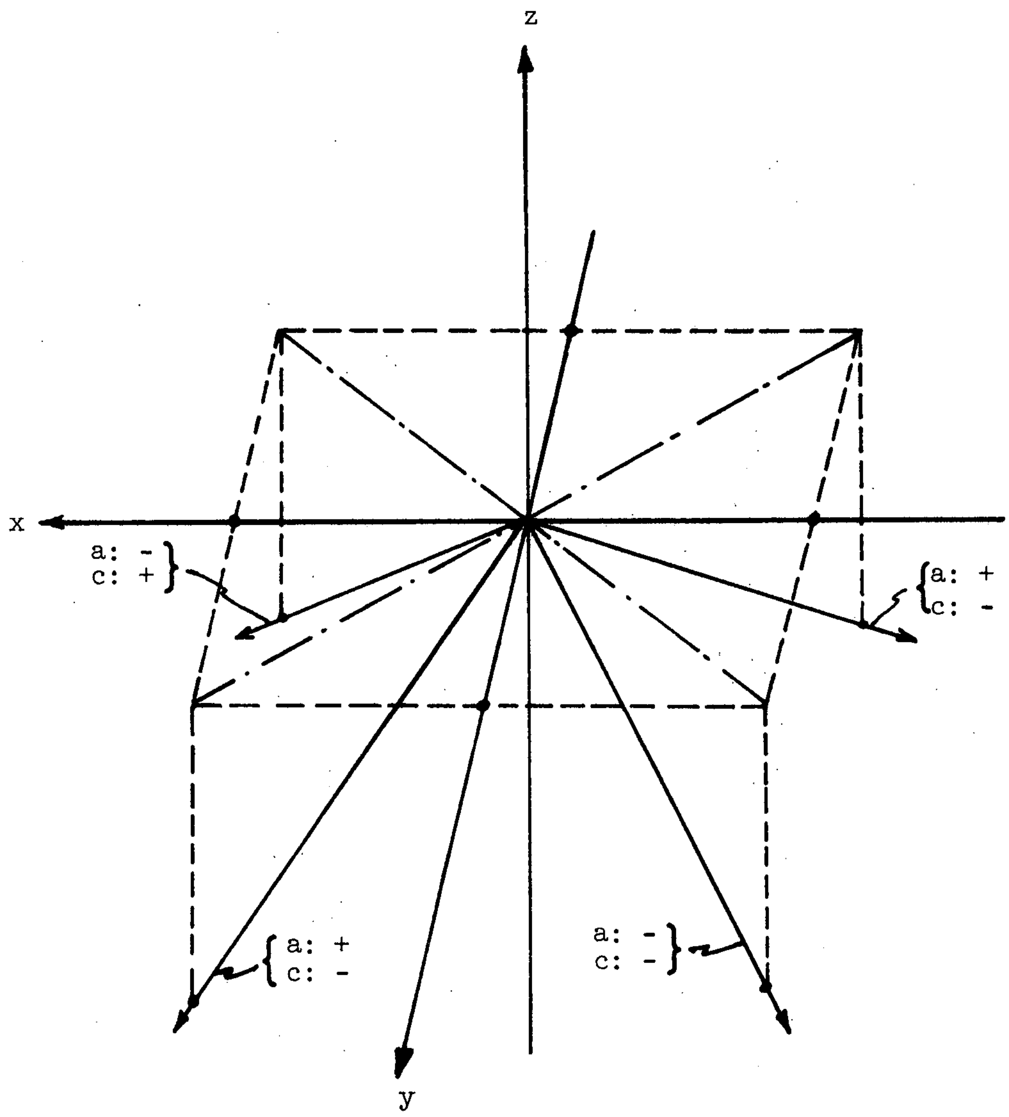

Fig. 3 Geometry of Four-Cone Intersection

$$\frac{x}{x_e} = \frac{y}{y_e} = \frac{z}{z_e} \tag{2-7}$$

The distance, d_e, from the satellite to earth center is given by

$$d_e = \sqrt{x_e^2 + y_e^2 + z_e^2} \tag{2-8}$$

The line from the satellite to the aircraft intersects the earth at the point (x_i, y_i, z_i), where

$$\left.\begin{aligned} x_i &= aT \\ y_i &= bT \\ z_i &= cT \\ T &= \left[S - \sqrt{S^2 - (d_e^2 - R_e^2)}\right] \\ S &= ax_e + by_e + cz_e \end{aligned}\right\} \tag{2-9}$$

The derivation of eq. (2-9) is found in Appendix A.

The angle, θ, between the satellite-earth center line and the satellite-aircraft line is given by

$$\theta = \cos^{-1} \frac{S}{d_e} \tag{2-10}$$

Once the user's altitude is determined (see Sec. 3.0) and the point on earth directly beneath the aircraft, $\left(x_{o_{AE}}, y_{o_{AE}}, z_{o_{AE}}\right)$ is found in the nominal coordinate system, the aircraft's latitude and longitude are then computed by the following equations:

$$\text{Lat.} = \sin^{-1} \frac{y_{o_{AE}}}{R_e} \tag{2-11}$$

$$\text{Long.} = 30^\circ \mp \cos^{-1} \frac{R_S + z_{o_{AE}}}{R_e \cos(\text{Lat.})}, \; x_{o_{AE}} \gtrless 0. \tag{2-12}$$

The point $(x_{o_{AE}}, y_{o_{AE}}, z_{o_{AE}})$ is referred to as the sub-aircraft position.

3.0 HEIGHT DETERMINATION METHOD USING MULTIPATH

The height determination method employed here is an iterative one which makes use of signals received by an aircraft after specular reflection off the earth's surface. Besides measuring the two phase differences discussed in Sec. 2.0, the user will also measure the time delay, Δt, between reception of direct signals and reflected signals.

It will be assumed that the specular reflection point, M, is in the satellite-aircraft-earth center plane, P. This is not an unreasonable assumption, since the boom length, D, is only 20λ (3.75 meters at L-band).

It will be necessary to consider another satellite-centered coordinate system, the u-v system, shown in Fig. 4, such that the u-v plane is the same as the plane, P, and such that the u-axis is directed towards the earth's center and the v-axis is directed towards the v-coordinate of the aircraft.

The details regarding the transformation from the x-y-z system to the u-v system, and vice versa, will be found in Appendix B.

The sub-satellite point (u_{AE}, v_{AE}), or more precisely the point $(x_{o_{AE}}, y_{o_{AE}}, z_{o_{AE}})$, determines the aircraft's latitude and longitude (eqs. 2-11 and 2-12).

The height determining procedure is based on the following manipulations:

Having determined values for θ and Δt via the phase difference and time delay measurements, the user then determines the point (u_i, v_i), which is the intersection of the satellite-aircraft line with the earth, and imagines a tangent to the earth at this point, as shown in Fig. 5. Since this is the first of several points of tangency*, it will be denoted (u_{i_1}, v_{i_1}). Using this tangent line, a satellite image point is found.**

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* It may be inefficient to start the iterative procedure with a plane tangent to (u_{i_1}, v_{i_1}). However, the simulation of the system shows that only three or four iterations are needed using this point; thus, it is not absolutely necessary to begin the procedure with another point.

** It is possible that the image method which is applicable to spheres would be useful.

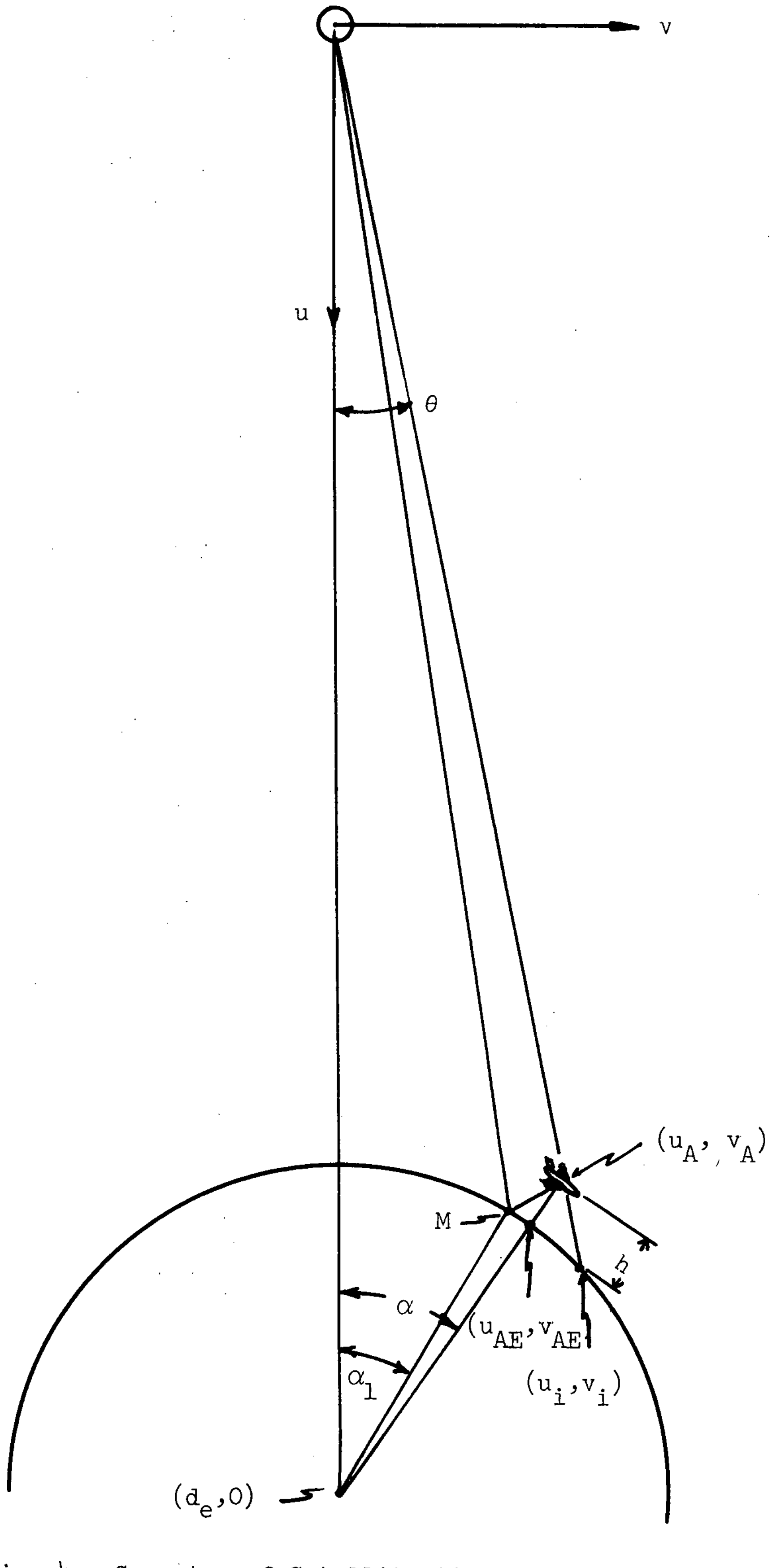

Fig. 4 Geometry of Satellite-Aircraft-Earth Center Plane

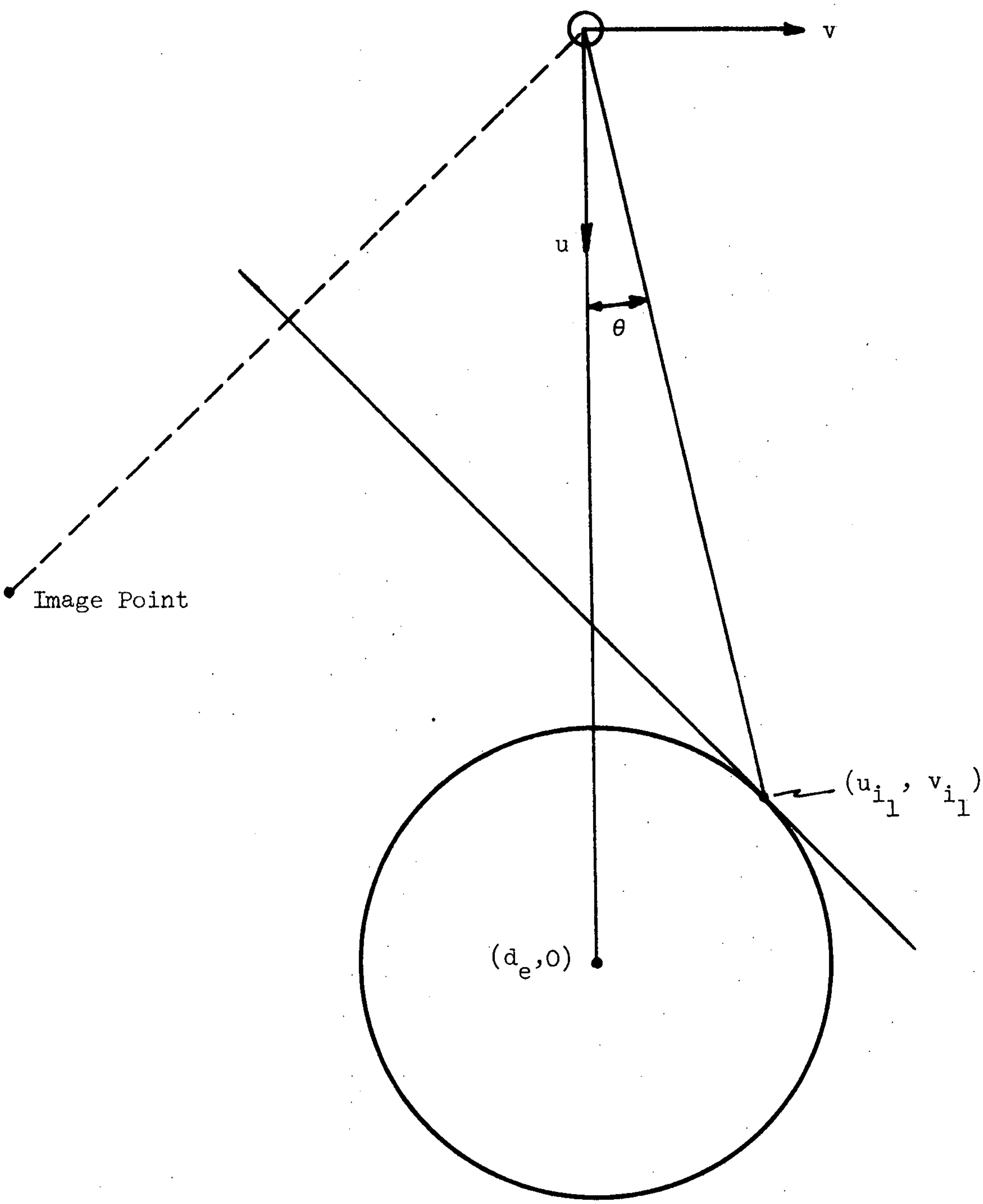

Fig. 5 Geometry of First Step of Iterative Procedure

The path length difference, $\Delta \ell$, between the direct and reflected signal paths is related to the arrival time delay between them by the following equation:

$$\Delta t = \frac{\Delta \ell}{c} \tag{3-1}$$

where c is the speed of light.

Consider, now, the hyperbola of Fig. 6. The difference between the focal radii, $\overline{P'F_1} - \overline{P'F_2}$, of any point, P', on a hyperbola is a constant, and is equal to the length of the transverse axis, ℓ. One can therefore imagine a hyperbola with transverse axis length equal to the measured path length difference, $\Delta \ell$, as given by eq. (3-1).

Consider, then, such a hyperbola, shown in Fig. 7, with foci at the satellite and satellite image point and with transverse axis equal to $\Delta \ell$. The intersection of this hyperbola with the satellite-aircraft line is the point (u_{A_1}, v_{A_1}) and is the first estimate of the aircraft's location. This point would be the true location, since it is on the satellite-aircraft line and since the observed path length difference at this point is $\Delta \ell$, were it not for the fact that reflection takes place off the tangent plane (line) in this approximation and not off the earth's surface.

A line is then imagined from the satellite through this planar point. This line intersects the earth at the point (u_{i_2}, v_{i_2}), as shown in Fig. 7; a new line, tangent to the earth at this point, is considered, and the process is repeated.

When the process has been taken through as many iterations as is necessary to satisfy required accuracies, the subaircraft point, (u_{AE}, v_{AE}), and the aircraft's altitude, h, are then determined (Fig. 4) by the following equations:

$$h_j = \sqrt{\left(d_e - u_{A_j}\right)^2 + v_{A_j}^{\;2}} - R_e \tag{3-2}$$

$$u_{AE_j} = \frac{R_e\, u_{A_j} + d_e\, h_j}{R_e + h_j} \tag{3-3}$$

$$v_{AE_j} = \frac{R_e v_{A_j}}{R_e + h_j}\,, \tag{3-4}$$

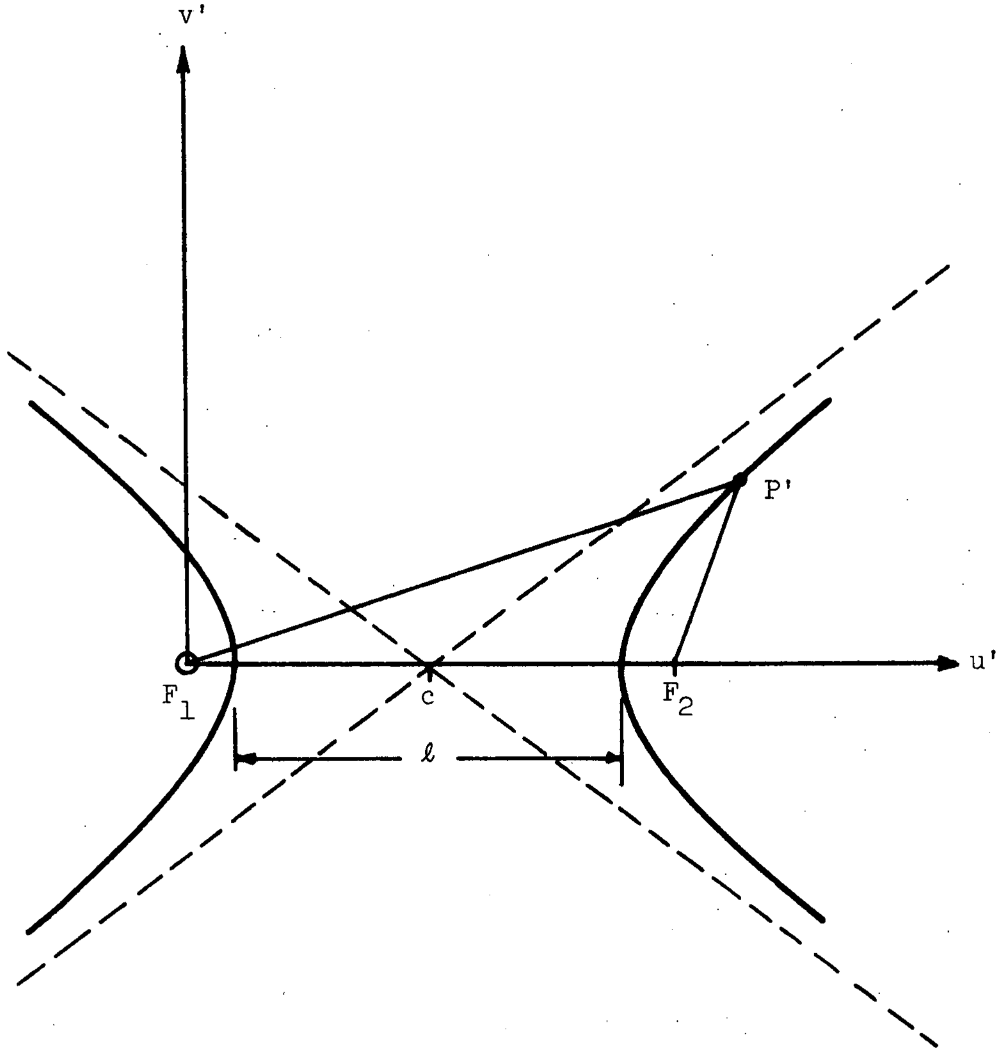

Fig. 6 Geometry of Hyperbola

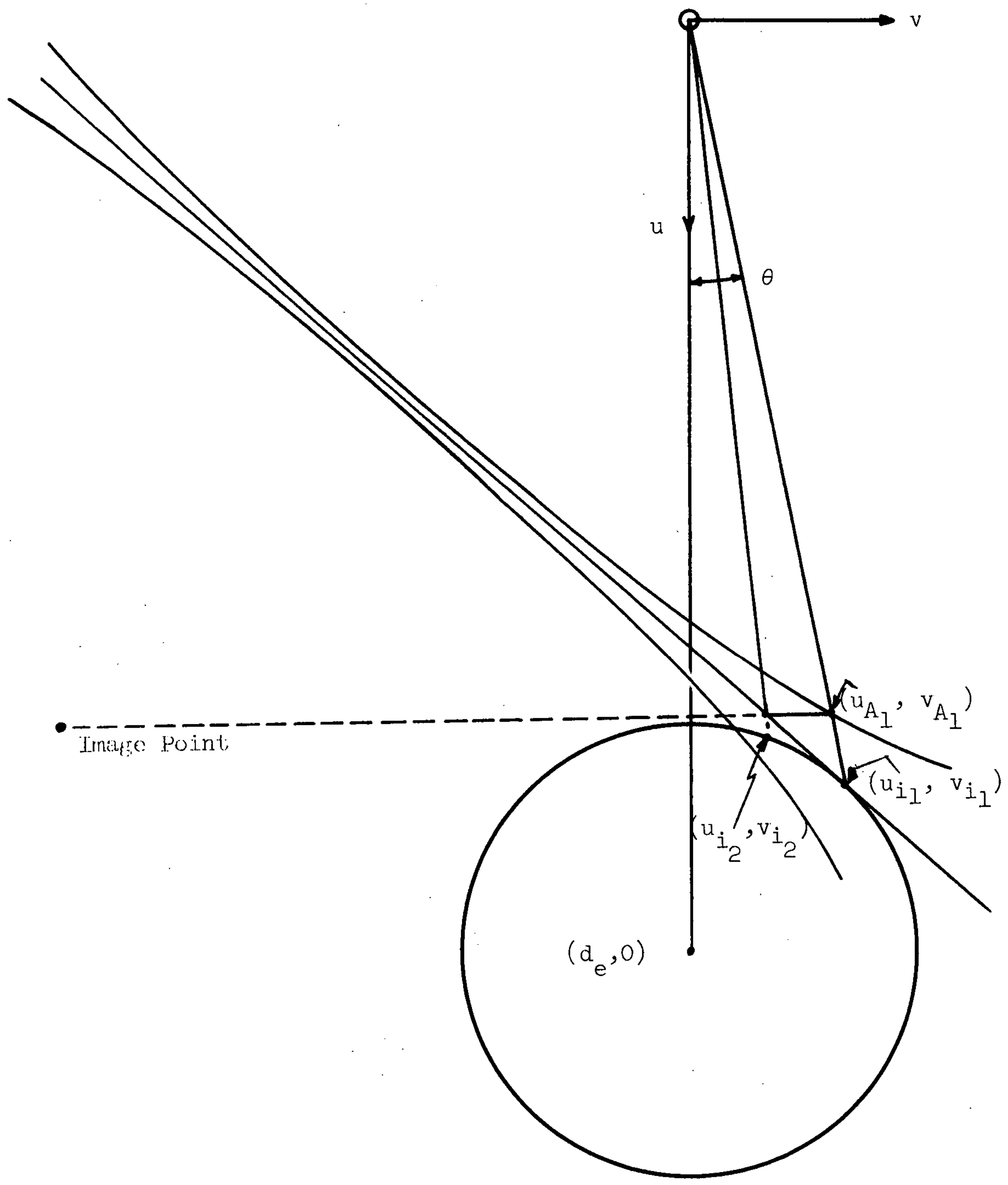

Fig. 7 Geometry of First Iteration

where the j refers to the values of the indicated parameters in the j^{th} iteration. The details of this procedure are found in Appendix A.

The point (u_{AE}, v_{AE}) is then converted back to the nominal coordinate system to become the point $(x_{o_{AE}}, y_{o_{AE}}, z_{o_{AE}})$, which is used to determine the aircraft's latitude and longitude from equations (2-11) and (2-12).

4.0 ERROR ANALYSIS

The altitude, latitude, and longitude of an aircraft may be written as functions of ψ_x, ψ_y, Δt, α, β, γ, a_o, b_o, and c_o. These measured parameters are independent, zero mean random variables with variances $\sigma_{\psi_x}^2$, $\sigma_{\psi_y}^2$, $\sigma_{\Delta t}^2$, σ_α^2, σ_β^2, σ_γ^2, $\sigma_{a_o}^2$, $\sigma_{b_o}^2$ and $\sigma_{c_o}^2$, respectively.

The variance of altitude errors is then given by

$$\sigma_{alt}^2 = \left[\frac{\partial\ alt}{\partial\ \psi_x}\right]^2 \cdot \sigma_{\psi_x}^2 + \left[\frac{\partial\ alt}{\partial \psi_y}\right]^2 \cdot \sigma_{\psi_y}^2 + \left[\frac{\partial\ alt}{\partial\ \Delta t}\right]^2 \cdot \sigma_{\Delta t}^2 + \left[\frac{\partial\ alt}{\partial\ \alpha}\right]^2 \cdot \sigma_\alpha^2$$

$$+ \left[\frac{\partial\ alt}{\partial\ \beta}\right]^2 \cdot \sigma_\beta^2 + \left[\frac{\partial\ alt}{\partial\ \gamma}\right]^2 \cdot \sigma_\gamma^2 + \left[\frac{\partial\ alt}{\partial\ a_o}\right]^2 \cdot \sigma_{a_o}^2 + \left[\frac{\partial\ alt}{\partial\ b_o}\right]^2 \cdot \sigma_{b_o}^2$$

$$+ \left[\frac{\partial\ alt}{\partial\ c_o}\right]^2 \cdot \sigma_{c_o}^2 \qquad (4\text{-}1)$$

and is a function of aircraft location.

Similar expressions may be written for the variance of latitude errors and of longitude errors.

The derivatives involved were all expressed analytically and were included in the computer program as part of the system evaluation. For instance, the altitude at the j^{th} iteration is given by

$$h_j = \sqrt{(d_e - u_{A_j})^2 + v_{A_j}^2} - R_e \qquad (4\text{-}2)$$

and the partial derivative of h_j with respect to, say α, is

$$\frac{\partial h_j}{\partial \alpha} = \frac{1}{R_e + h_j}\left[(d_e - u_{A_j}) \cdot \left(\frac{\partial d_e}{\partial\ \alpha} - \frac{\partial u_{A_j}}{\partial\ \alpha}\right) + v_{A_j} \cdot \frac{\partial v_{A_j}}{\partial\ \alpha}\right] \qquad (4\text{-}3)$$

Clearly, each of the three partial derivatives in eq. (4-3) now have to be evaluated. From

$$d_e = \sqrt{x_e^2 + y_e^2 + z_e^2} \tag{4-4}$$

the partial derivative is

$$\frac{\partial d_e}{\partial \alpha} = \frac{1}{d_e} \cdot \left[x_e \cdot \frac{\partial x_e}{\partial \alpha} + y_e \cdot \frac{\partial y_e}{\partial \alpha} + z_e \cdot \frac{\partial z_e}{\partial \alpha} \right] \tag{4-5}$$

and the partials in (4-5) now have to be evaluated, etc. The nine derivatives in (4-1), as well as those involved in the variance of both latitude and longitude errors, were all evaluated in this manner and the corresponding expressions made a part of the program which evaluated altitude, latitude and longitude.

The following values were used in the system evaluation:

$\sigma_{\psi_x} = \sigma_{\psi_y} = 50$ µrad [Ref. 1]

$\sigma_{\Delta t} = 0.1$ µs [Ref. 2]

$\sigma_{\alpha} = \sigma_{\beta} = \sigma_{\gamma} = 1.0$ µrad [Ref. 1]

$\sigma_{a_o} = \sigma_{b_o} = \sigma_{c_o} = 20$ meters [Ref. 3]

It should be pointed out that not all possible error sources were considered in this study. The earth, for instance, was taken to be a sphere, and reflection was assumed specular. These factors should be considered in any further work on this subject.

5.0 RESULTS AND CONCLUSIONS

The system was evaluated by considering various values of the input parameters (ψ_x, ψ_y, Δt, α, β, γ, a_o, b_o, c_o) corresponding to four altitudes at each of the North Atlantic locations denoted in Fig. 8 by the letters A through H. These particular points were examined since the coverage area under consideration extends from 40^oN to 60^oN and from 10^oW to 50^oW.

For each set of input quantities, the corresponding altitude, latitude, and longitude were computed, as well as the standard deviation of their errors. The results of these computations are given in Table 4-1.

Figures 9 and 10 are based on the values given in Table 4-1. Figure 9 shows σ_{ALT} plotted against relative longitude (that is, longitude relative to that of the nominal satellite position), while Fig. 10 has σ_{LAT} and σ_{LONG} vs relative longitude.

An examination of Table 4-1 and Figures 9 and 10 yields the following conclusions regarding aircraft position determination using multipath:

a. Altitude determination errors increase with increasing aircraft altitude and with increasing latitude.

b. The greatest 1σ altitude error was on the order of 65 meters and occurs at 60^oN at an altitude of 19 km. This figure is within the limits of a barometric altimeter (Ref. 2, 4).

c. Latitude errors and longitude errors increase with increasing latitude and with increasing relative longitude.

d. Position fixes can be made with 1σ accuracies of 1.5-3 nmi in latitude and 1-1.5 nmi in longitude in the North Atlantic coverage area.

A few words should be said regarding the convergence of the iterative procedure. A method is described in Appendix C in which Δt, h, u_{AE}, and v_{AE} are determined geometrically with θ and α_1 known (see Fig. 4). This computed value of Δt is used in the iterative procedure with the same θ. The values of h, u_{AE}, and v_{AE} computed here agree (to at least 14 places) with those found geometrically for each North Atlantic position under consideration (see Fig. 8 and Table 4-1). For all extent and purposes, then, the procedure could be said to converge in the area of interest.

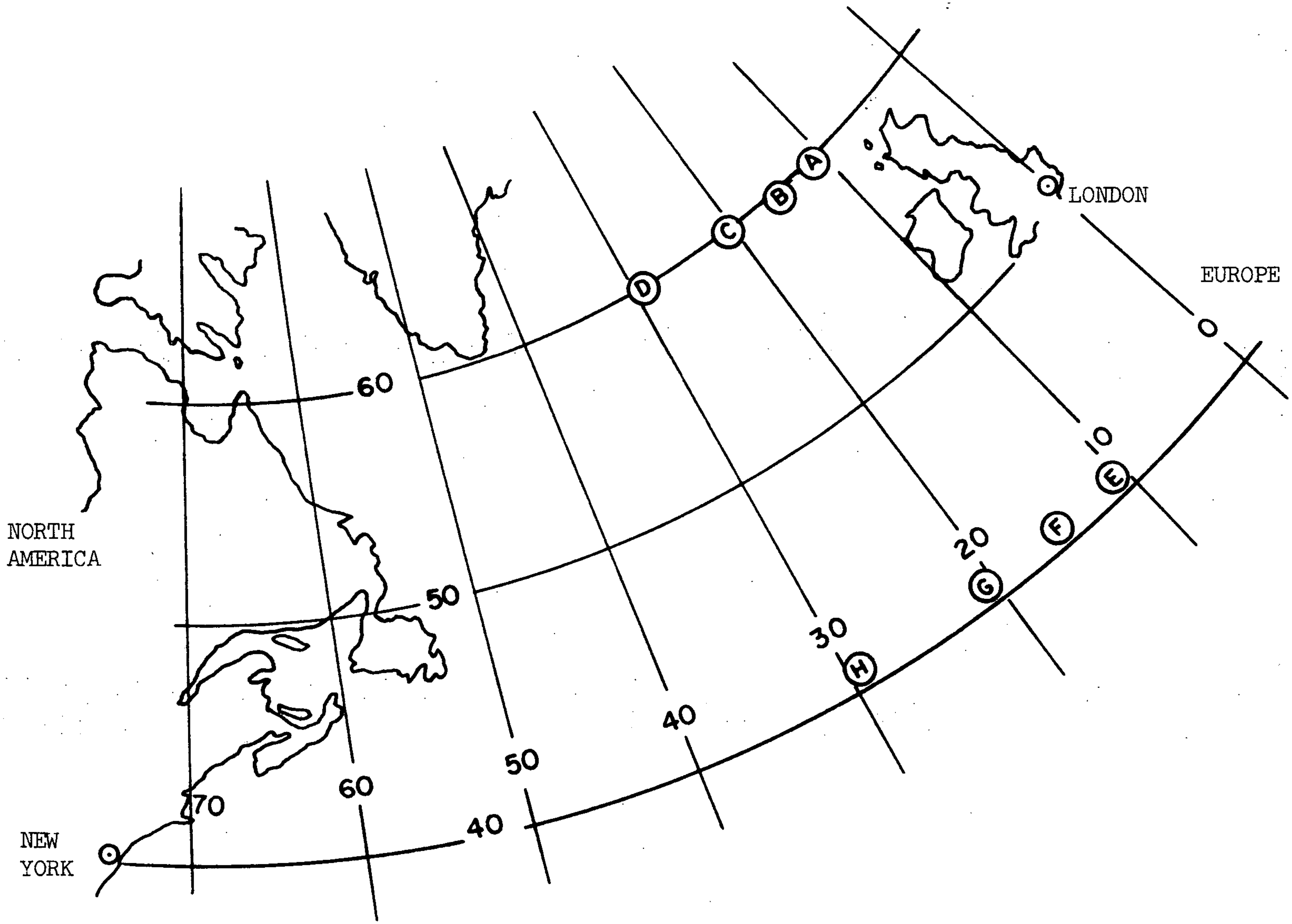

Fig. 8 Area of Interest in North Atlantic Showing Locations Considered in System Evaluation

Table 4-1

1-Sigma Errors

	Altitude (km)	σ_{ALT} (m)	σ_{LAT} (nmi)	σ_{LONG} (nmi)
A	18.967 14.440 10.686 7.734	65 54 49 47	2.83	1.50
B	18.903 14.484 10.738 7.473	61 52 47 45	2.82	1.31
C	18.653 14.377 10.840 7.665	56 50 46 43	2.78	1.18
D	18.852 14.233 10.729 7.508	55 49 45 41	2.78	1.09
E	18.929 14.222 10.681 7.130	26 25 25 24	1.51	1.18
F	18.912 14.377 10.966 7.548	25 24 24 23	1.49	1.11
G	18.856 14.439 10.674 7.346	24 23 23 23	1.48	1.06
H	18.918 14.153 10.897 7.635	24 23 22 22	1.47	1.03

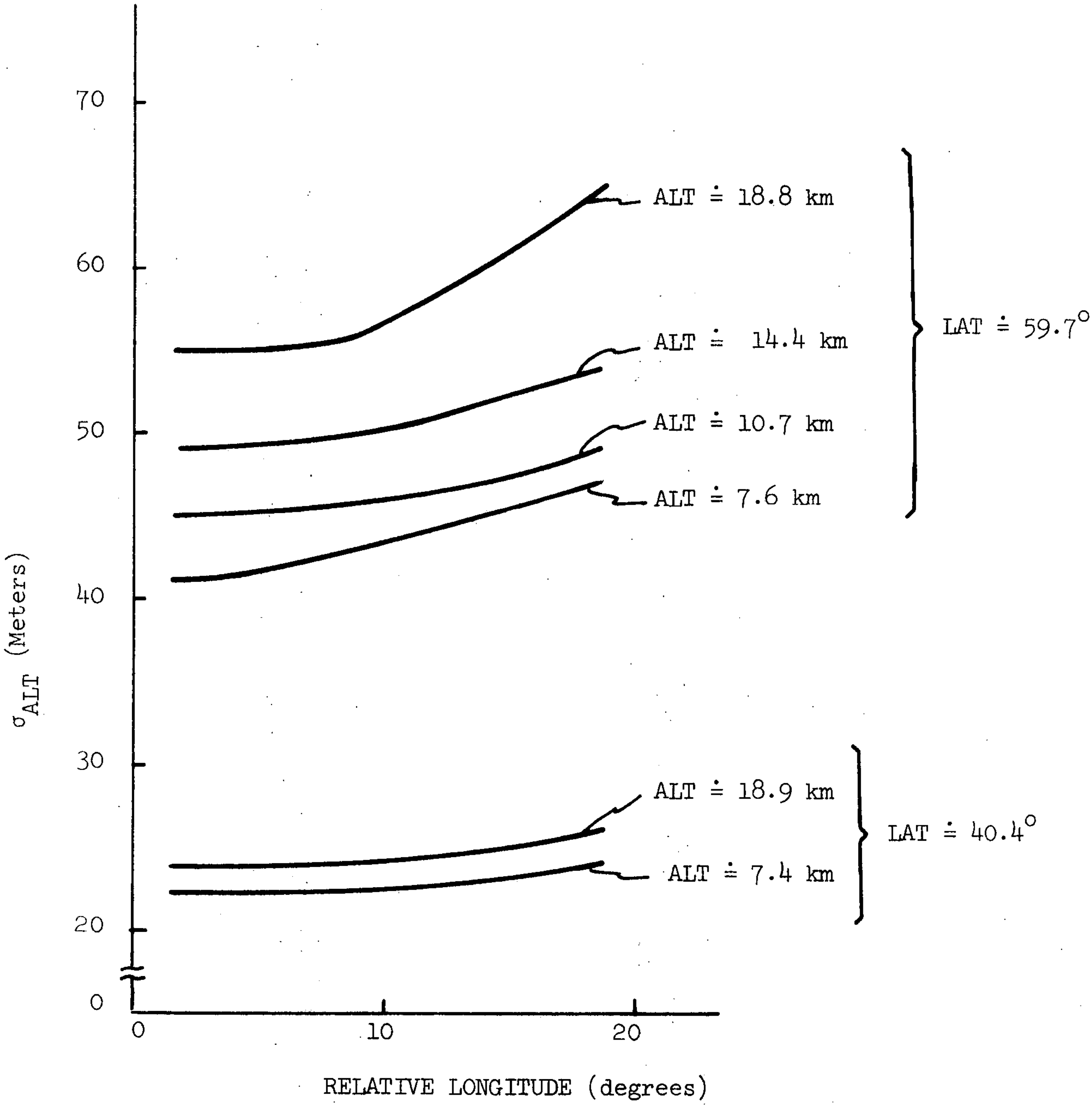

Fig. 9 1-Sigma Altitude Errors vs Relative Longitude

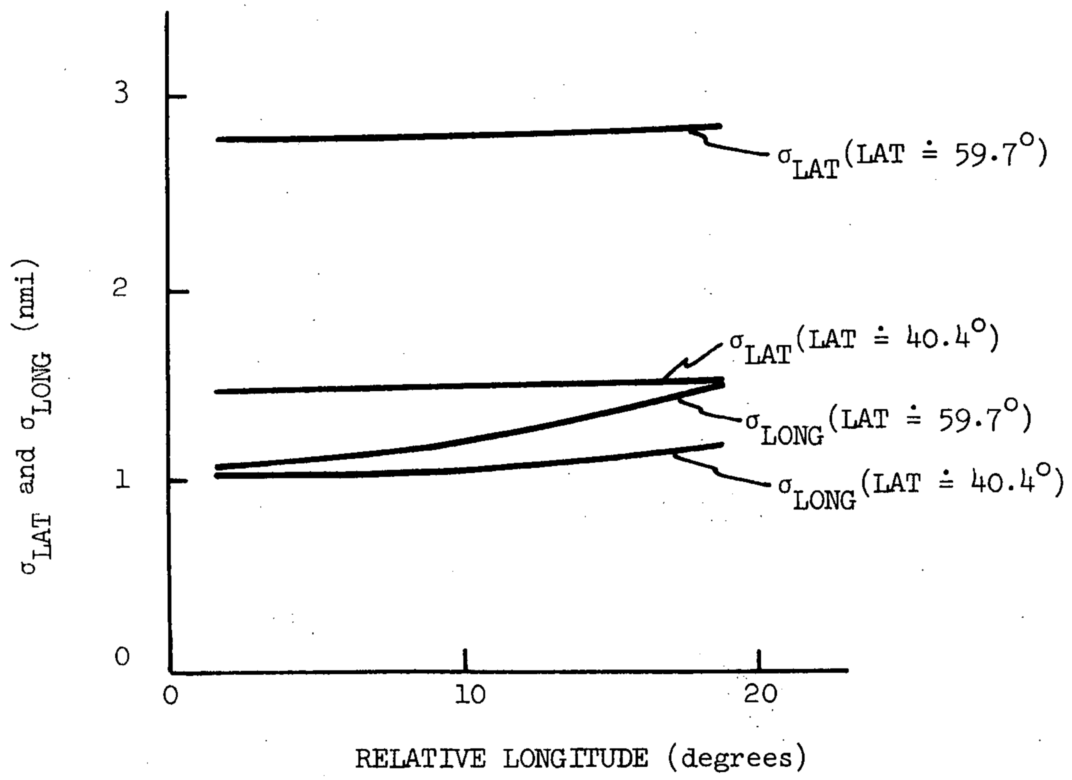

Fig. 10 1-Sigma Latitude and Longitude Errors vs Relative Longitude

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Appendix A

DERIVATION OF EQUATIONS

1. <u>Intersection of 2 Cones (Satellite-Aircraft Line)</u>

$$y^2 + z^2 - (\tan^2 \psi_x)\, x^2 = 0 \tag{A-1}$$

$$x^2 + z^2 - (\tan^2 \psi_y)\, y^2 = 0 \tag{A-2}$$

Solving (A-1) and (A-2) for z^2 and equating,

$$z^2 = (\tan^2 \psi_x)\, x^2 - y^2 = (\tan^2 \psi_y) y^2 - x^2$$

$$x^2(1 + \tan^2 \psi_x) = y^2(1 + \tan^2 \psi_y)$$

$$\frac{x^2}{\cos^2 \psi_x} = \frac{y^2}{\cos^2 \psi_y}$$

$$\pm \frac{x}{\cos \psi_x} = \frac{y}{\cos \psi_y} \tag{A-3}$$

From (A-2), $x^2 = (\tan^2 \psi_y) y^2 - z^2$

Then (A-1) becomes

$$y^2 + z^2 - (\tan^2 \psi_x)\left[(\tan^2 \psi_y) y^2 - z^2\right] = 0$$

$$y^2 + z^2 - (\tan^2 \psi_x)(\tan^2 \psi_y)\, y^2 + (\tan^2 \psi_x) z^2 = 0$$

$$y^2(1 - \tan^2 \psi_x \cdot \tan^2 \psi_y) + z^2(1 + \tan^2 \psi_x) = 0$$

$$y^2 = -\frac{1 + \tan^2 \psi_x}{1 - \tan^2 \psi_x \cdot \tan^2 \psi_y} z^2$$

$$= -\frac{1}{\cos^2\psi_x} \frac{z^2}{\left(1 - \dfrac{\sin^2\psi_x \sin^2\psi_y}{\cos^2\psi_x \cos^2\psi_y}\right)}$$

$$= -\frac{\cos^2 \psi_y}{\cos^2 \psi_x \cos^2 \psi_y - \sin^2 \psi_x \sin^2 \psi_y} z^2$$

$$\frac{y^2}{\cos^2\psi_y} = \frac{1}{\sin^2 \psi_x \sin^2 \psi_y - \cos^2 \psi_x \cos^2 \psi_y} z^2$$

$$= \frac{1}{(1 - \cos^2\psi_x)(1 - \cos^2\psi_y) - \cos^2\psi_x \cos^2\psi_y} z^2$$

$$\frac{y^2}{\cos^2\psi_y} = \frac{z^2}{1 - \cos^2 \psi_x - \cos^2 \psi_y}$$

$$\frac{y}{\cos \psi_y} = \pm \frac{z}{\sqrt{1 - \cos^2 \psi_x - \cos^2 \psi_y}} \tag{A-4}$$

$$\therefore \pm \frac{x}{\cos \psi_x} = \frac{y}{\cos \psi_y} = \pm \frac{z}{\sqrt{1 - \cos^2\psi_x - \cos^2\psi_y}} \tag{A-5}$$

Let

$$\left.\begin{aligned} a &= \pm \cos \psi_x \\ b &= \cos \psi_y \\ c &= \pm \sqrt{1 - \cos^2 \psi_x - \cos^2 \psi_y} \end{aligned}\right\} \tag{A-6}$$

(Note: $a^2 + b^2 + c^2 = 1$)

Then $$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \tag{A-7}$$

is the set of equations for the satellite-aircraft line.

2. Satellite-Earth Center Line

The equation of the line between (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2} \tag{A-8}$$

Here, satellite: (0, 0, 0)

earth center : (x_e, y_e, z_e)

$$\therefore \frac{x - 0}{0 - x_e} = \frac{y - 0}{0 - y_e} = \frac{z - 0}{0 - z_e}$$

or $$\frac{x}{x_e} = \frac{y}{y_e} = \frac{z}{z_e} \tag{A-9}$$

This, then, is the set of equations for the satellite-earth center line.

3. Satellite-Earth Center Distance

The equation for the distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d_e = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \tag{A-10}$$

$$\therefore d_e = \sqrt{x_e^2 + y_e^2 + z_e^2} \tag{A-11}$$

4. Intersection of Satellite-Aircraft Line with Earth

Equation of earth:

$$(x - x_e)^2 + (y - y_e)^2 + (z - z_e)^2 = R_e^2 \tag{A-12}$$

$$\text{Sat.-AC line:} \quad \frac{x}{a} = \frac{y}{b} = \frac{z}{c} \tag{A-7}$$

Substituting (A-7) into (A-12)

$$(x - x_e)^2 + \left(\frac{b}{a} x - y_e \right)^2 + \left(\frac{c}{a} x - z_e \right)^2 = R_e^2$$

or $$\frac{1}{a^2} (a^2 + b^2 + c^2)\, x^2 - \frac{2}{a} (ax_e + by_e + cz_e)x + (d_e^2 - R_e^2) = 0$$

$$\frac{1}{a^2}\, x^2 - \frac{2}{a} (ax_e + by_e + cz_e)x + (d_e^2 - R_e^2) = 0$$

Solving and denoting the root by x_i,

$$x_i = a \cdot \left\{ (ax_e + by_e + cz_e) - \sqrt{(ax_e + by_e + cz_e)^2 - (d_e^2 - R_e^2)} \right\}$$

Similarly,

$$y_i = b \cdot \left\{ (ax_e + by_e + cz_e) - \sqrt{(ax_e + by_e + cz_e)^2 - (d_e^2 - R_e^2)} \right\} \tag{A-13}$$

$$z_i = c \cdot \left\{ (ax_e + by_e + cz_e) - \sqrt{(ax_e + by_e + cz_e)^2 - (d_e^2 - R_e^2)} \right\}$$

Let
$$\begin{cases} S = ax_e + by_e + cz_e \\ T = S - \sqrt{S^2 - (d_e^2 - R_e^2)} \end{cases} \tag{A-14}$$

The point (x_i, y_i, z_i) is the intersection (on the satellite side of the earth) of the satellite-aircraft line with the earth.

5. Angle between Satellite-AC Line and Satellite-Earth Center Line

The angle, θ, between the line, $\frac{x}{a_1} = \frac{y}{b_1} = \frac{z}{c_1}$, and the line, $\frac{x}{a_2} = \frac{y}{b_2} = \frac{z}{c_2}$, is given by

$$\theta = \cos^{-1} \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}} \tag{A-15}$$

$$\therefore\ \theta = \cos^{-1} \frac{ax_e + by_e + cz_e}{\sqrt{a^2 + b^2 + c^2}\ \sqrt{x_e^2 + y_e^2 + z_e^2}} \tag{A-16}$$

or, from (A-6), (A-11) and (A-14),

$$\theta = \cos^{-1} \frac{S}{d_e} \tag{A-17}$$

6. Latitude, Longitude

From Fig. A-1

$$\sin \alpha_{lat} = \frac{y_{o_{AE}}}{R_e}$$

$$\therefore\ \alpha_{lat} = \sin^{-1} \frac{y_{o_{AE}}}{R_e} \tag{A-18}$$

Also

$$\cos\left(\Delta\alpha_{long}\right) = \frac{\cos \alpha}{\cos(\alpha_{lat})} \tag{A-19}$$

and

$$\cos \alpha = \frac{R_S + z_{o_{AE}}}{R_e} \tag{A-20}$$

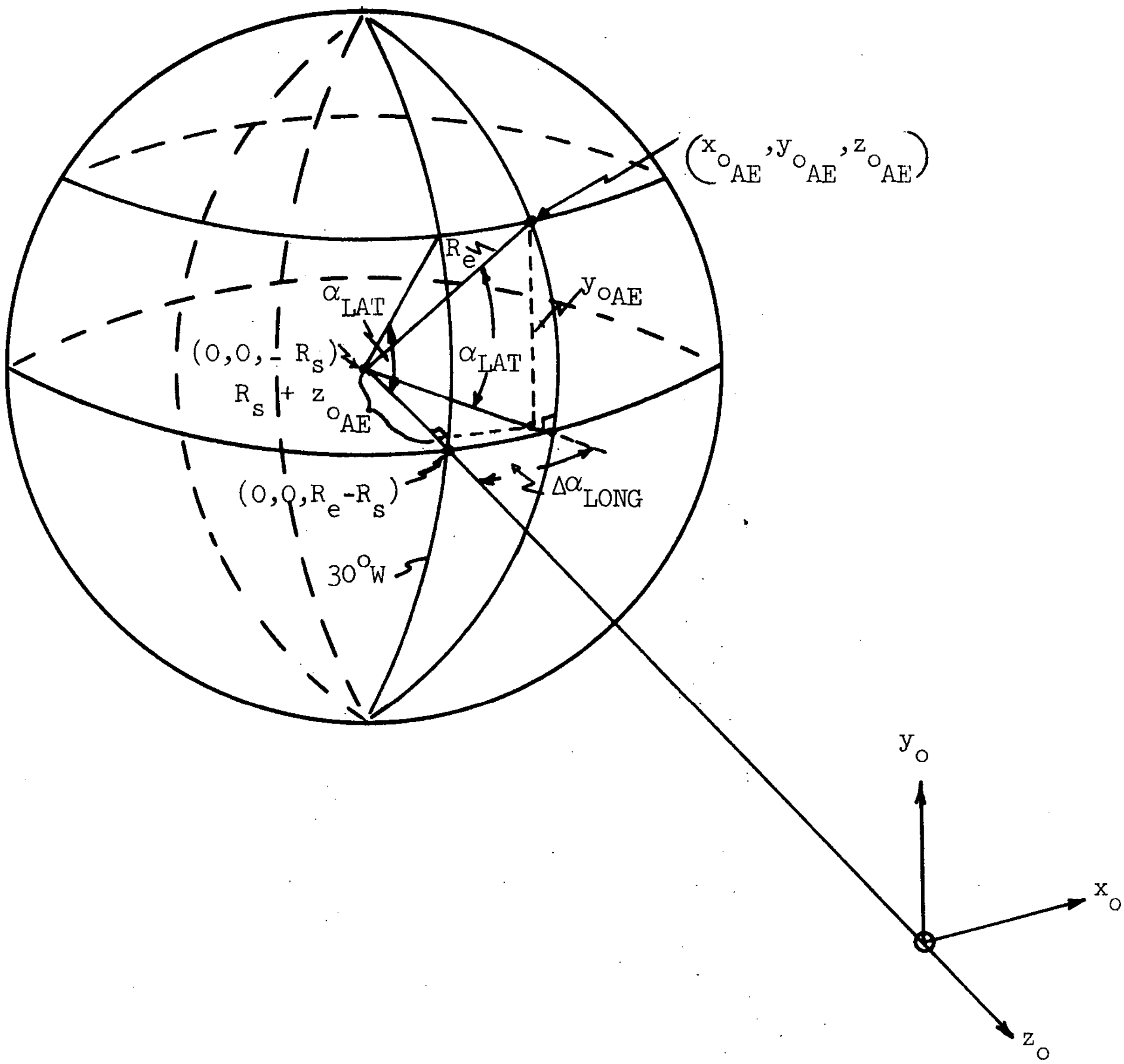


Fig. A-1 Geometry Relating Latitude and Longitude to Sub-Satellite Point

$$\therefore \quad \Delta\alpha_{long} = \cos^{-1} \frac{R_S + z_{o_{AE}}}{R_e \cos(\alpha_{lat})} \tag{A-21}$$

The nominal position of the satellite is at 30°W;

thus

$$\alpha_{long} = 30^{\circ}W \mp \Delta\alpha_{long}, \; x_{o_{AE}} \gtrless 0$$

or

$$\alpha_{long} = 30^{\circ} \mp \cos^{-1} \frac{R_S + z_{o_{AE}}}{R_e \cos(\alpha_{lat})}, \; x_{o_{AE}} \gtrless 0 \tag{A-22}$$

7. Iterative Procedure

Referring to Fig. A-2, the equation of the earth is given by

$$(u - d_e)^2 + v^2 = R_e^2 \tag{A-23}$$

and the equation of the satellite-aircraft line is

$$v = u \cdot \tan\theta \tag{A-24}$$

(a) Intersection of satellite-aircraft line with earth

Substituting eq. (A-24) into eq. (A-23) yields

$$(u - d_e)^2 + u^2 \cdot \tan^2\theta = R_e^2$$

$$u^2(1 + \tan^2\theta) - 2d_e u + (d_e^2 - R_e^2) = 0$$

$$u_{i_1} = \frac{2d_e - \sqrt{4d_e^2 - 4(1 + \tan^2\theta)(d_e^2 - R_e^2)}}{2(1 + \tan^2\theta)}$$

$$\left.\begin{aligned} \therefore \quad u_{i_1} &= \left[d_e - \sqrt{R_e^2 - \tan^2\theta(d_e^2 - R_e^2)} \right] \cdot \cos^2\theta \\ v_{i_1} &= u_{i_1} \cdot \tan\theta \end{aligned}\right\} \tag{A-25}$$

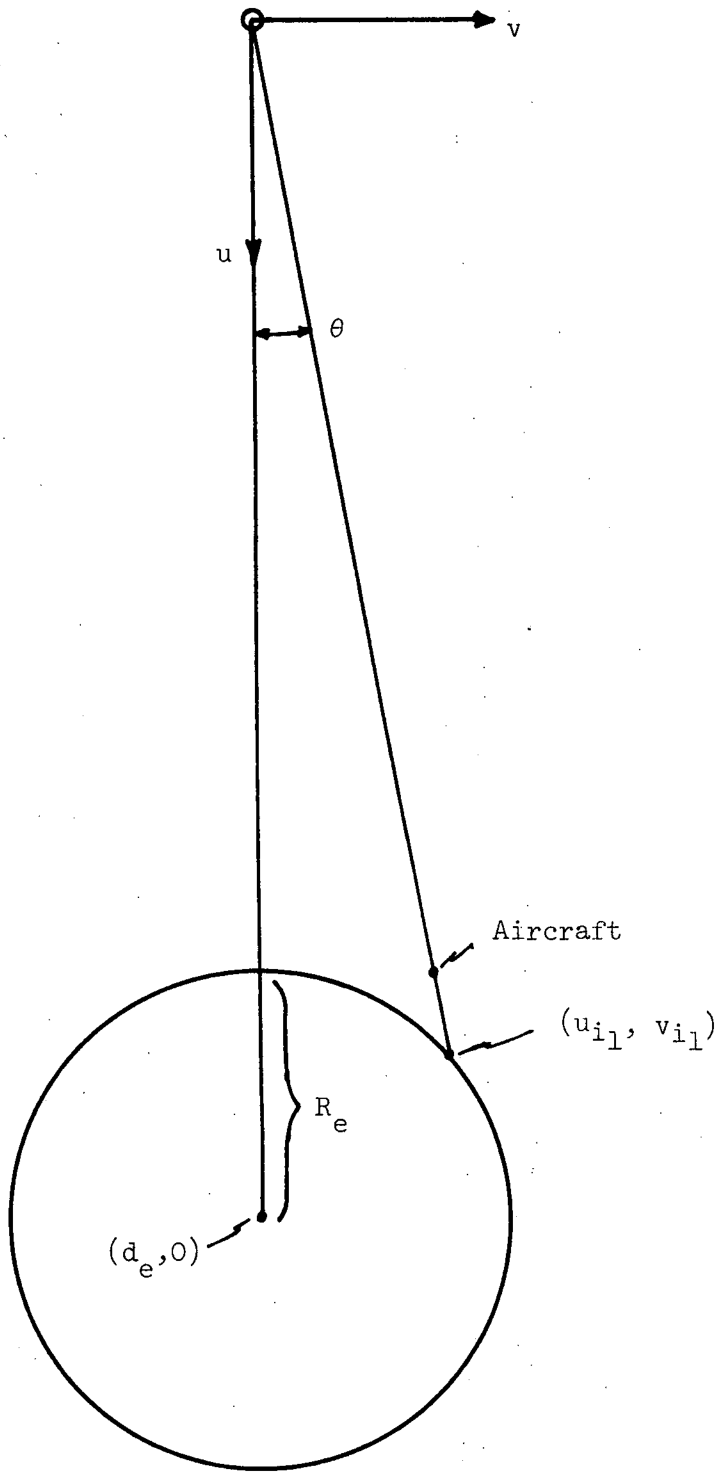


Fig. A-2 Intersection of Satellite-Aircraft Line with Earth

(The remainder of the derivations will use the general subscript, j.)

(b) Line tangent to earth at $\left(u_{i_j}, v_{i_j}\right)$

$$S_j = \text{slope of curve (earth) at} \left(u_{i_j}, v_{i_j}\right)$$

$$= \frac{d_e - u_{i_j}}{v_{i_j}} \tag{A-26}$$

as seen in Fig. A-3.

The equation for the line tangent to the earth at that point is given by

$$v - v_{i_j} = S_j(u - u_{i_j}) \tag{A-27}$$

or

$$v = \frac{d_e - u_{i_j}}{v_{i_j}} \cdot u - \frac{d_e u_{i_j} - \left(u_{i_j}^{\ 2} + v_{i_j}^{\ 2}\right)}{v_{i_j}} \tag{A-28}$$

(c) Coordinates of j^{th} satellite-image point

From Fig. A-4 it is seen that the slope of the satellite-j^{th} satellite image line is equal to the slope of the j^{th} tangent point-earth center line and has the value

$$-\frac{v_{i_j}}{d_e - u_{i_j}}$$

Therefore the equation of the satellite-j^{th} satellite image line is

$$v = -\frac{v_{i_j}}{d_e - u_{i_j}} \cdot u \tag{A-29}$$

The intersection of the satellite-j^{th} satellite image line and the j^{th} tangent line occurs at (u_{c_j}, v_{c_j}). To find this point the equations

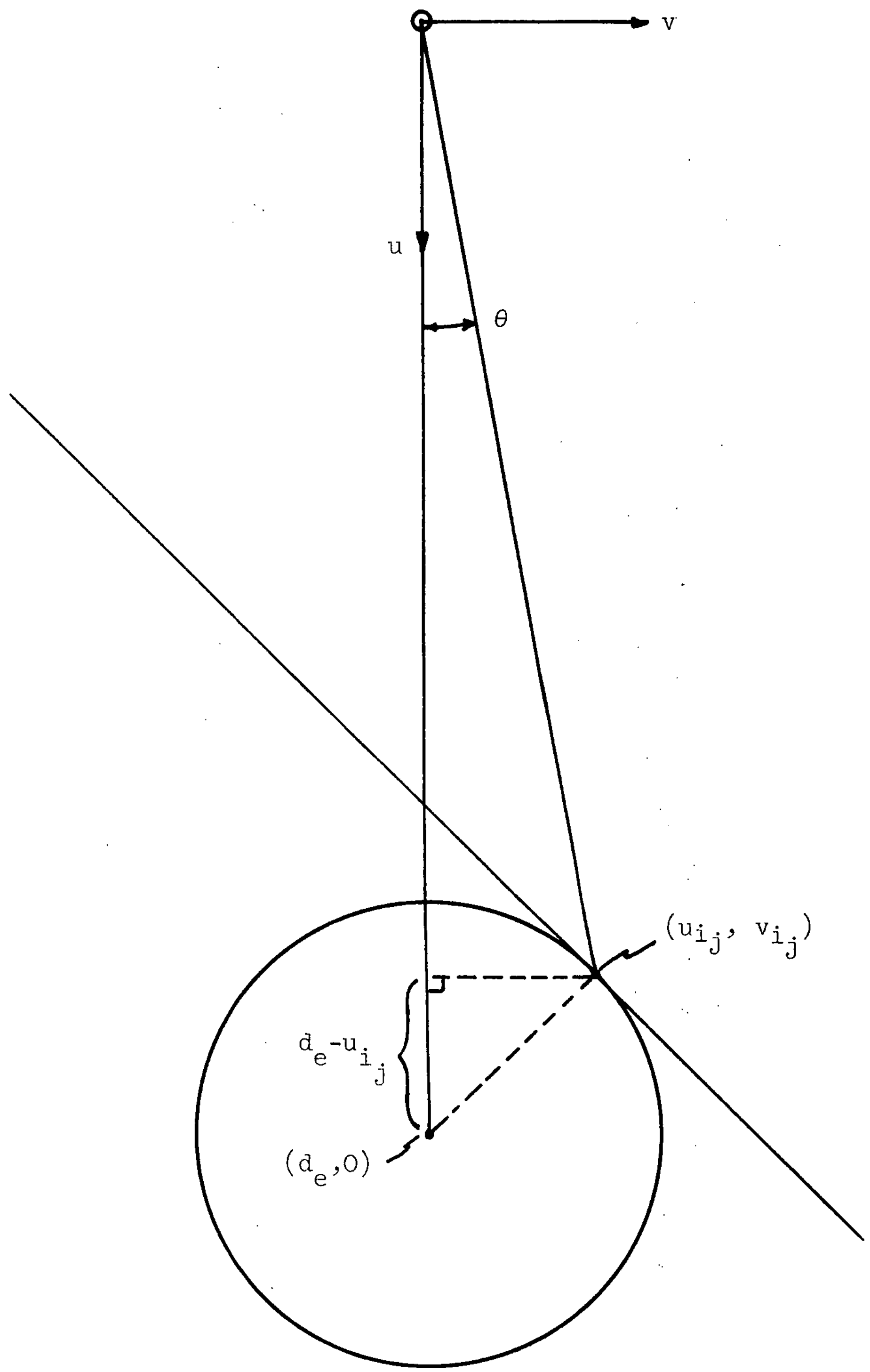


Fig. A-3 Geometry of 1st Tangent Line

for the two lines (eqs. A-28 and A-29) are equated

$$\frac{d_e - u_{i_j}}{v_{i_j}} \cdot u - \frac{d_e u_{i_j} - \left(u_{i_j}^{\ 2} + v_{i_j}^{\ 2}\right)}{v_{i_j}} = - \frac{v_{i_j}}{d_e - u_{i_j}} \cdot u$$

Solving for u,

$$\left.\begin{aligned} u_{c_j} &= (d_e - u_{i_j}) \cdot I_j \\ \text{and} \qquad v_{c_j} &= - v_{i_j} \cdot I_j , \end{aligned}\right\} \tag{A-30}$$

where

$$I_j = \frac{d_e u_{i_j} - \left(u_{i_j}^{\ 2} + v_{i_j}^{\ 2}\right)}{(d_e - u_{i_j})^2 + v_{i_j}^{\ 2}} \tag{A-31}$$

Since the tangent line bisects the satellite-satellite image line,

$$\left.\begin{aligned} u_{I_j} &= 2u_{c_j} \\ \text{and} \qquad v_{I_j} &= 2v_{c_j} \end{aligned}\right\} \tag{A-32}$$

or

$$\left.\begin{aligned} u_{I_j} &= 2(d_e - u_{i_j})I_j \\ \text{and} \qquad v_{I_j} &= - 2v_{i_j} I_j \end{aligned}\right\} \tag{A-33}$$

(d) Coordinates of j^{th} estimate of aircraft location

The angle, γ_j, between the satellite-earth center line and the satellite-j^{th} satellite image line is given by

$$\gamma_j = \tan^{-1} \frac{v_{i_j}}{d_e - u_{i_j}} \tag{A-34}$$

as shown in Fig. A-4.

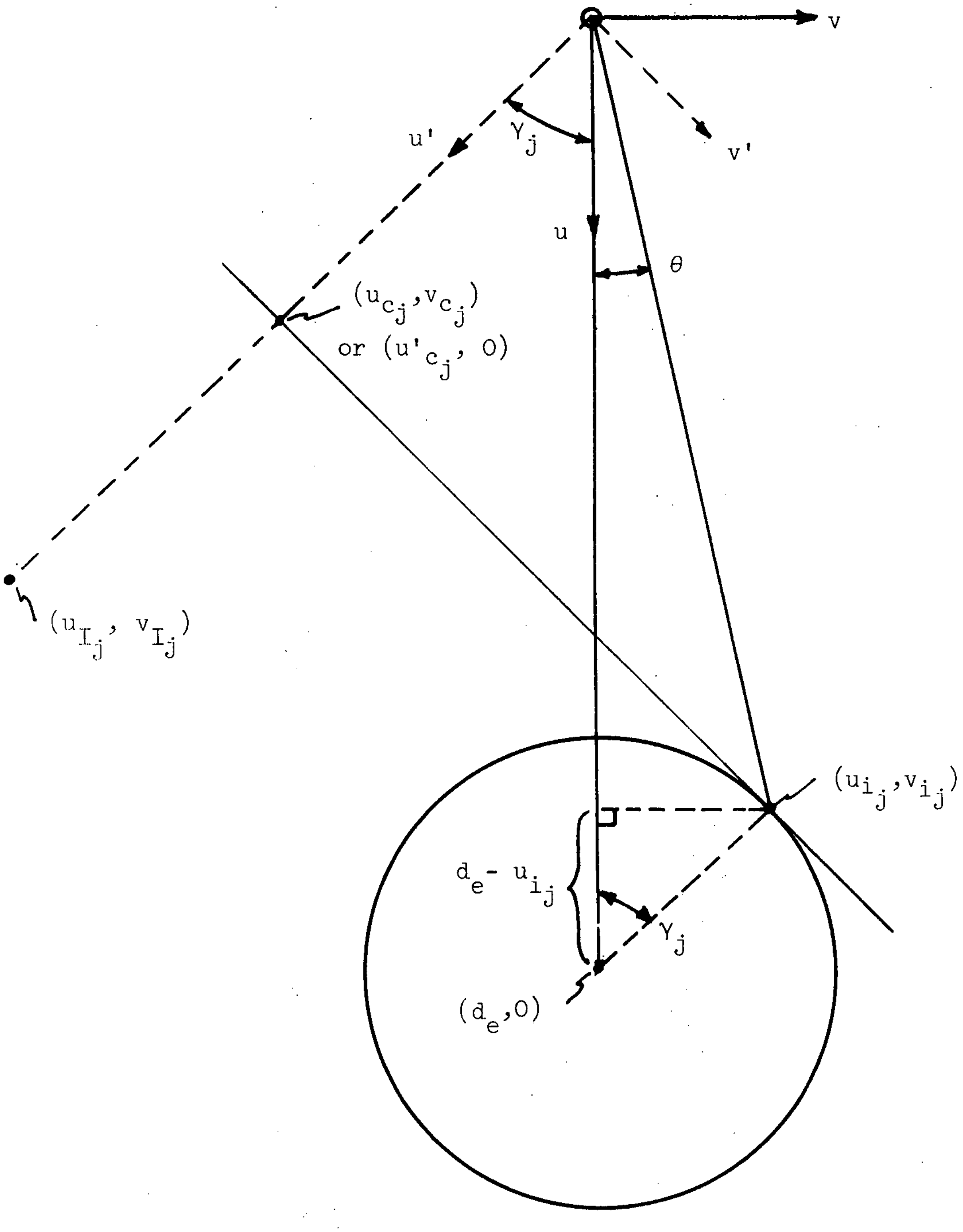


Fig. A-4 Geometry Relating Image Point to Tangent Line

Consider a new coordinate system, u'- v', as shown in Fig. A-5. The u'-axis is along the satellite-j^{th} satellite image line. The transformations from the u-v system to the u'-v' system, and vice versa, are given by:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} \cos \gamma_j & -\sin \gamma_j \\ \sin \gamma_j & \cos \gamma_j \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{A-35}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \gamma_j & \sin \gamma_j \\ -\sin \gamma_j & \cos \gamma_j \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Δt is the time delay between a signal received directly and a signal received by the aircraft after reflection. This is equivalent to a path length difference, $\Delta \ell$, where

$$\Delta \ell = c \cdot \Delta t \tag{A-36}$$

Consider a hyberbola whose principal axis (the axis which joints the two foci) is the u'-axis, whose foci are the satellite and the j^{th} satellite image, and whose center is $(u'_{cj}\ , 0)$. Now, the difference between the focal radii of any point on a hyperbola is a constant, and is equal to the length of the transverse axis (the portion of the principal axis included between the vertices). Consequently, this length will be set equal to the required path length difference, $\Delta \ell$, above.

The equation for this hyperbola is given by

$$\frac{(u' - u'_{c_j})^2}{Q_1^{\,2}} - \frac{(v')^2}{Q_2^{\,2}} = 1 \tag{A-37}$$

where $2Q_1 = \Delta \ell,$

$$\sqrt{Q_1^{\,2} + Q_2^{\,2}} = u'_{c_j}, \tag{A-38}$$

and $u'_{c_j} = u_{c_j} \cos \gamma_j - v_{c_j} \sin \gamma_j$ (A-35)

$$\therefore \quad Q_1 = \frac{\Delta \ell}{2}$$

and $$Q_2 = \sqrt{\left(u'_{c_j}\right)^2 - \left(\frac{\Delta \ell}{2}\right)^2} \tag{A-39}$$

Thus, the equation for the j^{th} hyperbola becomes

$$\frac{\left(u' - u'_{c_j}\right)^2}{\left(\frac{\Delta \ell}{2}\right)^2} - \frac{(v')^2}{\left(u'_{c_j}\right)^2 - \left(\frac{\Delta \ell}{2}\right)^2} = 1. \tag{A-40}$$

The satellite-aircraft line is given in the u'- v' system by

$$v' = \tan(\theta + \gamma_j) \cdot u' \tag{A-41}$$

The intersection of the satellite-AC line with the j^{th} hyperbola is the j^{th} estimate of aircraft location, shown in Fig. A-5:

$$\left.\begin{aligned} u'_{A_j} &= \frac{(u'_{c_j})^2 - \left(\frac{c}{2}\Delta t\right)}{u'_{c_j} \cos(\theta + \gamma_j) + \frac{c}{2}\Delta t} \cdot \cos(\theta + \gamma_j) \\ v'_{A_j} &= u'_{A_j} \cdot \tan(\theta + \gamma_j); \end{aligned}\right\} \tag{A-42}$$

or

$$\left.\begin{aligned} u_{A_j} &= \frac{\left(u'_{c_j}\right)^2 - \left(\frac{c}{2}\Delta t\right)^2}{u'_{c_j} \cos(\theta + \gamma_j) + \frac{c}{2}\Delta t} \cdot \cos\theta \\ v_{A_j} &= u_{A_j} \cdot \tan\theta \end{aligned}\right\} \tag{A-43}$$

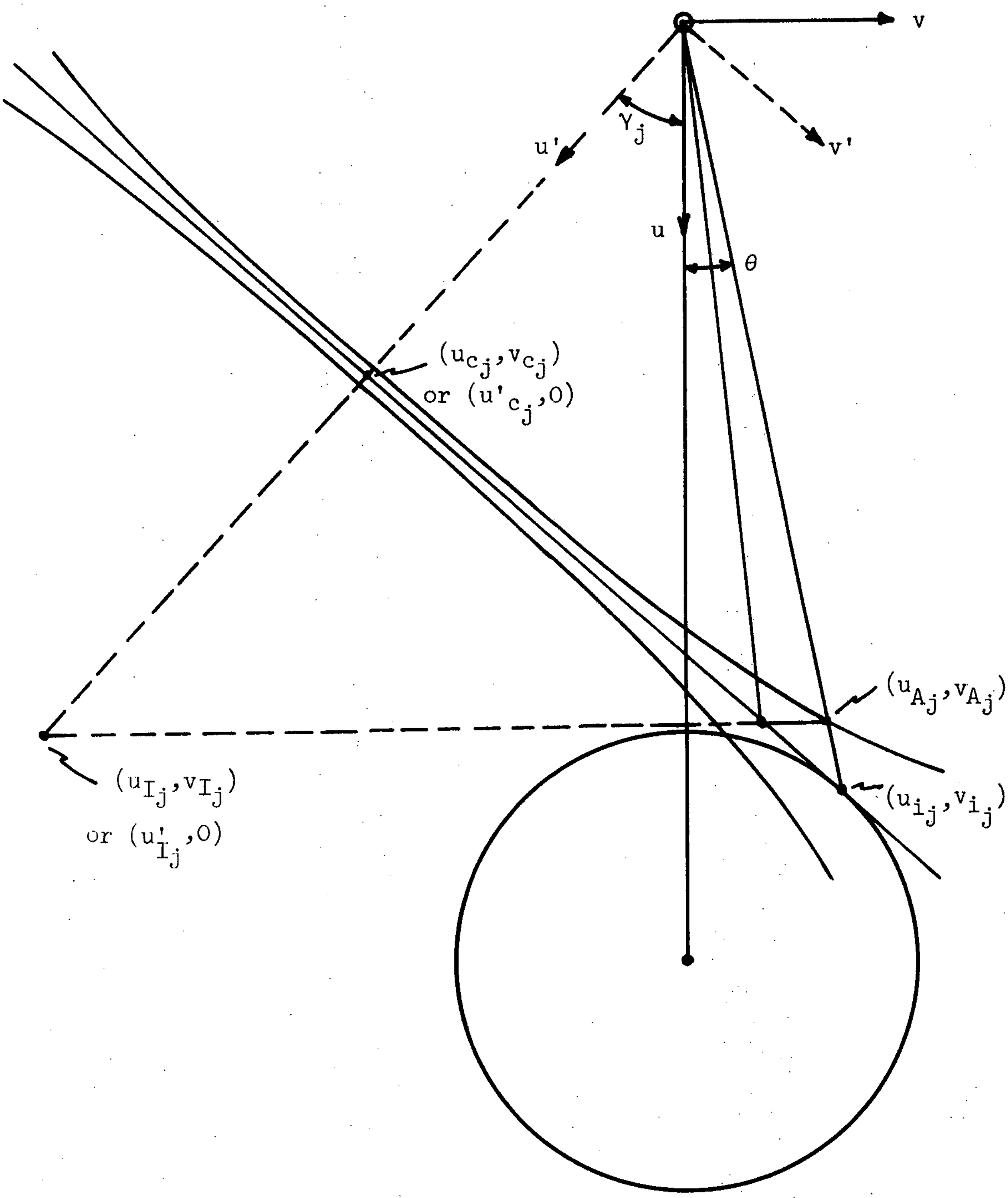


Fig. A-5 Geometry of Hyperbola

The line between the j^{th} satellite image point and the j^{th} aircraft location estimate is given by

$$\frac{u - u_{A_j}}{u_{A_j} - u_{I_j}} = \frac{v - v_{A_j}}{v_{A_j} - v_{I_j}}$$

or

$$v = \frac{v_{A_j} - v_{I_j}}{u_{A_j} - u_{I_j}} u - \frac{u_{I_j} v_{A_j} - u_{A_j} v_{I_j}}{u_{A_j} - u_{I_j}} \tag{A-44}$$

Let

$$m_j = \frac{v_{A_j} - v_{I_j}}{u_{A_j} - u_{I_j}}$$

and

$$b_j = \frac{u_{I_j} v_{A_j} - u_{A_j} v_{I_j}}{u_{A_j} - u_{I_j}} \tag{A-45}$$

Then,

$$v = m_j u - b_j \tag{A-46}$$

The intersection of this line with the j^{th} earth tangent line (eq. A-28) is the point (u_{t_j}, v_{t_j}):

$$\left.\begin{aligned} u_{t_j} &= \frac{d_e u_{i_j} - b_j v_{i_j} - \left(u_{i_j}^{2} + v_{i_j}^{2}\right)}{d_e - u_{i_j} - m_j v_{i_j}} \\ v_{t_j} &= m_j u_{t_j} - b_j \end{aligned}\right\} \tag{A-47}$$

Let

$$W_j = u_{t_j}^{2} + v_{t_j}^{2} \tag{A-48}$$

The line from the satellite through (u_{t_j}, v_{t_j}) is given by

$$v = \frac{v_{t_j}}{u_{t_j}} u \tag{A-49}$$

The intersection of this line with the earth (see Fig. A-6) is the point $\left(u_{i_{j+1}}, v_{i_{j+1}}\right)$:

$$\left.\begin{aligned} u_{i_{j+1}} &= \frac{d_e u_{t_j} - \sqrt{R_e^2 W_j - d_e^2 v_{t_j}^2}}{W_j} \cdot u_{t_j} \\ v_{i_{j+1}} &= \frac{v_{t_j}}{u_{t_j}} u_{i_{j+1}} \end{aligned}\right\} \qquad \text{(A-50)}$$

A new line is considered which is tangent to the earth at $\left(u_{i_{j+1}}, v_{i_{j+1}}\right)$ and the process is repeated, resulting in a $(j+1)^{th}$ (and better) estimate of aircraft location. When enough iterations have been taken, the aircraft's altitude and its location on the earth (subaircraft point) may be determined.

The distance between the aircraft location estimate $\left(u_{A_j}, v_{A_j}\right)$ and the earth center $(d_e, 0)$ is $\sqrt{\left(d_e - u_{A_j}\right)^2 + v_{A_j}^2}$. Consequently, the aircraft's altitude at the j^{th} iteration is

$$h_j = \sqrt{\left(d_e - u_{A_j}\right)^2 + v_{A_j}^2} - R_e \qquad \text{(A-51)}$$

The subaircraft point $\left(u_{AE_j}, v_{AE_j}\right)$ is the intersection of the aircraft location estimate-earth center line, given by

$$v = -\frac{v_{A_j}}{d_e - u_{A_j}} u + \frac{d_e v_{A_j}}{d_e - u_{A_j}} \qquad \text{(A-52)}$$

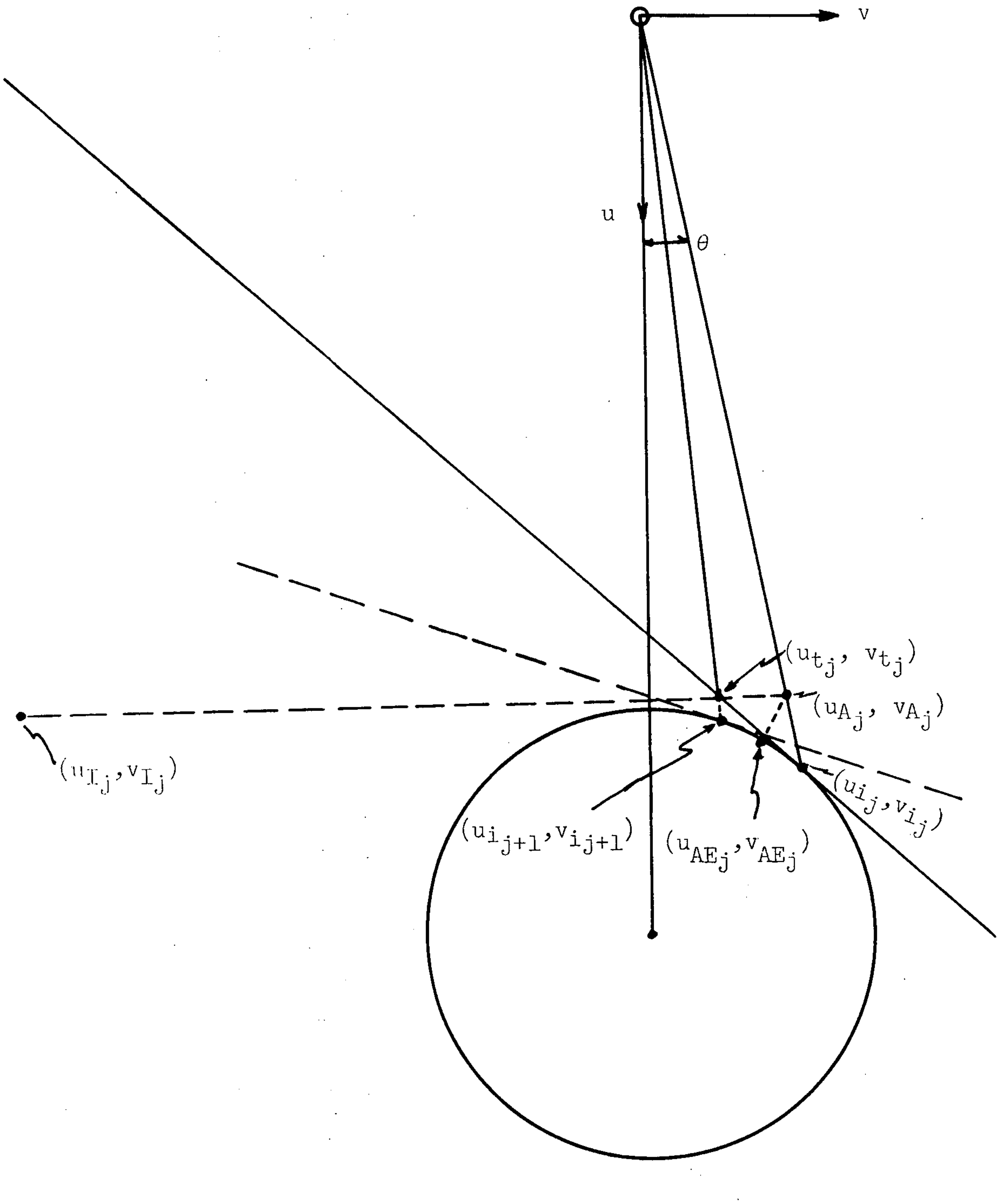


Fig. A-6 Geometry Showing Determination of Following Tangent Line and the earth.

$$\left.\begin{aligned} u_{AE_j} &= \frac{R_e u_{A_j} + d_e h_j}{R_e + h_j} \\ \\ v_{AE_j} &= \frac{R_e v_{A_j}}{R_e + h_j} \end{aligned}\right\} \qquad \text{(A-53)}$$

This point is converted back to the nominal coordinate system, the x_o- y_o- z_o system, by means of the coordinate transformation matrices, which are derived in Appendix B.

Appendix B

COORDINATE TRANSFORMATION MATRICES

1. $x_o - y_o - z_o / x - y - z$

The transformation from the nominal $x_o - y_o - z_o$ system to the body-fixed x-y-z system and vice versa are given by the following (see Fig. 1):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \nu_1 & \xi_1 & \eta_1 \\ \nu_2 & \xi_2 & \eta_2 \\ \nu_3 & \xi_3 & \eta_3 \end{pmatrix} \begin{pmatrix} x_o - a_o \\ y_o - b_o \\ z_o - c_o \end{pmatrix} \tag{B-1}$$

$$\begin{pmatrix} x_o \\ y_o \\ z_o \end{pmatrix} = \begin{pmatrix} \nu_1 & \nu_2 & \nu_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a_o \\ b_o \\ c_o \end{pmatrix} \tag{B-2}$$

where

$$\begin{aligned}
\nu_1 &= \cos\alpha \\
\nu_2 &= \sin\alpha \sin\gamma \\
\nu_3 &= \sin\alpha \cos\gamma \\
\xi_1 &= \sin\alpha \sin\beta \\
\xi_2 &= \cos\beta \cos\gamma - \cos\alpha \sin\beta \sin\gamma \\
\xi_3 &= -\cos\beta \sin\gamma - \cos\alpha \sin\beta \cos\gamma \\
\eta_1 &= -\sin\alpha \cos\beta \\
\eta_2 &= \sin\beta \cos\gamma + \cos\alpha \cos\beta \sin\gamma \\
\eta_3 &= -\sin\beta \sin\gamma + \cos\alpha \cos\beta \cos\gamma
\end{aligned} \tag{B-3}$$

α, β, and γ are the Euler angles involved.

2. x-y-z/u-v

The purpose of this transformation is to facilitate the use of the iterative altitude determining procedure by performing all geometric manipulations in one plane rather than in three dimensions. Thus, this transformation results in rotating the x-y-z system in such a way that two axes are in the satellite-aircraft-earth center plane and one axis is perpendicular to it.

The satellite-aircraft-earth center plane, P, is determined by those three points. Since the location of the aircraft is not known at this time, any point on the satellite-aircraft line will suffice. The equations for this line, derived in Appendix A, are

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

The point (a, b, c) is on the line. This point, together with the satellite, (0, 0, 0), and the earth center, (x_e, y_e, z_e), determine the equation for P in the following way:

$$\begin{vmatrix} x & y & z & 1 \\ 0 & 0 & 0 & 1 \\ x_e & y_e & z_e & 1 \\ a & b & c & 1 \end{vmatrix} = 0$$

or $$(cy_e - bz_e)\, x + (az_e - cx_e)y + (bx_e - ay_e)z = 0 \qquad \text{(B-4)}$$

The intersection of the plane, P, with the x-y plane is given by

$$x = m_1\, y, \qquad \text{(B-5)}$$

where $$m_1 = \frac{az_e - cx_e}{bz_e - cy_e} \quad . \qquad \text{(B-6)}$$

Let $$\xi = \tan^{-1} m, \qquad \text{(B-7)}$$

as shown in Fig. B-1.

The z-axis is rotated an amount ξ such that the new y-axis is in P. The new x-, y-, and z-axes are called the $\bar{x}$-, $\bar{y}$-, and $\bar{z}$-axes, respectively. The matrix D provides for the proper rotation:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = D \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where

$$D = \begin{pmatrix} \cos \xi & -\sin \xi & 0 \\ \sin \xi & \cos \xi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{B-8}$$

The intersection of P with the $\bar{x}$ - $\bar{z}$ plane is given by

$$\bar{x} = m_2 \bar{z}, \tag{B-9}$$

where

$$m_2 = \frac{ay_e - bx_e}{(cy_e - bz_e) \cos \xi - (az_e - cx_e) \sin \xi} \tag{B-10}$$

Let

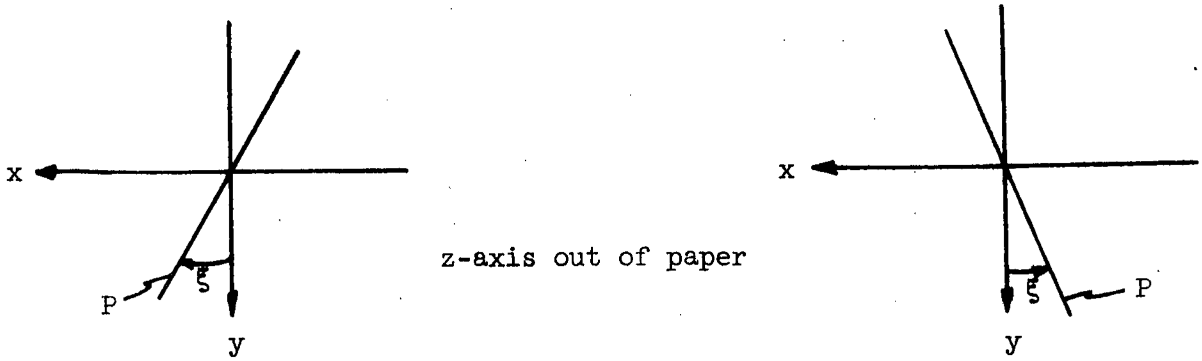
$$\eta = \tan^{-1} m_2, \tag{B-11}$$

as shown in Fig. B-2.

The $\bar{y}$-axis is rotated an amount η such that the new $\bar{z}$-axis is in P and the new $\bar{x}$-axis is perpendicular to P. The new $\bar{x}$-, $\bar{y}$-, and $\bar{z}$-axes are called the $\bar{\bar{x}}$-, $\bar{\bar{y}}$-, and $\bar{\bar{z}}$-axes, respectively. The matrix C provides for the proper rotation.

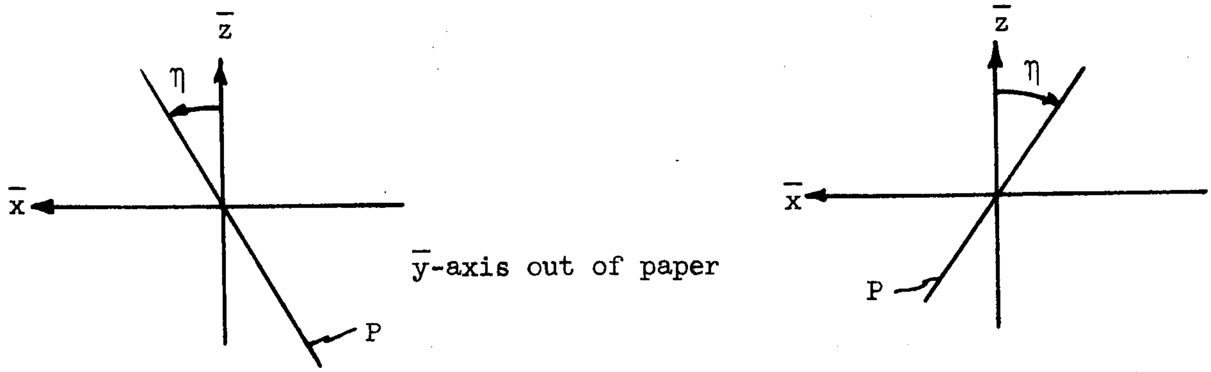
$$\begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \\ \bar{\bar{z}} \end{pmatrix} = C \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

where



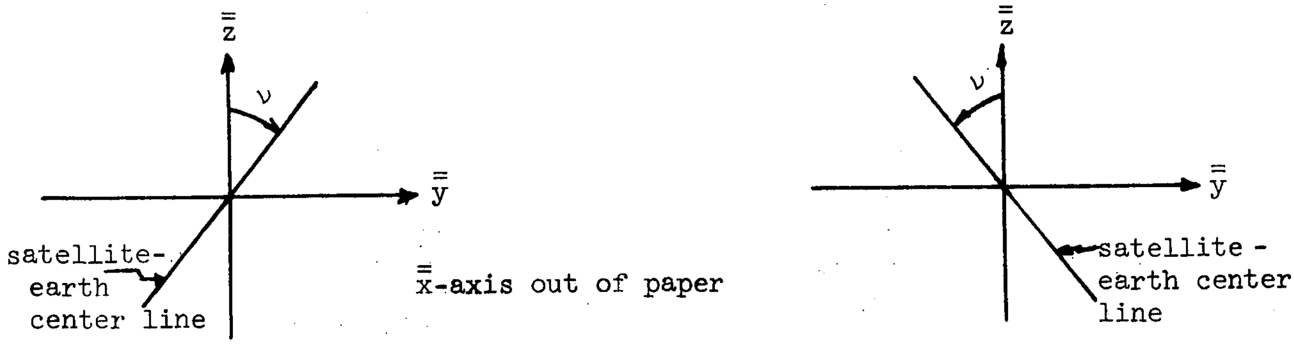
(a) $0 < m_1 < \infty$ (b) $-\infty < m_1 < 0$

Fig. B-1 Geometry Defining ξ



(a) $0 < m_2 < \infty$ (b) $-\infty < m_2 < 0$

Fig. B-2 Geometry Defining η



(a) $0 < m_3 < \infty$ (b) $-\infty < m_3 < 0$

Fig. B-3 Geometry Defining ν

$$C = \begin{pmatrix} \cos\eta & 0 & -\sin\eta \\ 0 & 1 & 0 \\ \sin\eta & 0 & \cos\eta \end{pmatrix} \tag{B-12}$$

The equation of the satellite-earth center line in the $\bar{\bar{x}}$-$\bar{\bar{y}}$-$\bar{\bar{z}}$ coordinate system is given by

$$\bar{\bar{y}} = m_3 \bar{\bar{z}}, \tag{B-13}$$

where $$m_3 = \frac{x_e \sin\xi + y_e \cos\xi}{x_e \cos\xi \sin\eta - y_e \sin\xi \sin\eta + z_e \cos\eta} \tag{B-14}$$

Let $$\nu = \tan^{-1} m_3 \tag{B-15}$$

as shown in Fig. B-3.

The $\bar{\bar{x}}$-axis is rotated an amount ν such that the new $\bar{\bar{z}}$-axis is along the satellite-earth center line, but with its positive direction pointed away from the earth. The new $\bar{\bar{x}}$-, $\bar{\bar{y}}$-, and $\bar{\bar{z}}$-axes are called the x'-, y'-, and z'-axes, respectively. The proper rotation is provided for in the matrix, B.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = B \begin{pmatrix} \bar{\bar{x}} \\ \bar{\bar{y}} \\ \bar{\bar{z}} \end{pmatrix}$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\nu & -\sin\nu \\ 0 & \sin\nu & \cos\nu \end{pmatrix} \tag{B-16}$$

Thus, the transformation from the x-y-z system to the x'-y'-z' system and vice versa is given by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{B-17}$$

where

$$A = B\ C\ D = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \tag{B-18}$$

and

$$\left.\begin{aligned}
r_{11} &= \cos \xi \cos \eta \\
r_{12} &= - \sin \xi \cos \eta \\
r_{13} &= - \sin \eta \\
r_{21} &= \sin \xi \cos \nu \sin \eta \sin \nu \\
r_{22} &= \cos \xi \cos \nu + \sin \xi \sin \eta \sin \nu \\
r_{23} &= - \cos \eta \sin \nu \\
r_{31} &= \sin \xi \sin \nu + \cos \xi \sin \eta \cos \nu \\
r_{32} &= \cos \xi \sin \nu - \sin \xi \sin \eta \cos \nu \\
r_{33} &= \cos \eta \cos \nu
\end{aligned}\right\} \tag{B-19}$$

The negative z'-axis is now called the u-axis (i.e., the u-axis coincides with the z'-axis but its positive direction is towards the earth), and the positive y'-axis is now called the v-axis, as shown in Fig. 4. In other words,

$$\left.\begin{aligned} u &= -z' \\ v &= y' \end{aligned}\right\} \tag{B-20}$$

Consequently, the transformation from the u-v system to the x-y-z system is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^T \begin{pmatrix} 0 \\ v \\ -u \end{pmatrix} \tag{B-21}$$

which reduces to

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r_{21} & r_{31} \\ r_{22} & r_{32} \\ r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} v \\ -u \end{pmatrix} \tag{B-22}$$

Appendix C

TEST OF ITERATIVE ALTITUDE DETERMINING TECHNIQUE

Consider Fig. C-1, which is the same as Fig. 4, except that it has additional angles marked. The values of Δt, u_{AE}, and v_{AE} will now be found, given θ, α_1, d_e, R_e and c (speed of light).

$$\begin{cases} d_1 \cos(\varphi + \alpha_1) = R_e \sin \alpha_1 & \text{(C-1)} \\ d_1 \sin(\varphi + \alpha_1) + R_e \cos \alpha_1 = d_e & \text{(C-2)} \end{cases}$$

Combining these two equations yields

$$\varphi = \tan^{-1} \frac{d_e \cos \alpha_1 - R_e}{d_e \sin \alpha_1} \tag{C-3}$$

From the law of sines,

$$\frac{d}{\sin(180^\circ - 2\varphi)} = \frac{d_1}{\sin(90^\circ - \theta + \varphi - \alpha_1)} \tag{C-4}$$

$$\frac{d_2}{\sin(\theta - 90^\circ + \varphi + \alpha_1)} = \frac{d_1}{\sin(90^\circ - \theta + \varphi - \alpha_1)} \tag{C-5}$$

Δt can now be found using (C-3), (C-4), and (C-5) from

$$\Delta t = \frac{d_1 + d_2 - d}{c}$$

$$= \frac{2}{c} \; \frac{(d_e \cos \alpha_1 - R_e)(\sin [\alpha_1 + \theta] - \cos \varphi)}{\cos(\alpha_1 + \theta - \varphi)} \tag{C-6}$$

From the law of sines,

$$\frac{d}{\sin \alpha} = \frac{d_e}{\sin(180^\circ - \theta - \alpha)} \tag{C-7}$$

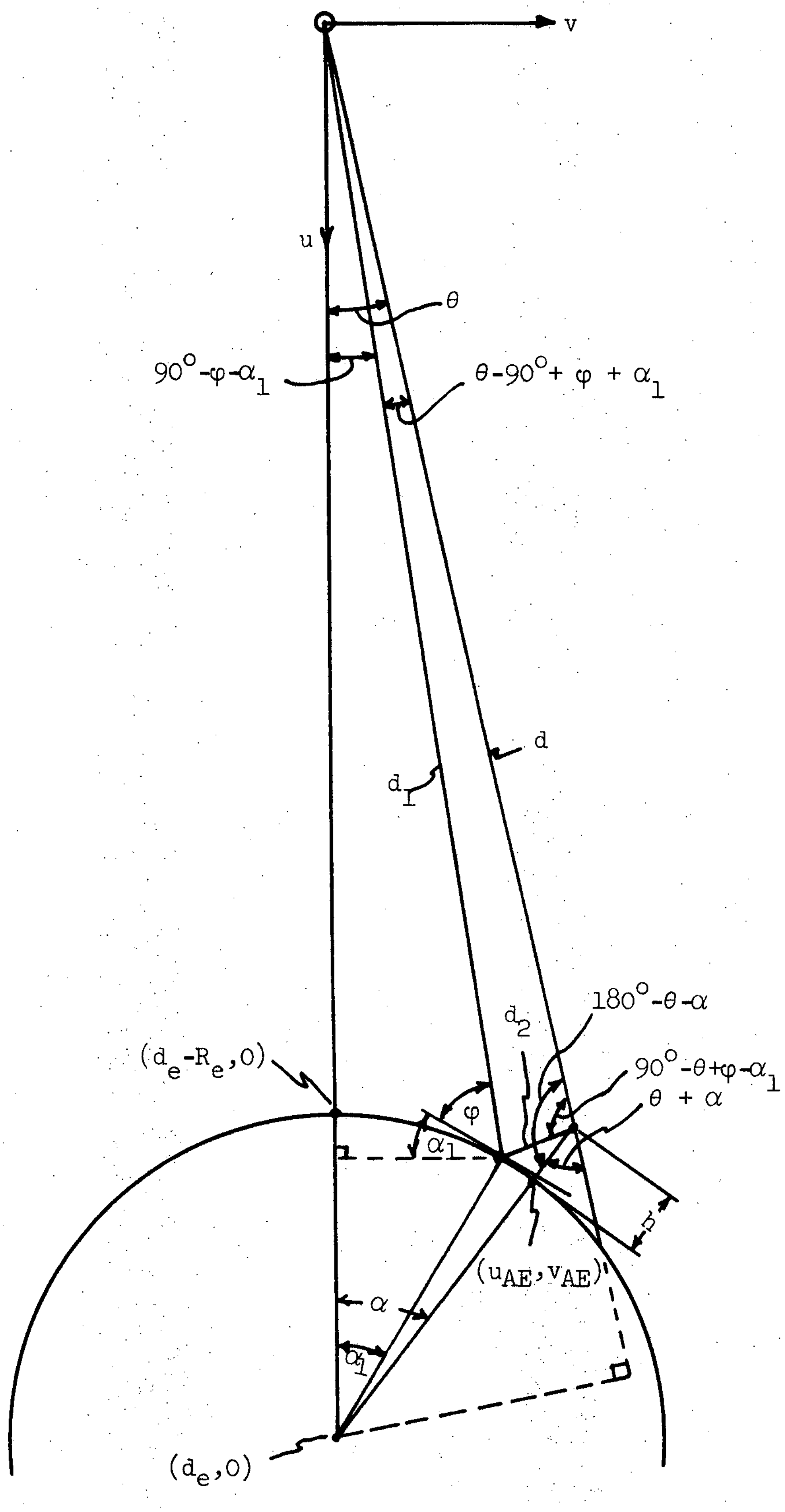


Fig. C-1 Multipath Geometry in u-v Plane

But from eq. (C-4),

$$\frac{d}{\sin\alpha} = \frac{1}{\sin\alpha}\, d_1 \frac{\sin(180^\circ - 2\varphi)}{\sin(90^\circ - \theta + \varphi - \alpha_1)} \tag{C-8}$$

Again, from the law of sines,

$$\frac{d_1}{\sin\alpha_1} = \frac{d_2}{\sin(90^\circ + \varphi)} \tag{C-9}$$

Combining (C-7), (C-8), and (C-9) yields

$$\alpha = \tan^{-1} \frac{2\sin\theta\sin\alpha_1\sin\varphi}{\cos(\alpha_1 + \theta - \varphi) - 2\cos\theta\sin\alpha_1\sin\varphi} \tag{C-10}$$

From the figure,

$$d_e \sin\theta = (R_e + h)\sin(\theta + \alpha) \tag{C-11}$$

Therefore,

$$h = \frac{d_e \sin\theta}{\sin(\alpha + \theta)} - R_e \tag{C-12}$$

Also, from the figure

$$u_{AE} = d_e - R_e\cos\alpha \tag{C-13}$$

$$v_{AE} = R_e \sin\alpha \tag{C-14}$$

Thus, considering the geometry of Fig. 4 (or Fig. C-1) and given θ, α_1, d_e, R_e and c, it is possible to compute the corresponding values of Δt, h, u_{AE} and v_{AE}. For instance, suppose

$$\left.\begin{aligned} \theta &= 8.0^\circ \\ \alpha_1 &= 58.5^\circ \\ d_e &= 42{,}237.92 \text{ km} \\ R_e &= 6371.26 \text{ km} \\ c &= 2.997925 \times 10^5 \text{km/s} \end{aligned}\right\} \tag{C-15}$$

Then, from equations (C-3), (C-6), (C-10), (C-12), (C-13), and (C-14):

$$\left.\begin{aligned} \Delta t &= 51.684414 \ \mu s \\ h &= 19.563931 \text{ km} \\ u_{AE} &= 38{,}946.876 \text{ km} \\ v_{AE} &= 5455.454 \text{ km} \end{aligned}\right\} \qquad \text{(C-16)}$$

Consider, now, the iterative altitude determining technique explained in Sec. 3.0. In this technique, the known quantities are θ, Δt, d_e, R_e, and c. If these quantities have the same values as in eq. (C-15)above, and Δt has the value computed in (C-16), the iterative technique yields

$$\left.\begin{aligned} h &= 19.563931 \text{ km} \\ u_{AE} &= 38{,}946.876 \text{ km} \\ v_{AE} &= 5455.454 \text{ km} \end{aligned}\right\} \qquad \text{(C-17)}$$

It is noted that the values computed via the iterative technique in eq. (C-17) are the same as those computed in (C-16).

For a second example, let

$$\left.\begin{aligned} \theta &= 6.22^\circ \\ \alpha_1 &= 39.5^\circ \\ \left.\begin{aligned} d_e &= \\ R_e &= \\ c &= \end{aligned}\right\} & \text{ same as above} \end{aligned}\right\} \qquad \text{(C-18)}$$

Using equations (C-3), (C-6), (C-10), (C-12), (C-13) and (C-14) yields

$$\left.\begin{aligned} \Delta t &= 48.522423 \text{ µs} \\ h &= 10.424577 \text{ km} \\ u_{AE} &= 37{,}328.483 \text{ km} \\ v_{AE} &= 4060.836 \text{ km} \end{aligned}\right\} \qquad \text{(C-19)}$$

With θ, d_e, R_e and c as above, and with Δt having the same value computed in (C-19), the iterative technique yields

$$\left.\begin{aligned} h &= 10.424577 \text{ km} \\ u_{AE} &= 37{,}328.483 \text{ km} \\ v_{AE} &= 4060.836 \text{ km} \end{aligned}\right\} \qquad \text{(C-20)}$$

Again, these values agree with those in (C-19).

This test of the iterative technique has produced the same result in every one of the many situations considered.

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13. ABSTRACT

In an angle-measuring navigation satellite system using a pair of crossed interferometers located on a satellite in synchronous orbit, three parameters are needed to determine a user's position unambiguously: the phase difference between received signals which had been transmitted by the two antennas on each of the two interferometers, and the user's altitude.

The two phase difference measurements yield a line of possible user locations, and the addition of the altitude measurement reduces this line to a single point. Instead of measuring altitude via a barometric altimeter, a method is proposed here which makes use of the navigation signals received after reflection off the earth's surface. The iterative procedure used here employs the arrival time difference between direct and reflected signals.

Based on previous calculations of errors in measuring the electrical parameters it is concluded that, for North Atlantic coverage and specular reflection, altitude measurements can be made with a 1-σ error of 65 meters.

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