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THE EQUILIBRIUM AND STABILITY OF THE GASEOUS COMPONENT OF THE GALAXY. III.

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ABSTRACT

The stability of a self-gravitating, non-rotating, isothermal gas layer threaded by a one-dimensional equipartition magnetic field, immersed in a rigid isothermal layer of stars, is considered with respect to waves with motions in and perpendicular to the $\vec{B}_e - \vec{g}_e$ plane, where $\vec{B}_e$ and $\vec{g}_e$ are the equilibrium magnetic and gravitational field vectors. When motions are perpendicular to the $\vec{B}_e - \vec{g}_e$ plane, the magnetic field hinders gravitational instability, increasing the minimum length necessary to produce instability by the factor $(1 + \alpha)^{1/2}$, where α is the ratio of magnetic pressure to gas pressure. When motions are in the $\vec{B}_e - \vec{g}_e$ plane, no such simple analytical solution is found. However, the resulting system of equations reduces to a single fourth order differential equation in $\Delta\phi_g$ (the perturbed gas potential) that defines an eigenvalue problem for the marginally unstable state when the four appropriate boundary conditions are considered.

I. INTRODUCTION

In Paper I of this series (Kellman 1972a) we investigated the equilibrium in the z direction (i.e. above the galactic plane) of a static, isothermal, plane-parallel layer of gas with equipartition magnetic and cosmic-ray components, immersed in an isothermal layer of stars. In Paper II (Kellman 1972b) we considered the gravitational stability of the gas layer with respect to plane and axially symmetric perturbations, neglecting the magnetic and cosmic-ray components, with the view toward explaining the large gas structures (1 - 2 kpc, 10^7 $M_\odot$) observed to be the principal elements of the gaseous component of spiral arms in the Galaxy (McGee and Milton 1964). It is our purpose here to modify the stability analysis of Paper II by including a one-dimensional equipartition magnetic field in the initial equilibrium state. We consider only the modes with motions (a) in and (b) perpendicular to the $\vec{B}_e$ - $\vec{g}_e$ plane, where $\vec{B}_e$ and $\vec{g}_e$ represent the equilibrium magnetic and gravitational field vectors. Paper IV (Kellman 1972c) will consider the effect of a combined magnetic field and cosmic-ray gas on the stability of the gas layer.

Field (1970) has investigated the stability of an infinite uniform self-gravitating gas (no stars or cosmic-rays) against disturbances propagating both along (case a) and perpendicular (case b) to a uniform magnetic field $\vec{B}_0$. For case (b), he found that the magnetic field increased the minimum length necessary to produce gravitational instability by the factor $(1 + 2\alpha)^{1/2}$, where α is the ratio of magnetic pressure to gas pressure. We will find a similar but somewhat smaller modification: $(1 + \alpha)^{1/2}$. For case (a), Field discovered two modes. The first is the simple Alfvén mode, and the waves travel with the Alfvén velocity $v_A = B_0/(4\pi\rho_0)^{1/2}$. The second is more subtle and divides into two cases, depending on the wavelength λ. When $\lambda < \lambda_J$, where $\lambda_J = (\pi \langle v_{tz}^2 \rangle / G\rho_0)^{1/2}$ is the Jeans' length, the solution is stable and gives acoustic or sound waves, and in the limit $\lambda << \lambda_J$, the waves travel at the sound speed. When $\lambda > \lambda_J$, the solution gives an unstable mode, the instability caused by self-gravitation of the gas. This is the well-known Jeans' instability.

Parker (1966) has considered the stability of a combined magnetic field, cosmic-ray gas, and thermal gas against disturbances propagating along the direction of the magnetic field (case a). Equipartition of energy between the components was assumed. However, self-gravitation of the gas layer was neglected; this explains why lengths significantly smaller than typical Jeans' lengths resulted.

II. STABILITY ANALYSIS

a) Motions in a Plane Perpendicular to the $\vec{B}_e - \vec{g}_e$ Plane

The relation between the vectors $\vec{B}_e$, $\vec{g}_e$, and $\vec{k}$ and the xyz coordinate system is shown in Figure 1. $\vec{k}$ is the wave propagation vector; $\vec{B}_e$ and $\vec{g}_e$ have been defined above. The case $\vec{B}_e \perp \vec{k}$ (case b) for an infinite uniform self-gravitating gas is particularly interesting (Field 1970). The longitudinal mode with $v_x = 0$, v_y finite propagates with phase velocity $v_p = \langle v_{tz}^2 \rangle^{1/2} (1 + 2\alpha)^{1/2}$, and is called, appropriately, the magnetosonic mode. The magnetic restoring forces are not due to curvature of the field lines as is true of the Alfvén mode, but to pressure gradients. The criterion for gravitational instability is $\lambda > (1 + 2\alpha)^{1/2} \lambda_J$; the presence of a magnetic field renders gravitational instability more difficult.

With these introductory remarks in mind, it seems appropriate to redefine the problem at hand. We seek to determine how a simple one-dimensional equipartition magnetic field (along x) effects the stability criterion of waves propagating perpendicular to $\vec{B}_e$ (along y) in an isothermal self-gravitating gas layer with infinite conductivity immersed in a rigid isothermal star layer.

The basic equations are the continuity, momentum, hydromagnetic, Poisson, and heat equation, and they are written as follows:

$$\frac{d}{dt} \rho_g + \rho_g \nabla \cdot \vec{v} = 0 \tag{1}$$

$$\nabla p_g + \rho_g \frac{d}{dt} \vec{v} - \frac{1}{4\pi} \vec{B} \cdot \nabla \vec{B} + \frac{1}{8\pi} \nabla B^2 + \rho_g \nabla \phi = 0 \tag{2}$$

$$\frac{\partial}{\partial t} \vec{B} - \nabla \times (\vec{v} \times \vec{B}) = 0 \tag{3}$$

$$4\pi G (\rho_g + \rho_*) - \nabla^2 (\phi_g + \phi_*) = 0 \tag{4}$$

$$p_g = \langle v_{tz}^2 \rangle \rho_g . \tag{5}$$

ρ, p, ϕ, and $\vec{B}$ are, respectively, the density, pressure, gravitational potential, and magnetic field strength. $\langle v^2_{tz} \rangle$ is the mean square z turbulent gas velocity dispersion. $\vec{v}$ is the gas velocity, a first order quantity. The subscripts g and * denote gas and star, respectively. The perturbations in ρ_g, p_g, ϕ_g, and $\vec{B}$ may be written as follows:

$$\rho_g = \rho_{eg} + \Delta \rho_g \tag{6}$$

$$p_g = p_{eg} + \Delta p_g = \langle v^2_{tz} \rangle \; \rho_g \tag{7}$$

$$\phi_g = \phi_{eg} + \Delta \phi_g \tag{8}$$

$$\phi = \phi_e + \Delta \phi_g = \phi_{eg} + \phi_{e*} + \Delta \phi_g \tag{9}$$

$$\vec{B} = \vec{B}_e + \Delta \vec{B}, \tag{10}$$

where the subscript 'e' denotes the equilibrium quantities and Δ denotes the perturbed quantities. Retaining terms only to first order in the perturbed quantities, equations (1)-(5) become

$$\frac{\partial}{\partial t} \Delta \rho_g + \vec{v} \cdot \nabla \rho_{eg} + \rho_{eg} \nabla \cdot \vec{v} = 0 \tag{11}$$

$$\langle v^2_{tz} \rangle \; \nabla \Delta \rho_g + \Delta \rho_g \nabla \phi_e + \rho_{eg} \frac{\partial \vec{v}}{\partial t} + \frac{1}{4\pi} \nabla (\vec{B}_e \cdot \Delta \vec{B}) + \rho_{eg} \nabla \Delta \phi_g = 0 \tag{12}$$

$$\frac{\partial}{\partial t} \Delta \vec{B} + \vec{v} \cdot \nabla \vec{B}_e + \vec{B}_e (\nabla \cdot \vec{v}) - (\vec{B}_e \cdot \nabla) \vec{v} = 0 \tag{13}$$

$$4\pi G \Delta \rho_g - \nabla^2 \Delta \phi_g = 0, \tag{14}$$

where we have introduced a well-known vector identity into the hydromagnetic equation.

The coefficients of the system (11)-(14) are all independent of t, x, and y, enabling us to Fourier analyze in these variables, in which case $\partial/\partial t \to n$, $\partial/\partial x \to ik_x$, and $\partial/\partial y \to ik_y$. However, since $\vec{k} = k_y \hat{e}_y + k_z \hat{e}_z$ for the case under study, $k_x = 0$, and equations (11)-(14), written in component form, become

$$n \Delta \rho_g + i\, k_y\, \rho_{eg}\, v_y + \left(\rho_{eg} \frac{\partial}{\partial z} + \frac{d}{dz} \rho_{eg}\right) v_z = 0 \tag{15}$$

$$i\, k_y \langle v_{tz}^2 \rangle \Delta \rho_g + n\, \rho_{eg}\, v_y + i\, k_y \frac{B_e}{4\pi} \Delta B_x + i\, k_y\, \rho_{eg} \Delta \phi_g = 0 \tag{16}$$

$$\langle v_{tz}^2 \rangle \frac{\partial}{\partial z} \Delta \rho_g + \Delta \rho_g \frac{d\phi_e}{dz} + n\, \rho_{eg}\, v_z + \frac{B_e}{4\pi} \frac{\partial}{\partial z} \Delta B_x + \frac{1}{4\pi} \Delta B_x \frac{d}{dz} B_e + \rho_{eg} \frac{\partial}{\partial z} \Delta \phi_g = 0 \tag{17}$$

$$i\, k_y\, B_e\, v_y + \left(B_e \frac{\partial}{\partial z} + \frac{d}{dz} B_e\right) v_z + n \Delta B_x = 0 \tag{18}$$

$$-4\pi G \Delta \rho_g + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \Delta \phi_g = 0. \tag{19}$$

Equations (16) and (17) are the y and z components of the momentum equation; equation (18) is the x component of the hydromagnetic equation. The y and z components of equation (13) are not written since they merely express the fact that $\Delta B_y = \Delta B_z = 0$.

If we write that

$$\frac{d}{d\,z}\,\rho_{eg} = f\,\rho_{eg}, \tag{20}$$

then it follows that

$$\frac{d}{d\,z}\,B_e = \frac{1}{2}\,f\,B_e, \tag{21}$$

since B_e^2/ρ_{eg} is assumed to be independent of z. Equation (20) is consistent with the presence of a stellar component. We proceed by deriving an expression for d/dz ϕ_e:

$$\frac{d}{d\,z}\,\phi_e = -\frac{1}{\rho_{eg}}\,\frac{d}{d\,z}\,(p_{eg} + B_e^2/8\,\pi) = -\frac{1}{\rho_{eg}}\,(\langle v_{tz}^2\rangle + v_A^2/2)\,\frac{d}{d\,z}\,\rho_{eg}$$

$$= -\frac{\langle v_{tz}^2\rangle}{\rho_{eg}}\,(1+\alpha)\,\frac{d}{d\,z}\,\rho_{eg} = -\,\langle v_{tz}^2\rangle\;f\;(1+\alpha), \tag{22}$$

where

$$\alpha = B_e^2/8\,\pi\,\rho_{eg}\;\langle v_{tz}^2\rangle = v_A^2/2\;\langle v_{tz}^2\rangle. \tag{23}$$

Equations (20)-(23) are substituted into the system (15)-(19), with the result that

$$n\,\frac{\Delta\,\rho_g}{\rho_{eg}} + i\,k_y\,v_y + \left(\frac{\partial}{\partial\,z} + f\right)\,v_z = 0 \tag{24}$$

$$\frac{\Delta\,\rho_g}{\rho_{eg}} + \frac{n}{i\,k_y\,\langle v_{tz}^2\rangle}\,v_y + 2\,\alpha\,\frac{\Delta\,B_x}{B_e} + \frac{\Delta\,\phi_g}{\langle v_{tz}^2\rangle} = 0 \tag{25}$$

$$\frac{1}{\rho_{eg}} \frac{\partial}{\partial z} \Delta \rho_g - f\,(1+\alpha) \frac{\Delta \rho_g}{\rho_{eg}} + \frac{n}{\langle v_{tz}^2 \rangle} v_z + \frac{2\,\alpha}{B_e} \frac{\partial}{\partial z} \Delta B_x$$

$$+ f\,\alpha \frac{\Delta B_x}{B_e} + \frac{\partial}{\partial z} \frac{\Delta \phi_g}{\langle v_{tz}^2 \rangle} = 0 \tag{26}$$

$$i\,k_y\,v_y + \left(\frac{\partial}{\partial z} + \frac{f}{2}\right) v_z + n \frac{\Delta B_x}{B_e} = 0 \tag{27}$$

$$\frac{-4\,\pi\,G}{\langle v_{tz}^2 \rangle} \Delta \rho_g + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \frac{\Delta \phi_g}{\langle v_{tz}^2 \rangle} = 0. \tag{28}$$

To proceed further, we need expressions for $1/B_e\ \partial/\partial z\,\Delta B_x$ and $1/\rho_{eg}\ \partial/\partial z\,\Delta\rho_g$:

$$\frac{1}{B_e} \frac{\partial}{\partial z} \Delta B_x = \frac{\partial}{\partial z} (\Delta B_x / B_e) - \Delta B_x \frac{d}{dz} (1/B_e) = \left(\frac{\partial}{\partial z} + \frac{f}{2}\right) \frac{\Delta B_x}{B_e} \tag{29}$$

$$\frac{1}{\rho_{eg}} \frac{\partial}{\partial z} \Delta \rho_g = \frac{\partial}{\partial z} (\Delta \rho_g / \rho_{eg}) - \Delta \rho_g \frac{\partial}{\partial z} \left(\frac{1}{\rho_{eg}}\right) = \left(\frac{\partial}{\partial z} + f\right) \frac{\Delta \rho_g}{\rho_{eg}}. \tag{30}$$

If we define ϵ, δ, and ψ by the equations

$$\epsilon = \Delta \rho_g / \rho_{eg} \tag{31}$$

$$\delta = \Delta B_x / B_e \tag{32}$$

$$\psi = \Delta \phi_g / \langle v_{tz}^2 \rangle \tag{33}$$

and employ equations (29) and (30), the system (24)-(28) reduces to

$$n \epsilon + i k_y v_y + \left(\frac{\partial}{\partial z} + f\right) v_z = 0 \quad (34)$$

$$\epsilon + \frac{n}{i k_y \langle v_{tz}^2 \rangle} v_y + 2 \alpha \delta + \psi = 0 \quad (35)$$

$$\left(\frac{\partial}{\partial z} - f \alpha\right) \epsilon + \frac{n}{\langle v_{tz}^2 \rangle} v_z + 2 \alpha \left(\frac{\partial}{\partial z} + f\right) \delta + \frac{\partial}{\partial z} \psi = 0 \quad (36)$$

$$i k_y v_y + \left(\frac{\partial}{\partial z} + \frac{f}{2}\right) v_z + n \delta = 0 \quad (37)$$

$$\frac{- 4 \pi G \rho_{eg}}{\langle v_{tz}^2 \rangle} \epsilon + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \psi = 0. \quad (38)$$

The usual way to proceed is to Fourier analyze in the remaining variable z and then set the determinant of the coefficients of the system (34)-(38) equal to zero so as to avoid the trivial solution for ϵ, v_y, v_z, δ, and ψ. A dispersion relation between n and k_y is then obtained, and each solution is called a mode. However, not all of the coefficients in this system are independent of z, rendering Fourier analysis in z a useless exercise since the dispersion relation would contain unknown integrals over ψ and other of the variables. An alternate approach is to derive from the system (34)-(38) a single differential equation in the unknown ψ. With the appropriate boundary conditions the equation would implicitly contain the desired dispersion relation.

We have been unable to reduce the system (34)-(38) to a single differential equation, and therefore will restrict the analysis to the neutral or marginally unstable state by setting n = 0:

$$\epsilon + 2\,\alpha\,\delta + \psi = 0 \tag{39}$$

$$\left(\frac{\partial}{\partial z} - f\,\alpha\right)\epsilon + 2\,\alpha\left(\frac{\partial}{\partial z} + f\right)\delta + \frac{\partial}{\partial z}\,\psi = 0 \tag{40}$$

$$\frac{-\,4\,\pi\,G\,\rho_{eg}}{\langle v_{tz}^2\rangle}\,\epsilon + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\psi = 0. \tag{41}$$

Equations (34) and (37) have not been written explicitly; they merely express that $v_y = v_z = 0$, the expected result that all velocities vanish in the marginally unstable state. To proceed, we differentiate equation (39) and subtract equation (40), with the result that

$$\delta = \epsilon/2. \tag{42}$$

Combining equations (39) and (42) it easily follows that

$$\epsilon = -\frac{\psi}{1+\alpha}, \tag{43}$$

and the Poisson equation (41) thus becomes

$$\frac{\partial^2\psi}{\partial z^2} + \left[\frac{4\,\pi\,G\,\rho_{eg}}{\langle v_{tz}^2\rangle\,(1+\alpha)} - k_y^2\right]\psi = 0. \tag{44}$$

Equation (13) in Paper II of this series (Kellman 1972b) defines the scale height H_g of the equilibrium gas layer in the z direction to be

$$H_g^2 = \frac{\langle v_{tz}^2 \rangle}{8 \pi G \rho_{ego}} , \tag{45}$$

where ρ_{ego} is the value of ρ_{eg} at the plane z = 0. Equation (44) may therefore be written

$$\frac{\partial^2 \psi}{\partial z^2} + \left[\frac{1}{2 H_g^2 (1 + \alpha)} \frac{\rho_{eg}(z)}{\rho_{ego}} - k_y^2 \right] \psi = 0, \tag{46}$$

with the boundary conditions

$$\frac{\partial}{\partial z} \psi (z = 0) = 0 \tag{47}$$

$$\lim_{|z| \to \infty} \psi = 0. \tag{48}$$

Equation (47) results because $\psi = \Delta \phi_g / \langle v_{tz}^2 \rangle$ is an even function of z; equation (48) results because $\Delta \phi_g$ is constrained to $\to 0$ as $|z| \to \infty$.

It is useful to recall that in Paper II we considered a problem similar in every respect to that considered here, except that $B_e = 0$. Therefore, as an obvious check on the validity of equations (46)-(48) we require that in the limit $B_e \to 0$ they reduce to the analogous set of equations in Paper II, equations (18)–(20). This requirement is indeed satisfied. In fact, equations (46)-(48) differ from equations (18)-(20) only by the factor $(1+\alpha)=(1+B_{eo}/8\pi\rho_{ego}\langle v_{tz}^2\rangle)$ in the second term of equation (46). Since $\alpha > 0$, the depth of the equivalent potential well is reduced, and from our discussion in Paper II the eigenvalue k_y decreases in magnitude. Therefore, the radius r_1 of the marginally unstable state in the symmetry plane (proportional to $1/k_y$) increases. This is entirely reasonable from a physical point of view; the presence of a magnetic field enhances the difficulty of gravitational instability to result from a given disturbance.

To calculate the radius of the marginally unstable state, we recall Figure 1 from Paper II, where r_1 is plotted as a function of $\langle v_{tz}^2\rangle^{1/2}$. We merely equate $\langle v_{tz}^2\rangle^{1/2}(1+\alpha)^{1/2}$ with $\langle v_{tz}^2\rangle^{1/2}$ and read off the value of r_1; this is equivalent to multiplying r_1 by $(1+\alpha)^{1/2}$ since r_1 is linearly related to $\langle v_{tz}^2\rangle^{1/2}$. As before, we choose ρ_{ego} = 1 H atom/cm^3 = 0.025 $M_\odot/pc^3$ (Weaver 1970), ρ_{*o} = 0.064 $M_\odot/pc^3$ (Luyten 1968), and $\langle v_{*z}^2\rangle^{1/2}$ = 18 km/sec (Woolley 1958). $\langle v_{tz}^2\rangle^{1/2}$ is allowed to vary between 1 and 20 km/sec. The results are displayed in Table 1 where we compare r_1 calculated with B_{eo} = 0μG, 3μG, and 5μG. The r_1 (0μG) are taken from Table 1 of Paper II. We see that the magnetic field increases its effect as $\langle v_{tz}^2\rangle^{1/2}$ decreases. For a typical interstellar value $\langle v_{tz}^2\rangle^{1/2}$ = 7.5 km/sec, a 3μG field increases r_1(0μG) by 15%; a 5μG field increases r_1 (0μG) by 44%.

In summary, if r_1(0μG) is the length for which a non-magnetic, self-gravitating, isothermal gas layer (immersed in a rigid isothermal star layer) just becomes gravitationally unstable, then the introduction of a one-dimensional equipartition magnetic field increases $r_1\,(0\,\mu G)$ by the factor $(1+\alpha)^{1/2} = (1 + v_A^2/2\ \langle v_{tz}^2\rangle)^{1/2}$, where the relation between $\vec{B}_e$, $\vec{g}_e$, and $\vec{k}$ is as shown in Figure 1. This result is similar to the modification $(1+2\alpha)^{1/2}$ induced by the presence of a uniform one-dimensional magnetic field in an infinite uniform self-gravitating gas (no stars) when disturbances propagate across $\vec{B}_e$ (Field 1970).

b) Motions in the $\vec{B}_e$ - $\vec{g}_e$ Plane

The relation between $\vec{B}_e$, $\vec{g}_e$, and $\vec{k}$ and the xyz coordinate system is shown in Figure 2. As in Section IIa, the basic equations are the continuity, momentum, hydromagnetic, Poisson, and heat equation. Only the momentum equation need be rewritten here, since we have chosen to write the magnetic force in a slightly different form:

$$\rho_g \frac{d}{dt}\vec{v} + \nabla p_g - \frac{1}{4\pi}(\nabla\times\vec{B})\times\vec{B} + \rho_g \nabla\phi = 0. \tag{49}$$

Introducing the perturbations from equations (6)-(10) and retaining terms only to first order in the perturbed quantities, equations (1), (3), (4), (5), and (49), written in component form, become

$$\frac{\partial}{\partial t}\Delta\rho_g + \rho_{eg}\frac{\partial}{\partial y}v_y + \left(\rho_{eg}\frac{\partial}{\partial z} + \frac{d}{dz}\rho_{eg}\right)v_z = 0 \tag{50}$$

$$\langle v_{tz}^2 \rangle \frac{\partial}{\partial x} \Delta \rho_g + \frac{B_e}{4\pi} \frac{\partial}{\partial x} \Delta B_y - \frac{B_e}{4\pi} \frac{\partial}{\partial y} \Delta B_x + \rho_{eg} \frac{\partial}{\partial x} \Delta \phi_g = 0 \tag{51}$$

$$\langle v_{tz}^2 \rangle \frac{\partial}{\partial y} \Delta \rho_g + \rho_{eg} \frac{\partial}{\partial t} v_y - \frac{1}{4\pi} \Delta B_z \frac{d}{dz} B_e + \rho_{eg} \frac{\partial}{\partial y} \Delta \phi_g = 0 \tag{52}$$

$$\langle v_{tz}^2 \rangle \frac{\partial}{\partial z} \Delta \rho_g + \Delta \rho_g \frac{d}{dz} \phi_e + \rho_{eg} \frac{\partial}{\partial t} v_z + \frac{B_e}{4\pi} \frac{\partial}{\partial z} \Delta B_y + \frac{1}{4\pi} \Delta B_y \frac{d}{dz} B_e$$

$$- \frac{B_e}{4\pi} \frac{\partial}{\partial y} \Delta B_z + \rho_{eg} \frac{\partial}{\partial z} \Delta \phi_g = 0 \tag{53}$$

$$\frac{\partial}{\partial t} \Delta B_x - B_e \frac{\partial}{\partial y} v_x = 0 \tag{54}$$

$$\frac{\partial}{\partial t} \Delta B_y + \left(B_e \frac{\partial}{\partial z} + \frac{d}{dz} B_e \right) v_z = 0 \tag{55}$$

$$\frac{\partial}{\partial t} \Delta B_z - B_e \frac{\partial}{\partial y} v_z = 0 \tag{56}$$

$$4\pi G \Delta \rho_g - \nabla^2 \Delta \phi_g = 0. \tag{57}$$

Equations (51), (52), and (53) are, respectively, the x, y, and z components of the momentum equation; equations (54), (55), and (56) are, respectively, the x, y, and z components of the hydromagnetic equation. Since we restrict the velocity vector to the y-z plane, $v_x = 0$ and from equation (54) it follows that $\Delta B_x = 0$.

The coefficients of the system (50)-(57) are all independent of t, x, and y, allowing us to Fourier analyze in these variables ($\partial/\partial t \to n$, $\partial/\partial x \to ik_x$, $\partial/\partial y \to ik_y$). However, since $\vec{k} = k_y \hat{e}_y + k_z \hat{e}_z$, $k_x = 0$ and equations (50)-(57) simplify to

$$n \Delta \rho_g + i k_y \rho_{eg} v_y + \left(\rho_{eg} \frac{\partial}{\partial z} + \frac{d}{dz} \rho_{eg}\right) v_z = 0 \tag{58}$$

$$i k_y \langle v_{tz}^2 \rangle \Delta \rho_g + n \rho_{eg} v_y - \frac{1}{4\pi} \Delta B_z \frac{d}{dz} B_e + i k_y \rho_{eg} \Delta \phi_g = 0 \tag{59}$$

$$\langle v_{tz}^2 \rangle \frac{\partial}{\partial z} \Delta \rho_g + \Delta \rho_g \frac{d}{dz} \phi_e + n \rho_{eg} v_z + \frac{B_e}{4\pi} \frac{\partial}{\partial z} \Delta B_y + \frac{1}{4\pi} \Delta B_y \frac{d}{dz} B_e$$

$$- i k_y \frac{B_e}{4\pi} \Delta B_z + \rho_{eg} \frac{\partial}{\partial z} \Delta \phi_g = 0 \tag{60}$$

$$\left(B_e \frac{\partial}{\partial z} + \frac{d}{dz} B_e\right) v_z + n \Delta B_y = 0 \tag{61}$$

$$i k_y B_e v_z - n \Delta B_z = 0 \tag{62}$$

$$4\pi G \Delta \rho_g - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \Delta \phi_g = 0. \tag{63}$$

If we write as before $d/dz\ \rho_{eg} = f \rho_{eg}$, then it follows that $d/dz\ B_e = 1/2\ fB_e$, since B_e^2/ρ_{eg} is assumed to be independent of z. These equations are consistent with the presence of a stellar component. We proceed by recalling equations (22), (29), and (30) for the quantities $d/dz\ \phi_e$, $1/B_e\ \partial/\partial z\ \Delta B_y$, and $1/\rho_{eg}\ \partial/\partial z\ \Delta\rho_g$.

Equations (58)-(63) now become

$$n \frac{\Delta \rho_g}{\rho_{eg}} + i\, k_y\, v_y + \left(\frac{\partial}{\partial z} + f \right) v_z = 0 \tag{64}$$

$$i\, k_y \frac{\Delta \rho_g}{\rho_{eg}} + \frac{n}{\langle v_{tz}^2 \rangle} v_y - f\, \alpha \frac{\Delta B_z}{B_e} + i\, k_y \frac{\Delta \phi_g}{\langle v_{tz}^2 \rangle} = 0 \tag{65}$$

$$\left(\frac{\partial}{\partial z} - f\, \alpha \right) \frac{\Delta \rho_g}{\rho_{eg}} + \frac{n}{\langle v_{tz}^2 \rangle} v_z + 2\, \alpha \left(\frac{\partial}{\partial z} + f \right) \frac{\Delta B_y}{B_e}$$

$$- i\, k_y\, 2\, \alpha \frac{\Delta B_z}{B_e} + \frac{\partial}{\partial z} \frac{\Delta \phi_g}{\langle v_{tz}^2 \rangle} = 0 \tag{66}$$

$$\left(\frac{\partial}{\partial z} + \frac{f}{2} \right) v_z + n \frac{\Delta B_y}{B_e} = 0 \tag{67}$$

$$i\, k_y\, v_z - n \frac{\Delta B_z}{B_e} = 0 \tag{68}$$

$$\frac{4 \pi G}{\langle v_{tz}^2 \rangle} \Delta \rho_g - \left(\frac{\partial^2}{\partial z^2} - k_y^2 \right) \frac{\Delta \phi_g}{\langle v_{tz}^2 \rangle} = 0. \tag{69}$$

As we found in Section IIa, the coefficients of the system (64)-(69) are not all independent of z, rendering Fourier analysis in z a useless exercise. We proceed as before by setting $n = 0$, which restricts the analysis to the marginally unstable state. If we make the substitutions

$$\epsilon = \Delta \rho_g / \rho_{eg} \tag{70}$$

$$\tau = \Delta B_y / B_e \tag{71}$$

$$\gamma = \Delta B_z/B_e \tag{72}$$

$$\psi = \Delta \phi_g / \langle v_{tz}^2 \rangle , \tag{73}$$

equations (64)-(69) become

$$i k_y \epsilon - f \alpha \gamma + i k_y \psi = 0 \tag{74}$$

$$\left(\frac{\partial}{\partial z} - f \alpha\right) \epsilon + 2 \alpha \left(\frac{\partial}{\partial z} + f\right) \tau - 2 i k_y \alpha \gamma + \frac{\partial}{\partial z} \psi = 0 \tag{75}$$

$$\frac{4\pi G \rho_{eg}}{\langle v_{tz}^2 \rangle} \epsilon - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \psi = 0. \tag{76}$$

Equations (64), (67), and (68) are not presented here; they merely express that $v_y = v_z = 0$, a result already expected for marginal stability.

We proceed by attempting to obtain from the system (74)-(76) a single differential equation in ψ, which together with the appropriate boundary conditions would implicitly contain the solutions for k_y. It appears at first glance that we have three equations (74)-(76) in the four unknowns ϵ, τ, γ, and ψ. τ and γ are not, however, independent. From equations (67) and (68) it follows that

$$\tau = - \frac{1}{i k_y} \left(\frac{\partial}{\partial z} + \frac{f}{2}\right) \gamma, \tag{77}$$

and equations (74)-(76) reduce to

$$i k_y - f \alpha \gamma + i k_y \psi = 0 \tag{78}$$

$$\left(\frac{\partial}{\partial z} - f\,\alpha\right)\epsilon - \frac{2\,\alpha}{i\,k_y}\left[\frac{\partial^2}{\partial z^2} + \frac{3}{2}\,f\,\frac{\partial}{\partial z} + \left(\frac{f^2}{2} - k_y^2\right)\right]\gamma + \frac{\partial}{\partial z}\,\psi = 0 \tag{79}$$

$$\frac{4\,\pi\,G\,\rho_{eg}}{\langle v_{tz}^2\rangle}\,\epsilon - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\psi = 0. \tag{80}$$

ϵ and γ are eliminated from the system (78)-(80), and the result is a single fourth order differential equation for ψ :

$$\frac{1}{f}\,\frac{\partial^4}{\partial z^4}\psi + \frac{\partial^3}{\partial z^3}\psi + \left(\frac{4\,\pi\,G\,\rho_{eg}}{f\langle v_{tz}^2\rangle} - \frac{2\,k_y^2}{f} + \frac{f\,\alpha}{2} + \frac{f}{2}\right)\frac{\partial^2}{\partial z^2}\psi$$
$$+\left(\frac{4\,\pi\,G\,\rho_{eg}}{\langle v_{tz}^2\rangle} - k_y^2\right)\frac{\partial\,\psi}{\partial\,z} + \left(\frac{2\pi f\,G\,\rho_{eg}}{\langle v_{tz}^2\rangle} - \frac{4\pi\,k_y^2\,G\,\rho_{eg}}{f\,\langle v_{tz}^2\rangle} - \frac{f\,\alpha\,k_y^2}{2}\right.$$
$$\left. - \frac{f\,k_y^2}{2} + \frac{k_y^4}{f}\right)\psi = 0. \tag{81}$$

The appropriate boundary conditions to consider with equation (81) are the following:

$$\frac{\partial}{\partial z}\,\psi\,(z = 0) = 0 \tag{82}$$

$$\frac{\partial^3}{\partial z^3}\,\psi\,(z = 0) = 0 \tag{83}$$

$$\lim_{|z|\to\infty}\psi = 0 \tag{84}$$

$$\lim_{|z|\to\infty}\frac{\partial^2\psi}{\partial z^2} = 0. \tag{85}$$

Equations (82) and (83) result because $\psi = \Delta\phi_g / \langle v_{tz}^2 \rangle$ is an even function of z. Equations (84) and (85) result because $\Delta \phi_g$ and $\partial^2/\partial z^2 \Delta\phi_g$ are constrained to $\rightarrow 0$ as $|z| \rightarrow \infty$.

Equations (81)-(85) form an eigenvalue problem similar to that described by equations (18)-(20) of Paper II and by equations (46)-(48) of this paper. Only certain discrete values of k_y will result in ψ that satisfy (a) equation (81) and (b) the boundary conditions imposed by equations (82)-(85). Each discrete k_y corresponds to a mode of the marginally unstable state; since $k_y \propto 1/\lambda_y$, where λ_y is a length in the y direction, each mode is characterized by a length λ_y. The numerical solution to this problem will be treated in a later paper of this series.

III. DISCUSSION

In Paper II we discussed in some detail the observations of McGee and Milton (1964) relating to the existence of large-scale structure in the gaseous component of spiral arms in the Galaxy. Typical dimensions were observed to be 10^7 $M_\odot$ and 1-2 Kpc. We also discussed Lin's (1970) recent proposal concerning the excitation of density waves in the galactic disk by classic Jeans' type instabilities in the gaseous component of the Galaxy beyond the corotation distance. The inclusion of an equipartition magnetic component to the stability analysis should improve its applicability to both of these topics.

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APPENDIX

MOTIONS IN THE $\vec{B}_e - \vec{g}_e$ PLANE (ALTERNATE DERIVATION): THE USE OF A VECTOR POTENTIAL

When $\vec{k}$ and $\vec{B}_e$ are constrained to lie along the y axis and $\vec{v}$ lies in the y-z plane (Figure 2), Parker (1966) and Field (1970) have found the vector potential $\vec{A}$ to be a useful quantity, where

$$\vec{B} = \nabla \times \vec{A}. \tag{A1}$$

Written in this way, the requirement $\nabla \cdot \vec{B} = 0$ is automatically satisfied. The hydromagnetic equation thus becomes

$$\nabla \times \left[\frac{\partial}{\partial t}\vec{A} - (\vec{v} \times \vec{B})\right] = 0, \tag{A2}$$

from which it follows that

$$\frac{\partial}{\partial t}\vec{A} = \vec{v} \times \vec{B} + \nabla S. \tag{A3}$$

S is an arbitrary scalar potential and will be set equal to zero. $\vec{A}$ is a useful quantity because it is a constant of the motion. To see this, we introduce the following vector identity:

$$\vec{v} \times \vec{B} = \vec{v} \times (\nabla \times \vec{A}) = -\vec{v} \cdot \nabla\vec{A} + \nabla(\vec{v} \cdot \vec{A}) - \vec{A} \cdot \nabla\vec{v} - \vec{A} \times (\nabla \times \vec{v}). \tag{A4}$$

$\vec{v} \cdot \vec{A}$ vanishes since $\vec{v}$ is constrained to the y-z plane and $\vec{A}$ has only an x component. Similarly, both $\vec{A} \cdot \nabla\vec{v}$ and $\vec{A} \times (\nabla \times \vec{v})$ vanish. Equation (A3) therefore becomes

$$\frac{\partial}{\partial t} \vec{A} + \vec{v} \cdot \nabla \vec{A} = 0, \tag{A5}$$

or more simply

$$\frac{d}{dt} \vec{A} = 0, \tag{A6}$$

and thus it follows that $\vec{A}$ is a constant of the motion.

In terms of $\vec{A}$, the continuity, momentum, hydromagnetic, Poisson, and heat equation may be written as follows:

$$\frac{\partial}{\partial t} \rho_g + \vec{v} \cdot \nabla \rho_g + \rho_g \nabla \cdot \vec{v} = 0 \tag{A7}$$

$$\nabla p_g + \rho_g \frac{d\vec{v}}{dt} + \frac{1}{4\pi} \nabla^2 A \, (\nabla A) + \rho_g \nabla \phi = 0 \tag{A8}$$

$$\frac{\partial}{\partial t} \vec{A} - \vec{v} \times \vec{B} = 0 \tag{A9}$$

$$4\pi G(\rho_g + \rho_*) - \nabla^2 (\phi_g + \phi_*) = 0 \tag{A10}$$

$$p_g = \langle v_{tz}^2 \rangle \rho_g, \tag{A11}$$

where $\vec{A} = A\hat{e}_x$ and

$$-\frac{1}{4\pi} (\nabla \times \vec{B}) \times \vec{B} = -\frac{1}{4\pi} [(\nabla \times \nabla \times \vec{A}) \times \vec{B}] =$$

$$\frac{1}{4\pi} (\nabla^2 \vec{A}) \times \vec{B} = \frac{1}{4\pi} \nabla^2 A \, (\nabla A) . \tag{A12}$$

Recalling the perturbations in ρ_g, p_g, ϕ_g, and $\vec{B}$ from equations (6)-(10) and writing the additional equation

$$\vec{A} = \vec{A}_e + \Delta\vec{A}, \tag{A13}$$

equations (A7)-(A11) become

$$\frac{\partial}{\partial t}\Delta\rho_g + \rho_{eg}\frac{\partial}{\partial y}v_y + \left(\rho_{eg}\frac{\partial}{\partial z} + \frac{d}{dz}\rho_{eg}\right) v_z = 0 \tag{A14}$$

$$\langle v_{tz}^2\rangle \nabla\Delta\rho_g + \Delta\rho_g \nabla\phi_e + \rho_{eg}\frac{\partial}{\partial t}\vec{v} + \frac{1}{4\pi}\left[(\nabla^2 A_e)(\nabla\Delta A)\right.$$

$$\left. + (\nabla^2\Delta A)(\nabla A_e)\right] + \rho_{eg}\nabla\Delta\phi_g = 0 \tag{A15}$$

$$\frac{\partial}{\partial t}\Delta\vec{A} - \vec{v}\times\vec{B}_e = 0 \tag{A16}$$

$$4\pi G\Delta\rho_g - \nabla^2\Delta\phi_g = 0. \tag{A17}$$

We have retained terms only to first order in the perturbed quantities.

The nonvanishing component of $\vec{v}\times\vec{B}_e$ is just $-B_e v_z \hat{e}_x$. In addition, we may write as before that $d/dz\ \rho_{eg} = f\rho_{eg}$ and $d/dz\ B_e = 1/2\ fB_e$. Since

$$\vec{B}_e = \nabla\times\vec{A}_e = \frac{d}{dz}A_e\,\hat{e}_y, \tag{A18}$$

it follows that

$$B_e = \frac{d}{dz}A_e \tag{A19}$$

and

$$\nabla^2 A_e = \frac{d^2}{dz^2} A_e = \frac{d}{dz} B_e = \frac{1}{2} f B_e . \tag{A20}$$

Because the coefficients of the system (A14)-(A17) are independent of x, y, and t, we Fourier analyze in these variables ($\partial/\partial x \rightarrow i k_x = 0$, $\partial/\partial y \rightarrow i k_y$, $\partial/\partial t \rightarrow n$), and equations (A14)-(A17) written in component form become

$$n \frac{\Delta\rho_g}{\rho_{eg}} + i k_y \; v_y + \left(\frac{\partial}{\partial z} + f\right) v_z = 0 \tag{A21}$$

$$\frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{i k_y \; \langle v_{tz}^2 \rangle} v_y + \frac{f\alpha}{B_e} \Delta A + \frac{\Delta\phi_g}{\langle v_{tz}^2 \rangle} = 0 \tag{A22}$$

$$\frac{1}{\rho_{eg}} \frac{\partial}{\partial z} \Delta\rho_g - f(1+\alpha) \frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{\langle v_{tz}^2 \rangle} v_z$$

$$+ \frac{2\alpha}{B_e} \left[\frac{f}{2} \frac{\partial}{\partial z} \Delta A + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \Delta A\right] + \frac{\partial}{\partial z} \frac{\Delta\phi_g}{\langle v_{tz}^2 \rangle} = 0 \tag{A23}$$

$$B_e v_z + n\Delta A = 0 \tag{A24}$$

$$4\pi G \Delta\rho_g - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \Delta\phi_g = 0. \tag{A25}$$

We have used equation (22) for d/dz ϕ_e. Equations (A22) and (A23) are, respectively, the y and z components of the momentum equation; equation (A24) is the x component of the hydromagnetic equation.

We recall equation (30) for $1/\rho_{eg}\ \partial/\partial z\ \Delta\rho_g$ and eliminate v_z from the system (A21)-(A25), with the result that

$$n\frac{\Delta\rho_g}{\rho_{eg}} + ik_y v_y - n\left(\frac{\partial}{\partial z} + f\right)\frac{\Delta A}{B_e} = 0 \tag{A26}$$

$$\frac{\Delta\rho_g}{\rho_{eg}} + \frac{n}{ik_y\langle v_{tz}^2\rangle} v_y + \frac{f\alpha}{B_e}\Delta A + \frac{\Delta\phi_g}{\langle v_{tz}^2\rangle} = 0 \tag{A27}$$

$$\left(\frac{\partial}{\partial z} - f\alpha\right)\frac{\Delta\rho_g}{\rho_{eg}} - \frac{n^2}{\langle v_{tz}^2\rangle B_e}\Delta A + \frac{2\alpha}{B_e}\left[\frac{f}{2}\frac{\partial}{\partial z} + \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\right]\Delta A + \frac{\partial}{\partial z}\frac{\Delta\phi_g}{\langle v_{tz}^2\rangle} = 0 \tag{A28}$$

$$\frac{4\pi G}{\langle v_{tz}^2\rangle}\Delta\rho_g - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right)\frac{\Delta\phi_g}{\langle v_{tz}^2\rangle} = 0. \tag{A29}$$

We proceed by substituting the variables

$$\epsilon = \Delta\rho_g/\rho_{eg} \tag{A30}$$

$$\psi = \Delta\phi_g/\langle v_{tz}^2\rangle \tag{A31}$$

into equations (A26)-(A29) and by restricting the analysis to the marginally unstable state ($n = 0$). Equations (A26)-(A29) reduce to

$$\epsilon + \frac{f\alpha}{B_e}\Delta A + \psi = 0 \tag{A32}$$

$$\left(\frac{\partial}{\partial z} - f\alpha\right) \epsilon + \frac{2\alpha}{B_e}\left(\frac{\partial^2}{\partial z^2} + \frac{f}{2}\frac{\partial}{\partial z} - k_y^2\right) \Delta A + \frac{\partial}{\partial z}\psi = 0 \tag{A33}$$

$$\frac{4\pi G \rho_{eg}}{\langle v_{tz}^2 \rangle} \epsilon - \left(\frac{\partial^2}{\partial z^2} - k_y^2\right) \psi = 0. \tag{A34}$$

ϵ and ΔA are eliminated from equations (A32)-(A34), and again the result is a single fourth order linear differential equation for ψ:

$$\frac{2}{f}\frac{\partial^4}{\partial z^4}\psi + \left(\frac{8\pi G \rho_{eg}}{f\langle v_{tz}^2 \rangle} - \frac{4k_y^2}{f} + f\alpha + f\right)\frac{\partial^2 \psi}{\partial z^2}$$

$$+\left(\frac{-8\pi G k_y^2 \rho_{eg}}{f\,\langle v_{tz}^2 \rangle} + \frac{4\pi G f \rho_{eg}}{\langle v_{tz}^2 \rangle} - f\alpha k_y^2 - f k_y^2 + \frac{2k_y^4}{f}\right)\psi = 0. \tag{A35}$$

This result may be compared with equation (81) derived above. The appropriate boundary conditions are expressed by equations (82)-(85).

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TABLE 1

RADIUS OF THE MARGINALLY UNSTABLE STATE AS A FUNCTION OF $\langle v_{tz}^2 \rangle^{1/2}$ AND B_{eo}

$\langle v_{tz}^2 \rangle^{1/2}$ (km/sec)	r_1 (B_{eo} = 0 μG) (kpc)	r_1 (B_{eo} = 3 μG) (kpc)	r_1 (B_{eo} = 5 μG) (kpc)
2.5	0.318	0.67	1.00
5.0	0.635	0.87	1.13
7.5	0.953	1.10	1.37
10.0	1.270	1.40	1.60
15.0	1.905	2.03	2.12
20.0	2.540	2.61	2.71

FIGURE CAPTIONS

1. The relation between $\vec{B}_e$, $\vec{g}_e$, and $\vec{k}$ and the xyz coordinate system when motions are perpendicular to the $\vec{B}_e - \vec{g}_e$ plane.

2. The relation between $\vec{B}_e$, $\vec{g}_e$, and $\vec{k}$ and the xyz coordinate system when motions are in the $\vec{B}_e - \vec{g}_e$ plane.

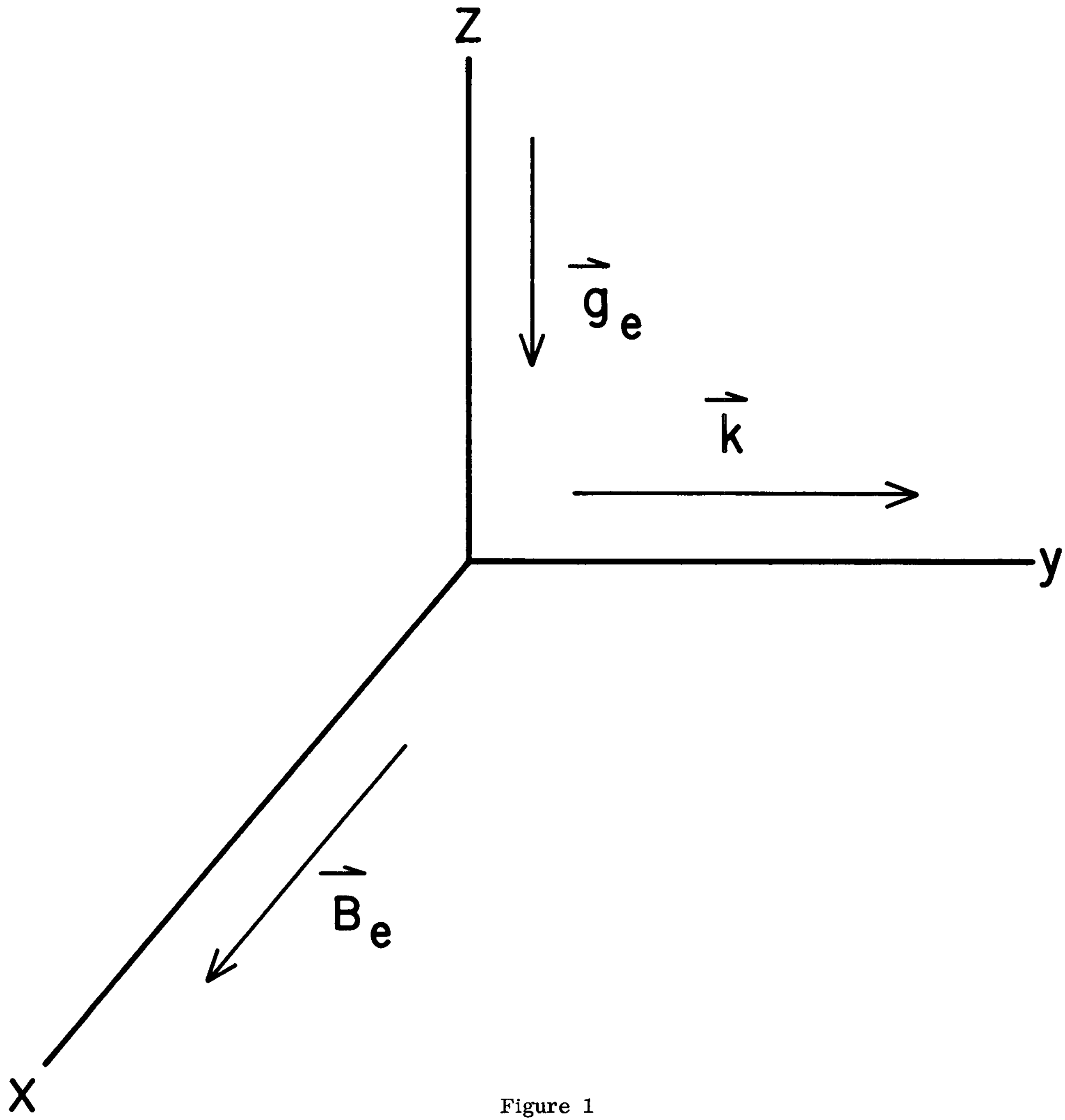

Figure 1

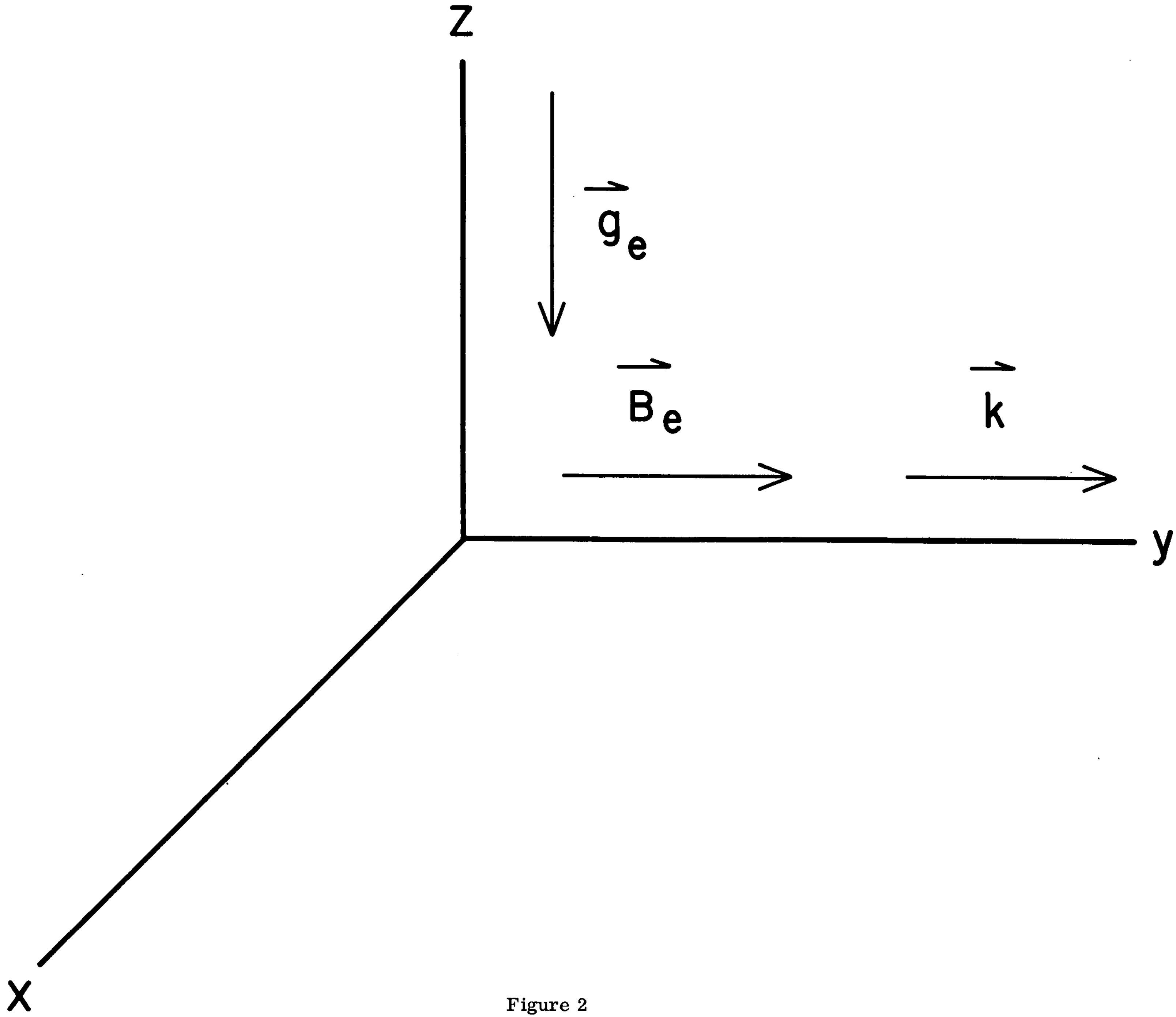

Figure 2