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ON LASHINSKY'S EQUATION $y'' + \alpha y' - ay + by^3 = 0$

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INTRODUCTION

The equation

$$y'' + \alpha y' - ay + by^3 = 0 \tag{1}$$

has been proposed by H. Lashinsky[1] as an analog to the Van der Pol equation[2] to describe nonlinear saturation of aperiodic instabilities such as the Benard instability in fluids[3] and related plasma instabilities[4].

In the Van der Pol equation, the saturation is described by a nonlinear dissipation term of the form $(-\alpha + \beta y + \gamma y^2)y'$. The instability corresponds to the negative sign of the coefficient α: This negative dissipation feeds energy into the system. The nonlinear terms saturate the instability.

In the Lashinsky equation, on the other hand, the saturation is described by a nonlinear restoring force of the form $(-a + by^2)y$. The instability itself corresponds to the negative sign of the coefficient a in the restoring force. The dissipation term consumes energy in this case.

OSCILLATORY SOLUTIONS

First we wish to determine whether equation (1) has periodic or oscillatory solutions for $a > 0$, $\alpha > 0$, $b > 0$. The equation (1) is a special form of the more general Liénard equation

$$y'' + f(y) \cdot y' + g(y) = 0 \tag{2}$$

which has been investigated extensively[5,6].

The Liénard Theorem[6,7,8] gives sufficient but not necessary conditions for the existence of a strictly periodic solution $y(t)$. The conditions to be satisfied are:

1. $f(y) = \alpha$, $g(y) = - ay + by^3$ analytic, which is satisfied for (1).
2. $g(y)$ an cdd function, $g(0) = 0$ and $yg(y) \geq 0$, so that a restoring force exists. This condition is also satisfied, depending on the values of the coefficients a and b.
3. $f(y)$ an even function and $f(0) < 0$ (negative damping at the origin, so that there is an unstable singular point at the origin). For (1) $f(0)$ is positive, however, and the damping term is not a source of energy.
4. $F(y) = \int_0^y f(y)dy = \alpha y$, $G(y) = \int_0^y g(y)dy = - \frac{a}{2} y^2 + b\frac{y^4}{4}$ go to ∞ as $y \to \infty$, which insures periodicity. This condition is satisfied.
5. $F(0) = 0$ has a unique root at y_o, $y_o > 0$ and is a monotonically increasing function for $y = y_o$. Since we have $y_o = 0$, the condition $y_o > 0$ is not satisfied.

Since all the conditions for periodicity are not satisfied, the Lashinsky equation cannot possess a periodic solution. Also the Bendixson Theorem[5], the Frommer Theorem[8], the Levinson-Smith Theorem[5,6] show that (1) has no periodic solutions in the sense

$$y(t + t_o) = y(t) \tag{3}$$

This result, however, does not preclude the existence of oscillatory solutions, i.e. solutions with several maxima and minima. Apparently these oscillatory solutions are damped.

In order to investigate these solutions, we use the phase plane method[8]. We write (1) in the form of two first order equations

$$y' = \upsilon \equiv P\,(\upsilon,\ y) \qquad (4)$$
$$\upsilon' = -\,\alpha\upsilon + ay - by^3 \equiv Q\,(\upsilon, y)$$

Critical (equilibrium) points are then given by $y' = 0$, $\upsilon' = 0$; these points are i.e.

$$y' = \upsilon = 0 \qquad\qquad y = \pm\sqrt{\frac{a}{b}}\ ,\ 0 \qquad (5)$$

so that $y'' = 0$ at these points. The characteristic equation that determines the nature of these critical points is then

$$\lambda^2 - (P_y + Q_\upsilon)\lambda + P_y Q_\upsilon - P_\upsilon Q_y = 0 \qquad (6)$$

From (4) we have $P_y = 0$, $P_\upsilon = 1$, $Q_y = a - 3by^2$, $Q_\upsilon = -\,\alpha$, so that (6) becomes

$$\lambda^2 + \alpha\lambda - a + 3by^2 = 0 \qquad (7)$$

which gives for the point $y = 0$, $\upsilon = 0$

$$\lambda_{1,2} = -\,\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + a} \qquad (8)$$

so that the origin of the phase plane is a saddle point (two real roots λ with opposite signs), so that this solution is unstable and $y \rightarrow \infty$,

$\upsilon \to \infty$ for $t \to \infty$. The critical points $y = \pm\sqrt{\frac{a}{b}}$, $\upsilon = 0$ give

$$\lambda_{1,2} = -\frac{\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} - 2a} \tag{9}$$

so that these points are either a stable focus (complex $\lambda_{1,2}$ with negative real parts) for $\alpha^2 < 8a$, indicating a damped oscillation, or a stable mode (both λ are real and negative) for $\alpha^2 > 8a$, indicating that no oscillation occurs and $y \to$ const or zero, $\upsilon \to 0$ for $t \to \infty$.

In the case given by Lashinsky[1], $\alpha \approx 10$, $a \approx 1$ and we have no oscillating solution, see section 3. In this section we investigate oscillatory solutions, assuming $\alpha \approx 0, 1, a \approx 1$. It is of interest that the value of b does not enter directly into this discussion which is based on a linearized stability analysis. This feature is noteworthy, since both terms $\alpha y'$ and $- ay$ are nonoscillatory and the only term that produces oscillations is by^3. The coefficient b enters the discussion only through (5).

In order to eliminate b completely, we make the transformation

$$y(x) = a^{1/2}\, b^{-1/2}\, u(t) \tag{10}$$

which gives

$$u'' + \alpha u' - au + au^3 = 0 \tag{11}$$

In the oscillatory case $\alpha^2 < 8a$ and the smaller α the weaker the dissipation. Neglecting $\alpha u'$ as a first approximation, by multiplying (11) by u' and integrating we find

$$\frac{u'^2}{2} - \frac{au^2}{2} + \frac{au^4}{4} = E \tag{12}$$

where E is an integration constant (the energy). In this conservative system no energy is lost and energy oscillates between kinetic and potential energy. If a small dissipative term $\alpha u'$ is now introduced, then E decreases slowly and E = const must be replaced by E(t). This procedure can be used to establish a perturbation method based on the averaging process[9].

From (12) we have

$$t(u) + \varphi = \int \frac{du}{\sqrt{2E + au^2 - \frac{au^4}{2}}} \tag{13}$$

where φ is another integration constant. The integral is an elliptic integral and may be solved by standard methods[10].

The inverse function u(t) - and also y(t) - may be expressed by Jacobi elliptic functions whose modulus k depends on the energy constant E[11,12]. If E varies slowly because of the dissipative term $\alpha u'$, then $k \to k(t)$ and we can use the averaging method we have described elsewhere[9].

NON-OSCILLATORY SOLUTIONS

This is the case discussed by H. Lashinsky[1]. For $\alpha \approx 10$, $a \approx 1$ the non-oscillatory condition $\alpha^2 > 8a$ is satisfied. We start from (11) using the transformation $\alpha t = \xi$, so that

$$\ddot{u} + \dot{u} - \mathcal{E}u + \mathcal{E}u^3 = 0 \tag{14}$$

where $\dot{u} = du/d\xi$ and $\mathcal{E} = a/\alpha^2 = 10^{-2}$. Now the perturbation problem is no longer singular (as it was for $\alpha^2 < 8a$) and we may set up a Poincare expansion of successive approximations

$$u(\xi) = u_o(\xi) + \mathcal{E}\, u_1(\xi) + \mathcal{E}^2\, u_2\,(\xi) + \ldots \tag{15}$$

Substituting into (14) we obtain

$$\ddot{u}_o + \dot{u}_o = 0 \tag{16}$$

$$\ddot{u}_1 + \dot{u}_1 = u_o - u_o^3 \tag{17}$$

$$\ddot{u}_2 + \dot{u}_2 = u_1 - 3\, u_o^2 u_1 \tag{18}$$

neglecting higher orders than $\mathcal{E}^2$. Integration yields

$$u_o(\xi) = Ae^{-\xi} + B \tag{19}$$

$$u_1(\xi) = - A\xi e^{-\xi} - A^3 \frac{1}{6} e^{-3\xi} - \frac{3A^2B}{2} e^{-2\xi} + 3AB^2\xi e^{-\xi} - (-B+B^3)\xi + Ce^{-\xi} + D \tag{20}$$

For $\xi \to \infty$, the terms containing $B\xi$, $B^3\xi$ tend to infinity, so that the expansion (15) does not converge for a finite number of terms. These terms are typical secular terms. In the oscillatory solution the secular terms are avoided by the averaging process[9]. In the present case the secular terms are eliminated by appropriate adjustment of the arbitrary integration constants (Lindstedt method[13]). To do this we put $B = 0$, $D = 0$. The terms $\xi e^{-\xi}$ etc. are not secular. For $u_2(\xi)$ we have

$$u_2(\xi) = \frac{7}{18} A^3 e^{-3\xi} + \frac{1}{2} A^3 \xi e^{-3\xi} + \frac{1}{40} A^5 e^{-5\xi} + \frac{A}{2} \xi^2 e^{-\xi} + A\xi e^{-\xi} - C\xi e^{-\xi} - \frac{A^2C}{2} e^{-3\xi} + Ee^{-\xi} + F \tag{21}$$

For $\xi \to \infty$ we now have from (15) and (19) - (21)

$$u(\infty) = \varepsilon^2 F, \quad u'(\infty) = 0, \quad u''(\infty) = 0 \tag{22}$$

so that from (14) $F = \varepsilon^{-2}$ follows.

The following two initial conditions are of physical interest:

$$u(0) = d, \qquad \dot{u}(0) = f$$

where

a) $f = 0$, d given; b) f given, $d = 0$.

We then have two equations for the three remaining constants A, C, E so that one constant may be chosen arbitrarily e.g. to give a better fit to data obtained from a numerical integration. However, in order to adjust a solution of the type (15) to the solution obtained from a numerical integration, higher powers of $\mathcal{E}$ than the second must be taken into account. Thus a straightforward numerical integration is simpler than the expansion (15). The expansion, however, gives more physical insight into the phenomena. The energy fed into the system at t = 0 is used to excite several modes - the higher the initial energy, the greater the number of modes that are excited. The higher the order of the mode, the more rapidly it decays, so that after a certain time only the asymptotic value 1 remains.

The result of the numerical integration for $\mathcal{E} = 10^{-2}$, f = 0, d = 0.2 are shown in Fig. 1; the values are given in Table 1. For $\mathcal{E} = 10^{-2}$, f = 0.2, d = 0 we find the values in Fig. 2 and Table 2.

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Table 1

x	y
0	0.20000
0.09999	0.20009
0.49999	0.20020
0.99999	0.20070
4.99986	0.20776
9.99969	0.21783
19.9973	0.23928
39.9863	0.28779
59.9757	0.34410
79.0148	0.40493
100.003	0.47911
120.992	0.55836
126.988 (a)	0.58143
139.981	0.63125
160.020	0.70539
180.009	0.77259
200.998	0.83272
219.987	0.87670
250.021	0.92699
280.004	0.95840
309.989	0.97684
343.984	0.98959
375.912	0.993722
400.000	0.99621

Table 2

x	y
0	0
0.04999	0.009754
0.09999	0.019033
0.99999	0.126628
4.99986	0.20461
9.99969 (b)	0.21584
19.9973	0.23713
39.863	0.285267
59.9753	0.34120
79.0148	0.40169
100.003	0.47556
120.992	0.55466
127.988 (a)	0.58156
139.981	0.62758
160.020	0.70149
180.009	0.76956
200.998	0.83022
219.987	0.87471
250.021	0.92570
280.004	0.95763
309.989	0.97639
349.984	0.98939
375.012	0.99360
400.000	0.99614

(a) Here we have a maximum of y'.
(b) Near here we have a minimum of y'.

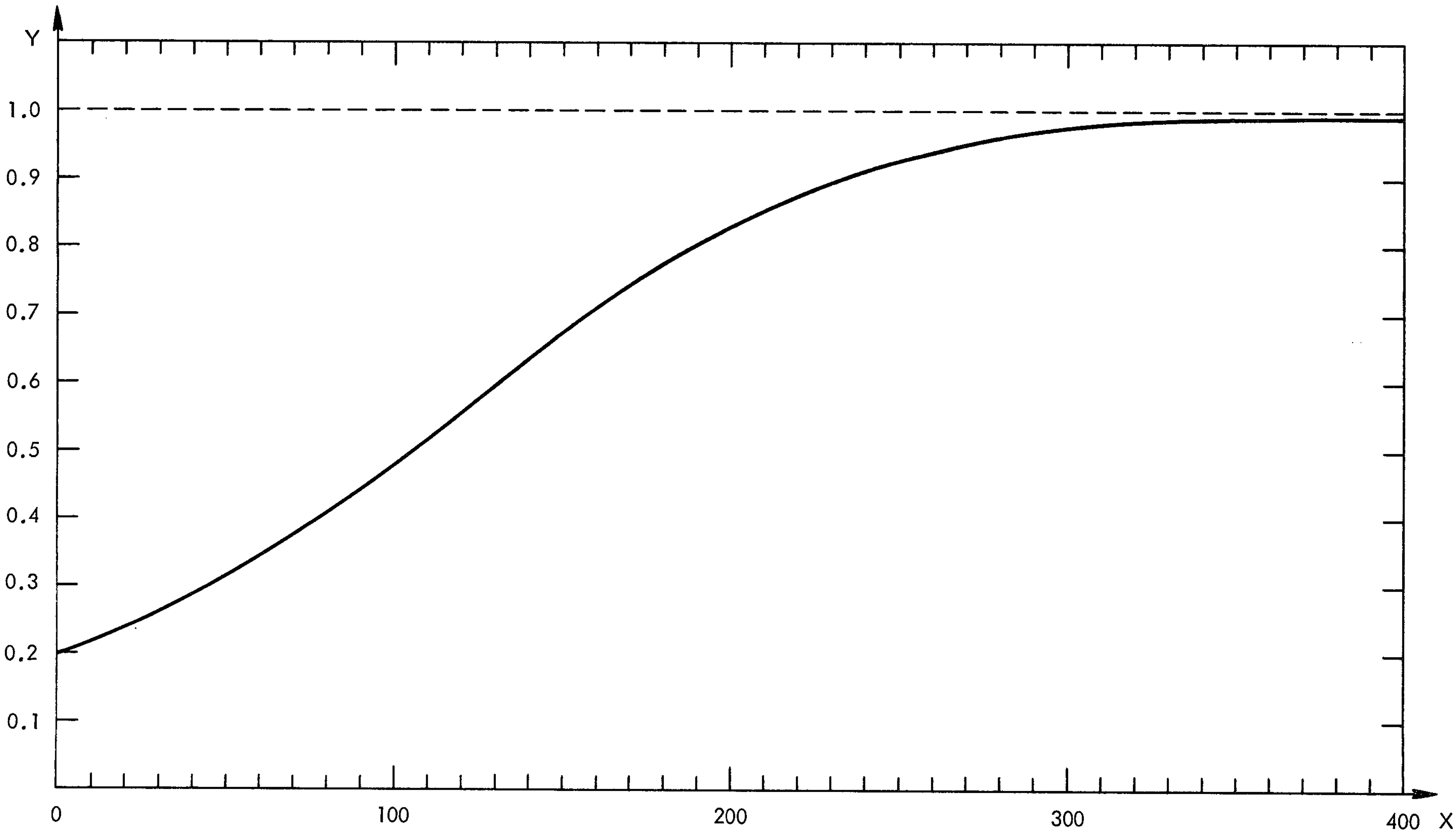

Figure 1.

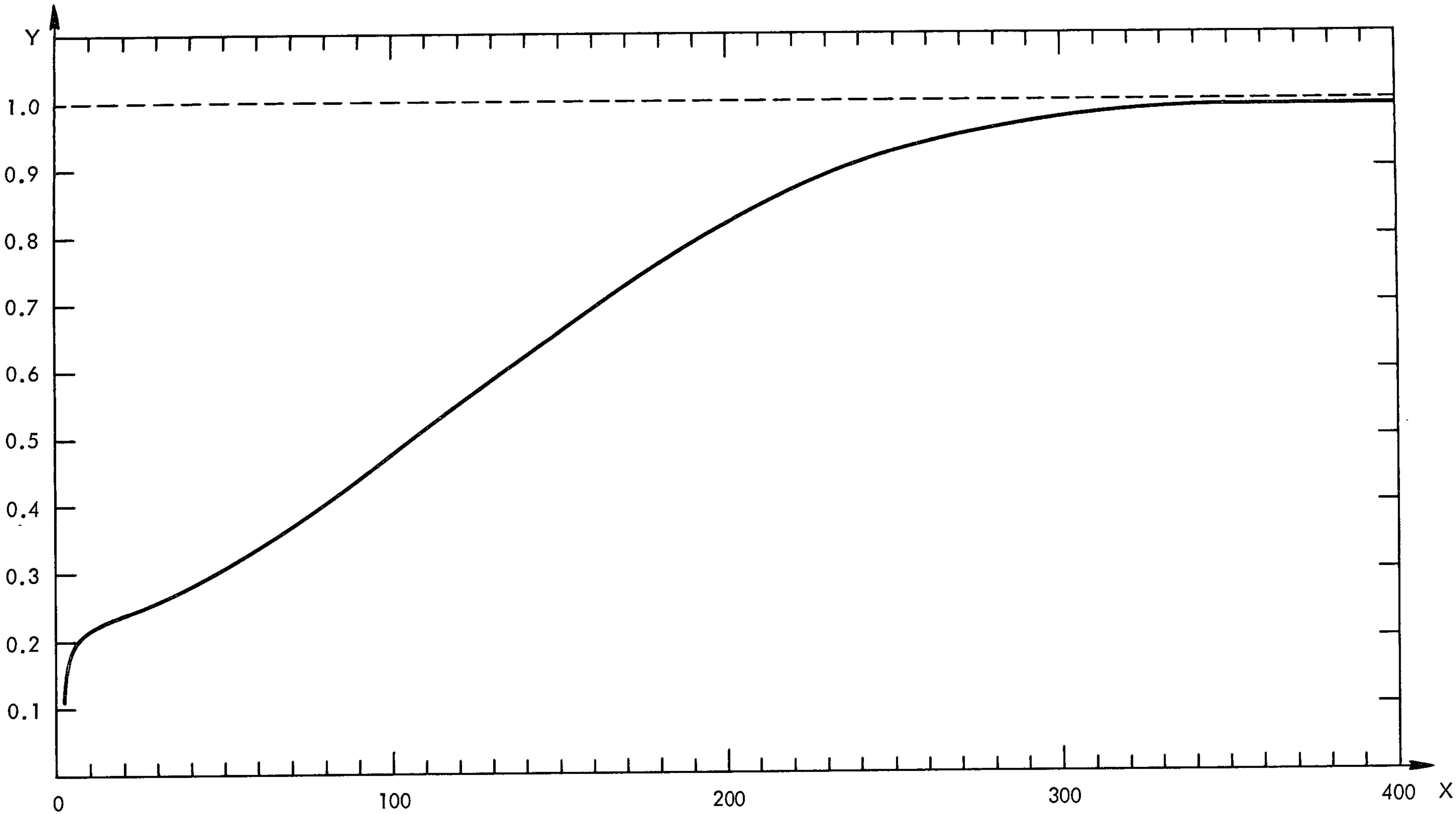

Figure 2.