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QUASI-LINEAR THEORY OF ELECTRON DENSITY AND TEMPERATURE FLUCTUATIONS WITH APPLICATION TO MHD GENERATORS AND MPD ARC THRUSTERS

by J. Marlin Smith
Lewis Research Center
Cleveland, Ohio

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QUASI-LINEAR THEORY OF ELECTRON DENSITY AND TEMPERATURE FLUCTUATIONS WITH APPLICATION TO MHD GENERATORS AND MPD ARC THRUSTERS

J. Marlin Smith, Aerospace Engineer
National Aeronautics and Space Administration
Lewis Research Center
Cleveland, Ohio

Abstract

Fluctuations in electron density and temperature coupled through Ohm's law are studied for an ionizable medium. The nonlinear effects are considered in the limit of a third order quasi-linear treatment. Equations are derived for the amplitude of the fluctuation. Conditions under which a steady state can exist in the presence of the fluctuation are examined and effective transport properties are determined. A comparison is made to previously considered second order theory. The effect of third order terms indicates the possibility of fluctuations existing in regions predicted stable by previous analysis.

Introduction

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Under conditions appropriate to MHD generators and MPD arc thrusters studies have shown that fluctuations in electron density and/or temperature can be unstable.[1,2] In both of these analyses only the linear effect of fluctuations was considered leading to the usual stability analysis dispersion relation from which the growth or damping of the fluctuation can be ascertained. In this paper we extend these analyses into the nonlinear regime in order to determine the amplitude of the fluctuation and its effect upon the properties of interest.

The oscillations under study primarily arise due to fluctuations in electron temperature. These temperature fluctuations can be amplified through increased Ohmic heating arising as a result of increased ionization and/or decreased collision frequency. These changes in frequency occur as a consequence of the initial electron temperature perturbation. The dispersion relation depicting this phenomena was derived in reference (2) by considering the time dependent electron density and temperature equations coupled through a generalized Ohm's law.

In this paper the nonlinear effects of the fluctuations are considered on the basis of a third order quasi-linear theory neglecting mode coupling. Even though the theory is strictly valid only in the neighborhood of the threshold of stability, the third order terms are included in an attempt to increase the applicability of the results. The analysis is carried out for those fluctuations which show the maximum rate of growth, i.e., first become unstable. Two limiting cases are considered. In the first limit, the ionization is assumed to be frozen. In this limit it is shown that the system can become unstable if electron collision frequencies decrease sufficiently rapidly with electron temperature. For the MHD generator, this mode does not appear to be important since operating conditions and working fluids of interest are such that the instability criterion is seldom satisfied. One notable exception may occur, however, when the gas is strongly preionized prior to entering the MHD channel. In this case, the ionization may be sufficient that Coulomb collisions dominate and the instability criterion is satisfied. This was mentioned in reference (3) as a possible source of the anomolous oscillations observed in these experiments with preionization. In reference (3) the oscillation was referred to as the static instability mentioned in reference (1) and is akin to our frozen flow limit. In this limit, the medium is assumed to be infinite and homogeneous and the results of the quasi-linear analysis are presented for a fluctuation propagating in the direction of maximum growth.

For the MPD arc thruster, the fluctuations are assumed to occur in the region of high current concentration in the throat of the device which is depicted by an annular region between two concentric cylinders as shown in figures 1 and 2. Since damping due to radiation and heat conduction are least prominent for long wavelengths, the most unstable waves are those with the longest wavelengths. In the case where the interelectrode distance is small compared to the mean circumference of the annular region, the longest wavelength is obviously nearly that of purely rotational propagation. The analysis of the MPD arc thruster is therefore simplified by considering only rotational propagation.

In the second limiting case, the ionization rate is assumed infinite. This case has been extensively analyzed with regard to MHD generators and was reviewed in reference (4). In that study, terms were retained to second order in fluctuating quantities. In our work, third order terms and some terms ignored in reference (4) as being small in the range of interest are retained. In the range of interest, i.e., the region in which stability is reached in the presence of fluctuations of finite amplitude, the effect of these additional terms is minor, at least in so far as present experimental data is interpretable. However, an interesting point arises in that these additional terms open the possibility of instability occurring in a region where linear theory predicts stability. In this case, instability can occur if a fluctuation of sufficient initial amplitude is present. Such fluctuations may be excited in the preionization region of the MHD duct.

For MPD arc thrusters in the purely rotational approximation the mode occurring in the infinite ionization rate limit is damped. For propagation at small angles from the rotational direction the mode can be unstable but requires large values of the Hall parameter for instability to occur. From the analysis of reference (2), these values appear to be much higher than those which occur in the high current throat area of present high pressure MPD thrusters. Therefore, it is not expected that this mode of instability plays an important role in the rotating spoke phenomena observed in these devices.

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Assumptions and Governing Equations

Assumptions

In those regimes where electrothermal disturbances predominate, the following assumptions, simplifications, and restrictions are appropriate:

1. The analysis is restricted to sufficiently short wavelengths such that the zeroth order properties do not vary appreciably over this distance, i.e., spatial gradients in zeroth order quantities are ignored.

2. Only fluctuations in space and/or time of the electron number density and temperature are considered. This is reasonable since the relatively lighter and more mobile electrons respond to disturbances of greatly different frequencies than do the heavier and less mobile neutral atoms and ions.

3. Only the propagation of magnetohydrodynamic disturbances is considered. This restriction implies neglecting the displacement current density in Maxwell's equations, ignoring induced magnetic field relative to the applied constant magnetic field, and assuming the plasma to be quasi-neutral.

4. Ion and neutral particle flow velocities are equal, i.e., ion slip is ignored. Furthermore, all heavy particle flow velocities are assumed equal to $\underline{V}_0$, the gas flow velocity.

5. Only propagation in planes perpendicular to the applied magnetic field is considered.

6. Ion and neutral particle temperatures are assumed equal and equal to T_0, the gas temperature.

For the above limitations the problem is completely determined in terms of the fluctuations of electron density and temperature and their coupling through Ohm's law and Maxwell's equations.

Maxwell's Equations

From Maxwell's equations under the assumption of quasi-charge neutrality and neglecting the induced magnetic field,

$$\bar{\nabla} \times \underline{E} = 0 \tag{1}$$

$$\bar{\nabla} \cdot \underline{j} = 0 \tag{2}$$

Electron Continuity Equation

It is assumed that in the region of interest, the ionization is dominated by electron-neutral atom ionizing collisions and by three body recombination. The electron density equation then takes the form

$$\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \bar{\nabla} n_e = n_e n_s \nu_i - n_e^3 \nu_r \tag{3}$$

It is noted that this equation depends upon the gas flow velocity $\underline{V}_0$, rather than the electron flow velocity, $\underline{V}_e$. This arises as a result of assumption (4) and the condition of quasi-charge neutrality.

Generalized Ohm's Law

The generalized Ohm's Law is

$$\underline{j} = \sigma \underline{E}^{**} - \frac{e}{m_e \nu} \underline{j} \times \underline{B}_0 \tag{4}$$

$$\underline{E}^{**} = \underline{E}^{*} + \frac{1}{n_e e} \bar{\nabla} p_e \tag{5}$$

where $\underline{E}^{*} = \underline{E} + \underline{V}_0 \times \underline{B}_0$ is the electric field in the frame of reference moving with the gas flow velocity $\underline{V}_0$ in the applied magnetic field $\underline{B}_0$, and $(1/n_e e)\bar{\nabla} p_e$ is the force due to electron temperature and density gradients.

The electrical conductivity, σ, is given by

$$\sigma = \frac{n_e e^2}{m_e \nu} \tag{6}$$

where the total electron momentum collision frequency is

$$\nu = \nu_{ec} + \nu_{es} + \nu_{ei} \tag{7}$$

The species collision frequencies are allowed the following general temperature dependence

$$\nu_{e\alpha} = A_{e\alpha} n_\alpha T_e^{a_{e\alpha}} \tag{8}$$

where α = c, s, or i corresponding to electron collisions with carrier gas atoms, seed atoms, or seed ions, respectively. $A_{e\alpha}$ and $a_{e\alpha}$ are arbitrary constants chosen to fit average collision frequency data over the temperature range of interest.

Electron Energy Equation

The electron energy equation is

$$\frac{3}{2} k n_e \left(\frac{\partial}{\partial t} T_e + \underline{v}_0 \cdot \bar{\nabla} T_e\right) - \frac{5}{2} \underline{j} \cdot \bar{\nabla} \frac{kT_e}{e}$$

$$= -2\nu_m n_e \frac{3}{2} k(T_e - T_0) + \underline{j} \cdot \underline{E}^{*}$$

$$- \left(kT_i + \frac{3}{2} kT_e\right)\left(\frac{\partial}{\partial t} n_e + \underline{v}_0 \cdot \bar{\nabla} n_e\right) \tag{9}$$

where ν_m is the collision frequency for energy transfer given by

$$\nu_m = \frac{m_e}{m_c} \nu_{ec} + \frac{m_e}{m_s} \nu_{es} + \frac{m_e}{m_i} \nu_{ei} \tag{10}$$

and the $\bar{\nabla} T_e$ contribution to the heat conduction term and radiation losses are neglected. These terms contribute damping terms to the dispersion relation which vary inversely with the wavelength relative to the wavelength independent elastic collision damping. Since, in quasi-linear theory, one is primarily restricted to the region in the neighborhood of the stability threshold, only those waves which are most unstable are of interest, i.e., the longest wavelengths. It is therefore assumed that at these wavelengths, the wavelength dependent terms are ignorably small and

hence for simplicity are not included in the analysis. The question of the relative importance of these terms has been extensively investigated for MHD generator type plasmas in references (5) and and (6).

Quasi-Linear Theory

The nonlinear problem is now treated by considering small departures from the stability threshold so that pertinent quantities can be written in the form

$$f(r,t) = \langle f \rangle + \tilde{f}(\underline{r},t) \tag{11}$$

where $\tilde{f}$ is the oscillatory function for which the time averaged value defined by the brackets $\langle \tilde{f} \rangle$ is zero and $\langle f \rangle$ is the average value of the quantity f. Furthermore, in keeping with our initial assumption $\langle f \rangle$ is spatially independent. The restriction of small departures from steady state then requires

$$\tilde{f} << \langle f \rangle \tag{12}$$

so that a solution may be formed as a power series expansion in $\tilde{f}$. In the cases considered here, the expansion is carried out to third order in the product of $\tilde{f}$'s. Furthermore, since we are considering small departures, only the fundamental mode is considered, i.e., mode coupling between harmonics or other fundamental modes is ignored.

The perturbation function is then taken to be of the form

$$\tilde{f}(\underline{r},t) = f_s(t)\sin(\underline{\ell} \cdot \underline{r} - \omega t) + f_c(t)\cos(\underline{\ell} \cdot \underline{r} - \omega t) \tag{13}$$

where $\underline{\ell}$ and ω are real, $\underline{\ell} \cdot \underline{B}_0 = 0$ (only consider propagation in plane perpendicular to $\underline{B}_0$), and the long term time dependence of the perturbation function is contained in the coefficients f_s and f_c which are assumed to be slowly varying over a period of the oscillation, τ, so that

$$\left.\begin{aligned} \langle f_s \rangle &\equiv \frac{1}{\tau}\int_t^{t+\tau} f_s(t')dt' \simeq f_s(t) \\ \langle f_c \rangle &\equiv \frac{1}{\tau}\int_t^{t+\tau} f_c(t')dt' \simeq f_c(t) \end{aligned}\right\} \tag{14}$$

The problem of interest is then one of determining under what circumstances, if any, that the amplitude of the fluctuation reaches a steady state value.

The general quasi-linear analysis of the set of equations (1) to (10) is complicated by the two time dependent equations (3) and (9). This allows two fundamental modes to propagate simultaneously. Therefore, in order to simplify the analysis, we consider two limiting cases. In the first case, oscillations in the electron number density are ignored. This is justified if either the ionizable material is completely ionized or the frequency of the oscillation is much greater than the ionization frequency. The electron number density like the gas dynamic properties is then specifiable by equations peculiar to the problem of interest.

In the second case we assume the ionization rate is infinite. In this case the left hand side of equation (3) can be neglected. The electron number density is then specified by

$$\frac{n_e^2}{n_{s0}} = \frac{\nu_i}{\nu_r} \tag{15}$$

It is further assumed that the electrons are initially in Saha equilibrium so that we are only concerned with small fluctuations from equilibrium. Under this condition, the principle of detail balance[7] can be invoked to relate the ionization coefficient, ν_i, to the recombination coefficient, ν_r, through the equilibrium constant. Then

$$\frac{n_e^2}{n_{s0}} = \frac{\nu_i}{\nu_r} \sim T_e^{3/2} \exp\left(-\frac{T_i}{T_e}\right) \tag{16}$$

This case has been extensively analyzed with regard to MHD generators.[4] In this paper, we extend this second order analysis to third order and also consider it with regard to high pressure MPD arc thrusters.

Frozen Ionization Rate Limit

This is the case where electron number density fluctuations are ignored. Before proceeding to the quasi-linear analysis, we consider the results previously obtained[2] from linear stability analysis. For these results extended to include a neutral carrier gas as well as an ionizable species the condition that the system be unstable is

$$1 + 2\left(1 - \frac{T_0}{T_{e0}}\right)\left[1 - \frac{(\underline{\ell} \cdot \underline{j}_0)^2}{\ell^2 j_0^2}\right] \times \left[a_{ec}\frac{\nu_{ec0}}{\nu_0} + a_{es}\frac{\nu_{es0}}{\nu_0} + a_{ei}\frac{\nu_{ei0}}{\nu_0}\right] < 0 \tag{17}$$

Let us first discuss the consequences of this stability criterion for combustion and nonequilibrium MHD generators and then consider the MPD thruster case. In the combustion generator inelastic collisions with the numerous molecular species in the combustion gas work fluid make it impossible to raise the electron temperature much above the gas temperature. In this case then, the factor $(1 - T_0/T_{e0})$ in the inequality (17) is in general, too small for the instability criterion to be satisfied.

In the nonequilibrium MHD generator operating on inert gases seeded with alkali metals the goal is to obtain an elevated electron temperature. An electron temperature approximately twice the gas temperature is typical. There, even for the most unstable direction of propagation, i.e., $\underline{\ell} \cdot \underline{j}_0 = 0$, the unstable condition requires the coefficient $\left[a_{ec}\frac{\nu_{ec0}}{\nu_0} + a_{es}\frac{\nu_{es0}}{\nu_0} + a_{ei}\frac{\nu_{ei0}}{\nu_0}\right] \lesssim -1$.

In general, over the temperature range and for neutral gases of interest, the temperature exponentials a_{ec} and a_{es} are positive or at least only slightly negative (e.g., see Maxwell-averaged cross sections for electron temperatures near 3000° K presented in ref. (8)). Therefore, in

order for the coefficient to be <-1, the plasma must be Coulomb collision dominated since $a_{ei} \simeq -3/2$. Since under normal operating conditions of MHD generators the collision frequency is neutral dominated, it appears that the range of importance of this mode of instability for these devices is small and can be avoided without great difficulty. The one exception to this could occur when the gas is preionized by strong electric fields prior to entering the MHD channel. In this case, high degrees of ionization can be reached and the resulting instability could propagate into the generator region. This has been mentioned in reference (3) as a possible source of the anomolous fluctuations observed in these experiments.

On the basis of the above discussion and the fact that MPD arc thrusters in general operate in the Coulomb collision dominated regime, the quasi-linear analysis in the limiting case of frozen ionization is developed only for $\nu \simeq \nu_{ei}$.

We first consider the equations governing the average and fluctuating electric current densities. Substituting quantities of the form of equation (11) into equation (4), we obtain

$$\underline{j} = (\langle\sigma\rangle + \tilde{\sigma})(\langle\underline{E}^*\rangle + \tilde{\underline{E}}^{**}) - (\langle\beta\rangle + \tilde{\beta})(\langle\underline{j}\rangle + \tilde{\underline{j}}) \times \hat{\underline{b}} \tag{18}$$

where to third order $\langle\underline{E}^{**}\rangle = \langle\underline{E}^*\rangle$, $\hat{\underline{b}}$ is a unit vector in the direction of $\underline{B}_0$ and the Hall parameter, β, is defined by

$$\beta \equiv \frac{eB_0}{m_e\nu} \tag{19}$$

Averaging equation (18) then gives an expression for the average current density.

$$\langle\underline{j}\rangle = \langle\sigma\rangle\langle\underline{E}^*\rangle + \langle\tilde{\sigma}\tilde{\underline{E}}^{**}\rangle - \langle\beta\rangle\langle\underline{j}\rangle \times \hat{\underline{b}} - \langle\tilde{\beta}\tilde{\underline{j}}\rangle \times \hat{\underline{b}} \tag{20}$$

The fluctuating part of the current density is then obtained by subtracting equation (20) from (18), forming the triple cross product $\underline{\ell} \times (\underline{\ell} \times \langle\underline{j}\rangle)$, and using equations (1) and (5), i.e., $\underline{\ell} \times \tilde{\underline{E}}^{**} = 0$, and equation (2), i.e., $\underline{\ell} \cdot \tilde{\underline{j}} = 0$ to obtain

$$\tilde{\underline{j}} = \frac{1}{\ell^2}\left\{\tilde{\sigma}\underline{\ell} \times (\underline{\ell} \times \langle\underline{E}^*\rangle) - \tilde{\beta}\underline{\ell} \times [\underline{\ell} \times (\langle\underline{j}\rangle \times \hat{\underline{b}})]\right\} \tag{21}$$

The fluctuating part of the field, $\tilde{\underline{E}}^{**}$, can be determined from the fluctuating current density equation obtained by subtracting equation (20) from (18), forming the dot product $\underline{\ell} \cdot \tilde{\underline{j}}$, using equation (1) so that $E = -\nabla\phi$, and equation (2), i.e., $\underline{\ell} \cdot \tilde{\underline{j}} = 0$ to obtain

$$\begin{aligned}\langle\sigma\rangle\ell^2\tilde{\underline{E}}^{**} = {} & -\tilde{\sigma}\underline{\ell}(\underline{\ell} \cdot \langle\underline{E}^*\rangle) + \underline{\ell}\tilde{\beta}\underline{\ell} \cdot (\langle\underline{j}\rangle \times \hat{\underline{b}}) \\ & + \underline{\ell}\langle\beta\rangle\underline{\ell} \cdot (\tilde{\underline{j}} \times \hat{\underline{b}}) \\ & + \text{higher order terms}\end{aligned} \tag{22}$$

Equation (20) can now be simplified by using equation (22) to eliminate $\langle\tilde{\sigma}\tilde{\underline{E}}^{**}\rangle$ and equation (21) to eliminate $\langle\tilde{\beta}\tilde{\underline{j}}\rangle$. We then obtain

$$\begin{aligned}\langle\underline{j}\rangle = {} & \langle\sigma\rangle\langle\underline{E}^*\rangle - \langle\beta\rangle\langle\underline{j}\rangle \times \hat{\underline{b}} \\ & - \frac{1}{\ell^2}\underline{\ell}\left[\frac{\langle\tilde{\sigma}^2\rangle}{\langle\sigma\rangle^2} + \left\langle\left(\frac{\tilde{\sigma}}{\langle\sigma\rangle} - \frac{\tilde{\beta}}{\langle\beta\rangle}\right)^2\right\rangle\langle\beta\rangle^2\right]\underline{\ell} \cdot \langle\underline{j}\rangle\end{aligned} \tag{23}$$

It is convenient at this point to define for later use the effective values of the conductivity and Hall parameter. These quantities are defined by

$$\langle\underline{j}\rangle \equiv \sigma_{eff}\langle\underline{E}^*\rangle - \beta_{eff}\langle\underline{j}\rangle \times \hat{\underline{b}} \tag{24}$$

It is obvious from equation (24) that

$$\sigma_{eff} = \frac{\langle\underline{j}\rangle^2}{\langle\underline{j}\rangle \cdot \langle\underline{E}^*\rangle} \tag{25}$$

$$\beta_{eff} = -\frac{\hat{\underline{b}} \cdot (\langle\underline{j}\rangle \times \langle\underline{E}^*\rangle)}{\langle\underline{j}\rangle \cdot \langle\underline{E}^*\rangle} \tag{26}$$

Substituting equation (23) into equations (25) and (26)

$$\sigma_{eff} = \frac{\langle\sigma\rangle}{1 + Z} \tag{27}$$

$$\beta_{eff} = \frac{\langle\beta\rangle + Z\tau}{1 + Z} \tag{28}$$

where

$$Z \equiv \left[\frac{\langle\tilde{\sigma}^2\rangle}{\langle\sigma\rangle^2} + \left\langle\left(\frac{\tilde{\sigma}}{\langle\sigma\rangle} - \frac{\tilde{\beta}}{\langle\beta\rangle}\right)^2\right\rangle\langle\beta\rangle^2\right]\frac{1}{(1 + \tau^2)} \tag{29}$$

$$\tau \equiv \tan\phi \equiv \tan\left[\cos^{-1}\frac{\underline{\ell} \cdot \langle\underline{j}\rangle}{\ell\langle j\rangle}\right] \tag{30}$$

Much of the above manipulation was from hindsight and for the purpose of obtaining the equations for the average current density, equation (23), and the fluctuating current density, equation (21), solely in terms of the fluctuating electrical conductivity, $\tilde{\sigma}$, and the fluctuating Hall parameter, $\tilde{\beta}$. The above equations are completely general, i.e., not restricted to either of our limiting cases. Specialization to these limiting cases arises through the dependence of $\tilde{\sigma}$ and $\tilde{\beta}$ upon $\tilde{n}_e$ and $\tilde{T}_e$ in these limits.

For the case under consideration

$$\tilde{n}_e = a_{ec} = a_{es} = 0;\ a_{ei} = -3/2$$

then from equations (6), (7), and (8)

$$\langle\sigma\rangle = \sigma_0\left(1 + \frac{3}{8}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right) \tag{31}$$

$$\begin{aligned}\frac{\tilde{\sigma}}{\langle\sigma\rangle} = {} & \frac{3}{2}\frac{\tilde{T}_e}{\langle T_e\rangle} + \frac{3}{8}\left(\frac{\tilde{T}_e^2}{\langle T_e\rangle^2} - \frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right) - \frac{1}{16}\frac{\tilde{T}_e^3}{\langle T_e\rangle^3} \\ & - \frac{9}{16}\frac{\tilde{T}_e}{\langle T_e\rangle}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2} + \ldots\end{aligned} \tag{32}$$

From equations (19), (7), (8), (31), and (32)

$$\frac{\langle\beta\rangle}{\beta_0} = \frac{\langle\sigma\rangle}{\sigma_0} \tag{33}$$

$$\frac{\tilde{\beta}}{\langle\beta\rangle} = \frac{\tilde{\sigma}}{\langle\sigma\rangle} \tag{34}$$

where σ_0 and β_0 are defined by equations (6) and (19), respectively, evaluated at the average electron temperature, $\langle T_e\rangle$.

We now consider the energy equation (9) by eliminating $\underline{j} \cdot \underline{E}^*$ in terms of $(1/\sigma)\underline{j}^2$ by taking the dot product of $\underline{j}$ with equation (4). It should then be obvious from equations (21), (23), (32), and (34) that the energy equation can be expanded in a power series in $\tilde{T}_e/\langle T_e\rangle$. Carrying out this expansion to third order in $\tilde{T}_e/\langle T_e\rangle$ and averaging the resulting equation we obtain, after considerable algebraic manipulation, the following equation for the average electron temperature

$$\frac{3}{2} kn_{e0} \frac{\partial}{\partial t} \langle T_e\rangle = -2 \frac{m_e}{m_a} n_{e0}\nu_0 \frac{3}{2} k(\langle T_e\rangle - T_0)$$
$$\times \left\{\left[1 + \frac{15}{8} \frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right] - \frac{\langle T_e\rangle}{(\langle T_e\rangle - T_0)} \frac{3}{2} \frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right\}$$
$$+ \frac{1}{\langle\sigma\rangle}\left[\langle\underline{j}\rangle^2 + \frac{1}{\ell^2}(\underline{\ell} \cdot \langle\underline{j}\rangle)^2 \frac{9}{4} \frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right] \tag{35}$$

At this point, it is appropriate to consider the essence of the analysis. First, it is to determine under what conditions, if any, the amplitude of a fluctuation which is linearly unstable can reach a steady state. The second purpose is to determine the values of the measured properties of the system in the presence of fluctuations. Since most measured properties are obtained with instruments that cannot resolve oscillations on the time scale of the fluctuation, they only measure time averaged properties (ideally).

It may be noted that for the third order the theory considered here, the pertinent equations for the average values, e.g., equation (31), depend upon the average of the square of the amplitude, $\langle\tilde{T}_e^2\rangle$, rather than the amplitude itself. Therefore, it is convenient to cast the equation for the amplitude, $\tilde{T}_e$, in this form. The equation for $\tilde{T}_e$ is obtained by subtracting equation (35) from equation (9), expanding to third order in $\tilde{T}_e/\langle T_e\rangle$, multiplying by $\tilde{T}_e$ and averaging to obtain

$$\frac{3}{2} kn_{e0}\langle T_e\rangle \frac{1}{2} \frac{\partial}{\partial t} \langle\tilde{T}_e^2\rangle = -2 \frac{m_e}{m_a} n_{e0}\nu_0 \frac{3}{2} k(\langle T_e\rangle - T_0)$$
$$\times\left\{-\frac{3}{2} - \frac{35}{16}\frac{3}{2}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2} + \frac{\langle T_e\rangle}{(\langle T_e\rangle - T_0)}\left[1 + \frac{15}{8}\frac{3}{2}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right]\right\}$$
$$\times\langle\tilde{T}_e^2\rangle + \frac{1}{\langle\sigma\rangle}\left\{\left[\frac{3}{2} - \frac{21}{32}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right]\langle\underline{j}\rangle^2\right.$$
$$\left. - 2\left[\frac{3}{2} + \frac{51}{32}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right]\frac{1}{\ell^2}(\underline{\ell} \cdot \langle\underline{j}\rangle)^2\right\}\langle\tilde{T}_e^2\rangle \tag{36}$$

which closes the set of equations describing the average properties of the system in terms of the average of the square of the amplitude of the fluctuation, $\langle\tilde{T}_e^2\rangle$.

Before proceeding to analyze the above equations, another property of interest, namely, the frequency of the fluctuation is determined. This quantity is obtained by forming the equation for the amplitude of the fluctuation, $\tilde{T}_e$, as in the above derivation, but instead of multiplying by $\tilde{T}_e$ and averaging, we multiply by $1/\ell^2\, \underline{\ell} \cdot \nabla\tilde{T}_e$ and average. The frequency in the frame of reference moving with the gas flow velocity is then

$$\omega^* = \omega + \underline{\ell} \cdot \underline{v}_0 = \frac{\underline{\ell} \cdot \langle\underline{j}\rangle}{n_{e0}e} \tag{37}$$

which was the expression previously determined from linear stability analysis.[2] In other words, the frequency of the oscillation depends upon the amplitude of the oscillation only through $\langle\underline{j}\rangle$.

<u>The Effect of Fluctuations on Ohm's Law</u>. The conductivity and Hall parameter are often indirectly determined in experiments by first measuring the electric current density and electromagnetic fields. Then Ohm's Law is applied to determine the conductivity and Hall parameter. In the presence of fluctuations it is then obvious from equation (24) that the measured values are the effective values. Clearly, a discrepancy between theory and experiment arises when one calculates the conductivity and Hall parameter in their classical form, i.e., equations (6) and (19) evaluated at the measured electron temperature. It is therefore of interest to compare the ratio of effective to classical values of these parameters.

From equations (27), (29), (30), and (31) we obtain

$$\frac{\sigma_{eff}}{\sigma_0} = \frac{\left(1 + \frac{3}{8}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right)}{\left[1 + \frac{9}{4}\frac{1}{(1+\tau^2)}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right]} \tag{38}$$

In figure 3, this ratio is plotted as a function of the amplitude function $\langle\tilde{T}_e^2\rangle/\langle T_e\rangle^2$ for various angles of propagation. The interesting point to note is that the effective conductivity is enhanced by the presence of the fluctuation for values of $|\tau| > \sqrt{5}$ but is reduced for values less than $\sqrt{5}$.

From equations (28), (29), (30), (33), and (31), we obtain

$$\beta_{eff} = \frac{\beta_0\left(1 + \frac{3}{8}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right) + \frac{9}{4}\frac{\tau}{(1+\tau^2)}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}}{1 + \frac{9}{4}\frac{1}{(1+\tau^2)}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}} \tag{39}$$

Comparison of equations (39) and (38) shows that in the limiting cases $\tau = 0$ and $|\tau| = \infty$,

$$\frac{\beta_{eff}}{\beta_0} = \frac{\sigma_{eff}}{\sigma_0} \tag{40}$$

At intermediate values of τ, β_{eff}/β_0 depends upon the value of β_0. Therefore, the dividing point between enhancement and degradation of the Hall parameter is not simply a function of the angle of propagation as it was for the conductivity ratio. The plot of β_{eff}/β_0 for the limiting cases is therefore identical to the σ_{eff}/σ_0 plot in figure 3, except that the neutral point $\beta_{eff}/\beta_0 = 1$ occurs for values of τ given by

$$\tau = -\frac{3}{\beta_0}\left[1 \pm \sqrt{1 + \frac{5}{9}\beta_0^2}\right] \tag{41}$$

These values of τ are plotted in figure 4 for various values of β_0. The angles of propagation for which $\beta_{eff} = \beta_0$ are restricted to $0 \le \tau \le \sqrt{5}$ and $-\infty \le \tau \le -\sqrt{5}$. The asymptotic values $\tau = \pm\sqrt{5}$ occur in the limit $\beta_0 \to \infty$. At the limiting point equality (40) again holds as can be seen by taking the limit $\beta_0 \to \infty$ in equation (39) and comparing the results to equation (38).

The Effect of Fluctuations for the Case of Maximum Growth Rate. We are interested in determining whether a steady state can exist in the presence of an infinitesimal oscillation, and if so, under what circumstance the oscillation occurs and what its amplitude is as a function of experimentally measured parameters.

We first consider the amplitude equation (36) for the direction of propagation which results in the maximum growth rate of the disturbance. Since in this direction, the fluctuation is first unstable and grows most rapidly, it is assumed that for small displacements from the stability threshold this is the observed disturbance. For the Coulomb dominated regime considered here ($a_{ec} = a_{es} = 0$, $a_{ei} = -3/2$) this direction, as can be seen from the linear stability term of equation (17), occurs for propagation perpendicular to the average current density, i.e., $\underline{\ell} \cdot \langle \underline{j} \rangle = 0$. For this case, the amplitude function is given by

$$\frac{1}{2}\frac{\partial}{\partial t}\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2} = 2\frac{m_e}{m_a}\nu_0\left[\left(2 - 3\frac{T_0}{\langle T_e \rangle}\right) + \frac{3}{2}\frac{29}{8}\left(\frac{2}{29} - \frac{T_0}{\langle T_e \rangle}\right)\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}\right]\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2} \tag{42}$$

where we have used equation (35) with $\partial/\partial t\ \langle T_e \rangle = 0$ to eliminate $(1/\langle \sigma \rangle)\langle \underline{j} \rangle^2$ and have neglected terms greater than second order in $\langle \tilde{T}_e^2 \rangle/\langle T_e \rangle^2$. We note that the linear term is just the stability condition obtained from the linear analysis and obviously is damping until the instability point, $T_0/\langle T_e \rangle \le 2/3$, is reached. Beyond this point the linear term causes a growth in the amplitude of the oscillation until the nonlinear term, which is damping for $T_0/\langle T_e \rangle > 2/29$, offsets the linear growth term. A steady state is reached when

$$\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2} = -\frac{\left(2 - 3\frac{T_0}{\langle T_e \rangle}\right)}{\frac{3}{2}\frac{29}{8}\left(\frac{2}{29} - \frac{T_0}{\langle T_e \rangle}\right)} \tag{43}$$

Obviously in the case in which $T_0/\langle T_e \rangle < 2/29$ the nonlinear term contributes to the growth and the thermal perturbation is unconditionally unstable. However, it is obvious from equation (43) that as $T_0/\langle T_e \rangle \to 2/29$, $\langle \tilde{T}_e^2 \rangle/\langle T_e \rangle^2 \to \infty$ which violates the quasi-linear condition of small amplitude oscillations, i.e., $\langle \tilde{T}_e^2 \rangle/\langle T_e \rangle^2 \ll 1$. Therefore, this point is not considered further.

The amplitude function, $\langle \tilde{T}_e^2 \rangle/\langle T_e \rangle^2$, as given by equation (43), can be calculated by determining the temperature ratio $T_0/\langle T_e \rangle$ from equation (35). For $\underline{\ell} \cdot \langle \underline{j} \rangle = 0$, the steady state condition $\partial/\partial t \langle T_e \rangle = 0$, and either the MPD thruster geometry shown in figure 2 or the MHD Hall generator configuration, this equation reduces to

$$\left(1 - \frac{T_0}{\langle T_e \rangle}\right)\left\{1 + \left[\frac{15}{8} - \frac{3}{2}\frac{1}{\left(1 - \frac{T_0}{\langle T_e \rangle}\right)}\right]\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}\right\} = \frac{T_0}{\langle T_e \rangle}\left(1 - \frac{3}{8}\frac{\langle \tilde{T}_e^2 \rangle}{\langle T_e \rangle^2}\right)\frac{\langle \beta \rangle^2}{\left(1 + \langle \beta \rangle^2\right)} R \tag{44}$$

where

$$R \equiv \frac{\frac{1}{2} m_a \left(\frac{E}{B_0}\right)^2}{\frac{3}{2} kT_0} \tag{45}$$

is the ratio of the ion drift energy to its random kinetic energy, $E = |\langle \underline{E}^* \rangle|$ and B_0 are the magnitude of the applied electric and magnetic fields, respectively. In deriving equation (44), we have also used equations (23), (31), and (33).

Equations (43) and (44) can be solved simultaneously for given values of the Hall parameter $\langle \beta \rangle$ and the parameter R to determine the amplitude function $\langle \tilde{T}_e^2 \rangle/\langle T_e \rangle^2$. These values are shown in figure 5, where the amplitude function has been plotted as a function of the Hall parameter $\langle \beta \rangle$ for various values of the parameter R. The first thing to be noted from this figure is that for a given value of R, there is no instability, i.e., no physically exceptable solutions exist, until a critical value of $\langle \beta \rangle$ is reached corresponding to the onset of instability. This critical value of $\langle \beta \rangle$ increases with decreasing values of R. Also to be noted from figure 5 is that the amplitude function reaches an asymptotic value as $\langle \beta \rangle \to \infty$, the magnitude of which depends upon the parameter R.

The Effect of Fluctuations for the Case of a Purely Rotational Disturbance. In the previous section the medium was taken to be infinite and homogeneous. As explained in the Introduction in the case where the disturbance is confined to the annular region between two concentric cylindrical electrodes, the longest wavelength and hence the most unstable wavelength (due to radiation and heat

conduction damping) is approximately that which corresponds to pure rotation. In this section, we therefore consider the effect of a purely rotational fluctuation. This was the case considered in reference 2.

The amplitude equation (36) to second order in $\langle\tilde{T}_e^2\rangle/\langle T_e\rangle^2$ reduces, after considerable algebraic manipulation, to

$$\frac{1}{2}\frac{\partial}{\partial t}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2} = 2\frac{m_e}{m_a}\nu_0\frac{1}{(1+\beta_0^2)}\left\{\left[2 - 3\frac{T_0}{\langle T_e\rangle} - \beta_0^2\right]\right.$$

$$+\frac{3}{2}\frac{29}{8}\left[\frac{2}{29} - \frac{T_0}{\langle T_e\rangle} + \frac{23}{29}\beta_0^2 - \frac{53}{29}\beta_0^2\frac{T_0}{\langle T_e\rangle} - \frac{3}{29}\beta_0^4\right]$$

$$\left.\times\frac{1}{(1+\beta_0^2)}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right\}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2} \qquad (46)$$

Unfortunately the dependence of equation (46) upon $T_0/\langle T_e\rangle$ and β_0^2 is such that it is not possible to determine a steady state amplitude of the fluctuation for the same assumptions used in the linearized theory of reference (2). In order to obtain the values of the local gas dynamic properties required in the theory from the available experimental data, a number of assumptions had to be made in the analysis of reference (2). One of these was to take the local gas temperature, T_0, to be sufficiently small so that it could be ignored in the onset region where the frequency of the oscillation was evaluated. If this is done here, the nonlinear term of equation (46) becomes

$$\frac{3}{2}\frac{29}{8}\left[\frac{2}{29} + \frac{23}{29}\beta_0^2 - \frac{3}{29}\beta_0^4\right]\frac{1}{(1+\beta_0^2)}\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2} \qquad (47)$$

which is positive and hence destabilizing even for the maximum value of β_0^2 for which the linear term in equation (46) is destabilizing, i.e., $\beta_0^2 = 2$. Therefore, a more complete steady state solution is required and/or local gas properties must be measured experimentally in order to ascertain the validity of this assumption.

However, a second interpretation is possible. Since, in reference (2), the rotational frequency was calculated at what was assumed to be the upstream onset point, it could well be that the amplitude of the oscillation is growing at this point, reaching its quasi-steady value downstream of the point in question. In this case, the amplitude may indeed be small at the point at which the frequency was calculated so that nonlinear effects can be ignored. The effect of nonlinearity then manifests itself downstream in the growth of the oscillation.

Of further interest is the fact that the linear and nonlinear coefficients in equation (46),

$$C_0 \equiv 2 - 3\frac{T_0}{\langle T_e\rangle} - \beta_0^2 \qquad (48)$$

$$C_1 \equiv \frac{3}{2}\frac{29}{8}\left[\frac{2}{29} - \frac{T_0}{\langle T_e\rangle} + \frac{23}{29}\beta_0^2 - \frac{53}{29}\beta_0^2\frac{T_0}{\langle T_e\rangle} - \frac{3}{29}\beta_0^4\right]$$

$$\times\frac{1}{(1+\beta_0^2)} \qquad (49)$$

respectively, can be positive or negative, depending upon the magnitude of $T_0/\langle T_e\rangle$ and β_0^2. In figure 6, the curves of $C_0 = 0$ and $C_1 = 0$ are plotted as functions of $T_0/\langle T_e\rangle$ and β_0^2. For both coefficients, points $(\beta_0^2, T_0/\langle T_e\rangle)$ above their respective curves result in negative values of the coefficient and hence damp the fluctuation. However, for points $(\beta_0^2, T_0/\langle T_e\rangle)$ below their respective curves, the coefficients are positive so that the terms contribute to the growth of the fluctuation. As a result, the $T_0/\langle T_e\rangle$ against β_0^2 plane is divided into four regions having the following characteristics:

1. Region I - Both coefficients are damping and hence fluctuation is unconditionally damped.

2. Region II - Linear term damping, nonlinear term growing. In this case the usual linear stability theory would predict a damped disturbance. However, the growth contribution of the nonlinear term indicates that, for disturbances with initial magnitude of the amplitude function $\langle\tilde{T}_e^2\rangle/\langle T_e\rangle^2 > |C_0/C_1|$, the disturbances grow unstably. Therefore, in this region fluctuations are damped for $\langle\tilde{T}_e^2\rangle/\langle T_e\rangle^2 < |C_0/C_1|$ and grow unstably for $\langle T_e^2\rangle/\langle T_e\rangle^2 > |C_0/C_1|$.

3. Region III - Both coefficients are positive and hence any disturbance grows unstably.

4. Region IV - Linear term growing, nonlinear term damping. Disturbance grows or damps until a stable equilibrium point is reached with amplitude function $\langle\tilde{T}_e^2\rangle/\langle T_e\rangle^2 = |C_0/C_1|$.

Whether the above regions exist under actual experimental conditions only further experimentation can answer. Furthermore, the effect of higher order terms in those regions in which the theory predicts the unstable growth of the disturbance is not known. In these regions, therefore, a fully nonlinear theory must be developed.

Infinite Ionization Rate Limit

In the case of infinite ionization rate the electron number density is related to the electron temperature by Saha's equation (16). Taking the square root of equation (16) and expanding the right hand side in a power series in $\tilde{T}_e/\langle T_e\rangle$ we obtain for the ratio of the fluctuating density to its average value

$$\frac{\tilde{n}_e}{\langle n_e\rangle} = \left[\frac{1}{2}\left(\frac{3}{2} + \frac{T_i}{\langle T_e\rangle}\right)\frac{\tilde{T}_e}{\langle T_e\rangle} - \frac{1}{8}\left(\frac{3}{2} + \frac{T_i}{\langle T_e\rangle} - \frac{T_i^2}{\langle T_e\rangle}\right)\right.$$

$$\times\left(\frac{\tilde{T}_e^2}{\langle T_e\rangle^2} - \frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\right) - \frac{1}{2}\left(\frac{3}{2} + \frac{T_i}{\langle T_e\rangle}\right)$$

$$\times\frac{1}{8}\left(\frac{T_i^2}{\langle T_e\rangle^2} - \frac{T_i}{\langle T_e\rangle} - \frac{3}{4}\right)\frac{\langle\tilde{T}_e^2\rangle}{\langle T_e\rangle^2}\frac{\tilde{T}_e}{\langle T_e\rangle}$$

$$+\frac{1}{32}\left(\frac{5}{4} + \frac{5}{2}\frac{T_i}{\langle T_e\rangle} - 5\frac{T_i^2}{\langle T_e\rangle^2} + \frac{3}{2}\frac{T_i^3}{\langle T_e\rangle^3}\right)$$

$$\left.\times\frac{\tilde{T}_e^3}{\langle T_e\rangle^3} + \ldots\right] \qquad (50)$$

The important thing to be noted is that for devices of interest in this paper the average electron temperature is in general much less than the temperature corresponding to the ionization energy, i.e.,

$$\frac{T_i}{\langle T_e\rangle} \gg 1 \qquad (51)$$

so that

$$\frac{\tilde{n}_e}{\langle n_e\rangle} \gg \frac{\tilde{T}_e}{\langle T_e\rangle} \qquad (52)$$

Therefore throughout the subsequent analysis terms the order of $\tilde{T}_e/\langle T_e\rangle$ are neglected with respect to $\tilde{n}_e/\langle n_e\rangle$.

The equations governing the average current density $\langle\underline{j}\rangle$ and the fluctuating current density $\tilde{\underline{j}}$ are equations (23) and (21), respectively. These equations are specialized to the present limiting case by specifying the dependence of $\tilde{\sigma}$ and $\tilde{\beta}$ upon $\tilde{T}_e/\langle T_e\rangle$ and $\tilde{n}_e/\langle n_e\rangle$ in this limit. Ignoring $\tilde{T}_e/\langle T_e\rangle$ with respect to $\tilde{n}_e/\langle n_e\rangle$ we obtain from equations (6), (7), and (8)

$$\frac{\tilde{\sigma}}{\langle\sigma\rangle} = (1 - A)\frac{\tilde{n}_e}{\langle n_e\rangle} - A(1 - A)\left(\frac{\tilde{n}_e^2}{\langle n_e\rangle^2} - \frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2}\right)$$

$$+ A^2(1 - A)\frac{\tilde{n}_e^3}{\langle n_e\rangle^3} + A(1 - A)^2\frac{\tilde{n}_e}{\langle n_e\rangle}\frac{\tilde{n}_e^2}{\langle n_e\rangle^2} + \ldots \qquad (53)$$

from equations (19), (7), and (8)

$$\frac{\tilde{\beta}}{\langle\beta\rangle} = -A\frac{\tilde{n}_e}{\langle n_e\rangle} + A^2\left(\frac{\tilde{n}_e^2}{\langle n_e\rangle^2} - \frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2}\right) - A^3\frac{\tilde{n}_e^3}{\langle n_e\rangle^3}$$

$$+ A^3\frac{\langle\dot{r}_e\rangle}{\langle n_e\rangle}\frac{\tilde{n}_e^2}{\langle n_e\rangle^2} + \ldots \qquad (54)$$

where

$$A \equiv \left[\frac{\nu_{ei0}}{\nu_0} - \frac{\nu_{es0}}{\nu_0}\frac{\langle n_e\rangle}{(n_{s0} - \langle n_e\rangle)}\right] \qquad (55)$$

and we have ignored the electron number density dependence of the logarithmic factor in the Coulomb cross section.

Expressions for the average electron number density and the average of the square of the fluctuating electron number density can be obtained by expanding the electron energy equation in a power series in $\tilde{n}_e/\langle n_e\rangle$. This expansion is carried out to third order in $\tilde{n}_e/\langle n_e\rangle$, neglecting terms of the order of $\tilde{T}_e/\langle T_e\rangle$ with respect to $\tilde{n}_e/\langle n_e\rangle$. Since the method is analogous to that considered in the previous limiting case to determine $\langle T_e\rangle$ and $\langle\tilde{T}_e^2\rangle$ only the result is presented here. For the amplitude function we obtain

$$\frac{\partial}{\partial t}\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2} - \Phi_1\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2} - (\Phi_2 - \Phi_3)\left(\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2}\right)^2 \qquad (56)$$

where

$$\Phi_1 \equiv -[(1 + A) + 2K] + \left[(1 - A)\right.$$

$$\left. - 2(1 - A)\frac{1}{(1 + \tau^2)} - 2\beta_0\frac{\tau}{(1 + \tau^2)}\right] \qquad (57)$$

$$\Phi_2 \equiv [(1 + A) + 2K]Z' \qquad (58)$$

$$\Phi_3 \equiv \frac{1}{2}(A - 3)Z' - [2K + (A - A_m)]\frac{5A_m}{4}$$

$$+ [(1 + A) + 2K]\left[(1 + A_m)(A + 2K) - A_m^2\right]$$

$$+ A\left(1 + \frac{1}{2}A\right)[(1 + A) + 2K] \qquad (59)$$

$$Z = [(1 - A)^2 + \langle\beta\rangle^2]\frac{1}{(1 + \tau^2)}\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2} \equiv Z'\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2} \qquad (60)$$

$$2K \equiv \frac{1}{\frac{1}{2}\left(\frac{3}{2} + \frac{T_i}{\langle T_e\rangle}\right)}\frac{\langle T_e\rangle}{(\langle T_e\rangle - T_0)} + (A_m - A) \qquad (61)$$

$$A_m = \frac{m_e}{m_s}\left[\frac{\nu_{ei0}}{\nu_{m0}} - \frac{\nu_{es0}}{\nu_{m0}}\frac{\langle n_e\rangle}{(n_{s0} - \langle n_e\rangle)}\right] \qquad (62)$$

The coefficient Φ_1, of the linear term is proportional to the stability term obtained from linear wave analysis. Its consequences are reviewed in reference (4). It is found that the oscillation is most unstable for propagation at the angle given by

$$\tau_m = -\frac{1 - A}{\langle\beta\rangle} - \sqrt{\left(\frac{1 - A}{\langle\beta\rangle}\right)^2 + 1} \qquad (63)$$

The linear term is then found to be positive and hence growing for a Hall parameter

$$\beta > \beta_c \equiv 2\sqrt{(1 + K)(A + K)} \qquad (64)$$

The coefficient Φ_2 is the result obtained

in reference (9) and reviewed in reference (4). This result was obtained for an expansion of the energy equation to second order in $\tilde{n}_e/\langle n_e\rangle$ and under the assumption that Z' was large in the region of interest so that nonlinear terms not involving Z' could be ignored. These previously ignored terms as well as third order terms are contained in the coefficient Φ_3.

In figures (7) and (8) the values of $\sigma_{eff}/\langle\sigma\rangle$ and $\beta_{eff}/\langle\beta\rangle$ are plotted, respectively, as a function of $\langle\beta\rangle$ for the theory of reference (5), the present third order theory, and for the experimental data of reference (10) for $\langle j\rangle = 2$ amps/cm^2. The theoretical curves are calculated by the method of reference (9) in which a value of the critical Hall parameter, β_c, is chosen to best fit the experimental data of figures 7 and 8. This then allows the energy loss factor K to be calculated from equation (63). All other parameters are calculated from the experimental data of reference (10). The amplitude function, $\langle\tilde{n}_e^2\rangle/\langle n_e\rangle^2$, is then calculated from equation (56) for $\partial/\partial t\,\langle\tilde{n}_e^2\rangle/\langle n_e\rangle = 0$. $\beta_{eff}/\langle\beta\rangle$ and $\sigma_{eff}/\langle\sigma\rangle$ are then obtained from equations (27) and (28), respectively, using equations (53) and (54).

It is seen from figures 7 and 8 that the additional terms derived in this paper do not have a major effect upon the interpretation of the experimental data, at least as it is presently interpreted. The agreement with effective Hall parameter is slightly better and the agreement with effective conductivity is slightly worse. The interesting point, however, is that the terms contained in Φ_3 can contribute a negative value to the coefficient of the nonlinear term. This opens the possibility of instability occurring in a region where linear theory predicts stability, i.e., $\Phi_1 < 0$. In this case, instability can occur for $\Phi_3 > \Phi_2$ if the initial amplitude of the fluctuation is

$$\frac{\langle\tilde{n}_e^2\rangle}{\langle n_e\rangle^2} > \frac{|\Phi_1|}{|\Phi_3| - |\Phi_2|}$$

In figure 9 we have plotted (for the conditions of fig. 4 of ref. (5), i.e., $\beta_c = 2$, $A_m = A = 1$) the values of $\langle\tilde{n}_e^2\rangle/\langle n_e\rangle^2$ for an unstable solution in the linear stable region $\beta \leq \beta_c = 2$. Obviously for these conditions the initial amplitude must be large in order for the system to be unstable. One means by which an initially large amplitude fluctuation could occur would be propagation of fluctuations into the MHD generator from a preionizer. This may also be an explanation of the anomolous results obtained in the experiments of reference (3) during the application of preionization.

It should be emphasized that the prediction of instability in the linearly stable region by the present theory is only a possibility since it occurs as the nonlinear term begins to dominate. Higher order terms may provide sufficient damping to stabilize this region. This question can only be answered by considering the full nonlinear problem.

In MPD arc thrusters limited by geometry to approximately rotational propagation, it has been shown in reference (2) that the linear stability term Φ_1, requires large values of β_0 (the order of the ratio of the mean circumference of the annular region to its width) for instability to occur. Since the analysis indicated that the values of β_0 in the throat area, where the instability is assumed to occur, were of order one it does not appear that this mode of instability is of importance in these devices.

Concluding Remarks

The nonlinear effects of fluctuations in the electron properties of a partially ionized gas have been considered on the basis of a third order quasi-linear theory neglecting mode coupling. The two limiting cases of frozen flow and infinite ionization rate have been investigated with application to MHD generator and high pressure MHD arc thrusters operation.

For MHD generators oscillations occurring in the frozen flow limit are damped for normal operating conditions. However, where strong preionization is used prior to entering the MHD duct such oscillations can be excited and this may explain the anomolous fluctuations observed in the experiments of reference (3). The effect of oscillations in the infinite ionization limit has been previously studied on the basis of a second order quasi-linear theory.(9) A comparison of these results to the third order theory developed in this paper show little difference in as far as the interpretation of present experimental data is possible. However, the present theory indicates the possibility of fluctuations occurring in the region which is linearly stable provided fluctuations of a finite initial amplitude are present in the system. In the MHD generator case these may occur as a result of strong preionization.

In high pressure MPD arcs geometrically confined between two concentric cylindrical electrodes it is argued that the most unstable oscillation occurs for nearly pure rotation propagation. In this case oscillations arising in the infinite ionization limit are damped at least for high pressures. In the frozen flow limit oscillation can occur. Based on the assumption of zero gas temperature in the onset region, which was used in reference (2) to reduce the existing experimental data for comparison with linear theory, the quasi-linear theory predicts unstable growth of the fluctuation. Therefore, as a consequence of this assumption, a "steady state" fluctuation of finite amplitude is not predicted by quasi-linear theory. The validity of the zero gas temperature assumption requires further experimental information or a more complete steady state theory.

Symbols

A, A_m	defined by eq. (55) and (62), respectively
$A_{e\alpha}, a_{e\alpha}$	defined by eq. (8)
$\underline{B}$	magnetic field vector
$\underline{\hat{b}}$	unit vector in magnetic field direction
C_0, C_1	defined by eq. (48) and (49), respectively

$\underline{E}$	electric field vector
e	magnitude of the charge of an electron
$\underline{j}$	electric current density vector
k	Boltzmann constant
$\underline{\ell}$	wave vector
m	particle mass
n	particle number density
p	pressure
R	defined by eq. (45)
$\underline{r}$	position vector
T	temperature of neutrals and ions
T_e	temperature of electrons
T_i	temperature equivalent of ionization potential
t	time coordinate
$\underline{v}$	gas flow velocity vector
Z	defined by eq. (29)
β	Hall parameter
ν	total electron momentum collision frequency
ν_i	ionization coefficient
ν_{ij}	momentum collision frequency between i^{th} and j^{th} particles
ν_m	total electron energy collision frequency
ν_r	recombination coefficient
σ	electrical conductivity
τ	period of oscillation in eq. (14). In rest of text it is defined by eq. (30)
Φ_1, Φ_2, Φ_3	defined by eq. (57) through (59)
ϕ	angle between $\underline{\ell}$ and $\langle \underline{j} \rangle$
ω	wave frequency

Subscripts:

a	heavy particles (neutral atoms and ions)
c	carrier (nonionizable) atoms
e	electron
eff	effective value
i	ions
$\underline{\ell}$	component parallel to the vector $\underline{\ell}$
S	seed (ionizable) atoms
x,y,z	components of x, y, z coordinate system
0	zeroth order value

Superscripts:

~	defined by eq. (11)
*	value in coordinate system moving with gas

Brackets:

$\langle \cdot \cdot \cdot \rangle$	average value of quantity within brackets

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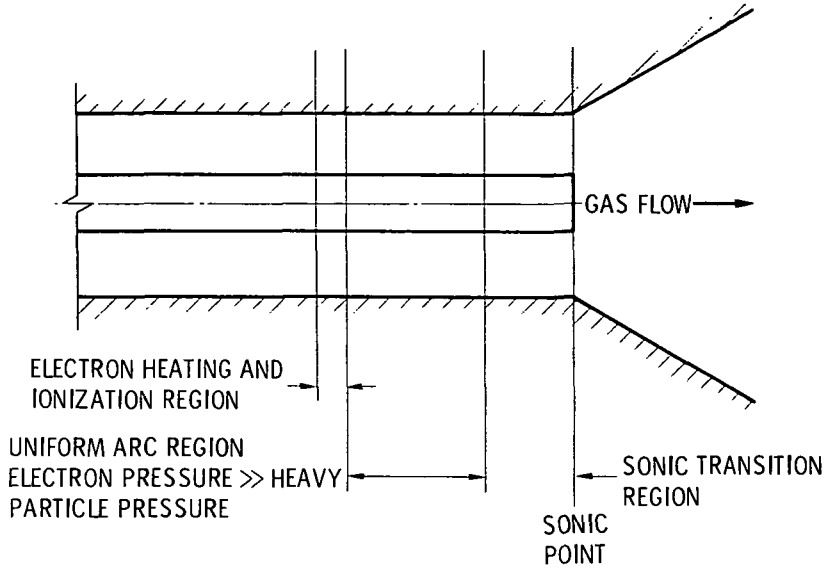
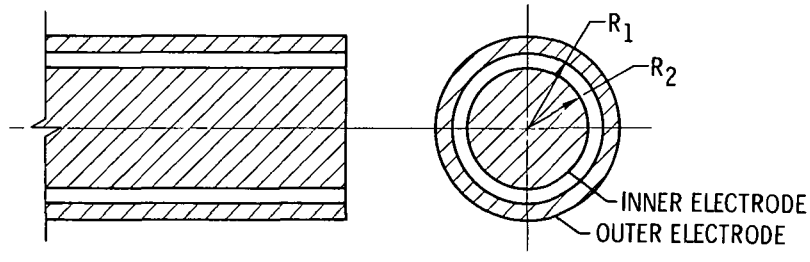
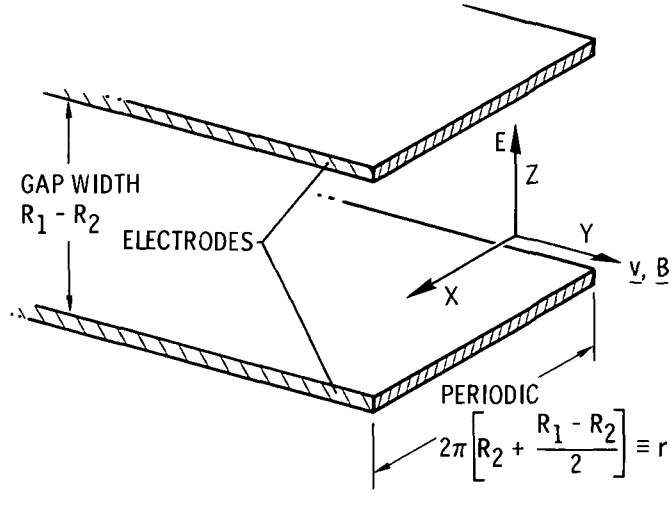


Figure 1. - Theoretical model of MPD thruster.



(A) ASSUMED GEOMETRY.



(B) APPROXIMATE GEOMETRY USED FOR ANALYSIS.

Figure 2. - Thruster geometry.

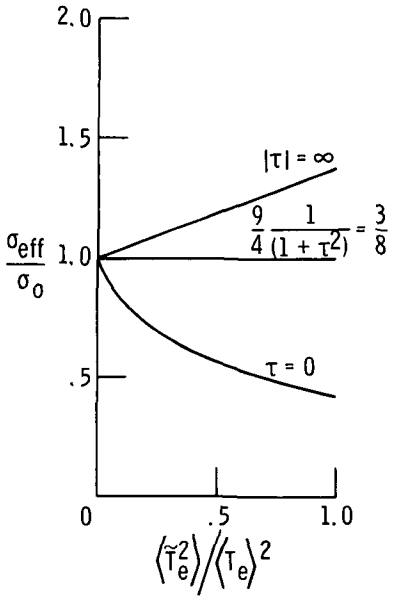


Figure 3. - Ratio of effective to zeroth order electrical conductivity as a function of amplitude function for various angles of propagation.

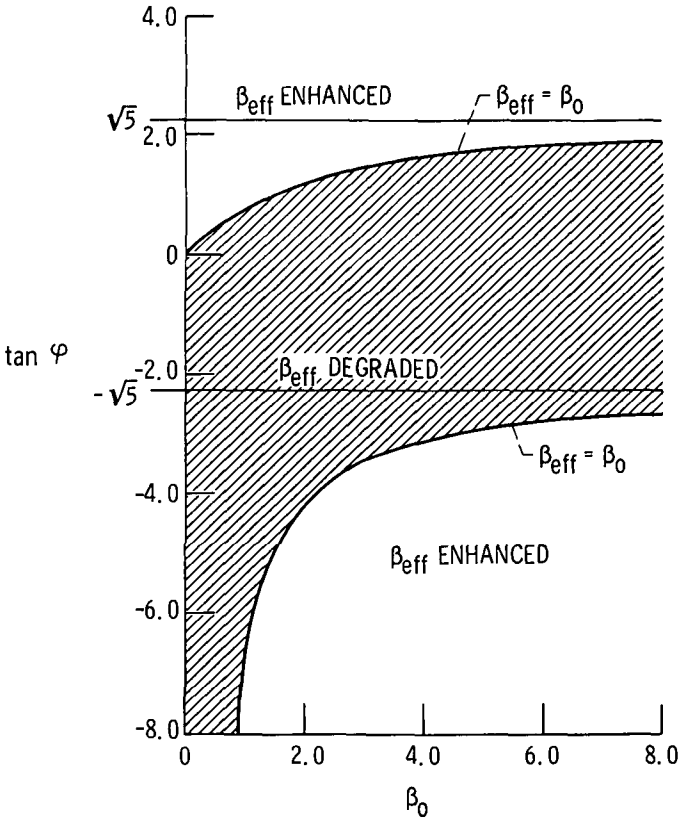


Figure 4. - Value of tan φ at which $\beta_{eff} = \beta_0$ as a function of β_0.

E-6660

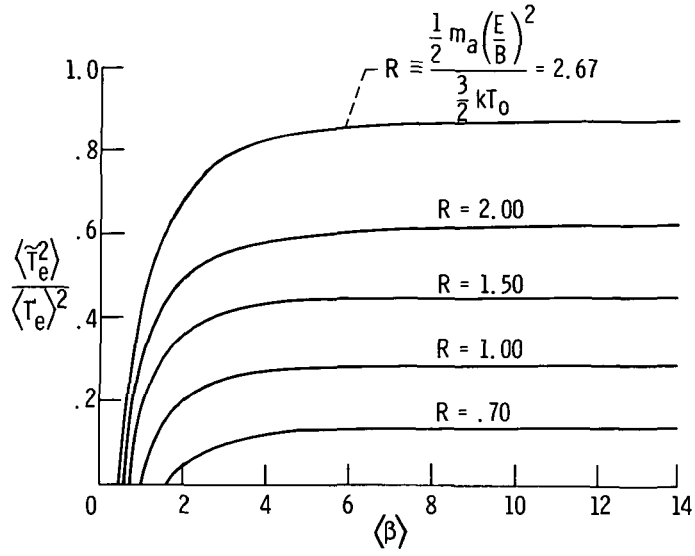


Figure 5. - Amplitude function as a function of the Hall parameter.

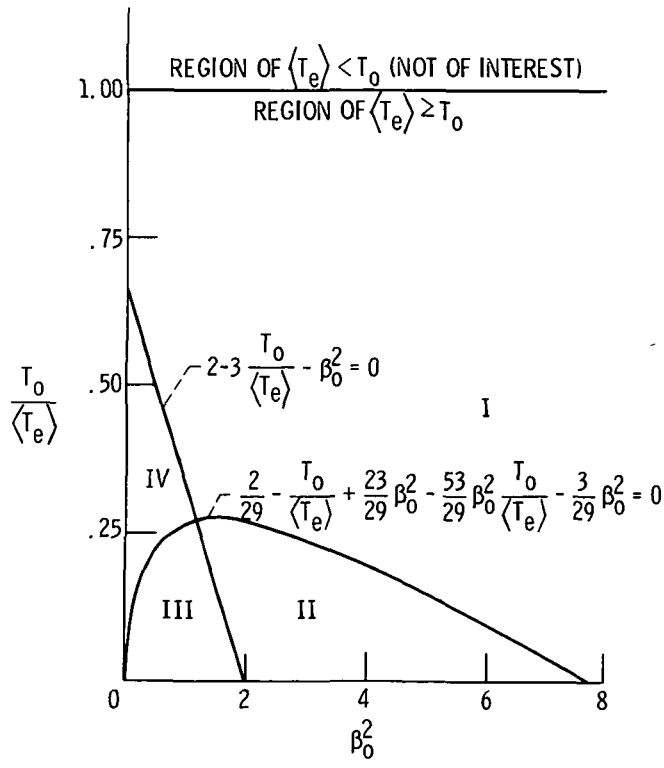


Figure 6. - Damping and growth regions for rotation disturbance.

E-6660

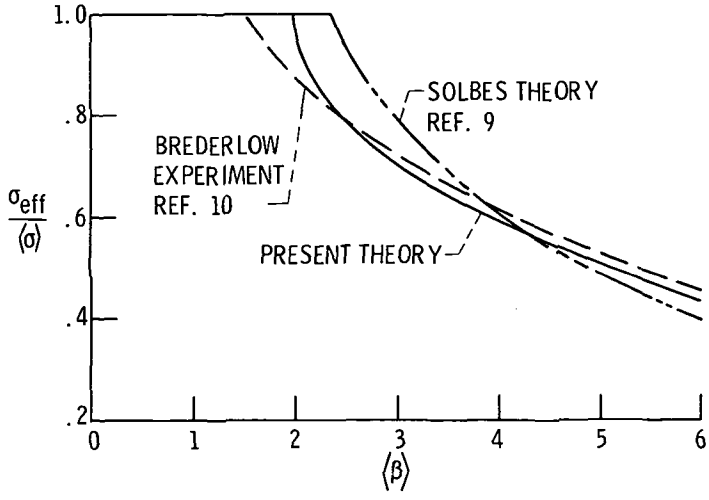


Figure 7. - Ratio of effective to average conductivity versus average Hall parameter.

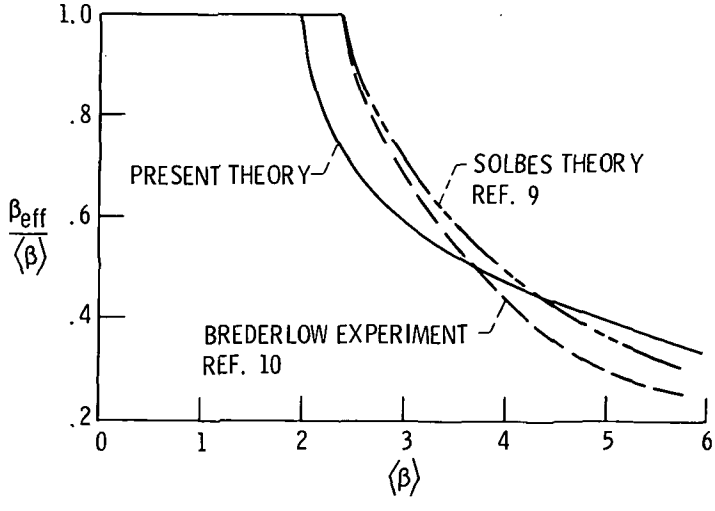


Figure 8. - Ratio of effective to average Hall parameter versus average Hall parameter.

E-6660

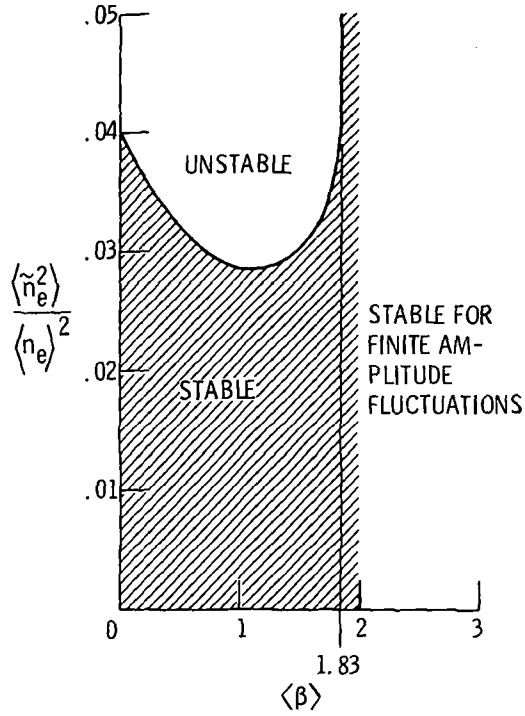


Figure 9. - Amplitude function versus average Hall parameter.