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A MULTIPLE-IMPULSE FUNCTION FOR ORBITAL TRANSFER AND ITS DERIVATIVES

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SUMMARY

A multiple-impulse function is represented as a sequence of single-impulse functions. The single-impulse transfer which yields the velocity change required to transfer from a specified initial orbit to a partially specified final orbit is developed. Analytic derivatives of the function are obtained for use in optimization techniques. A four-impulse transfer is outlined. The analytic derivatives may allow more efficient optimization than numerical derivatives do.

INTRODUCTION

The calculation of the impulsive velocity change required to transfer between two general orbits is a frequently encountered problem in celestial mechanics. Orbital trim, planetary deboost, and orbital rendezvous problems are examples which may require one or more transfer maneuvers to achieve the desired objectives. In most cases, it is also desirable to minimize the fuel expended during the maneuver. A comprehensive survey of previous work in impulsive transfers has been made by Gobetz and Doll (ref. 1). In particular, the two-impulse transfer between inclined elliptical orbits has been studied intensively by McCue (ref. 2) and Lee (ref. 3). However, in some cases, it is desirable to satisfy constraints on the intermediate or final orbits and thus additional impulses may be required. The purpose of this paper is to develop a multiple-impulse function which describes the transfer between two general orbits. In addition, for minimization purposes, the partial derivatives of the function are taken with respect to the independent variables.

The technique is to consider the multiple-impulse transfer to be a sequence of coupled single impulses; thereby, the orbit elements of the intermediate conics are made independent variables. The equations which describe the single-impulse transfer are found to be a function of only four variables on the second orbit; thus, the single-impulse maneuver consists of transferring from a given orbit to an orbit which is only partially specified. The multiple-impulse function can then be minimized over the space defined by four independent variables from each intermediate conic.

The impulse function is developed for a single-impulse transfer. Analytic derivatives of the impulse function are presented and, as an example, the derivatives required to minimize a four-impulse transfer are given.

SYMBOLS

a	semimajor axis, km
e	eccentricity
f	true anomaly, deg
i	inclination, deg
n	number of impulses
$\hat{P}$	unit radius vector lying in orbit plane and pointing toward periapsis
$\hat{Q}$	unit radius vector lying in orbit plane and pointing toward 90° true anomaly
$\hat{R}$	unit vector along line of intersection between two noncoplanar orbits
r	magnitude of radius at a given true anomaly, km
$\hat{r},\hat{h},\hat{n}$	radial, horizontal, and normal components of velocity
V	velocity magnitude, km/sec
ΔV	magnitude of change in velocity required to transfer between two orbits, km/sec
$\hat{W}$	unit vector normal to orbit plane and completing the PQW right-handed triad
β	angle between two velocity vectors, deg
γ	flight-path angle, deg
θ	angle between two orbit planes, deg

μ gravitational constant, km^3/sec^2

Ω right ascension of ascending node, deg

ω argument of periapsis, deg

Subscripts:

0,1 refers to the first or second orbit, respectively

T total

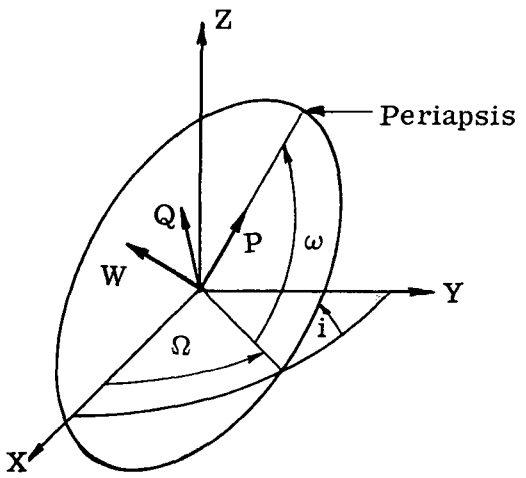
Bars over quantities denote vectors.

ANALYSIS

Derivation of the Impulse Function for a Single-Impulse Transfer

The single-impulse transfer between two noncoplanar intersecting Keplerian orbits can be determined analytically if five elements are known on the first orbit and four elements are specified on the second orbit. It is arbitrarily chosen to specify the classical elements $(a_0, e_0, i_0, \omega_0, \Omega_0)$ on the initial orbit; and the second orbit is specified by the orientation angles $(i_1, \omega_1, \Omega_1)$ and one of the shape parameters (for example, a_1). Of course, other choices are possible and would result in a different formulation.

The first requirement is to find the line of intersection of the two orbits which is uniquely determined by i_0, Ω_0, i_1, and Ω_1. It is convenient to perform a coordinate transformation from the XYZ set of axes to the PQW system as shown in sketch (a).



Sketch (a) - Transformation of XYZ to PQW.

In the PQW system, $\hat{P}$ is a unit vector pointing in the direction of periapsis, $\hat{Q}$ is a unit vector lying in the plane of the orbit and pointing toward a true anomaly of 90°, and $\hat{W}$ is a unit vector, normal to the orbit plane, which completes the right-handed triad. By rotating through the angles Ω, i, and ω, the resulting transformation (ref. 4) is

$$\left.\begin{aligned}
\hat{P} &= (\cos\omega\cos\Omega - \sin\omega\sin\Omega\cos i)\hat{i} + (\cos\omega\sin\Omega + \sin\omega\cos\Omega\cos i)\hat{j} \\
&\quad + (\sin\omega\sin i)\hat{k} \\
\hat{Q} &= (-\sin\omega\cos\Omega - \cos\omega\sin\Omega\cos i)\hat{i} + (-\sin\omega\sin\Omega + \cos\omega\cos\Omega\cos i)\hat{j} \\
&\quad + (\cos\omega\sin i)\hat{k} \\
\hat{W} &= (\sin\Omega\sin i)\hat{i} + (-\cos\Omega\sin i)\hat{j} + (\cos i)\hat{k}
\end{aligned}\right\} \tag{1}$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the corresponding unit vectors in the XYZ coordinate system.

The unit vector $\hat{R}$, along the line of intersection of the two orbits, is determined by

$$\hat{R} = \frac{\hat{W}_0 \times \hat{W}_1}{\left|\hat{W}_0 \times \hat{W}_1\right|} \tag{2}$$

where

$$\hat{W}_0 = \left(\sin i_0 \sin\Omega_0\right)\hat{i} + \left(-\sin i_0 \cos\Omega_0\right)\hat{j} + \left(\cos i_0\right)\hat{k}$$

(normal to first orbit) and

$$\hat{W}_1 = \left(\sin i_1 \sin\Omega_1\right)\hat{i} + \left(-\sin i_1 \cos\Omega_1\right)\hat{j} + \left(\cos i_1\right)\hat{k}$$

(normal to second orbit) are known from equation (1).

Now,

$$\hat{W}_0 \times \hat{W}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin i_0 \sin\Omega_0 & -\sin i_0 \cos\Omega_0 & \cos i_0 \\ \sin i_1 \sin\Omega_1 & -\sin i_1 \cos\Omega_1 & \cos i_1 \end{vmatrix}$$

Expanding this expression results in

$$\hat{W}_0 \times \hat{W}_1 = \begin{bmatrix} -\sin i_0 \cos i_1 \cos \Omega_0 + \sin i_1 \cos i_0 \cos \Omega_1 \\ -\sin i_0 \cos i_1 \sin \Omega_0 + \sin i_1 \cos i_0 \sin \Omega_1 \\ \sin i_0 \sin i_1 \sin \Delta\Omega \end{bmatrix} \tag{3}$$

where $\Delta\Omega = \Omega_1 - \Omega_0$. The magnitude of $\hat{W}_0 \times \hat{W}_1$ is given by

$$\begin{aligned} \left|\hat{W}_0 \times \hat{W}_1\right| = \Big(&\sin^2 i_0 \cos^2 i_1 \cos^2 \Omega_0 - 2 \sin i_0 \sin i_1 \cos i_0 \cos i_1 \cos \Omega_0 \cos \Omega_1 \\ &+ \sin^2 i_1 \cos^2 i_0 \cos^2 \Omega_1 + \sin^2 i_0 \cos^2 i_1 \sin^2 \Omega_0 \\ &-2 \sin i_0 \sin i_1 \cos i_0 \cos i_1 \sin \Omega_0 \sin \Omega_1 + \sin^2 i_1 \cos^2 i_0 \sin^2 \Omega_1 \\ &+ \sin^2 i_0 \sin^2 i_1 \sin^2 \Delta\Omega\Big)^{1/2} \end{aligned}$$

Simplifying this expression results in

$$\left|\hat{W}_0 \times \hat{W}_1\right| = \left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2} \tag{4}$$

Substituting equations (3) and (4) into equation (2) yields the expression for a unit vector in the XYZ system along the line of intersection between the two orbits:

$$\hat{R} = \frac{1}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \begin{bmatrix} -\sin i_0 \cos i_1 \cos \Omega_0 + \sin i_1 \cos i_0 \cos \Omega_1 \\ -\sin i_0 \cos i_1 \sin \Omega_0 + \sin i_1 \cos i_0 \sin \Omega_1 \\ \sin i_0 \sin i_1 \sin \Delta\Omega \end{bmatrix} \tag{5}$$

There are two points where an impulsive maneuver can take place, that is, in the direction of either positive or negative $\hat{R}$. Consider first the case of the transfer point lying in the positive $\hat{R}$ direction. The transfer-point true anomaly on each orbit must be determined.

For the first orbit

$$\hat{P}_0 \cdot \hat{R} = \left|\hat{P}_0\right|\left|\hat{R}\right| \cos f_0$$

or

$$\cos f_0 = \hat{P}_0 \cdot \hat{R} \tag{6}$$

Substituting the expression for $\hat{P}_0$ from equation (1), and equation (5) into equation (6), and simplifying yields

$$\cos f_0 = \frac{-\cos \omega_0\left(\sin i_0 \cos i_1 - \sin i_1 \cos i_0 \cos \Delta\Omega\right) + \sin \omega_0 \sin i_1 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{7}$$

In order to resolve the ambiguity involved with the cosine, an expression for $\sin f_0$ is required.

By definition,

$$\hat{P}_0 \times \hat{R} = \hat{W}_0 \left|\hat{P}_0\right| \left|\hat{R}\right| \sin f_0$$

or

$$\hat{W}_0 \sin f_0 = \hat{P}_0 \times \hat{R}$$

From equation (2),

$$\hat{W}_0 \sin f_0 = \hat{P}_0 \times \frac{\hat{W}_0 \times \hat{W}_1}{\left|\hat{W}_0 \times \hat{W}_1\right|} = \frac{\hat{P}_0 \times \left(\hat{W}_0 \times \hat{W}_1\right)}{\left|\hat{W}_0 \times \hat{W}_1\right|}$$

Using the identity

$$\overline{A} \times \left(\overline{B} \times \overline{C}\right) = \left(\overline{A} \cdot \overline{C}\right)\overline{B} - \left(\overline{A} \cdot \overline{B}\right)\overline{C}$$

where $\overline{A}$, $\overline{B}$, and $\overline{C}$ are any vectors yields

$$\hat{W}_0 \sin f_0 = \frac{\left(\hat{P}_0 \cdot \hat{W}_1\right)\hat{W}_0 - \left(\hat{P}_0 \cdot \hat{W}_0\right)\hat{W}_1}{\left|\hat{W}_0 \times \hat{W}_1\right|}$$

But $\hat{P}_0 \cdot \hat{W}_0 = 0$; therefore,

$$\hat{W}_0 \sin f_0 = \frac{\left(\hat{P}_0 \cdot \hat{W}_1\right)\hat{W}_0}{\left|\hat{W}_0 \times \hat{W}_1\right|}$$

or

$$\sin f_0 = \frac{\hat{P}_0 \cdot \hat{W}_1}{\left|\hat{W}_0 \times \hat{W}_1\right|} \tag{8}$$

Substituting equation (4) and the expressions for $\hat{P}_0$ and $\hat{W}_1$ from equation (1) into equation (8) and simplifying yields

$$\sin f_0 = \frac{\sin \omega_0\left(\sin i_0 \cos i_1 - \sin i_1 \cos i_0 \cos \Delta\Omega\right) + \cos \omega_0 \sin i_1 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{9}$$

In like manner, the equations defining the true anomaly of intersection on the second orbit are

$$\cos f_1 = \frac{\cos \omega_1\left(\cos i_0 \sin i_1 - \sin i_0 \cos i_1 \cos \Delta\Omega\right) + \sin \omega_1 \sin i_0 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{10}$$

and

$$\sin f_1 = \frac{-\sin \omega_1\left(\cos i_0 \sin i_1 - \sin i_0 \cos i_1 \cos \Delta\Omega\right) + \cos \omega_1 \sin i_0 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 - \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{11}$$

The second point where an impulsive maneuver can take place is in the direction of negative $\hat{R}$. The true anomaly of intersection on the first orbit is given by

$$\cos f_0 = \hat{P}_0 \cdot (-\hat{R})$$

and

$$\hat{W}_0 \sin f_0 = \hat{P}_0 \times (-\hat{R})$$

Substituting and simplifying as in equations (7) and (9) yields

$$\cos f_0 = \frac{\cos \omega_0\left(\cos i_1 \sin i_0 - \sin i_1 \cos i_0 \cos \Delta\Omega\right) - \sin \omega_0 \sin i_1 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{12}$$

and

$$\sin f_0 = \frac{-\sin \omega_0\left(\cos i_1 \sin i_0 - \sin i_1 \cos i_0 \cos \Delta\Omega\right) - \cos \omega_0 \sin i_1 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{13}$$

Similarly, the equations defining the true anomaly of intersection on the second orbit are

$$\cos f_1 = \frac{-\cos \omega_1\left(\cos i_0 \sin i_1 - \sin i_0 \cos i_1 \cos \Delta\Omega\right) - \sin \omega_1 \sin i_0 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{14}$$

and

$$\sin f_1 = \frac{\sin \omega_1\left(\cos i_0 \sin i_1 - \sin i_0 \cos i_1 \cos \Delta\Omega\right) - \cos \omega_1 \sin i_0 \sin \Delta\Omega}{\left[1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2\right]^{1/2}} \tag{15}$$

In other words,

$$\left(f_0\right)_{-R} = \left(f_0\right)_{+R} + 180^\circ$$

and

$$\left(f_1\right)_{-R} = \left(f_1\right)_{+R} + 180^\circ$$

For an impulsive maneuver to take place at the point of intersection, the magnitudes of the radii on the first and second orbit must be equal. Therefore, a constraint equation at the point of intersection is

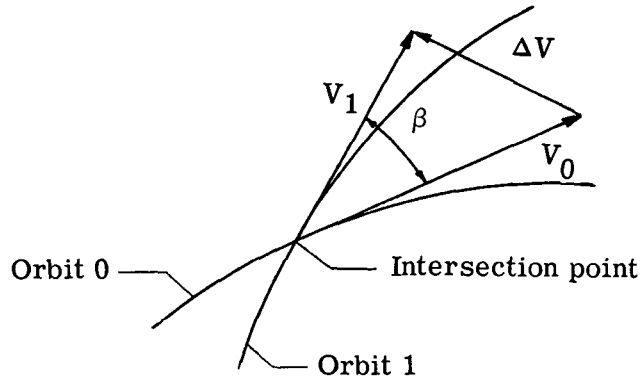
$$\frac{a_0\left(1 - e_0^2\right)}{1 + e_0 \cos f_0} = \frac{a_1\left(1 - e_1^2\right)}{1 + e_1 \cos f_1} \tag{16}$$

The elements a_0, e_0, f_0, f_1, and a_1 or e_1 are known. Therefore, equation (16) relates the unknown element in the second orbit $\left(e_1 \text{ or } a_1\right)$ to known parameters.

Next, an expression for the magnitude of the vector difference in velocity (ΔV) between the two orbits is required. From sketch (b), using the law of cosines,

$$(\Delta V)^2 = V_0^2 + V_1^2 - 2V_0V_1 \cos \beta \tag{17}$$

where β is the angle between the two velocity vectors.



Sketch (b) - Geometry of ΔV.

At the maneuver point, by using the equations

$$V = \left[2\mu\left(\frac{1}{r} - \frac{1}{2a}\right)\right]^{1/2}$$

and

$$r = \frac{a\left(1 - e^2\right)}{1 + e \cos f}$$

the expressions for the magnitude of the velocity on the first and second orbit are

$$V_0 = \left[\frac{\mu\left(1 + 2e_0 \cos f_0 + e_0^2\right)}{a_0\left(1 - e_0^2\right)}\right]^{1/2} \tag{18}$$

and

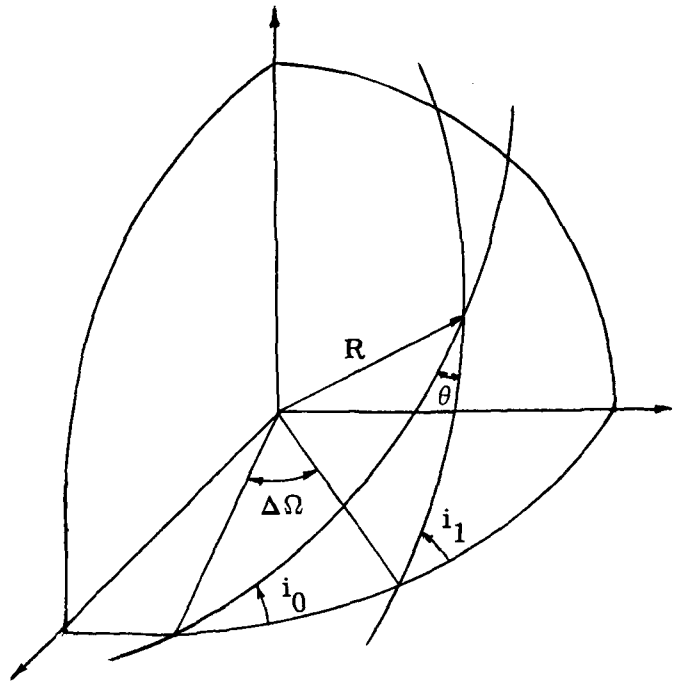
$$V_1 = \left[\frac{\mu\left(1 + 2e_1 \cos f_1 + e_1^2\right)}{a_1\left(1 - e_1^2\right)}\right]^{1/2} \tag{19}$$

From sketch (c), by using the law of cosines for spherical triangles, the angle between the orbit planes θ may be found from

$$\cos \theta = -\cos i_0 \cos\left(180^\circ - i_1\right) + \sin i_0 \sin\left(180^\circ - i_1\right) \cos \Delta\Omega$$

or

$$\cos \theta = \cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega \tag{20}$$



Sketch (c) - Computation geometry for θ.

From sketch (d), the velocity vectors may be written as

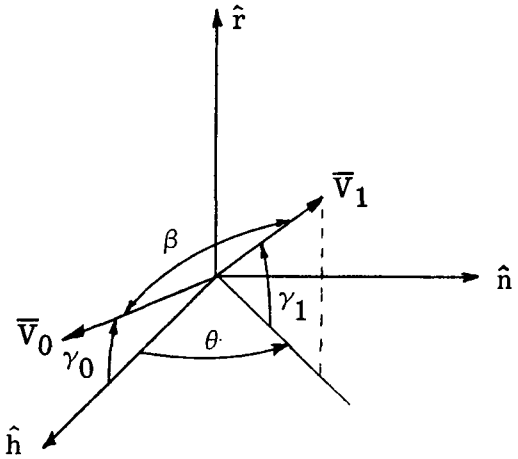
$$\overline{V}_0 = V_0 \begin{bmatrix} \sin \gamma_0 \\ \cos \gamma_0 \\ 0 \end{bmatrix}$$

and

$$\overline{V}_1 = V_1 \begin{bmatrix} \sin \gamma_1 \\ \cos \gamma_1 \cos \theta \\ \cos \gamma_1 \sin \theta \end{bmatrix}$$

where γ_0 and γ_1 are the flight-path angles on the first and second orbits. The angle between the velocity vectors β may now be found from

$$\cos \beta = \frac{\overline{V}_0 \cdot \overline{V}_1}{|V_0 V_1|} = \cos \theta \cos \gamma_0 \cos \gamma_1 + \sin \gamma_0 \sin \gamma_1 \tag{21}$$



Sketch (d) - Velocity vectors in the radial, horizontal, and normal coordinate system.

Substituting the flight-path angle relations from reference 4

$$\sin \gamma = \frac{e \sin f}{\left(1 + 2e \cos f + e^2\right)^{1/2}}$$

and

$$\cos \gamma = \frac{1 + e \cos f}{\left(1 + 2e \cos f + e^2\right)^{1/2}}$$

and equation (20) into equation (21) yields

$$\cos \beta = \frac{\left(1 + e_0 \cos f_0\right)\left(1 + e_1 \cos f_1\right)\left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) + \left(e_0 \sin f_0\right)\left(e_1 \sin f_1\right)}{\left[\left(1 + 2e_0 \cos f_0 + e_0^2\right)\left(1 + 2e_1 \cos f_1 + e_1^2\right)\right]^{1/2}} \tag{22}$$

Substituting equations (18), (19), and (22) into equation (17) results in the required expression for ΔV:

$$\Delta V = \left\{\frac{\mu}{a_0\left(1 - e_0^2\right)}\left(1 + 2e_0 \cos f_0 + e_0^2\right) + \frac{\mu}{a_1\left(1 - e_1^2\right)}\left(1 + 2e_1 \cos f_1 + e_1^2\right)\right.$$
$$- \frac{2\mu}{\left[a_0\left(1 - e_0^2\right)a_1\left(1 - e_1^2\right)\right]^{1/2}}\left[\left(1 + e_0 \cos f_0\right)\left(1 + e_1 \cos f_1\right)\left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)\right.$$
$$\left.\left. + e_0 e_1 \sin f_0 \sin f_1\right]\right\}^{1/2} \tag{23}$$

The set of equations (7), (9) to (16), and (23) are those required for an analytic solution to the problem of a single-impulse transfer between two noncoplanar orbits. Two solutions result from the positive and negative cases for $\hat{R}$. In addition, if e_1 is a function of a_1 in the constraint equation (16), two solutions arise from the quadratic in e_1. All the multiple solutions may not be physically possible. For example, an arbitrarily specified a_1 may result in a maximum radius on the second orbit which is less than the smallest radius on the initial orbit. That is, the constraint equation (16) cannot be satisfied. In addition, if the central body has a finite radius, a region of mathematically possible orbits must be rejected since they intercept the surface of the planet. In the event no practical solutions for the transfer exist, a more realistic set of variables $(a_1, i_1, \omega_1, \Omega_1)$ on the second orbit must be chosen. In the event of multiple practical solutions for the transfer, a choice may be based on mission considerations, such as choosing the transfer which results in the lowest ΔV.

Partial Derivatives of the Single-Impulse Function

In orbital transfer problems, it is usually desirable to minimize ΔV with respect to the control parameters. Many of the minimization algorithms require partial derivatives of ΔV with respect to the independent variables. Finite-difference approximations to the partial derivatives are commonly employed. However, in complex multiple-impulse problems, it may take relatively large amounts of computer time to develop the approximations. If exact partial derivatives are available, the minimization algorithms are generally more efficient and less computer time is used. Therefore, analytic partial derivatives of equation (23) have been developed. It is assumed that a, e, i, ω, and Ω are known on the initial orbit and that a, i, ω, and Ω are known on the final orbit. Eccentricity of the final orbit is a dependent variable defined in the constraint equation (16).

For convenience, define the following intermediate quantities as

$$A_0 = \mu\left[a_0\left(1 - e_0^2\right)\right]^{-1} \qquad A_1 = \mu\left[a_1\left(1 - e_1^2\right)\right]^{-1}$$

$$B_0 = 1 + 2e_0 \cos f_0 + e_0^2 \qquad B_1 = 1 + 2e_1 \cos f_1 + e_1^2$$

$$C = -2\mu\left[a_0\left(1 - e_0^2\right)a_1\left(1 - e_1^2\right)\right]^{-1/2}$$

$$D = \left(1 + e_0 \cos f_0\right)\left(1 + e_1 \cos f_1\right)\left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) + e_0 e_1 \sin f_0 \sin f_1$$

Thus, equation (23) can be written as

$$\Delta V = \left(A_0 B_0 + A_1 B_1 + CD\right)^{1/2}$$

or

$$(\Delta V)^2 = A_0 B_0 + A_1 B_1 + CD$$

The required derivatives can be expressed in the form

$$\frac{\partial \Delta V}{\partial \alpha} = \frac{1}{2\Delta V} \frac{\partial (\Delta V)^2}{\partial \alpha}$$

$$= \frac{1}{2\Delta V}\left(B_0 \frac{\partial A_0}{\partial \alpha} + A_0 \frac{\partial B_0}{\partial \alpha} + B_1 \frac{\partial A_1}{\partial \alpha} + A_1 \frac{\partial B_1}{\partial \alpha} + C \frac{\partial D}{\partial \alpha} + D \frac{\partial C}{\partial \alpha}\right) \tag{24}$$

where α represents any one of the variables a_0, e_0, i_0, ω_0, Ω_0, a_1, i_1, ω_1, and Ω_1.

It is necessary now to develop the partial derivatives of the following six intermediate parameters:

(1) $\frac{\partial A_0}{\partial \alpha}$: Let

$$A_0 = \mu\left[a_0\left(1 - e_0^2\right)\right]^{-1}$$

then

$$\frac{\partial A_0}{\partial a_0} = -\mu a_0^{-2}\left(1 - e_0^2\right)^{-1} = -\frac{A_0}{a_0}$$

Similarly,

$$\frac{\partial A_0}{\partial e_0} = \frac{2A_0 e_0}{1 - e_0^2}$$

and

$$\frac{\partial A_0}{\partial i_0} = \frac{\partial A_0}{\partial \omega_0} = \frac{\partial A_0}{\partial \Omega_0} = \frac{\partial A_0}{\partial \alpha_1} = 0$$

where α_1 represents a_1, i_1, ω_1, and Ω_1.

(2) $\frac{\partial A_1}{\partial \alpha}$: Let

$$A_1 = \mu\left[a_1\left(1 - e_1^2\right)\right]^{-1}$$

then

$$\frac{\partial A_1}{\partial a_0} = 2\mu e_1 a_1^{-1}\left(1 - e_1^2\right)^{-2} \frac{\partial e_1}{\partial a_0} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial a_0}$$

Similarly,

$$\frac{\partial A_1}{\partial e_0} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial e_0} \qquad \frac{\partial A_1}{\partial i_0} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial i_0}$$

$$\frac{\partial A_1}{\partial \omega_0} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial \omega_0} \qquad \frac{\partial A_1}{\partial \Omega_0} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial \Omega_0}$$

$$\frac{\partial A_1}{\partial a_1} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial a_1} - \frac{A_1}{a_1} \qquad \frac{\partial A_1}{\partial i_1} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial i_1}$$

$$\frac{\partial A_1}{\partial \omega_1} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial \omega_1} \qquad \frac{\partial A_1}{\partial \Omega_1} = \frac{2e_1 A_1}{1 - e_1^2} \frac{\partial e_1}{\partial \Omega_1}$$

The $\frac{\partial e_1}{\partial \alpha}$ remain to be determined.

(3) $\frac{\partial B_0}{\partial \alpha}$: Let

$$B_0 = 1 + 2e_0 \cos f_0 + e_0^2$$

Then

$$\frac{\partial B_0}{\partial a_0} = 2e_0 \frac{\partial \cos f_0}{\partial a_0} = 0$$

since

$$\frac{\partial \cos f_0}{\partial a_0} = 0$$

by inspection of equation (7). Similarly,

$$\frac{\partial B_0}{\partial e_0} = 2\left(e_0 + \cos f_0\right) \qquad \frac{\partial B_0}{\partial i_0} = 2e_0 \frac{\partial \cos f_0}{\partial i_0}$$

$$\frac{\partial B_0}{\partial \omega_0} = 2e_0 \frac{\partial \cos f_0}{\partial \omega_0} \qquad \frac{\partial B_0}{\partial \Omega_0} = 2e_0 \frac{\partial \cos f_0}{\partial \Omega_0}$$

$$\frac{\partial B_0}{\partial a_1} = 0 \qquad \frac{\partial B_0}{\partial i_1} = 2e_0 \frac{\partial \cos f_0}{\partial i_1}$$

$$\frac{\partial B_0}{\partial \omega_1} = 0 \qquad \frac{\partial B_0}{\partial \Omega_1} = 2e_0 \frac{\partial \cos f_0}{\partial \Omega_1}$$

The $\frac{\partial \cos f_0}{\partial \alpha}$ remain to be determined.

(4) $\frac{\partial B_1}{\partial \alpha}$: Let

$$B_1 = 1 + 2e_1 \cos f_1 + e_1^2$$

then

$$\frac{\partial B_1}{\partial a_0} = 2e_1 \frac{\partial \cos f_1}{\partial a_0} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial a_0} = 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial a_0}$$

since

$$\frac{\partial \cos f_1}{\partial a_0} = 0$$

by inspection of equation (10). Similarly,

$$\frac{\partial B_1}{\partial e_0} = 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial e_0} \qquad \frac{\partial B_1}{\partial i_0} = 2e_1 \frac{\partial \cos f_1}{\partial i_0} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial i_0}$$

$$\frac{\partial B_1}{\partial \omega_0} = 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial \omega_0} \qquad \frac{\partial B_1}{\partial \Omega_0} = 2e_1 \frac{\partial \cos f_1}{\partial \Omega_0} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial \Omega_0}$$

$$\frac{\partial B_1}{\partial a_1} = 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial a_1} \qquad \frac{\partial B_1}{\partial i_1} = 2e_1 \frac{\partial \cos f_1}{\partial i_1} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial i_1}$$

$$\frac{\partial B_1}{\partial \omega_1} = 2e_1 \frac{\partial \cos f_1}{\partial \omega_1} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial \omega_1} \qquad \frac{\partial B_1}{\partial \Omega_1} = 2e_1 \frac{\partial \cos f_1}{\partial \Omega_1} + 2\left(\cos f_1 + e_1\right)\frac{\partial e_1}{\partial \Omega_1}$$

The $\frac{\partial \cos f_1}{\partial \alpha}$ remain to be determined.

(5) $\frac{\partial C}{\partial \alpha}$: Let

$$C = -2\mu\left[a_0\left(1 - e_0^2\right)a_1\left(1 - e_1^2\right)\right]^{-1/2}$$

then

$$\frac{\partial C}{\partial a_0} = \mu\left[a_0\left(1 - e_0^2\right)a_1\left(1 - e_1^2\right)\right]^{-3/2}\left[a_1\left(1 - e_0^2\right)\left(1 - e_1^2\right) - 2e_1a_0a_1\left(1 - e_0^2\right)\frac{\partial e_1}{\partial a_0}\right]$$

$$= C\left(\frac{e_1}{1 - e_1^2}\frac{\partial e_1}{\partial a_0} - \frac{1}{2a_0}\right)$$

Similarly,

$$\frac{\partial C}{\partial e_0} = C\left(\frac{e_1}{1 - e_1^2}\frac{\partial e_1}{\partial e_0} + \frac{e_0}{1 - e_0^2}\right)$$

$$\frac{\partial C}{\partial i_0} = \frac{Ce_1}{1 - e_1^2}\frac{\partial e_1}{\partial i_0}$$

$$\frac{\partial C}{\partial \omega_0} = \frac{Ce_1}{1 - e_1^2}\frac{\partial e_1}{\partial \omega_0}$$

$$\frac{\partial C}{\partial \Omega_0} = \frac{Ce_1}{1 - e_1^2}\frac{\partial e_1}{\partial \Omega_0}$$

$$\frac{\partial C}{\partial a_1} = C\left(\frac{e_1}{1 - e_1^2}\frac{\partial e_1}{\partial a_1} - \frac{1}{2a_1}\right)$$

$$\frac{\partial C}{\partial i_1} = \frac{Ce_1}{1 - e_1^2}\frac{\partial e_1}{\partial i_1}$$

$$\frac{\partial C}{\partial \omega_1} = \frac{Ce_1}{1 - e_1^2} \frac{\partial e_1}{\partial \omega_1}$$

$$\frac{\partial C}{\partial \Omega_1} = \frac{Ce_1}{1 - e_1^2} \frac{\partial e_1}{\partial \Omega_1}$$

(6) $\frac{\partial D}{\partial \alpha}$: Let

$$D = \left(1 + e_0 \cos f_0\right)\left(1 + e_1 \cos f_1\right)\left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)$$
$$+ e_0 e_1 \sin f_0 \sin f_1$$

Define the intermediate terms as

$$D_0 = 1 + e_0 \cos f_0$$

$$D_1 = 1 + e_1 \cos f_1$$

$$D_2 = \cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega$$

then

$$\frac{\partial D}{\partial a_0} = \left(D_0 D_2 \cos f_1 + e_0 \sin f_0 \sin f_1\right) \frac{\partial e_1}{\partial a_0}$$

$$\frac{\partial D}{\partial e_0} = D_2\left(D_0 \cos f_1 \frac{\partial e_1}{\partial e_0} + D_1 \cos f_0\right) + \sin f_0 \sin f_1\left(e_0 \frac{\partial e_1}{\partial e_0} + e_1\right)$$

$$\frac{\partial D}{\partial i_0} = D_0\left[D_1\left(\cos i_0 \sin i_1 \cos \Delta\Omega - \sin i_0 \cos i_1\right) + D_2\left(e_1 \frac{\partial \cos f_1}{\partial i_0} + \cos f_1 \frac{\partial e_1}{\partial i_0}\right)\right]$$
$$+ D_2 D_1 e_0 \frac{\partial \cos f_0}{\partial i_0} + e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial i_0} + \sin f_1 \frac{\partial \sin f_0}{\partial i_0}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial i_0}\right]$$

$$\frac{\partial D}{\partial \omega_0} = D_2\left[D_0\left(e_1 \frac{\partial \cos f_1}{\partial \omega_0} + \cos f_1 \frac{\partial e_1}{\partial \omega_0}\right) + D_1 e_0 \frac{\partial \cos f_0}{\partial \omega_0}\right]$$
$$+ e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial \omega_0} + \sin f_1 \frac{\partial \sin f_0}{\partial \omega_0}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial \omega_0}\right]$$

$$\frac{\partial D}{\partial \Omega_0} = D_0\left[D_1 \sin i_0 \sin i_1 \sin \Delta\Omega + D_2\left(e_1 \frac{\partial \cos f_1}{\partial \Omega_0} + \cos f_1 \frac{\partial e_1}{\partial \Omega_0}\right)\right]$$
$$+ D_1 D_2 e_0 \frac{\partial \cos f_0}{\partial \Omega_0} + e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial \Omega_0} + \sin f_1 \frac{\partial \sin f_0}{\partial \Omega_0}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial \Omega_0}\right]$$

$$\frac{\partial D}{\partial a_1} = \left(D_0 D_2 \cos f_1 + e_0 \sin f_0 \sin f_1\right)\frac{\partial e_1}{\partial a_1}$$

$$\frac{\partial D}{\partial i_1} = D_0\left[D_1\left(\sin i_0 \cos i_1 \cos \Delta\Omega - \cos i_0 \sin i_1\right) + D_2\left(e_1 \frac{\partial \cos f_1}{\partial i_1} + \cos f_1 \frac{\partial e_1}{\partial i_1}\right)\right]$$
$$+ D_1 D_2 e_0 \frac{\partial \cos f_0}{\partial i_1} + e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial i_1} + \sin f_1 \frac{\partial \sin f_0}{\partial i_1}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial i_1}\right]$$

$$\frac{\partial D}{\partial \omega_1} = D_2\left[D_0\left(e_1 \frac{\partial \cos f_1}{\partial \omega_1} + \cos f_1 \frac{\partial e_1}{\partial \omega_1}\right) + D_1 e_0 \frac{\partial \cos f_0}{\partial \omega_1}\right]$$
$$+ e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial \omega_1} + \sin f_1 \frac{\partial \sin f_0}{\partial \omega_1}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial \omega_1}\right]$$

$$\frac{\partial D}{\partial \Omega_1} = D_0\left[-D_1 \sin i_0 \sin i_1 \sin \Delta\Omega + D_2\left(e_1 \frac{\partial \cos f_1}{\partial \Omega_1} + \cos f_1 \frac{\partial e_1}{\partial \Omega_1}\right)\right]$$
$$+ D_1 D_2 e_0 \frac{\partial \cos f_0}{\partial \Omega_1} + e_0\left[e_1\left(\sin f_0 \frac{\partial \sin f_1}{\partial \Omega_1} + \sin f_1 \frac{\partial \sin f_0}{\partial \Omega_1}\right) + \sin f_0 \sin f_1 \frac{\partial e_1}{\partial \Omega_1}\right]$$

The $\frac{\partial \sin f_0}{\partial \alpha}$ and $\frac{\partial \sin f_1}{\partial \alpha}$ are to be determined.

The preceding equations involve derivatives of eccentricity on the second orbit with respect to the elements on the first and second orbit. Rewriting the constraint equation (16) yields

$$a_1\left(1 - e_1^2\right)\left(1 + e_0 \cos f_0\right) = a_0\left(1 - e_0^2\right)\left(1 + e_1 \cos f_1\right)$$

When E is defined as

$$E = a_0\left(1 - e_0^2\right) \cos f_1 + 2a_1 e_1\left(1 + e_0 \cos f_0\right)$$

this equation can be differentiated implicitly to obtain the following derivatives:

$$\frac{\partial e_1}{\partial a_0} = \frac{1}{E}\left[-\left(1 - e_0^2\right)\left(1 + e_1 \cos f_1\right)\right]$$

$$\frac{\partial e_1}{\partial e_0} = \frac{1}{E}\left[a_1\left(1 - e_1^2\right) \cos f_0 + 2a_0e_0\left(1 + e_1 \cos f_1\right)\right]$$

$$\frac{\partial e_1}{\partial i_0} = \frac{1}{E}\left[a_1e_0\left(1 - e_1^2\right)\frac{\partial \cos f_0}{\partial i_0} - a_0e_1\left(1 - e_0^2\right)\frac{\partial \cos f_1}{\partial i_0}\right]$$

$$\frac{\partial e_1}{\partial \omega_0} = \frac{1}{E}\left[a_1e_0\left(1 - e_1^2\right)\frac{\partial \cos f_0}{\partial \omega_0}\right]$$

$$\frac{\partial e_1}{\partial \Omega_0} = \frac{1}{E}\left[a_1e_0\left(1 - e_1^2\right)\frac{\partial \cos f_0}{\partial \Omega_0} - a_0e_1\left(1 - e_0^2\right)\frac{\partial \cos f_1}{\partial \Omega_0}\right]$$

$$\frac{\partial e_1}{\partial a_1} = \frac{1}{E}\left[\left(1 - e_1^2\right)\left(1 - e_0 \cos f_0\right)\right]$$

$$\frac{\partial e_1}{\partial i_1} = \frac{1}{E}\left[a_1e_0\left(1 - e_1^2\right)\frac{\partial \cos f_0}{\partial i_1} - a_0e_1\left(1 - e_0^2\right)\frac{\partial \cos f_1}{\partial i_1}\right]$$

$$\frac{\partial e_1}{\partial \omega_1} = \frac{1}{E}\left[-a_0e_1\left(1 - e_0^2\right)\frac{\partial \cos f_1}{\partial \omega_1}\right]$$

$$\frac{\partial e_1}{\partial \Omega_1} = \frac{1}{E}\left[a_1e_0\left(1 - e_1^2\right)\frac{\partial \cos f_0}{\partial \Omega_1} - a_0e_1\left(1 - e_0^2\right)\frac{\partial \cos f_1}{\partial \Omega_1}\right]$$

Finally, it is necessary to obtain derivatives of equations (7), (9), (10), and (11). (This method assumes that the transfer takes place in the direction of positive $\hat{R}$. For negative $\hat{R}$ (eqs. (12) to (15)) simply interchange the initial sign on the derivatives.)

Define

$$F = 1 - \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right)^2$$

By inspection of equation (7),

$$\frac{\partial \cos f_0}{\partial a_0} = \frac{\partial \cos f_0}{\partial e_0} = \frac{\partial \cos f_0}{\partial a_1} = \frac{\partial \cos f_0}{\partial \omega_1} = 0$$

$$\frac{\partial \cos f_0}{\partial i_0} = -\frac{1}{F}\left[\sin i_1 \sin \Delta\Omega\left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) \sin f_0\right]$$

$$\frac{\partial \cos f_0}{\partial \omega_0} = \sin f_0$$

$$\frac{\partial \cos f_0}{\partial \Omega_0} = \frac{1}{F}\left[\sin i_1 \left(\cos i_0 \sin i_1 - \sin i_0 \cos i_1 \cos \Delta\Omega\right) \sin f_0\right]$$

$$\frac{\partial \cos f_0}{\partial i_1} = \frac{1}{F}\left(\sin i_0 \sin \Delta\Omega \sin f_0\right)$$

$$\frac{\partial \cos f_0}{\partial \Omega_1} = -\frac{\partial \cos f_0}{\partial \Omega_0}$$

By inspection of equation (9),

$$\frac{\partial \sin f_0}{\partial a_0} = \frac{\partial \sin f_0}{\partial e_0} = \frac{\partial \sin f_0}{\partial a_1} = \frac{\partial \sin f_0}{\partial \omega_1} = 0$$

$$\frac{\partial \sin f_0}{\partial i_0} = \frac{1}{F}\left[\sin i_1 \sin \Delta\Omega \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) \cos f_0\right]$$

$$\frac{\partial \sin f_0}{\partial \omega_0} = -\cos f_0$$

$$\frac{\partial \sin f_0}{\partial \Omega_0} = -\frac{1}{F}\left[\sin i_1 \left(\sin i_1 \cos i_0 - \cos i_1 \sin i_0 \cos \Delta\Omega\right) \cos f_0\right]$$

$$\frac{\partial \sin f_0}{\partial i_1} = -\frac{1}{F}\left(\sin i_0 \sin \Delta\Omega \cos f_0\right)$$

$$\frac{\partial \sin f_0}{\partial \Omega_1} = -\frac{\partial \sin f_0}{\partial \Omega_0}$$

By inspection of equation (10),

$$\frac{\partial \cos f_1}{\partial a_0} = \frac{\partial \cos f_1}{\partial e_0} = \frac{\partial \cos f_1}{\partial \omega_0} = \frac{\partial \cos f_1}{\partial a_1} = 0$$

$$\frac{\partial \cos f_1}{\partial i_0} = -\frac{1}{F}\left(\sin i_1 \sin \Delta\Omega \sin f_1\right)$$

$$\frac{\partial \cos f_1}{\partial \Omega_0} = -\frac{1}{F}\left[\sin i_0 \left(\sin i_0 \cos i_1 - \cos i_0 \sin i_1 \cos \Delta\Omega\right) \sin f_1\right]$$

$$\frac{\partial \cos f_1}{\partial i_1} = \frac{1}{F}\left[\sin i_0 \sin \Delta\Omega \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) \sin f_1\right]$$

$$\frac{\partial \cos f_1}{\partial \omega_1} = \sin f_1$$

$$\frac{\partial \cos f_1}{\partial \Omega_1} = -\frac{\partial \cos f_1}{\partial \Omega_0}$$

By inspection of equation (11),

$$\frac{\partial \sin f_1}{\partial a_0} = \frac{\partial \sin f_1}{\partial e_0} = \frac{\partial \sin f_1}{\partial \omega_0} = \frac{\partial \sin f_1}{\partial a_1} = 0$$

$$\frac{\partial \sin f_1}{\partial i_0} = \frac{1}{F}\left(\sin i_1 \sin \Delta\Omega \cos f_1\right)$$

$$\frac{\partial \sin f_1}{\partial \Omega_0} = \frac{1}{F}\left[\sin i_0 \left(\sin i_0 \cos i_1 - \cos i_0 \sin i_1 \cos \Delta\Omega\right) \cos f_1\right]$$

$$\frac{\partial \sin f_1}{\partial i_1} = -\frac{1}{F}\left[\sin i_0 \sin \Delta\Omega \left(\cos i_0 \cos i_1 + \sin i_0 \sin i_1 \cos \Delta\Omega\right) \cos f_1\right]$$

$$\frac{\partial \sin f_1}{\partial \omega_1} = -\cos f_1$$

$$\frac{\partial \sin f_1}{\partial \Omega_1} = -\frac{\partial \sin f_1}{\partial \Omega_0}$$

All these relations can be substituted into equation (24) to obtain the desired derivatives of ΔV. It is prohibitive to write these equations explicitly. However, they are readily programed for the computer. The analytic derivatives have been checked with good agreement against central-difference numerical derivatives.

Generalization to Multiple-Impulse Transfer

The multiple-impulse transfer is treated as a sequence of single-impulse transfers. The first orbit is specified by a, e, i, ω, and Ω, and the final orbit is specified by a, i, ω, and Ω, eccentricity being a dependent variable. On each intermediate conic there are four control variables (a, i, ω, and Ω) and one dependent parameter (e). By choosing values of the control variables, the total velocity change is the sum of the single impulses. For minimization purposes, the derivatives of total ΔV with respect to the

control variables are obtained by chaining together the derivatives obtained in the previous section for the single-impulse transfer.

As an example, consider a four-impulse transfer problem. Let α_i represent any of the four independent parameters of the ith intermediate orbit (i = 1,2,3). The total velocity change is the sum of the four impulses and may be written as

$$\Delta V_T = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4 \tag{25}$$

where

$$\Delta V_1 = \Delta V_1(\alpha_1)$$

$$\Delta V_2 = \Delta V_2(\alpha_1,\alpha_2,e_1)$$

$$\Delta V_3 = \Delta V_3(\alpha_2,\alpha_3,e_2)$$

$$\Delta V_4 = \Delta V_4(\alpha_3,e_3)$$

and

$$e_1 = e_1(\alpha_1)$$

$$e_2 = e_2(\alpha_1,\alpha_2,e_1)$$

$$e_3 = e_3(\alpha_2,\alpha_3,e_2)$$

The partial derivatives of ΔV_T with respect to the independent elements are

$$\frac{\partial \Delta V_T}{\partial \alpha_1} = \frac{\partial \Delta V_1}{\partial \alpha_1} + \left(\frac{\partial \Delta V_2}{\partial \alpha_1} + \frac{\partial \Delta V_2}{\partial e_1}\frac{\partial e_1}{\partial \alpha_1}\right) + \left[\frac{\partial \Delta V_3}{\partial e_2}\left(\frac{\partial e_2}{\partial \alpha_1} + \frac{\partial e_2}{\partial e_1}\frac{\partial e_1}{\partial \alpha_1}\right)\right] + \left[\frac{\partial \Delta V_4}{\partial e_3}\frac{\partial e_3}{\partial e_2}\left(\frac{\partial e_2}{\partial \alpha_1} + \frac{\partial e_2}{\partial e_1}\frac{\partial e_1}{\partial \alpha_1}\right)\right] \tag{26}$$

$$\frac{\partial \Delta V_T}{\partial \alpha_2} = \frac{\partial \Delta V_2}{\partial \alpha_2} + \left(\frac{\partial \Delta V_3}{\partial \alpha_2} + \frac{\partial \Delta V_3}{\partial e_2}\frac{\partial e_2}{\partial \alpha_2}\right) + \left[\frac{\partial \Delta V_4}{\partial e_3}\left(\frac{\partial e_3}{\partial \alpha_2} + \frac{\partial e_3}{\partial e_2}\frac{\partial e_2}{\partial \alpha_2}\right)\right] \tag{27}$$

$$\frac{\partial \Delta V_T}{\partial \alpha_3} = \frac{\partial \Delta V_3}{\partial \alpha_3} + \left(\frac{\partial \Delta V_4}{\partial \alpha_3} + \frac{\partial \Delta V_4}{\partial e_3}\frac{\partial e_3}{\partial \alpha_3}\right) \tag{28}$$

Thus, to relate any two consecutive orbits, the following derivatives are necessary: $\frac{\partial \Delta V_{i+1}}{\partial e_i}$, $\frac{\partial \Delta V_{i+1}}{\partial \alpha_i}$, $\frac{\partial \Delta V_{i+1}}{\partial \alpha_{i+1}}$, $\frac{\partial e_{i+1}}{\partial e_i}$, $\frac{\partial e_{i+1}}{\partial \alpha_i}$, and $\frac{\partial e_i}{\partial \alpha_i}$. The derivatives are obtained from equation (24) and ΔV_{i+1} represents the velocity change required to transfer from the ith to the $(i+1)$th orbit. The extension of this formulation to n-impulses simply involves $n - 1$ equations similar to equations (25) to (28).

CONCLUDING REMARKS

A multiple-impulse function has been developed as a series of single impulses. Analytic derivatives of the function with respect to the independent variables have been presented. An example of a four-impulse transfer is given and could be expanded to n-impulses by the same techniques. The development of analytic derivatives of this impulse function removes the dependency of optimization algorithms on numerical derivatives. Analytic derivatives may permit much more efficient optimization techniques.

Langley Research Center,
National Aeronautics and Space Administration,
Hampton, Va., January 18, 1972.

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