DESIGN OF RECURSIVE DIGITAL FILTERS
HAVING SPECIFIED PHASE
AND MAGNITUDE CHARACTERISTICS

by
Robert E. King
Langley Research Center

and
Gregory W. Condon
Langley Directorate,
U.S. Army Air Mobility R&D Laboratory

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1972
A method for a computer-aided design of a class of optimum filters, having specifications in the frequency domain of both magnitude and phase, is described. The method, an extension to the work of Steiglitz, uses the Fletcher-Powell algorithm to minimize a weighted squared magnitude and phase criterion. Results using the algorithm for the design of filters having specified phase as well as specified magnitude and phase compromise are presented.
DESIGN OF RECURSIVE DIGITAL FILTERS HAVING SPECIFIED
PHASE AND MAGNITUDE CHARACTERISTICS

By Robert E. King
Langley Research Center

and

Gregory W. Condon
Langley Directorate, U.S. Army Air Mobility R&D Laboratory

SUMMARY

A method for a computer-aided design of a class of optimum filters, having specifications in the frequency domain of both magnitude and phase, is described. The method, an extension to the work of Steiglitz, uses the Fletcher-Powell algorithm to minimize a weighted squared magnitude and phase criterion. Results using the algorithm for the design of filters having specified phase as well as specified magnitude and phase compromise are presented.

INTRODUCTION

Recursive filters, wherein the output sequence is both a function of the input as well as past output samples, are commonly used in digital signal processing and analysis. Such digital filters in many applications offer distinct advantages of precision and versatility over their continuous or analog counterparts. There exist a number of design procedures for implementing digital filters (see ref. 1) each one of which strives to attain some analogy between discrete and continuous systems. Transform methods such as the matched-z, bilinear-z, and standard-z which lead to specific property invariances are available (see ref. 2) to the designer familiar with continuous filter design.

For frequency-domain synthesis (see refs. 3 and 4), realization is normally by means of cascade or parallel combinations of pole and zero pairs in the complex plane. The synthesis problem is, in fact, reduced to one of approximation since the filter topology is generally specified. In none of the available design procedures, which can yield filters having excellent magnitude-frequency characteristics, however, do the resultant filters, in themselves, have particularly useful phase characteristics. Indeed, in striving for particular magnitude characteristics by using any of the available design methods, there is no control over the filter phase properties.
In practice, it is often desirable to specify a digital filter in the frequency domain by its phase (see ref. 5) or even a compromise between magnitude and phase. The procedure in this paper meets these requirements through the use of an iterative computer-aided design leading to an optimum set of parameters for a specified filter topology and is an extension of the technique described by Steiglitz (see ref. 6) for determining the optimum coefficients of a cascade filter having magnitude specifications alone. The extension makes possible the design of a new class of digital filters having the prescribed phase characteristics.

**SYMBOLS**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>filter multiplier</td>
</tr>
<tr>
<td>$D_k^i$</td>
<td>denominator of $i$th stage of $H(z)$ at $\Omega_k$</td>
</tr>
<tr>
<td>$E_k^M$</td>
<td>magnitude error at $\Omega_k$</td>
</tr>
<tr>
<td>$E_k^\phi$</td>
<td>phase error at $\Omega_k$</td>
</tr>
<tr>
<td>$\vec{e}_k$</td>
<td>error vector at $\Omega_k$</td>
</tr>
<tr>
<td>$\partial \vec{e}_k / \partial A$</td>
<td>derivative of error vector at $\Omega_k$ with respect to zero frequency gain</td>
</tr>
<tr>
<td>$f_k$</td>
<td>frequency at $k$th specification point, Hz</td>
</tr>
<tr>
<td>$f_s$</td>
<td>sampling frequency, Hz</td>
</tr>
<tr>
<td>$H(z)$</td>
<td>unity gain discrete transfer function</td>
</tr>
<tr>
<td>$</td>
<td>H_k</td>
</tr>
<tr>
<td>$\bar{H}_k$</td>
<td>conjugate of $H(z)$ at $\Omega_k$</td>
</tr>
<tr>
<td>$\partial</td>
<td>H_k</td>
</tr>
<tr>
<td>$I(\ )$</td>
<td>imaginary part of quantity</td>
</tr>
<tr>
<td>i, ..., N</td>
<td>denotes filter stage</td>
</tr>
</tbody>
</table>
\( \vec{J}_k \)  
Jacobian at \( \Omega_k \), 
\[
\left[ A^* \left| \frac{\partial H_k}{\partial \vec{p}} \right| \frac{\partial \phi_k}{\partial \vec{p}} \right]
\]

\( k \)  
sample point

\( M_k \)  
specification magnitude at \( \Omega_k \)

\( N_k^i \)  
umerator of ith stage of \( H(z) \) at \( \Omega_k \)

\( \vec{p} \)  
parameter vector

\( \vec{p}_i \)  
set of filter parameters for the ith stage, \( a_i, b_i, c_i, \) and \( d_i \)

\( q_1^i(k) \)  
first system state of ith stage at kth sample point

\( q_2^i(k) \)  
second system state of ith stage at kth sample point

\( R( \cdot ) \)  
real part of quantity

\( u_1(k) \)  
input to ith stage at kth sample point

\( V \)  
criterion functional, that is, \( V(A,\vec{p}) \)

\( V_k \)  
criterion functional at \( \Omega_k \), that is, \( V_k(A,\vec{p}) \)

\( \hat{V} \)  
reduced criterion functional, that is, \( V(A^*,\vec{p}) \)

\( \partial V/\partial A \)  
slope of criterion functional with respect to zero frequency gain

\( \partial V_k/\partial \vec{e}_k \)  
gradient vector of criterion functional at \( \Omega_k \) with respect to error vector at \( \Omega_k \)

\( \vec{W}_k \)  
weighting matrix at \( \Omega_k \)

\( W_k^M \)  
magnitude weighting at \( \Omega_k \)

\( W_k^\phi \)  
phase weighting at \( \Omega_k \)

\( w^i(k) \)  
dummy variable of ith stage at kth sample point
DISCUSSION

The Filter Form

The fundamental advantages of the N-stage cascade canonical form of recursive digital filter whose signal flow graph is shown in figure 1 and which is described by the product operator

\[ Y(z) = A \left( \prod_{i=1}^{N} \frac{1 + a_i z^{-1} + b_i z^{-2}}{1 + c_i z^{-1} + d_i z^{-2}} \right) \]

are (1) its relative insensitivity to perturbations in the denominator coefficients, an important consideration in digital filters, especially of high order and particularly where finite register lengths (see ref. 1) are involved; (2) its simplicity of implementation; and (3) the simplicity of factoring the filter operator to determine its roots. This form has found extensive application in practical filters for signal processing, and a version employing serial arithmetic (ref. 7) is commercially available.
For completeness, an alternative description of the filter is given in terms of the system states $q_1^i$ and $q_2^i$ and clearly demonstrates the recursive nature of the filter. The set of difference equations describing the filter and required in developing a computer algorithm is presented. Thus, for the $i$th stage in figure 1 at the $k$th sample point

$$w^i(k) = A_1 u_1(k) - c_1 q_1^i(k) - d_1 q_2^i(k)$$

$$q_1^i(k + 1) = w^i(k)$$

$$q_2^i(k + 1) = q_1^i(k)$$

$$y_1(k) = w^i(k) + a_1 q_1^i(k) + b_1 q_2^i(k)$$

where

$$u_1(k) = y_{i-1}(k)$$

is the input to the $i$th stage and is identical to the output of the $(i - 1)$ stage and

$$A_1 = \begin{cases} A & (i = 1) \\ 1 & (i \neq 1) \end{cases}$$

The Synthesis Problem

The design problem considered in this paper can be stated as follows: When the magnitude and phase specifications ($M_k$ and $\theta_k$, respectively) at the $k$th fractional Nyquist frequencies $f_{ik} = 2f_k/f_s$ (where $f_s$ is the sampling frequency in Hz) are known, determine the set of optimum parameters $p^*$ of an $N$-stage cascade filter having the form of equation (1) so that the resultant digital filter will have a minimum sum squared magnitude and phase error for all specified frequencies.

By constraining the filter topology, the optimum synthesis problem becomes one of parametric optimization with respect to a given criterion of fit. The composite criterion which can weight the magnitude and phase requirements independently and as functions of frequency is chosen as the inner product

$$V(A,p) = \sum_k \left< \bar{e}_k, \bar{w}_k \bar{e}_k \right> = \sum_k V_k$$

(2)
where

$$
\vec{e}_k = \begin{bmatrix} A |H_k| - M_k \\ \phi_k - \theta_k \\ E_k^M \\ E_k^\phi \end{bmatrix} = \begin{bmatrix} E_k^M \\ E_k^\phi \end{bmatrix}
$$

is the error vector and

$$
\vec{w}_k = \begin{bmatrix} W_k^M & 0 \\ 0 & \lambda W_k^\phi \end{bmatrix}
$$

is the diagonal weighting matrix. Clearly, $V(A, \vec{p})$ is a nonlinear function of the parameter vector $\vec{p} = (a_1, b_1, c_1, d_1, \ldots, a_N, b_N, c_N, d_N)^T$, which involves the $4N$ filter coefficients, and of the filter multiplier $A$.

The Minimization Algorithm

Through formal differentiation of the criterion function (eq. (2)) with respect to the multiplier $A$, the minimization procedure can be slightly simplified to that of finding the minimum of a reduced functional $\tilde{V}(\vec{p}) = V(A^*, \vec{p})$ involving only $4N$ parameters. Thus

$$
\frac{\partial V}{\partial A} = \sum_k \left( \frac{\partial \vec{e}_k}{\partial A} \right) \frac{\partial V_k}{\partial \vec{e}_k} = 2 \sum_k \begin{bmatrix} |H_k| W_k^M \\ 0 \end{bmatrix} \vec{e}_k
$$

and $\frac{\partial V}{\partial A} = 0$ implies

$$
2 \sum_k |H_k| W_k^M \left( A^* |H_k| - M_k \right) = 0
$$

or

$$
A^* = \frac{\sum_k |H_k| W_k^M M_k}{\sum_k |H_k|^2 W_k^M}
$$

(3)

An additional necessary condition for existence of an extremum is that the gradient vector be zero; thereby, the optimum parameter vector $\vec{p}^*$ is obtained. From equation (2)
\[
\frac{\partial \hat{V}}{\partial \bar{p}} = 2 \sum_k \left\langle \bar{J}_k, \bar{W}_k \bar{e}_k \right\rangle
\]  

(4)

where the \((4N \times 2)\) Jacobian \(\bar{J}_k\) is

\[
\bar{J}_k^T = \mathbf{v}_p \bar{e}_k = \begin{bmatrix} \mathbf{A}^* \frac{\partial |H_k|}{\partial \bar{p}} & \frac{\partial \phi_k}{\partial \bar{p}} \end{bmatrix}^T
\]

(5)

Clearly, each element of the gradient vector is the sum of two weighted functions of the magnitude and phase error. By writing

\[
|H_k|^2 = H_k \bar{H}_k
\]

where \(\bar{H}_k\) is the conjugate of \(H_k\) evaluated at the fractional frequency \(\Omega_k\), it is readily shown (see ref. 6), where \(\bar{p}_i\) is the set of filter parameters for the \(i\)th stage, that

\[
\frac{\partial |H_k|}{\partial \bar{p}_i} = \frac{1}{|H_k|} R \left( \frac{\bar{H}_k \frac{\partial H_k}{\partial \bar{p}_i}}{\bar{p}_i} \right)
\]

For the cascaded filter topology in terms of the elements of \(\bar{p}_i\),

\[
\frac{\partial |H_k|}{\partial a_i} = |H_k| R \left( \frac{z_k^{-1}}{N_k^i} \right)
\]

\[
\frac{\partial |H_k|}{\partial b_i} = |H_k| R \left( \frac{z_k^{-2}}{N_k^i} \right)
\]

\[
\frac{\partial |H_k|}{\partial c_i} = -|H_k| R \left( \frac{z_k^{-1}}{D_k^i} \right)
\]

and

\[
\frac{\partial |H_k|}{\partial d_i} = -|H_k| R \left( \frac{z_k^{-2}}{D_k^i} \right)
\]
where, with \( z_k = e^{j\pi \Omega_k} \),

\[
N_k^i = N^i(z_k) = 1 + a_i z_k^{-1} + b_i z_k^{-2}
\]

and

\[
D_k^i = D^i(z_k) = 1 + c_i z_k^{-1} + d_i z_k^{-2}
\]

By letting

\[
H_k = |H_k| e^{j\phi_k}
\]

it follows that

\[
\phi_k = I(\log_e H_k)
\]

whence

\[
\frac{\partial \phi_k}{\partial \theta} = I\left( \frac{\partial}{\partial \theta} \log_e H_k \right) = I\left( \frac{1}{H_k} \frac{\partial H_k}{\partial \theta} \right)
\]

which takes on a particularly simple form for the cascade topology. For the ith stage parameters, in fact,

\[
\frac{\partial \phi_k}{\partial a_i} = I\left( \frac{1}{N_k^i} \right)
\]

\[
\frac{\partial \phi_k}{\partial b_i} = I\left( \frac{1}{N_k^i} \right)
\]

\[
\frac{\partial \phi_k}{\partial c_i} = -I\left( \frac{1}{D_k^i} \right)
\]
and
\[
\frac{\partial \phi_k}{\partial d_1} = -I \left( \frac{z_{-2}}{D_k} \right)
\]

The special case of a one-stage (N = 1) filter is illustrated. Here

\[
H_k = A \frac{1 + az_{-1}^k + bz_{-2}^k}{1 + cz_{-1}^k + dz_{-2}^k}
\]

\[
\hat{V} = \sum_k \left( A^* |H_k| - M_k \right)^2 W_k^M + \lambda \sum_k \left( \phi_k - \theta_k \right)^2 W_k^\phi
\]

and
\[
\frac{\partial \hat{V}}{\partial a} = 2 \sum_k \left( E_k^M W_k^M \frac{\partial |H_k|}{\partial a} + \lambda E_k^\phi W_k^\phi \frac{\partial \phi_k}{\partial a} \right) = \sum_k \left[ Q_k^M R \left( \frac{z_{-1}^k}{N_k} \right) + \lambda R_k^\phi \left( \frac{z_{-1}^k}{N_k} \right) \right]
\]

Similarly,
\[
\frac{\partial \hat{V}}{\partial b} = \sum_k \left[ Q_k^M R \left( \frac{z_{-2}^k}{N_k} \right) + \lambda R_k^\phi \left( \frac{z_{-2}^k}{N_k} \right) \right]
\]
\[
\frac{\partial \hat{V}}{\partial c} = \sum_k \left[ Q_k^M R \left( \frac{z_{-1}^k}{D_k} \right) + \lambda R_k^\phi \left( \frac{z_{-1}^k}{D_k} \right) \right]
\]
\[
\frac{\partial \hat{V}}{\partial d} = \sum_k \left[ Q_k^M R \left( \frac{z_{-2}^k}{D_k} \right) + \lambda R_k^\phi \left( \frac{z_{-2}^k}{D_k} \right) \right]
\]

where
\[
Q_k^M = 2 E_k^M W_k^M |H_k|
\]
and

\[ R_k^\phi = 2E_k^\phi W_k^\phi \]

are the weighted errors. It is obvious that the frequency intervals of the input data (specifications) need not be uniform and may, in fact, be intentionally unequal to allow for nonuniform frequency weighting.

**Complementary Root Reflection and Stability**

In deriving the frequency response of a discrete operator by letting \( z_k \) lie on the unit circle \( \Gamma \), it is possible to take advantage of a unique property of the discrete transform pertaining to its magnitude when a root lying outside the unit circle is imaged or reflected into the unit circle. It is easy to show that the magnitude of a phasor \( z - z_0 \), where \( z_0 \) is a root of the discrete transform lying outside the unit circle, is equal to

\[ |z - z_0| = |z_0| \left| z - \frac{1}{z_0} \right| ; z \in \Gamma \]

Since \( z_0 \) has been assumed to be outside the unit circle, \( 1/z_0 \) must be inside, the term \( |z_0| \) correcting for magnitude changes. Thus, if in the optimization procedure a pole should stray outside the unit circle and thereby lead to an unstable filter, root reflection guarantees stability with no magnitude change. There is no analogous simple identity for the phase of a reflected root. Experience with the procedure has shown that provided the design requirements can be met by means of a stable filter, that is, that a feasible solution exists, an optimum will indeed be found through repeated application of root reflection.

**The Computer Algorithm**

A complete listing of the filter design algorithm, which is an adaptation of the program written by Steiglitz, is given in the appendix. The main program is termed STGZ3 which calls four principal subroutines: (1) FUNCT performs the functional and gradient computation for each iteration as well as putting out the final optimum parameters and plots, (2) FLPWL is a Fletcher-Powell conjugate gradient routine, (3) INSIDE computes root reflection, and (4) ROOTS determines the poles and zeros of the filter. Single-precision arithmetic has been employed.

When minimization of the functional has been attained in the first pass or the minimization algorithm has iterated 300 times, a test is made to ascertain that all the roots are within the unit circle, a necessary requirement for the poles for stability reasons and for the zeros to insure minimum phase. If the design should result in an unstable
configuration, the roots are reflected about the unit circle and minimization is resumed in a second pass. If a minimum does indeed exist and all the roots then lie within the unit circle, the program computes and prints out the frequency response and commences plotting.

Minimization is deemed to be achieved when the absolute difference in functionals between successive iterations \( \epsilon = |\hat{V}_{\text{new}} - \hat{V}_{\text{old}}| \) or the norm of the gradient vector falls below preassigned limits. Convergence is generally fast for magnitude or phase filters but can be very slow for the case of compromise filters.

When the design specifications cannot be met after \( \text{LIM} \) iterations (see appendix), the program will stop; this situation indicates that the optimum could not be found and the resultant characteristic which may be unusable is plotted. Generally, feasible designs have been determined in less than 2000 iterations.

Minimization of the criterion function does not guarantee determination of a global minimum but rather determination of a local minimum. Depending upon the parameter vector utilized for initialization of the algorithm computation, different minima may be achieved. Experience has shown that stage-by-stage optimization, that is, utilization of the \( i \)-th stage optimum parameter vector as the initial parameter vector for the \( (i + 1) \)-th stage of an \( N \)-stage filter, yields lower minimum values of the criterion function than does single-pass optimization.

APPLICATIONS

Linear-Phase Filter

This example considers a digital filter having application as a phase discriminator with a linear phase characteristic and arbitrary magnitude characteristic and is shown in figure 2. In this example all magnitude weights were set to zero and all phase weights to unity, the multiplier \( A \) being arbitrarily made unity since it has no effect on the phase characteristic.

The phase requirements were \( \theta_k = 1 - 2\Omega_k \quad (0 \leq \Omega_k \leq 1) \), and a two-stage filter was specified. When an initial parameter vector \( \bar{\mathbf{p}} = (0, 0, 0, 0.25, 0, 0, 0, 0)^T \) was used, the algorithm converged to the optimum, with \( \epsilon = 10^{-4} \), in 52 iterations and a Control Data 6600 computer time of 14 seconds. The optimum parameter values computed were to four places

\[
A = 1.0
\]
\[
a_1 = 0 \quad b_1 = -0.9871 \quad c_1 = 0 \quad d_1 = 0.0395
\]
\[
a_2 = 0 \quad b_2 = -0.9871 \quad c_2 = 0 \quad d_2 = -0.0127
\]
It is interesting to note that the phase requirements were met to within $0.008\pi$ radian for approximately 95 percent of the frequency range.

**Constant-Phase Filters**

Two cases were considered to obtain filters having constant phases of $-\pi/2$ and $\pi/2$ radians over a frequency range $0.3 \leq \Omega \leq 0.7$. As in the previous case, the form of the magnitude characteristic was of no concern; hence, zero magnitude weighting was specified. With the same initial parameter state used in the previous example, the first case (lag network) optimized in 1673 iterations and 42 seconds to yield a hyperbolic magnitude characteristic and phase errors of less than $0.0003\pi$ radian throughout the specified band.

The computed parameters for the lag case were

\[
A = 1.0 \\
a_1 = 0.5580 \quad b_1 = -0.1857 \quad c_1 = -0.4752 \quad d_1 = 0.0363 \\
a_2 = 0.5580 \quad b_2 = -0.1857 \quad c_2 = -0.3712 \quad d_2 = -0.5686
\]

The positive phase filter (lead network), however, took only 165 iterations and 17 seconds to yield the desired phase characteristic with errors nowhere exceeding $0.001\pi$ radian in the specified band.

The optimum filter parameters for this second case were determined to be

\[
A = 1.0 \\
a_1 = -0.4768 \quad b_1 = -0.1548 \quad c_1 = 0.5022 \quad d_1 = -0.1082 \\
a_2 = -0.4768 \quad b_2 = -0.1548 \quad c_2 = 0.4515 \quad d_2 = -0.2008
\]

It is noted that for both cases, the phase weights outside the specified band were set to zero, and thereby allowed for arbitrary phase in these regions. Figures 3(a) and 3(b) show the resultant frequency characteristics for the lag and lead cases, respectively, of two-stage filters. The combination of the two filters, although they have antagonistic magnitude characteristics, suggests the possibility of a phase-splitting digital network.
Limited-Band Constant-Gain Linear-Phase Filter

The third example demonstrates a compromise design of a digital filter having constant-magnitude and linear-phase characteristics, over a limited frequency band, typical of phase discriminators. Here, except for $\lambda = 0$, the specifications were stated as

$$M_k = \begin{cases} 
1 & (0.3 \leq \Omega_k \leq 0.7) \\
0 & \text{(Elsewhere)}
\end{cases}$$

$$\theta_k = \begin{cases} 
1 - 2\Omega_k & (0.3 \leq \Omega_k \leq 0.7) \\
0 & \text{(Elsewhere)}
\end{cases}$$

Equal error and frequency weights were employed and the effects of changes in $\lambda$ are shown in figure 4 for a two-stage design. Figure 4(a) shows the case of $\lambda = 0$, that is, a magnitude-only filter being specified, and coincidentally yields the linear-phase-filter characteristic derived in the first example. (See fig. 2.) Figures 4(b) and 4(c) show the magnitude and phase characteristics for the cases of $\lambda = 10$ and $\lambda = 1000$, respectively. The increasing weight on phase and resultant degradation in the magnitude characteristic are shown. The optimum parameters were

$\lambda = 0$:

$$A = 0.2063$$

$$a_1 = 0.0000 \quad b_1 = -1.0000 \quad c_1 = 0.0000 \quad d_1 = 0.1539$$

$$a_2 = 0.0000 \quad b_2 = -1.0000 \quad c_2 = 0.0000 \quad d_2 = 0.1539$$

$\lambda = 10$:

$$A = 0.3658$$

$$a_1 = -0.9754 \quad b_1 = 0.7300 \quad c_1 = 0.4529 \quad d_1 = 0.7211$$

$$a_2 = 0.8632 \quad b_2 = 0.5632 \quad c_2 = -0.6119 \quad d_2 = 0.7443$$
\[ \lambda = 1000: \]

\[
A = 0.4232
\]

\[
a_1 = -1.1739 \quad b_1 = 0.8489 \quad c_1 = 0.7596 \quad d_1 = 0.6691
\]

\[
a_2 = 1.1739 \quad b_2 = 0.8489 \quad c_2 = -0.7596 \quad d_2 = 0.6691
\]

**Low-Pass Zero-Phase Filter**

The fourth example considers a compromise filter, having two and three stages, with specifications that are intentionally conflicting. A filter described by

\[
M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) \\
0.5 & (\Omega_k = 0.5) \\
0.0 & (0.5 < \Omega_k \leq 1.0)
\end{cases}
\]

\[
\theta_k = \begin{cases} 
0 & (0.0 \leq \Omega_k \leq 0.5) \\
\text{Unspecified} & (\text{Elsewhere})
\end{cases}
\]

is specified.

Figures 5 and 6 show the results for the two- and three-stage designs, respectively, with figures 5(a) and 6(a) showing the magnitude-only (\(\lambda = 0\)) case. The degradation in the magnitude characteristics when greater emphasis is placed on the phase specifications is evident in figures 5(b) and 6(b) for \(\lambda = 10\) and in figures 5(c) and 6(c) for \(\lambda = 1000\). Comparison of figure 6 with figure 5 demonstrates the improvement brought about by increasing the number of stages. The optimum parameters for the two-stage filter were

\[ \lambda = 0: \]

\[
A = 0.1196
\]

\[
a_1 = 1.0240 \quad b_1 = 1.0000 \quad c_1 = -0.1713 \quad d_1 = 0.7676
\]

\[
a_2 = 1.0240 \quad b_2 = 1.0000 \quad c_2 = -0.5324 \quad d_2 = 0.2286
\]
\( \lambda = 10: \)

\[
A = 0.4879
\]

\[
a_1 = 0.2018 \quad b_1 = 0.6684 \quad c_1 = 0.3560 \quad d_1 = 0.4612
\]

\[
a_2 = 0.6597 \quad b_2 = 0.4335 \quad c_2 = 0.0806 \quad d_2 = 0.7671
\]

\( \lambda = 1000: \)

\[
A = 0.5343
\]

\[
a_1 = 0.0205 \quad b_1 = 0.7169 \quad c_1 = -0.0836 \quad d_1 = 0.6255
\]

\[
a_2 = 0.6286 \quad b_2 = 0.7905 \quad c_2 = 0.2123 \quad d_2 = 0.6681
\]

The optimum parameters for the three-stage filter were

\( \lambda = 0: \)

\[
A = 0.0510
\]

\[
a_1 = 0.8537 \quad b_1 = 1.0000 \quad c_1 = -0.1068 \quad d_1 = 1.0000
\]

\[
a_2 = 0.8537 \quad b_2 = 1.0000 \quad c_2 = -0.4046 \quad d_2 = 0.5990
\]

\[
a_3 = 0.8537 \quad b_3 = 1.0000 \quad c_3 = -0.6799 \quad d_3 = 0.2069
\]

\( \lambda = 10: \)

\[
A = 0.5109
\]

\[
a_1 = 1.3302 \quad b_1 = 0.5515 \quad c_1 = -0.1731 \quad d_1 = 0.8097
\]

\[
a_2 = 0.6844 \quad b_2 = 0.7157 \quad c_2 = 1.1880 \quad d_2 = 0.5850
\]

\[
a_3 = -0.0373 \quad b_3 = 0.7012 \quad c_3 = 0.3825 \quad d_3 = 0.5262
\]
\[ \lambda = 1000: \]

\[
\begin{align*}
A &= 0.4515 \\
\alpha_1 &= 1.5107 \\
\beta_1 &= 0.5286 \\
\gamma_1 &= -0.1771 \\
\delta_1 &= 0.8972 \\
\alpha_2 &= 0.5825 \\
\beta_2 &= 0.7490 \\
\gamma_2 &= 1.3094 \\
\delta_2 &= 0.4191 \\
\alpha_3 &= -0.1663 \\
\beta_3 &= 0.7485 \\
\gamma_3 &= 0.2002 \\
\delta_3 &= 0.6393
\end{align*}
\]

A three-stage design of this example is used to demonstrate the existence of two distinct local minima, dependent upon the initial parameter vector. In the first case, a single-pass optimization was accomplished with \( \hat{p} = (0, 0, 0, 0.25, 0, 0, 0, 0)^T \) for the initial parameter vector and resulted in the optimum filter shown in figure 6(a). In the second case, a stage-by-stage optimization was accomplished by utilizing the optimum parameter vector from a two-stage design for the initial parameter vector of a three-stage design and resulted in the optimum filter shown in figure 7. Comparison of these results demonstrates the existence of two distinct local minima, the stage-by-stage minimization yielding superior results.

**CONCLUDING REMARKS**

A method has been developed for a computer-aided design of cascade canonical digital filters having prescribed magnitude or phase characteristics or a compromise between the two. The method, which uses an unconstrained minimization algorithm, allows for arbitrary error and frequency weighting. Representative designs of phase and compromise filters have demonstrated the utility of the technique. Although convergence is generally fast for magnitude phase filters, it may be slow for the case of compromise filters.

Langley Research Center, National Aeronautics and Space Administration, Hampton, Va., February 17, 1972.
APPENDIX

PROGRAM LISTING

This appendix contains a program listing written for the Control Data 6600 computer at the Langley Research Center, Hampton, Virginia, and is an adaptation of that written by Kenneth Steiglitz at Princeton University for the design of specified magnitude-only filters.

```
PROGRAM STG73 INOUT, INPUT, TAP5 = INPUT, TAPF6 = CUTOLT, PUNCH
000003 EXTERNAL FUNCT
000003 DIMENSION M(144), XI(144), G(144)
000003 C(KK), W1(1001), W1(1001), ALAMCA, FR, W1M(1001)
000003 CTYP(1001), XTYP
000003 CMMKY/RW11/CALL, KCALL, LIN
000003 CALL CALCMP
000003 CALL LFR1Y
000005 WRITE(*,-11)
000011 51 FORMAT(* INPUT DATA*)
000011 M=0
000012 3C N=M+1
000014 REAL X(M), Y(M), PHASED(M), WTM(M), WTP(M)
000031 21 FFMAT(510,5)
000031 WRITE(*,72) X(M), Y(M), PHASED(M), WTM(M), WTP(M)
000051 22 FORMAT(* I=1,13, N=**, M=** F 7.4,** Y=** F 7.4,** PHASED=**, F 7.4,**
000051 C =** WTM=**, F 7.4,** WTP=**, F 7.4,)
000051 IF (M=1,1,0,1,1.1,16)
000054 DO 15 J=1,16
000056 1E XI(1)=0.00
000061 XI(4)=.25
000062 96 IFAC(5,99)L1,L2,L3,L4,FSP,MAX,ALAMCA,FR,KTYP
000004 60 FORMAT(215.5F10.5,1)
000016 IF (FR .LT. .001) FR=1.0
000017 H=4.0
0000113 IF (FFX,5) CXX,XX
0000117 488 CONTINUE
0000117 N=10**M,11,L1,L2,L3,L4,FSP,MAX,FR,ALAMCA
0000141 61 FORMAT(* L=1,12, L=**, M=**, F10.5,** EST=**, F10.5,** FFS=**, F10.5,**
0000141 C =**, MAX=**, F10.5,** FREQUENCY=**, F10.5,** LAMBDA=**, F10.5,**
0000141 ICALL=0
0000142 4F KCALL=0
0000143 IF (FLPWL(IFUNCT,N,X,F,G,FST,FPS,EC,IFP,H)
0000155 CALL FNPNS(N,X)
0000157 CALL INSIDE1(N,X,KFLAG)
0000159 WRITE(*,261EEK,KFLAG,ICALL,KCALL
0000176 IF (FLPWL(IFUNCT,F,G,H,MAX)
0000176 IF (KFLAG.NE.0.AND.ICALL.LT.FRST) GO TO 5R
0000213 CALL KRTSIN(X)
0000215 ICALL=-10
0000215 CALL FUNCT(N,X,F,G,HMAX)
0000222 STOP
0000223 CALL CALP1T(C,*,*,*,)
0000230 END
```
APPENDIX – Continued

```plaintext
000427      DC  427 J=1. X
000428      J=J+1  #4
000429      1=1. #4
000430      G(J+1)=G(J+2)+6#7CLR
000431      G(J+2)=G(J+2)+0#7CLR
000432      U=U#7Y7771رع(1,F(1,1))
000433      G(J+1)=G(J+1)+#7CLR
000434      G(J+2)=G(J+2)+#7CLR
000435      4#7CONTINUE #14
000436      1=4  #1,
000437      J=J+1  #4
000438      G(J+1)=G(J+1)+T#7AIMA+I7CLR/TUM(1,1))
000439      G(I+2)=G(I+2)+#7AY/@7CLR(17U771
000440      4#7CONTINUE #7
000441      F=V#7#7AIMA#7V2
000442      ICALL=ICALL+1
000443      KCALL=KCALL+1
000444      IF ICALL<GT N?7CC13RTURN
000445      IF (ICALL<10)100, EQ. ICALL<1)WRITE (6,2)=ICALL,F,(G(I),J=1,N)
000446      IF (ICALL<30)30, 0, E4>F15,9/1.0, X, 4F15,9)
000447      IF ICALL<GT 300) GO TO 450
000448      ICALL=ICALL+1
000449      GO TO 449
000450      IF ICALL<GT LIM#7241) GO TO 446
000451      RETURN
000452      1.1.1.1.1.1. RETURN
000453      445 WRITE(6,50)
000454      50 FORMAT(*, FINAL FUNCTION VALUE =#,15.9)
000455      445 WRITE(6,51)
000456      51 FORMAT(*, 6#715,9)
000457      445 WRITE(6,52)
000458      52 FORMAT(*, 6#715,9)
000459      445 WRITE(6,54)
000460      54 FORMAT(*, 6#715,9)
000461      445 WRITE(6,54)
000462      55 FORMAT(*, 6#715,9)
000463      445 WRITE(6,54)
000464      56 FORMAT(*, 6#715,9)
000465      445 WRITE(6,54)
000466      57 FORMAT(*, 6#715,9)
000467      445 WRITE(6,54)
000468      58 FORMAT(*, 6#715,9)
000469      445 WRITE(6,54)
000470      59 FORMAT(*, 6#715,9)
000471      445 WRITE(6,54)
000472      60 FORMAT(*, 6#715,9)
000473      445 WRITE(6,54)
000474      61 FORMAT(*, 6#715,9)
000475      445 WRITE(6,54)
000476      62 FORMAT(*, 6#715,9)
000477      445 WRITE(6,54)
000478      63 FORMAT(*, 6#715,9)
000479      445 WRITE(6,54)
000480      64 FORMAT(*, 6#715,9)
000481      445 WRITE(6,54)
000482      65 FORMAT(*, 6#715,9)
000483      445 WRITE(6,54)
000484      66 FORMAT(*, 6#715,9)
000485      445 WRITE(6,54)
000486      67 FORMAT(*, 6#715,9)
000487      445 WRITE(6,54)
000488      68 FORMAT(*, 6#715,9)
000489      445 WRITE(6,54)
000490      69 FORMAT(*, 6#715,9)
000491      445 WRITE(6,54)
000492      70 FORMAT(*, 6#715,9)
000493      445 WRITE(6,54)
000494      71 FORMAT(*, 6#715,9)
000495      445 WRITE(6,54)
000496      72 FORMAT(*, 6#715,9)
000497      445 WRITE(6,54)
000498      73 FORMAT(*, 6#715,9)
000499      445 WRITE(6,54)
000500      74 FORMAT(*, 6#715,9)
000501      445 WRITE(6,54)
000502      75 FORMAT(*, 6#715,9)
000503      445 WRITE(6,54)
000504      76 FORMAT(*, 6#715,9)
000505      445 WRITE(6,54)
000506      77 FORMAT(*, 6#715,9)
000507      445 WRITE(6,54)
000508      78 FORMAT(*, 6#715,9)
000509      445 WRITE(6,54)
000510      79 FORMAT(*, 6#715,9)
000511      445 WRITE(6,54)
000512      80 FORMAT(*, 6#715,9)
000513      445 WRITE(6,54)
000514      81 FORMAT(*, 6#715,9)
000515      445 WRITE(6,54)
000516      82 FORMAT(*, 6#715,9)
000517      445 WRITE(6,54)
000518      83 FORMAT(*, 6#715,9)
000519      445 WRITE(6,54)
000520      84 FORMAT(*, 6#715,9)
000521      445 WRITE(6,54)
000522      85 FORMAT(*, 6#715,9)
000523      445 WRITE(6,54)
000524      86 FORMAT(*, 6#715,9)
000525      445 WRITE(6,54)
000526      87 FORMAT(*, 6#715,9)
000527      445 WRITE(6,54)
000528      88 FORMAT(*, 6#715,9)
000529      445 WRITE(6,54)
000530      89 FORMAT(*, 6#715,9)
000531      445 WRITE(6,54)
000532      90 FORMAT(*, 6#715,9)
000533      445 WRITE(6,54)
000534      91 FORMAT(*, 6#715,9)
000535      445 WRITE(6,54)
000536      92 FORMAT(*, 6#715,9)
000537      445 WRITE(6,54)
000538      93 FORMAT(*, 6#715,9)
000539      445 WRITE(6,54)
000540      94 FORMAT(*, 6#715,9)
000541      445 WRITE(6,54)
000542      95 FORMAT(*, 6#715,9)
000543      445 WRITE(6,54)
000544      96 FORMAT(*, 6#715,9)
000545      445 WRITE(6,54)
000546      97 FORMAT(*, 6#715,9)
000547      445 WRITE(6,54)
000548      98 FORMAT(*, 6#715,9)
000549      445 WRITE(6,54)
000550      99 FORMAT(*, 6#715,9)
```
```
APPENDIX – Continued

001221 IF(FHASS=YMMAX3) 301,3C1,4C1 0191
001226 4C1 YMMAK1=PHASE 0152
001230 3C1 CONTINUE 0193
001230 IF(FHASS=YMMIN3) 402,3C2,3C2 0194
001233 4C2 YMMIN3=PHASE 0195
001235 3C0 CONTINUE 0196
001235 IF(AI=YMMAX11) 300,3C0,4C0 0197
001240 4C0 YMMAX11=A1 0198
001242 3C0 CONTINUE 0199
001242 GEN WRITE(6,52)REC,PHASE,AL 0200
001251 52 FORMAT(* W=R,F15.8,* PHASE/F15.8,* YH=R,F15.8) 0201
0202 SCALE COMPUTATIONS
001261 IYMAX11=YMMAK1,7999 0203
001264 YMAX1=FMAX(IYMAX11) 0204
001266 IF(HMAX.EQ.0) GO TO 332 0205
001267 YMAX1=HMAX 0206
001270 332 CONTINUE 0207
001270 IYMAX3=YMAX1+7999 0208
001273 YMAX3=FMAX(IYMAX3) 0209
001275 IYMAX2=YMAX1+7999 0210
001280 YMAX2=FMAX(IYMAX2) 0211
001291 AXM=DF/F(PS/2) 0212
001303 AXM=IHMAGNITUDOS 0213
001304 AWF=30.*FR 0214
001306 NFR=200 0215
C MAGNITUDE (COMPUTED) - FREQUENCY PLOT 0216
CALL INP3PLI6,NFR,FMAGA1,AMAG1,AL,FR(0),YMMAK1,C.5:-10,AXM:-10, 0217
AYM,01 0218
C MAGNITUDE (DESIRFED) - FREQUENCY PLOT 0219
001327 CALL INP3PLI6,NFR,FMAGA1,AMAG1,AL,FR(F0),YMMAK1,C.5:-10,AXM:-10,AYM,11) 0220
CALL INP3PLI6,NFR,FMAGA1,AMAG1,AL,FR(0),YMMAK1,C.5:-10,AXM:-10, 0221
AYM,11 0222
C PHASE-FREQUENCY PLOT 0223
001351 CALL 1VP3PLI6,AF, YHFEA1,PHASEX,1.0,FR(0),YMMAK1,YMAX3,0.5:-10, 0224
AYM,01 0225
C PHASE (COMPUTED) - FREQUENCY PLOT 0226
001371 CALL INP3PLI6,1,1.0,1,FHASS1,1.0,FR(0),YMMAK1,YMAX3,1.0,10, 0227
AYM,11 0228
C THETA1 0229
C THETA1 0229

SUBSTITUTE FPLMU(FUNCT,A,X,F,G,FR,PS,LITIT,IFR,K1) 0230
001371 IF(M'=51) 1111,11,1,11 0231
001371 1 (K1,N11,L11,LF) 0232
CALL KALLI1,LF 0233
001371 IF(X'CALL,GF,700) GO TO 722 0234
001371 722 IF(11) 11 0235
001371 10T1-N 0236
001371 917 KFR=0 0237
001371 K1=0 0238
001360 K1=0 0239
001360 K1=0 0240
001360 K1=0 0241
001360 K1=0 0242
001372 723 IF(K1) 5,5,5,52 0243
001372 5 I1=L1,1,N 0244
001372 5 K1=K1 0245
001372 5 K1=K1 0245
APPENDIX – Continued

00041 4 \( H(KL) = 0.00 \)
00044 4 \( K = KL + 1 \)
00072 5 \( KOUNT = KOUNT + 1 \)
00114 \( \text{WRITE} (", 501) \quad KOUNT \)
00117 501 \( \text{PRINT}(9, \quad KOUNT = 1,15) \)
00101 \( \text{DOF} = F \)
00114 \( I < J = 1, N \)
00117 \( K = N \)
00118 \( H(K) = G(J) \)
00114 \( K = N \)
00118 \( H(K) = X(J) \)
00121 \( K = 1 + 13 \)
00122 \( T = C, 0 \)
00123 \( N = 1 = 1, U \)
00124 \( T = I - G(1) = N(1) \)
00151 \( \text{IF}(I = J) = N, 7, 7 \)
00134 \( \text{IF}(K = I - 1) \)
00137 \( I < K = 1 \)
00141 \( \text{CONTINUE} \)
00144 \( I = J = T \)
00150 \( \text{DY} = C, 0.0 \)
00150 \( \text{HNRM} = 0.00 \)
00151 \( \text{GNRM} = 0.00 \)
00153 \( \text{DO} 10 \quad J = 1, N \)
00154 \( \text{HNRM} = \text{HNRM} + \text{ABS}(H(J)) \)
00160 \( \text{GNRM} = \text{GNRM} + \text{ABS}(G(J)) \)
00163 \( 10 \quad \text{DY} = \text{DY} + H(J) * G(J) \)
00173 \( \text{IF}(\text{DY}) 11, 51, 51 \)
00174 \( 11 \quad \text{IF} (\text{HNRM} / \text{GNRM} - \text{EPS}) 51, 51, 12 \)
00200 12 \( \text{FY} = F \)
00201 \( \text{ALFA} = 2.00 * (\text{FST} - F) / \text{DY} \)
00204 \( \text{AMDA} = 1.00 \)
00205 \( \text{IF} (\text{ALFA}) 15, 15, 13 \)
00207 13 \( \text{IF} (\text{ALFA} = \text{AMDA}) 14, 15, 15 \)
00212 14 \( \text{AMCA} = \text{ALFA} \)
00214 15 \( \text{ALFA} = 0.00 \)
00215 16 \( \text{FX} = \text{FY} \)
00216 \( \text{DX} = \text{DY} \)
00227 \( \text{DC} 17 \quad J = 1, N \)
00222 17 \( X(I) = X(I) + \text{AMCA} * H(I) \)
00231 \( \text{CALL} \text{FUNCTION}(X, F, G, HMAX) \)
00243 \( \text{IF}(\text{KCALL} . GT. 300) \quad \text{GO TO} 724 \)
00246 \( \text{IF} (\text{ICALL} . LT. 1) \quad \text{GO TO} 724 \)
00251 \( \text{GO TO} 918 \)
00251 724 \( \text{IFR} = 3 \)
00253 \( \text{return} \)
00253 918 \( \text{FY} = F \)
00254 \( \text{DY} = C, 0.0 \)
00255 \( \text{DO} 18 \quad I = 1, N \)
00257 18 \( \text{DY} = \text{DY} + G(I) * H(I) \)
00266 \( \text{IF}(\text{DY}) 19, 46, 22 \)
00267 19 \( \text{IF} (\text{FY} = \text{FX}) 20, 22, 22 \)
00272 20 \( \text{AMDA} = \text{AMCA} + \text{ALFA} \)
00274 \( \text{ALFA} = \text{AMDA} \)
00275 \( \text{ERROR} = 1.0 \)
00276 \( \text{IF} (\text{HNRM} * \text{AMDA} - \text{ERROR}) 16, 16, 21 \)
00302 21 \( \text{IFR} = 2 \)
00304 \( \text{return} \)
00304 22 \( T = 0.00 \)
00305 23 \( \text{IF} (\text{AMDA}) 24, 36, 24 \)
00306 24 \( Z = 3.00 * \text{(FX} - \text{FY}) / \text{AMDA} * \text{DX} * \text{DY} \)
00314 \( \text{ALFA} = \text{AMAX}(\text{ABS}(Z), \text{ARSDX} \), \text{ARSDY}) \)
00326 \( \text{CALFA} = Z / \text{ALFA} \)

21
APPENDIX – Continued

000327     $\text{DALFA}=\text{CALFA} \cdot \text{DALFA} \cdot \text{DX} / \text{ALFA} \cdot \text{CY} / \text{ALFA}
000333     \text{IF(CALFA) 51,25,25}
000335     25 $\text{w}=\text{ALFA} \cdot \text{SORT(ICALFA)}$
000340     \text{ALFA} \cdot (\text{DY} \cdot \text{w} - \text{T} \cdot \text{AMBDA} / (\text{DY} \cdot \text{w} - \text{DX}))
000351     \text{DO 26 I}=1,\text{N}
000356     26 $\text{X} = \text{X}\{\text{I} \} + \text{H} \cdot \text{ALFA} \cdot \text{H} \{\text{I}\}
000366     \text{CALL FUNCTN}\{\text{X}, \text{F}, \text{G}, \text{H}\}
000400     \text{IF(KCALL.GT.300) GO TO 725}
000403     \text{IF(KCALL.GT.310) GO TO 725}
000406     \text{GO TO 919}
000408     725 \text{IF} = 3
000410     \text{RFLPN}
000419     915 \text{IF(F-FX) 27,27,29}
000419     27 \text{IF(F-FY) 36,36,29}
000416     28 $\text{DALFA}=0,00$
000417     \text{DC 29 I}=1,\text{N}
000421     29 $\text{DALFA}=\text{DALFA} \cdot \text{G} \{\text{I}\} \cdot \text{H} \{\text{I}\}$
000430     \text{IF(CALFA) 30,33,33}
000431     30 $\text{IF(F-FX) 32,31,33}$
000434     31 $\text{IF(FX-CALFA) 32,36,32}$
000436     32 $\text{FX}=\text{F}$
000437     $\text{DX}=\text{CALFA}$
000440     $\text{T}=\text{ALFA}$
000442     $\text{AMBDA}=\text{ALFA}$
000443     \text{GO TO 23}
000443     33 $\text{IF(FC-F) 35,34,35}$
000445     34 $\text{IF(FY-CALFA) 35,36,35}$
000447     35 $\text{FY}=\text{F}$
000450     $\text{DY}=\text{CALFA}$
000451     $\text{AMBDA}=\text{AMBDA}=\text{ALFA}$
000454     \text{GO TO 22}
000443     36 $\text{DC 37 J}=1,\text{N}$
000443     $\text{K} = \text{N+J}$
000457     $\text{H} \{\text{K} \} = \text{G} \{\text{I} \} \cdot \text{H} \{\text{K} \}$
000463     $\text{K} = \text{K+1}$
000464     37 $\text{H} \{\text{K} \} = \text{X} \{\text{I} \} \cdot \text{H} \{\text{K} \}$
000477     \text{IF(ODF+F)PS) 51,38,38}
000478     38 $\text{IFR}=0$
000477     \text{IF(KOUNT-N) 42,39,39}
000501     39 $\text{T}=0,00$
000502     $\text{Z}=0,00$
000504     \text{DC 40 J}=1,\text{N}
000504     $\text{K} = \text{N+J}$
000506     $\text{W}=\text{H} \{\text{K} \}$
000508     $\text{K} = \text{K+1}$
000511     \text{T}=T \cdot \text{EPS} \{\text{H} \{\text{K} \} \}$
000514     40 $\text{Z} = \text{Z} + \text{W} \{\text{H} \{\text{K} \} \}$
000524     40 $\text{IF(HVR-F)PS) 41,41,42}$
000526     41 $\text{IF(T-FPS) 54,53,54}$
000534     42 $\text{IF(KOUNT-LIMIT) 43,5,53}$
000534     43 $\text{ALFA} = \text{G} \{\text{F} \}$
000535     \text{DC 47 J}=1,\text{N}
000537     $\text{K} = \text{J+1}$
000540     $\text{W}=\text{W},00$
000542     \text{DC 49 L}=1,\text{N}
000543     \text{K} = \text{N+1}$
000544     \text{W} = \text{W} + \text{H} \{\text{K} \} \cdot \text{H} \{\text{K} \}$
000552     \text{IF(L-J) 44,45,46}$
000554     44 $\text{K} = \text{K+1}$
000557     \text{DC TO 46}$
000557     \text{GO TO 46}$
000561     \text{CONTINUE}$
000564     $\text{K} = \text{N+1}$
000565     $\text{ALFA}=\text{ALFA} \cdot \text{W} \{\text{H} \{\text{K} \} \}$

22
SUBROUTINE INSIDE(N,X,KFLAG)
DIMENSION X(16)
J=-1
KFLAG=0
10 J=J+2
000006 IF(J.GT.N)RETURN
000006 E=-500*X(J)
000016 C=X(J+1)
000020 DISC=E*B-C
000022 IF(DISC.LE.0.00)GOTO20
C.......REAL ROOTS
000024 DISC= SQRT(DISC)
000025 R1=B+DISC
000027 R2=B-DISC
000031 CR1= ABS(R1)
000033 DR2= ABS(R2)
000035 IF(DR1.LE.1.00.AND.DR2.LE.1.00)GOTO10
000051 KFLAG=1
000051 IF(DR1.GT.1.00)R1=1.00/R1
000054 IF(DR2.GT.1.00)R2=1.00/R2
000060 X(J)=1.00*(R1+R2)
000064 X(J+1)=R1*R2
000066 GOTO10
C.......COMPLEX ROOTS
20 IF(C.GT.1.00)GOTO10
000071 KFLAG=1
000071 C=1.00/C
000072 X(J+1)=C
000074 X(J)=X(J)*C
000076 GOTO10
000077 END
APPENDIX - Continued
APPENDIX – Concluded

```plaintext
SUBROUTINE ROOTS(N,X)
  DIMENSION X(16)
  WRITE(6,40)
  40 FORMAT(* ROOTS/6X,*,REAL*,11X,*,IMAG*,11X,*,REAL*,11X,*,IMAG*)
  J=1
  10 J=J+2
  IF(J.GT.N)RETURN
  B=-.500*X(J)
  C=X(J+1)
  DISC=B*B-C
  IF(DISC.LE.0.00)GOTO20
  DISC=SOR(T(DISC)
  R1=B+DISC
  R2=B-DISC
  WRITE(6,30)R1,R2
  30 FORMAT(* *,F15.8,15X,F15.8)
  GOTO10
C*****REAL ROOTS
  DISC=SOR(T(-1.00*DISC)
  DISCM=-1.00*DISC
  WRITE(6,50)DISCM
  50 FORMAT(* *,F15.8)
  GOTO10
END
```

REFERENCES


Figure 1.- Signal flow graph of cascaded digital filter.
\[ \theta_k = 1 - 2\Omega_k \quad (0 \leq \Omega_k \leq 1) \]

\[ M_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \]

**Figure 2.** Two-stage linear-phase filter.
\( \theta_k = \begin{cases} 
-\pi/2 & (0.3 \leq \Omega_k \leq 0.7) \\
\text{Unspecified} & \text{(elsewhere)} 
\end{cases} \)

\( M_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \)

(a) Lag filter.

Figure 3.- Two-stage constant-phase filters.
\[ \theta_k = \begin{cases} \pi/2 & (0.3 \leq \Omega_k \leq 0.7) \\ \text{Unspecified} & \text{(elsewhere)} \end{cases} \]

\[ M_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \]

(b) Lead filter.

Figure 3.- Concluded.
\[ M_k = \begin{cases} 
1 & (0.3 \leq \Omega_k \leq 0.7) \\
0 & \text{(elsewhere)} 
\end{cases} \]

\[ \theta_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \]

(a) Unspecified phase filter. \( \lambda = 0. \)

Figure 4. - Two-stage limited-band constant-gain filters.
\[ M_k = \begin{cases} 
1 & (0.3 \leq \Omega_k \leq 0.7) \\
0 & \text{(elsewhere)} 
\end{cases} \]

\[ \theta_k = \begin{cases} 
1 - 2\Omega_k & (0.3 \leq \Omega_k \leq 0.7) \\
0 & \text{(elsewhere)} 
\end{cases} \]

(b) Linear-phase filter. \( \lambda = 10 \).

Figure 4. - Continued.
\[ M_k = \begin{cases} 1 & (0.3 \leq \Omega_k \leq 0.7) \\ 0 & \text{(elsewhere)} \end{cases} \]

\[ \theta_k = \begin{cases} 1-2\Omega_k & (0.3 \leq \Omega_k \leq 0.7) \\ 0 & \text{(elsewhere)} \end{cases} \]

(c) Linear-phase filter. \( \lambda = 1000 \).

Figure 4.- Concluded.
\[ \theta_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \]

\[ M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) \\
0.5 & (\Omega_k = 0.5) \\
0.0 & (0.5 < \Omega_k \leq 1.0) 
\end{cases} \]

(a) Unspecified-phase filter. \( \lambda = 0 \).

Figure 5. - Two-stage low-pass filters.
$$\theta_k = \begin{cases} 
0 & (0.0 \leq \Omega_k \leq 0.5) 
\text{Unspecified} & (\text{elsewhere}) 
\end{cases}$$

$$M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) 
0.5 & (\Omega_k = 0.5) 
0.0 & (0.5 < \Omega_k \leq 1.0) 
\end{cases}$$

(b) Zero-phase filter. $\lambda = 10$.

Figure 5.- Continued.
\[ \theta_k = \begin{cases} 
0 & (0.0 \leq \Omega_k \leq 0.5) \\
\text{Unspecified} & (\text{elsewhere}) 
\end{cases} \]

\[ M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) \\
0.5 & (\Omega_k = 0.5) \\
0.0 & (0.5 < \Omega_k \leq 1.0) 
\end{cases} \]

(c) Zero-phase filter. \( \lambda = 1000 \).

Figure 5. - Concluded.
\( \theta_k = \text{Unspecified} \quad (0 \leq \Omega_k \leq 1) \)

\[ M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) \\
0.5 & (\Omega_k = 0.5) \\
0.0 & (0.5 < \Omega_k \leq 1.0) 
\end{cases} \]

(a) Unspecified-phase filter. \( \lambda = 0. \)

Figure 6.- Three-stage low-pass filters.
\( \theta_k = \begin{cases} 
0 & (0.0 \leq \Omega_k \leq 0.5) \\
\text{Unspecified} & \text{elsewhere} 
\end{cases} \)

\( M_k = \begin{cases} 
1.0 & (0.0 \leq \Omega_k < 0.5) \\
0.5 & (\Omega_k = 0.5) \\
0.0 & (0.5 < \Omega_k \leq 1.0) 
\end{cases} \)

(b) Zero-phase filter. \( \lambda = 10 \).

Figure 6.- Continued.
\[ \theta_k = \begin{cases} 0 & (0.0 \leq \Omega_k \leq 0.5) \\ \text{Unspecified} & \text{(elsewhere)} \end{cases} \]

\[ M_k = \begin{cases} 1.0 & (0.0 \leq \Omega_k < 0.5) \\ 0.5 & (\Omega_k = 0.5) \\ 0.0 & (0.5 < \Omega_k \leq 1.0) \end{cases} \]

(c) Zero-phase filter. \( \lambda = 1000. \)

Figure 6.- Concluded.
Figure 7. - Three-stage low-pass filter.
"The aeronautical and space activities of the United States shall be conducted so as to contribute ... to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—National Aeronautics and Space Act of 1958

**NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS**

**TECHNICAL REPORTS:** Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

**TECHNICAL NOTES:** Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

**TECHNICAL MEMORANDUMS:** Information receiving limited distribution because of preliminary data, security classification, or other reasons.

**CONTRACTOR REPORTS:** Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

**TECHNICAL TRANSLATIONS:** Information published in a foreign language considered to merit NASA distribution in English.

**SPECIAL PUBLICATIONS:** Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

**TECHNOLOGY UTILIZATION PUBLICATIONS:** Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Technology Surveys.

*Details on the availability of these publications may be obtained from:*

**SCIENTIFIC AND TECHNICAL INFORMATION OFFICE**

**NATIONAL AERONAUTICS AND SPACE ADMINISTRATION**

Washington, D.C. 20546