OPTIMIZED SOLUTION OF KEPLER'S EQUATION

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This document presents a detailed description of KEPLER, an IBM 360 computer program used for the solution of Kepler's equation for eccentric anomaly. KEPLER employs a second-order Newton-Raphson differential correction process, and it is faster than previously developed programs by an order of magnitude.
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1.0 INTRODUCTION

1.1 General Description of KEPLER

KEPLER is an IBM 360 computer program used to solve Kepler’s equation for eccentric anomaly:

\[ E = M + e \sin E. \]

The double precision input to the program consists of mean anomaly \( M \) (in radians) and eccentricity of the orbit \( e \). The double precision output from the program consists of eccentric anomaly \( E \) (in radians), \( \sin E \), and \( \cos E \).

1.2 KEPLER and KEPLR1

KEPLER has been developed by the authors as a replacement for KEPLR1, a similar program contained in the definitive orbit determination system (DODS) used at Goddard Space Flight Center to determine orbits for NASA’s scientific satellites. KEPLR1 was not a hastily coded and formulated program designed for early replacement as soon as DODS became operational. On the contrary, KEPLR1 was developed after a rather extensive research effort that recommended a particular algorithm, the Myles Standish algorithm.* This algorithm was then implemented by Federal Systems Division of IBM under contract to GSFC (see Appendix A). KEPLR1 has been in productive use since May 1968.

KEPLER was developed as a part of an effort to optimize the execution times of frequently called subprograms in DODS. The initial thought was simply to recode KEPLR1 in assembly language coding (ALC), since, like most other programs in DODS, KEPLR1 was coded in FORTRAN, and previous experience with other programs in DODS led the authors to anticipate a 20 to 30 percent speed improvement from a FORTRAN-to-ALC recoding. However, a little analysis of the formulation used in KEPLR1 led the authors to believe that a new analytic approach to the age-old problem was

in order. As a result, they engaged in a research effort of their own* and developed a completely new computer program, KEPLER.

KEPLER not only solves Kepler's equation for eccentric anomaly but also outputs accurate values for the sine and cosine of the eccentric anomaly. This is important since in almost every case for which DODS calls on KEPLR1 to solve Kepler's equation for \( E \), it then uses \( E \) as the input argument to the DSIN(X) and DCOS(X) functions. These function calls are unnecessary when KEPLER is used since \( \sin E \) and \( \cos E \) are part of its output.

1.3 Outline of Remainder of This Document

Section 2 of this document is a description of the relative performance of KEPLER versus that of KEPLR1. Section 3 is a detailed design and performance description of the newly developed program, KEPLER. Section 4 presents the mathematical derivation of the principal formulas used in KEPLER, namely the second-order Newton-Raphson differential correction of eccentric anomaly. Section 5 summarizes the significance of the use of KEPLER and its called module in DODS, DPSC (double precision sine/cosine), and recommends the development of related programs.

Appendix A is a module performance and design description of KEPLR1 prepared for GSFC by IBM. Appendixes B and C list the test programs used to compare KEPLER with KEPLR1. Appendix D lists KEPLR1 (in FORTRAN) and KEPLER and its called program, DPSC (in ALC).

2.0 A COMPARISON OF KEPLER AND KEPLR1

This section deals with accuracy, speed, core storage requirement, and reentrant properties of the KEPLER program. The DODS module KEPLR1 is used as a benchmark for comparison.

2.1 Accuracy of KEPLER

Appendix B lists a test program, TEST 1, which is used to exercise both KEPLR1 and KEPLER over a full range of the input arguments, \( M \) and \( e \). The maximum error produced by each program over full ranges of the arguments is obtained by substitution of the solution for eccentric anomaly back into Kepler's equation:

\[
\text{KEPLR1 error} = |E - (M + e \sin E)| = 0.50 \times 10^{-15}
\]

\[
\text{KEPLER error} = |E - (M + e \sin E)| = 0.44 \times 10^{-15}.
\]

2.2 Execution Times

Appendix C lists a test program, TEST 2, which times the execution of KEPLER and KEPLR1 on GSFC's IBM 360/95 computer. Full ranges of \( e \) and \( M \) are used in this test. The average execution

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times reported are based on 400 000 test cases:

Average KEPLR1 execution time = 314 μs*

Average KEPLER execution time = 74 μs.

2.3 Core Requirements

KEPLR1 requires 820 bytes of core for itself and 620 bytes of core for its called module, IHCLSCN, the standard, release 19 FORTRAN library (FORTLIB) double precision sine/cosine routine. KEPLER requires 440 bytes of core for itself and 2360 bytes of core for its called module, DPSC.

2.4 Reentrancy of KEPLER

Both KEPLER and its called module, DPSC, are reentrant and, therefore, are candidates for the high-speed system link pack area. Neither KEPLR1 nor its called module are reentrant. Therefore, they may not be stored in the high-speed system link pack area.

2.5 Optimal Execution Times for KEPLER

The favorable ratio of execution times reported in Section 2.2 (KEPLR1/KEPLER = 314/74 = 4.24) will be enhanced by factors of 2, 3, and 4, on the IBM 360/95, 360/75, and 360/65, respectively, when KEPLER and DPSC are located in the high-speed system link pack area and KEPLR1 and its called module are located in low-speed core. Under these optimal conditions, the execution time ratios would be 8.48, 12.72, and 16.96, respectively.

Furthermore, when KEPLER and DPSC are located in the system link pack area, several concurrent jobs could be calling on KEPLER, and only one copy of KEPLER would be in core. (The core storage requirement of KEPLER would be charged to system overhead and not to a particular job.)

3.0 KEPLER–MODULE PERFORMANCE AND DESIGN DESCRIPTION

3.1 Language

KEPLER is written in ALC in order to reduce both execution time and core storage. The core storage savings are incidental; the main reason for the use of ALC is to produce a faster executing program. It is estimated that the use of ALC for KEPLER is responsible for about one-third of the improvement in execution time when that program is used in place of KEPLR1.

3.2 Module Size

KEPLER requires 440 bytes of core storage.

*This figure includes the time required to calculate sin E and cos E. Without these calculations, the average execution time for KEPLR1 is 266 μs.
3.3 Purpose of KEPLER

KEPLER solves Kepler's equation for eccentric anomaly when mean anomaly and eccentricity are known. The sine and cosine of eccentric anomaly are output by this module.

3.4 Linkage Information

3.4.1 Calling Sequence

KEPLER is invoked via the following call statement:

\[
\text{CALL KEPLER}(MA, ECC, IERR, OUT),
\]

where

- \(MA\) (input) \(= M\), mean anomaly (in radians);
- \(ECC\) (input) \(= e\), eccentricity;
- \(IERR\) (output) = error code
  \(= 0\), if no error
  \(= 1\), if \(e\) is negative or greater than 0.99;
- \(OUT\) (output) = 3 word matrix
  \(OUT(1)\) = \(E\), eccentric anomaly
  \(OUT(2)\) = \(\sin E\)
  \(OUT(3)\) = \(\cos E\).

The qualities \(M\), \(e\), \(E\), \(\sin E\), and \(\cos E\) are double precision floating point numbers. The error code, \(IERR\), is a full word integer.

3.4.2 Called Modules

KEPLER calls on the reentrant DPSC module. It uses the ALC entry point, SINCOS, which inputs \(x\) in FR0 and outputs \(\sin x\) in FR0 and \(\cos x\) in FR2. (Note: FR0 is floating point register 0 and FR2 is floating point register 2.)

3.4.3 Calling Modules

The following modules in DODS call on KEPLER:

- \(NEGEN0\), \(DCCON0\),
- \(EPTRB0\), \(CNVRT0\),
- \(ELCON0\), \(UNCAL0\).

3.5 Functional Analysis

3.5.1 Module Component 1, Main Program
3.5.1.1 Method

Kepler's equation, \( E = M + e \sin E \), is solved for \( E \) by use of a second-order Newton-Raphson iterative algorithm. The derivation of this algorithm is discussed in Section 4 of this document. The iterative algorithm is enhanced by four features of KEPLER:

1. The mean anomaly \( M \) is reduced to a value between \(-\pi\) and \(+\pi\), and the resultant sign is saved. Then, the absolute value of \( M \) is used to solve Kepler's equation. After a solution is obtained, \( E \) is set equal to \( 2\pi - E \) if the reduced value of \( M \) is negative.

2. A highly efficient initial estimate algorithm is used to generate \( E' \), a starting value for the iterative process. This algorithm is discussed in Section 3.5.2.

3. The SINCOS entry in the DPSC module is used to calculate simultaneously \( \sin E' \) and \( \cos E' \).

4. Sum formulas are used to calculate \( \sin (E' + C) \) and \( \cos (E' + C) \) whenever \( C \) becomes small enough that first- or second-order approximations of \( \sin C \) and \( \cos C \) are tolerable.

3.5.1.2 Main Program Algorithm

Step 1: Error exit if \( e \) is negative or greater than 0.99.

Step 2: Reduce \( M \) modulo \( 2\pi \) to range \([-\pi, +\pi]\).
   Save sign of \( M \) and set \( M = |M| \).

Step 3: \( E' = \text{ESTIMATE} (M, e) \). \hfill (See module component 2.)

Step 4: Calculate \( \sin E' \) and \( \cos E' \) via SINCOS routine.

Step 5: \( F = M + e \sin E' - E' \). \hfill (linear correction)

Step 6: \( D = 1 - e \cos E' \). \hfill (first-order derivative)

Step 7: \( D' = D + 0.5Fe \sin E'/D \). \hfill (second-order derivative)

Step 8: \( C = F/D' \). \hfill (second-order correction)

Step 9: \( E'' = E' + C \); store as \( E' \) \hfill (enhanced \( E' \))

Step 10: If \( |C| > 10^{-5} \), return to Step 4.

Step 11: If \( |C| < 10^{-8} \), skip to Step 13.

Step 12: \[
\begin{align*}
\sin E'' &= \sin (E' + C) = (1 - 0.5C^2) \sin E' + C \cos E' \\
\cos E'' &= \cos (E' + C) = (1 - 0.5C^2) \cos E' - C \sin E'.
\end{align*}
\]
   Replace \( \sin E' \) with \( \sin E'' \), replace \( \cos E' \) with \( \cos E'' \), and return to Step 5.

Step 13: \[
\begin{align*}
\sin E'' &= \sin (E' + C) = \sin E' + C \cos E' \\
\cos E'' &= \cos (E' + C) = \cos E' - C \sin E'.
\end{align*}
\]
   \hfill (first-order sums formulas)

Step 14: If sign of reduced \( M \) is positive, skip to Step 16.

Step 15: Set \( \sin E' = -\sin E' \) and \( E' = 2\pi - E' \).
Step 16: Output \( E = E', \sin E = \sin E', \) and \( \cos E = \cos E' \), with \( E, \sin E, \) and \( \cos E \) accurate to 15 decimal places.

Step 17: Return to calling program.

3.5.1.3 Explanation of Algorithm

Step 1: If \( e \) is out of range, no output other than error code is generated.

Step 2: The reduction of \( M \) to the range \([-\pi, +\pi]\) and the saving of its sign have several advantages: (1) it increases the precision of the calculation of the linear correction in Step 5 since \( M \) and \( E' \) will not exceed \( \pi \); (2) it simplifies the estimation function (Step 3) since \( M \) is constrained to the range \([0, \pi]\); (3) it provides a convenient test for \( E \) being output in the range \([0, 2\pi]\) (Steps 14-15).

Step 3: The estimation function is treated in detail in Section 3.5.2. The purpose of this function is to provide a sufficiently accurate initial estimate of eccentric anomaly to ensure (1) that the differential correction process defined in Steps 4-9 converges and (2) that this convergence takes place in a minimum number of iterations.

Steps 4-9: This sequence of steps constitutes one second-order Newton-Raphson iteration. This algorithm is considerably more accurate than a first-order differential correction and its use has two basic advantages: (1) it converges in fewer iterations than the first-order correction and (2) the convergence tolerance \( e \) used to terminate the second-order differential correction process may be larger than that for the first-order correction because the correction is more accurate. (Other programs use a convergence tolerance of \( 5 \times 10^{-12} \), but KEPLER is able to maintain accuracy with a convergence tolerance of \( 10^{-8} \).)

Step 10: If the absolute value of the correction \( C \), is greater than \( 10^{-5} \), Steps 4-8 are repeated. That is, the lengthy SINCOS routine is reexecuted in Step 4 to provide accurate values for \( \sin (E' + C) \) and \( \cos (E' + C) \).

Step 11: If the absolute value of \( C \) is less than \( 10^{-8} \), sufficient convergence is obtained to guarantee that \( E \) is accurate to 15 significant digits. Step 12 is skipped when this condition is met.

Step 12: When the absolute value of \( C \) is between \( 10^{-5} \) and \( 10^{-8} \), the algorithm iterates, but the lengthy sine/cosine calculation (Step 4) is replaced by the second-order sum formulas in Step 12. The largest truncated term in these sum formulas is \( C^3 / 3! \). This means that when \( C < 10^{-5} \), the sum formulas have a relative accuracy of \( 0.167 \times 10^{-15} \), which is slightly more accurate than the original calculation of \( \sin E' \) and \( \cos E' \) in Step 4.

Step 13: When convergence takes place (\( C < 10^{-8} \)), \( \sin E' \) and \( \cos E' \) are updated by the first-order sum formulas in Step 13. The largest truncated term in the first-order sum formula is \( C^2 / 2! \). This means that when \( C < 10^{-8} \), the sum formulas have a relative accuracy of \( 0.5 \times 10^{-16} \), which again is more accurate than the original calculation of \( \sin E' \) and \( \cos E' \).
Step 14: If $M$ is positive after being reduced to the range $[-\pi, +\pi]$, Step 15 is skipped (sin $E > 0$, $E < \pi$).

Step 15: If $0 > M > -\pi$, sin $E'$ is set negative, and $E'$ is set equal to $2\pi - E'$.

Step 16: The quantities $E$, sin $E$, and cos $E$ are sequentially output in a 3 word matrix.

Step 17: The program is concluded.

3.5.2 Module Component 2, Initial Estimate

3.5.2.1 Method

As stated in the preceding section, the main purpose of the initial estimate algorithm is to provide a starting value for the differential correction process defined in Steps 4-9 of the main program. There is an obvious tradeoff between time and accuracy in this initial estimate algorithm. As the initial estimate is made more accurate, the number of times Steps 4-9 of the main program must be executed is reduced. There are, however, constraints on this tradeoff.

When $E'$ is as accurate as one part in $10^5$, the time-consuming sine/cosine calculation step in the main program will be executed only once. Therefore, it is desirable to generate $E'$ to this degree of accuracy for most combinations of the input parameters $e$ and $M$. However, it would be uneconomical to spend too much time trying to achieve a greater overall accuracy since, regardless of the accuracy achieved, the full sine/cosine calculation must be executed at least once in order to further refine $E'$ and to generate the sin $E$ and cos $E$ output.

The algorithm used by KEPLER for the initial estimate was selected only after a large number of alternative algorithms were tested and proven to be less efficient.* The name abbreviated Newton-Raphson is given to this algorithm because it represents a first-order Newton-Raphson correction in truncated precision. It possesses two desirable properties: (1) it is executed in a minimal amount of time (37 \text{\mu s} on the IBM 360/75) and (2) for most combinations of $M$ and $e$, it achieves the desired accuracy of one part in $10^5$.

3.5.2.2 Initial Estimate Algorithm (Abbreviated Newton-Raphson)

The abbreviated Newton-Raphson algorithm consists of two steps.

Step A: \[ \overline{E} = M + ez, \quad 0 < M < \pi, \]

where $z$ is a linear estimate of sin $M$:

\[ z = 0.75 M, \quad M \leq \pi/2; \]

\[ z = 0.75 (\pi - M), \quad M > \pi/2. \]

Step B: \[ E' = \overline{E} + \frac{M + e \sin \overline{E} - \overline{E}}{1 - e \cos \overline{E}}. \]

where $\sin E$ and $\cos E$ are calculated from the first two terms of the Maclaurin expansions:

\[
\sin x = x - \frac{x^3}{3!}
\]

\[
\cos x = 1 - \frac{x^2}{2!}
\]

Step B involves several substeps:

1. Set $x = \overline{E}$ and $S = 1$ ($S = \text{SWITCH}$).
2. If $x > \pi/2$, set $x = \pi - x$ and $S = 2$.
3. If $x > \pi/4$, set $x = \pi/2 - x$ and $S = -S$.
4. Calculate $\sin E = x - \frac{x^3}{6}$ and $\cos E = 1 - \frac{x^2}{2}$.
5. If $S$ is negative, exchange $\sin E$ and $\cos E$ and set $S = -S$.
6. If $S = 2$, set $\cos E = -\cos E$.
7. $E' = \overline{E} + \frac{M + e \sin E - \overline{E}}{1 - e \cos \overline{E}}$.

3.5.2.3 Relation of Component 2 to Main Program

The initial estimate function (component 2) is linearly coded as Step 3 of the main program (component 1). Component 2 does not have a separate entry point in KEPLER.

3.5.3 Flowcharts

No flowcharts are provided for the main program or the initial estimation function since KEPLER's source code and source code comments directly conform to the logic outlined in Sections 3.5.1.2 and 3.5.2.2.

3.6 Restrictions and Limitations

If the input value of $e$ is negative or greater than 0.99, no output other than error code is generated.

3.7 Storage Tables External to KEPLER

None.

3.8 Input/Output Device Requirements

None.
3.9 Error Conditions and Recovery

The error code, IERR, is examined after execution. If IERR = 1, the input value of \( e \) is out of range, and hence no other output can be expected.

3.10 Module Design Technique

KEPLER is reentrant and, therefore, may be loaded in the system link pack area. KEPLER is optimized for fast execution; it is several times faster than existing modules.

3.11 Test Procedures and Results

The speed and accuracy of KEPLER have been verified by the test programs given in Appendices B and C. The results of these tests are summarized in Sections 2.1 and 2.2.

4.0 SECOND-ORDER NEWTON-RAPHSON METHOD

Deutsch applies the Newton-Raphson method to the problem of solving Kepler's equation (Reference 1, pp. 24–25). He extends the procedure to include second-order effects, but the final equation in the development includes an error in sign, as will be noted later.

If

\[ f(E) = E - M - e \sin E , \]

then,

\[ f'(E) = 1 - e \cos E . \]

Let

\[ \Delta E = E_1 - E_0 . \]

Then,

\[ \Delta E = \frac{-(E_0 - M - e \sin E_0)}{1 - e \cos E_0} + O[(\Delta E)^2] . \]

To obtain an expression valid to terms of order \((\Delta E)^2\), Deutsch proceeds as follows:

\[
M = (E_0 + \Delta E) - e \sin (E_0 + \Delta E) \\
= E_0 + \Delta E - e(\sin E_0 \cos \Delta E + \sin \Delta E \cos E_0) \\
= E_0 + \Delta E - e\left\{ \sin E_0 \left[ 1 - \frac{(\Delta E)^2}{2} \right] + \Delta E \cos E_0 \right\} ;
\]

then,

\[
\frac{e \sin E_0}{2} (\Delta E)^2 + (1 - e \cos E_0)\Delta E + (E_0 - M - e \sin E_0) = 0 .
\]
Let
\[ x = \frac{1}{\Delta E}, \]
\[ A = E_0 - M - e \sin E_0, \]
\[ B = 1 - e \cos E_0, \]
\[ C = \frac{e \sin E_0}{2}. \]

We have then
\[ Ax^2 + Bx + C = 0 \]
\[ x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \]

Hence,
\[ \Delta E = \frac{2A}{-B \pm \sqrt{B^2 - 4AC}} \]
\[ \approx \frac{2A}{-B \pm (B - 2AC/B)}. \]

Following Deutsch, we adopt the minus sign as the appropriate choice in the denominator;
\[ \Delta E = \frac{E_0 - M - e \sin E_0}{-(1 - e \cos E_0) + (1/2)(E_0 - M - e \sin E_0)e \sin E_0(1 - e \cos E_0)^{-1}}, \]
\[ \Delta E = \frac{M - E_0 + e \sin E_0}{1 - e \cos E_0 + (1/2)(M - E_0 + e \sin E_0)e \sin E(1 - e \cos E_0)^{-1}}, \]
\[ \Delta E = \frac{M - E_0 + e \sin E_0}{1 - e[\cos E_0 - (1/2)(M - E_0 + e \sin E_0) \sin E_0(1 - e \cos E_0)^{-1}]} \]

(The minus sign within the brackets in the denominator which precedes (1/2) is incorrectly given as a plus sign in Reference 1.)

In general, for functions \( f(E) \) for which the relevant derivatives exist, we obtain from a Taylor's series expansion:
\[ \Delta E = \frac{f(E_0)}{-f'(E_0) + f(E_0)f''(E_0)/2f'(E_0)}, \]
where terms through \((\Delta E)^2\) have been included.
5.0 CONCLUSIONS AND RECOMMENDATIONS

5.1 Effect of KEPLER on DODS Performance

The use of KEPLER in DODS results in a relatively insignificant enhancement of that system because, prior to the use of KEPLER, DODS was spending less than three percent of its time solving Kepler’s equation. Therefore, even if KEPLER were one hundred times faster than KEPLR1, the time saving would not be highly significant in DODS operation.

5.2 Effect of the DPSC Module on KEPLER

Of more significance to DODS is the concurrent development of DPSC, with ALC entry points SINCOS, DSINX, and DCOSX and FORTRAN entry points, DPSC, DSIN, and DCOS. KEPLER uses only the SINCOS entry point. The use of this efficient subroutine to simultaneously calculate \( \sin E' \) and \( \cos E' \) accounts for about one-third of KEPLER’s enhancement of DODS. Use of ALC and the improved mathematical model account for the other two-thirds.

5.3 Effect of DPSC on DODS

The inclusion of the DSIN and DCOS entry points in the DPSC module will enhance DODS considerably more than the inclusion of KEPLER alone. Every sine and cosine calculation in DODS will be executed more efficiently since the DSIN and DCOS entry points in DPSC will override the DSIN and DCOS entry points in the FORTLIB module, IHCLSCN.

5.4 Recommended Use of Simultaneous Sine/Cosine Routine

The DODS formulation contains many situations in which the calculation of both the sine and cosine of a given angle is required. The simultaneous sine/cosine entry points (DPSC and SINCOS) in the DPSC module are now available to DODS programmers who are optimizing DODS modules (such as KEPLER) that call for the calculation of both \( \sin(x) \) and \( \cos(x) \). The FORTRAN subprogram call,

\[
\text{CALL DPSC (X, SC),}
\]

takes about one-half as much time to execute as the separate function calls to IHCLSCN:

\[
\text{SC(1) = DSIN(X)}
\]
\[
\text{SC(2) = DCOS(X).}
\]

(When DSIN and DCOS entry points are in the DPSC module, the time enhancement is reduced from a factor of 2 to a factor of 1.5.)

5.5 Recommended Use of ALC Entry Points in DPSC

The ALC entry points in the DPSC module enable an ALC program to execute register-to-register sine/cosine functions that bypass the highly indirect FORTRAN convention of passing to the function program the address of the address of the argument in general register 1. Besides saving time (the ALC
functions are about seven percent faster), the ALC entry points make it possible for some calling programs to be written in reentrant code, without using the time-consuming GETMAIN macro. This is possible since the ALC functions do not require the storage of an argument list and use only the last eight bytes of the save area, which they can share with the calling program. KEPLER is a good example of a second-order reentrant program sharing its save area with the SINCOS routine.

5.6 Expansion of the DPSC Module

Because of the frequency of calls to mathematical functions in DODS (and other production programs run on GSFC computers), the authors are developing a series of reentrant modules to replace the most frequently called function subprogram modules in FORTLIB. The following function subprograms, called TRIGPACK, are all either completed or nearing completion:

\[
\begin{align*}
\text{DSQRT}(X), & \quad \text{DTAN}(X), \quad \text{DATAN2}(X, Y), \\
\text{DSIN}(X), & \quad \text{DCOT}(X), \quad \text{DASIN}(X), \\
\text{DCOS}(X), & \quad \text{DATAN}(X), \quad \text{DACOS}(X).
\end{align*}
\]

The single precision counterparts of these double precision function subprograms are also nearing completion.

Besides the standard FORTRAN entry points, these modules all contain corresponding ALC entry points (FORTRAN name with an X suffix—DTANX, for example). The ALC entry points assume that the argument is already in floating point register 0. (For the case of the double argument in the DATAN2X function, the arguments are assumed to be in floating point registers 0 and 2.) The ALC entry points will permit the development of a large number of reentrant second-order subroutines since only the last eight bytes of the save area are used and the storage of an argument list is not required as a prelude to the subroutines execution.

5.7 Element Conversion Module

In conjunction with the development of KEPLER and an optimized reentrant TRIGPACK (Section 5.6), the authors are recoding a DODS module called ELCONO, which contains two inverse subprograms: one for converting position and velocity vectors to osculating Keplerian elements, and the other for performing the reverse of this transformation. This third-order module, written in ALC, calls on KEPLER and the ALC entry points SINCOS, DSQRTX, DSINX, DATANX, DATAN2X, DACOSX, and DTANX in the TRIGPACK modules. It also calls on ALC entry points, VCROSSX, VDOTX, and XDOTX in a newly developed vector package. KEPLER and the SINCOS entry in DPSC are the heart of this newly optimized module.
ACKNOWLEDGMENT

Miss Anne Bomford provided invaluable computer programming support in the development of the programs presented in this document.

REFERENCE

Appendix A*

Definitive Orbit Determination System—Model 1
(Module Performance and Design)

5. MODULE NAME: KEPLR1—SOLUTION OF KEPLER'S EQUATION FOR ECCENTRIC ANOMALY.

5.1 LANGUAGE
FORTRAN IV

5.2 MODULE SIZE
The source deck of KEPLR1 consists of 18 executable FORTRAN statements and requires 820 bytes of core storage.

5.3 PURPOSE
KEPLR1 solves Kepler's equation for eccentric anomaly given mean anomaly and eccentricity by the Myles Standish algorithm.

5.4 INTERFACE INFORMATION

5.4.1 LINKAGE DEFINITION
Linkage to this module requires the following CALL statement: CALL KEPLR1 (MA, ECC, ERRC, E2). See Table 1 for the definition of the calling sequence arguments.

5.4.2 INTERFACE BLOCK DIAGRAM

Table 1. Calling Sequence Arguments

<table>
<thead>
<tr>
<th>Argument Name</th>
<th>Analytic Symbol</th>
<th>I/O</th>
<th>Description</th>
<th>Units</th>
<th>Format*</th>
<th>Limits Min/Max</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>MA</td>
<td>M</td>
<td>I</td>
<td>Mean anomaly</td>
<td>Radians</td>
<td>LF</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>ECC</td>
<td>e</td>
<td>I</td>
<td>Eccentricity</td>
<td>-</td>
<td>LF</td>
<td>0 - 1</td>
<td>1</td>
</tr>
<tr>
<td>ERRC</td>
<td>O</td>
<td>O</td>
<td>Error code</td>
<td>-</td>
<td>LI</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>E2</td>
<td>E</td>
<td>O</td>
<td>Eccentric</td>
<td>Radians</td>
<td>LF</td>
<td>-</td>
<td>1</td>
</tr>
</tbody>
</table>

*Format Key

LF - Long Form Floating Point
LI - Long Form Integer

5.4.3 INTERFACE BLOCK DIAGRAM NARRATIVE
None

5.4.4 CALLED MODULES
None

5.4.5 CALLING MODULES
ELCON0 - Elements Conversion Package
DCCON0 - DC Control
CNVRT0 - CONVERT Control
UNCAL0 - Unknown Calculation
CONEG0 - CONVERT Normal Equations
NEGEN0 - DC Normal Equations
EPTRB0 - EPHEM Tape Record Builder

5.5 FUNCTIONAL ANALYSIS

5.5.1 MODULE COMPONENT 1: Solve Kepler's equation

5.5.1.1 Method:
Given the mean anomaly, M, and the eccentricity, e, the algorithm for
computing the eccentric anomaly, \( E \), will be:

1. Set error code = 0  
   Set limit of number of iterations, \( \text{MAX} = 10 \)

2. Set \( E = 0 \)  
   If \( M = 0 \), go to Step 13  
   If \( M \neq 0 \), go to Step 3

3. \( E_0 = M + e \sin M \)  
   Set number of iterations = 1

4. \( F = E_0 - (e \sin E_0) - M \)

5. \( D = 1.0 - [ e \cos (E_0 - 0.5F) ] \)

6. \( E = E_0 - F/D \)

7. If \( |E_0 - E| - \text{TOL} \leq 0 \), go to Step 13; otherwise continue to Step 8.

8. Add 1 to number of iterations

9. If (number of iterations - \( \text{MAX} \)) \leq 0, continue; otherwise go to Step 12

10. \( E_0 = E \)

11. Return to Step 4

12. Set error code = 4

13. Modulo \( E \) by \( 2 \pi \)

14. Return to calling program

The limit of iterations through Steps 4 to 11 is \( \text{MAX} = 10 \). If this number is exceeded, the error code is set to 4.

TOL is the tolerance at which the last significant digit of the difference between the previous calculated eccentric anomaly and the present calculated anomaly is allowed. TOL allows an error of \( \pm 5 \times 10^{-15} \)

5.5.1.2 Source and Type of Inputs:
The mean anomaly, \( M \), and the eccentricity, \( e \), will be obtained by this module from the calling sequence of the CALL statement which transfers control to this function. The format of \( M \) and \( e \) will be long form floating-point.

5.5.1.3 Destination and Type of Outputs:
Output of this module will be the eccentric anomaly, \( E \), in long form floating-point format. Also the error code, \( \text{ERRC} \), in long integer format will be returned to the calling sequence of the CALL statement which transfers control to this function.
5.5.1.4 Component Level Flowcharts

```
| ENTER KEPLR1 | 1B |
|------------------|
| SET ITER=1 |
| 1C |
| IS MEAN ANOMALY=0? NO YES |
| 1D |
| SET ECCENTRIC ANOMALY=0 |
| 1E |
| E0 = M + e sin M |
| 1F |
| F = E0 - (e sin E0) / M |
| 1G |
| D = 1 - (e cos (E0 - 5F)) |
| 1H |
| E = E0 - F / D |
| 1I |
| IS |I0 - E| ≤ TOL ≤ 0? |
| 1J |
| ITER = ITER + 1 |
| 1K |
| IS ITER MAX ≤ 0? |
| 1L |
| SET ERROR CODE=4 |
| 1M |
| E0 = E |
| 1N |
| RETURN |
| 1O |
```
Component Level Flowchart Description

1A - Values for mean anomaly, M, and eccentricity, e, are transferred to the module via the calling sequence. The eccentric anomaly, E, and the error code, ERRC, are returned via the calling sequence.

1B - The counter for the number of iterations is set to 1.

1C - Test to see if mean anomaly is equal to zero.

1D - Sets the eccentric anomaly, E, equal to zero when the mean anomaly, M, is equal to zero and returns to calling module.

1E - Computes the initial eccentric anomaly, E₀.

1F - Computes the expression E₀ - (e sin E₀) - M using the initial computed value of E.

1G - Computes the expression 1.0 - [e cos (E₀ - 0.5F)] using the initial computed value of E₀ and F computed in 1F.

1H - Computes a more accurate value for eccentric anomaly, E = E₀ - F/D.

1I - Test to see if eccentric anomaly has been determined. If the value of the expression |E₀ - E| - TOL is equal to or less than zero, E has been determined. If E has been determined control is returned to the calling module.

1J - Increase number of iterations by 1 when eccentric anomaly has not been determined.

1K - Test to see if number of iterations is less than or equal to the maximum number of iterations allowed.

1L - If the number of iterations exceeds the maximum, set error code equal to 4. Return control to calling module.

1M - If the number of iterations meets the test, make the initial eccentric anomaly, E₀, equal to the computed eccentric anomaly, E, and return to 1F.

5.6 RESTRICTIONS AND LIMITATIONS
None

5.7 STORAGE TABLES EXTERNAL TO MODULE
None

5.8 INPUT/OUTPUT DEVICE REQUIREMENTS
None
5.9 ERROR CONDITIONS AND RECOVERY
Check to see that number of iterations does not exceed the maximum.
If so, set error code, ERRC = 4.

5.10 MODULE DESIGN TECHNIQUE
5.10.1 MODULARITY REQUIREMENT
None

5.10.2 EXPANDABILITY REQUIREMENT
None

5.10.3 PARAMETERIZATION
The maximum number of iterations to determine the eccentric anomaly was set equal to 10.

5.10.4 SPECIAL FEATURES
None

5.11 TESTING PROCEDURES AND RESULTS
5.11.1 UNIT TEST DRIVER DESIGN AND IMPLEMENTATION
Test 1 - Input data—eccentricity and mean anomaly are read from data cards, subroutine KEPLR1 is executed and the results are printed.
Test 2 - The eccentricity is varied from 0 to 1 by increments of .05 for all values of eccentric anomaly from 0 to 360 degrees, incremented by 15 degrees in order to compute values of mean anomaly. Subroutine KEPLR1 is called for all values of mean anomaly and corresponding eccentricity, and the results are printed out.
5.11.1.1 Unit Test Driver Flowchart

ENTER

WRITE HEADING INPUT OUTPUT PARAMS 2A

CONVERSION FACTOR FOR DEGREES TO RADIANS 2B

READ M, \( \alpha \), E IN DEG 2C

CONVERT M TO RADIANS 2D

COMPUTE E FROM KEPLER 1 2E

CONVERT E TO DEGREES 2F

CONVERT COMPUTED E TO DEGREES 2G

WRITE M, \( \alpha \), E (INPUT), E (COMPUTED) 2H

HAS ALL DATA BEEN READ?

RETURN

YES NO 2I
CONVERT TO RADIANS
COMPUTE MEAN ANOMALY, M = E - e sin E
COMPUTE E FROM KEPLER
CONVERT M AND BOTH E' S TO DEGREES

START

3A
3B
3C
3D
3E
3F

WRITE HEADING

4A
4B
4C
4D
4E
4F

INCREMENT E BY .05
INCREMENT E BY 15 DEGREES
4G

COMPUTE DIFFERENCE IN ECCENTRIC ANOMALY VALUES
WRITE M, BOTH E'S AND DIFFERENCE

IS ECCENTRIC ANOMALY ≤ 360°

STOP

IS e ≤ 1

NO
YES

YES
NO
5.11.2 BENCHMARK TESTING
None

5.11.3 TEST RESULTS AND ACCURACY
This item is not covered at the module level; it is included in the system evaluation document.*

5.12 GLOSSARY
A glossary of internal symbols associated with quantities having analytic significance is given in Table 2.

5.13 REFERENCES
Memorandum from I. Cole to IBM; 1 August 1967

Table 2. Internal Symbols

<table>
<thead>
<tr>
<th>Program Symbol</th>
<th>Analytic Symbol</th>
<th>Description of Term</th>
<th>Units</th>
<th>Format</th>
<th>Limits Min/Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>E₀</td>
<td>Eccentricity anomaly</td>
<td>Radians</td>
<td>LF</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Compare value</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ITER</td>
<td>ITER</td>
<td>Number of iterations</td>
<td>-</td>
<td>LI</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Completed</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MAX</td>
<td>MAX</td>
<td>Limit of number of</td>
<td>-</td>
<td>LI</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>iterations</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TOL</td>
<td>TOL</td>
<td>Tolerance for convergence</td>
<td>-</td>
<td>LF</td>
<td></td>
</tr>
</tbody>
</table>

*Working document in use at GSFC.
Appendix B

Test 1 and Results

C TEST 1
C TEST PROGRAM TO TIME TWO KEPLER PROGRAMS
C PROGRAMMER ANNE BOMFORD

REAL*M, E, EA(3), DE

500 FORMAT('1',//,'//49X,'EXECUTION TIMES FOR KEPLER AND KEPLR1',//)
WRITE(6,500)
E=1D-03
ITIME =2
CALL INTIMO(1)
DO 3 I=1,20
M=1D-02
DO 2 J=1,200
DO 1 K=1,100
1 CALL KEPLER(M, E, IRR, EA)
2 M=M+6.28D-02
3 E=E+.49D-01
CALL INTIMO(ITIME)
ITIME=ITIME/400000

506 FORMAT('//30X,'AVERAGE EXECUTION TIME FOR KEPLER ON 360/95',1X,I3,
11X,'MICROSECONDS'//)
WRITE(6,506) ITIME
E=1D-03
ITIME=2
CALL INTIMO(1)
DO 6 I=1,20
M=1D-02
DO 5 J=1,200
DO 4 K=1,100
4 CALL KEPLR1(M, E, IRR, EA)
5 M=M+6.28D-02
6 E=E+.49D-01
CALL INTIMO(ITIME)
ITIME=ITIME/400000

507 FORMAT('//30X,'AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95',1X,I3,
11X,'MICROSECONDS'//)
508 WRITE(6,507) ITIME
E=1D-03
ITIME=2
CALL INTIMO(1)
DO 10 I=1,20
M=1D-02
DO 11 J=1,200
DO 12 K=1,100
10 CALL KEPLR1(M, E, IRR, EA)
12 EA(2)=DSIN(E2)
11 M=M+6.28D-02
10 E=E+.49D-01
CALL INTIMO(ITIME)
ITIME=ITIME/400000

607 FORMAT('//30X,'AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95',1X,I3,
11X,'MICROSECONDS(WITH SINCOS)')
WRITE(6,607) ITIME
RETURN
END

EXECUTION TIMES FOR KEPLER AND KEPLR1

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

EXECUTION TIMES FOR KEPLER AND KEPLR1

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS

AVERAGE EXECUTION TIME FOR KEPLER ON 360/95 74 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 314 MICROSECONDS
AVERAGE EXECUTION TIME FOR KEPLR1 ON 360/95 266 MICROSECONDS
Appendix C

Test 2 and Results

TEST 2

TEST PROGRAM TO TEST VALIDITY OF TWO KEPLER PROGRAMS

PROGRAMMER ANNE BOMFORD

REAL*8 M, E, EA(3), DE
REAL*8 A, B, ERR, MAXERR
REAL*8 E2

800 FORMAT('!*','/49X,'VALIDITY TEST FOR KEPLER AND KEPLR1','//')
WRITE(6,800)
DO 501 K=1,2
MAXERR=0.0000
E=1D-03
DO 802 I=1,100
M=1D-02
DO 803 J=1,1000
IF(K.EQ.2) GO TO 505
CALL KEPLER(M, E, IRR, EA)
ERR=ABS(EA(1)-M-E*EA(2))
IF(ERR.LE.MAXERR) GO TO 803
MAXERR=ERR
A=M
B=E
GO TO 803
505 CALL KEPLR1(M, E, IRR, E2)
EA(1)=E2
EA(2)=DSIN(E2)
ERR=ABS(EA(1)-M-E#EA(2))
IF(ERR.LE.MAXERR) GO TO 803
MAXERR=ERR
A=M
B=E
803 M=M+.628D-02
802 E=E+.980D-02
IF(K.EQ.2) GO TO 808
806 FORMAT('!*','/20X,'MAXIMUM ERROR FOR KEPLER='*,D10.3,' M='*,E10.3,' E='*,E10.3)
WRITE(6,806) MAXERR,A,B
GO TO 501
807 FORMAT('!*','/20X,'MAXIMUM ERROR FOR KEPLR1='*,D10.3,' M='*,E10.3,' E='*,E10.3)
808 WRITE(6,807) MAXERR,A,B
501 CONTINUE
RETURN
END

******************************************************************************
VALIDITY TEST FOR KEPLER AND KEPLR1
MAXIMUM ERROR FOR KEPLER=0.444D-15 M=0.520D 01 E=0.951D 00
MAXIMUM ERROR FOR KEPLR1=0.500D-15 M=0.264D 01 E=0.941D 00
******************************************************************************

******************************************************************************
SUBROUTINE KEPLR1(MA,ECC,ERRC,E2)

C THIS SUBROUTINE IS USED TO SOLVE KEPLER'S EQUATION
C FOR ECCENTRIC ANOMALY GIVEN MEAN ANOMALY AND ECCENTRICITY
C BY THE MYLES STANDISH ALGORITHM.
C IN THE CALLING SEQUENCE MA IS THE MEAN ANOMALY, ECC IS
C THE ECCENTRICITY, ERRC IS THE ERROR CODE FOR NUMBER OF
C ITERATIONS AND ER IS THE ECCENTRIC ANOMALY.
INTEGER ERRC,MAX,ITER,EC
DOUBLE PRECISION MA,ECC,E1,E2,F,D,ABS
REAL*8 TOL/.05D-10/,PI2/6.283185307179586/
MAX=10
ERRC=0
E2=0.0D0
IF(MA) 3,13,3
E1=MA+ECC*DSIN(MA)
ITER=1
F=E1-(ECC*DSIN(E1))-MA
D=1.0DO-(ECC*DCOS(E1-0.5DO*F))
E2=E1-F/D
C TEST FOR CONVERGENCE
IF(DABS(E1-E2)-TOL) 13,13,8
ITER=ITER + 1
C TEST FOR NUMBER OF ITERATIONS
IF(ITER-MAX) 10,10,12
E1=E2
GO TO 4
13 E2=DMOD(E2,PI2)
IF(E2.LT.0) E2=PI2+E2
RETURN
END

KEPLER CSECT
using *,15
STM 14,4,12(13) save registers
LM 1,4,0(1) argument addresses M,E,IERR,EA
* step one
LD 2,0(2) E
LTDR 2,+2 test e for minus
BM ERROR
CE 2,+E'*.99' test e for size
BL 2,+12 ok if less than *.99
ERROR MV1 3(3),1 * set error code =1
B exit * and exit
MV1 3(3),0 zero error code (no error)
* STEP TWO
  LD 4,01(1) M HOLD FOR SIGN
  LPDR 4,4 M ABSOLUTE
  CD 0,TWOPI OK IF LESS THAN 2PI
  BL 2*20 *
  LDR 6,0 * OTHERWISE
  DD 6,TWOPI * REDUCE M
  AW 6,ZERO14 * TO MODULO 2PI
  MD 6,TWOPI *
  SDR 0,6 *
  CD 0,PI *
  BL 2+12 * IF X GREATER THAN PI
  LCDR 4,4 * AND REVERSE SIGN
  LCDR 0,0 * AND SET X = 2PI-X
  AD 0,TWOPI *
  STE 4,52(13) SIGN OF M SAVED
  STD 0,6(13) M ABSOLUTE BETWEEN 0 AND PI

* STEP THREE-A
  CE 4,E1.570963' *
  BL 2+10 *
  LCDR 0,0 *(3/4)*X X = M M BETWEEN 0-PI/2
  HDR 0,0 *
  HDR 4,0 *
  ADR 0,4 *
  MER 0,2 E*SIN(E)
  AD 0,56(13) EA = M + E*SIN(M), 1ST ESTIMATE
  STD 0,014 HOLD AS EA (POSITIVE)

* STEP THREE-B
  QA 4,E1.570963' *
  LA 1 S=1 (ASSUMES EA IN QUAD1)
  CE 0,E1.570963' *
  BL QUAD1 OK 1ST QUADRANT
  LA 0,2 OTHERWISE SET S=2 *
  LCDR 0,0 *
  AD 0,PI EA= PI-EA * EA LESS THAN PI/2

QUAD1
  CE 0,E1.7853981 *
  BL OCT1 OK 1ST OCTANT
  LCR 0,0 *
  LCR 0,0 *
  ME 4,E6.666667' -X3/6
  AD 2,ONEX 1-X2/2 = COS(EA)
  ADR 0,4 X-X3/6 = SIN(EA)
  LTR 0,0 TEST S FOR + OR -
  BP 2+12 OK 1ST OCTANT
  LCR 0,0 *
  LDR 4,2 *
  LDR 2,0 * EXCHANGE
  LDR 0,4 *
  BCT 0,**+6 -COS(EA) - IN QUAD 1
  LCDR 2,2 -COS(EA) + IN QUAD 2
ME 0,0(2)
ME 2,0(2)
AD 2,ONEX 1-E*COS(EA) =D 1ST ORDER
AD 0,56(13) M+E*SIN(EA)
SD 0,0(4) M+E*SIN(EA) -EA =F
DER 0,2 F/D =C 1ST ORDER
AD 0,0(4) EA+C
STD 0,0(4) STORE AS EA ESTIMATE (POSITIVE)

* STEP FOUR
L 15,=V(SINCOS)
BALR 14,15
USING *,14

* STEP FIVE
ITERATE STD 0,8(4) SIN(EA)
MD 0,0(2) E*SIN(EA)
HDR 4,0 .5*E*SIN(EA) HOLD
AD 0,56(13) M+E*SIN(EA)
SD 0,0(4) M+E*SIN(EA) -EA =F

* STEP SIX
LDR 6,2 -COS(EA)
MD 6,0(2) -E*COS(EA)
AD 6,ONEX 1-E*COS(EA) =D 1ST ORDER

* STEP SEVEN
MER 4,0 F*.5*E*SIN(EA)
DER 4,6 F*.5*E*SIN(EA)/D
ADR 6,4 D +F*.5*E*SIN(EA)/D = D 2ND ORDER

* STEP EIGHT
DDR 0,6 F/D = C 2ND ORDER
LDR 4,0 SAVE C

* STEP NINE
AD 0,0(4) EA+C
STD 0,0(4) SAVE AS ENHANCED EA

* STEP TEN
LPER 6,4 C ABSOLUTE
CE 6,=E'1.0E-5' * RETURN FOR FULL SINCOS AND ITERATE
BCR 2,15 * IF C GREATER THAN .00001

* STEP ELEVEN
CE 6,=E'1.0E-8' * CONVERGENCE
BL OUT * WHERE C LESS THAN .0000001

* STEP TWELVE
LDR 0,4 -C 2ND ORDER SUMS FORMULAE *
HDR 6,4 C/2 C BETWEEN 10**-5,10**-8 *
MD 0,6 -C*C/2 *
AD 0,ONEX 1-C*C/2 = COS(C) *
LDR 6,4 C = SIN(C) *
MDR 6,2 SIN(C)*COS(EA) *
MDR 2,0 COS(C)*COS(EA) *
MD 4,8(4) SIN(C)*SIN(EA) *
MD 0,8(4) COS(C)*SIN(EA) *
ADR 0,6 SIN(EA+C)=SIN(C)*COS(EA)+COS(C)*SIN(EA) *
SDR 2,4 COS(EA+C)=COS(C)*COS(EA)-SIN(C)*SIN(EA) *
BR 14 ITERATE *

* STEP THIRTEEN
OUT LDR 0,4 C 1ST ORDER SUMS FORMULAE *
MDR 0,2 C*COS(EA) C LESS THAN 10**-8 *

31
\[
\sin(EA + C) = \sin(EA) + C \cdot \cos(EA) \\
\cos(EA + C) = \cos(EA) - C \cdot \sin(EA)
\]

* STEP FOURTEEN

\[
\text{TM} 52(13),128 \\
\text{BZ PLUS} \\
\text{CHECK SIGN OF M PLUS OK IF M POSITIVE}
\]

* STEP FIFTEEN

\[
\text{LCDR 0,0} \\
\text{OTHERWISE COMPLIMENT SIN(EA)}
\]

* STEP SIXTEEN

\[
\text{PLUS STD 0,8(4) OUTPUT SIN(EA)} \\
\text{STD 2,16(4) OUTPUT COS(EA)}
\]

* STEP SEVENTEEN

\[
\text{EXIT LM 14,12(13) RESTORE REGISTERS} \\
\text{BR 14 AND RETURN}
\]

**BEGINNING**

\[
\text{TWOPI DC D'6.2831853071795864' 2PI} \\
\text{PI DC X'413243F6A8885A31' PI} \\
\text{ZERO14 DC X'E000000000000000' } \\
\text{ONEX DC X'40FFFFFFFFFFFFF'}
\]

\[
\text{END } =E'9.99' \\
\text{=E'1.57' } \\
\text{=E'1.5707963' } \\
\text{=E'.7853981' } \\
\text{=E'.6666667' } \\
\text{=V(SINCOS) } \\
\text{=E'1.E-5' } \\
\text{=E'1.E-8'}
\]

**INTIMO**

\[
\text{START 0} \\
\text{BC 15,12(15) BRANCH AROUND CONSTANTS} \\
\text{DC X'7' ESTABLISH A HALF-WORD BOUNDARY} \\
\text{DC CL7' INTIMO' NAME} \\
\text{STM 14,12,12(13) SAVE THE REGISTERS} \\
\text{BALR 12,0 BASE REGISTER} \\
\text{USING *,12} \\
\text{LR 3,1} \\
\text{L 7,0(3) CHECK THE INDICATOR} \\
\text{CLI 3(7),X'2' BRANCH IF SECOND ENTRY} \\
\text{BC 8,INTER} \\
\text{STIMER TASK,UINTVL=INTER SET THE TIMER} \\
\text{LM 14,12,12(13) RESTORE THE REGISTERS} \\
\text{MVI 12(13),X'FF' INDICATE CONTROL RETURNED} \\
\text{BCR 15,14 RETURN TO CALLING PROGRAM} \\
\text{INTER} \\
\text{L 5,INTER LOAD MAXIMUM TIME} \\
\text{TTIMER} \\
\text{SR 5,0 DETERMINE ELAPSED TIME} \\
\text{M 4,TWSIX CONVERT UNITS TO MICRO SECONDS} \\
\text{ST 5,0(7) STORE TIME IN RETURN LOCATION} \\
\text{LM 14,12,12(13) RESTORE REGISTERS} \\
\text{MVI 12(13),X'FF' INDICATE CONTROL RETURNED} \\
\text{BCR 15,14 RETURN TO CALLING PROGRAM} \\
\text{DS 0F} \\
\text{INTER DC F'2147483647'} \\
\text{TWSIX DC F'26'} \\
\text{END}
\]
DPSC
CSECT
USING *,15
STM 14,2,12(13)  SAVE REGS
LM 1,2,0(1) X,SC ADDRESSES
LD 0,0(1) X TO FRO
LA 15,SINCOS SINCOS ADD
BALR 14,15 SIN TO FRO, COS TO FRO
STD 0,0(2) OUTPUT SIN(X)
STD 2,8(2) OUTPUT COS(X)
LM 14,2,12(13) RESTORE REGS
BR 14 AND RETURN
ENTRY SINCOS
USING *,15
SINCOS
LDR 6,0 SIGN OF SIN(A)
LPDR 0,0 X ABSOLUTE
CD 0,TW0PI *
BL **20 * IF X GREATER THAN 2PI
LDR 2,0 * REDUCE
DO 2,TW0PI * TO
AW 2,ZERO14 * MODULO
MD 2,TW0PI * TW0PI
SDR 0,2 *
CD 0,PI *
BL **12 * IF X GREATER THAN PI
L0DR 6,6 * REVERSE SIGN OF SINE
L0DR 0,0 * AND SET X = 2PI-X
AD 0,TW0PI *
LA 0,2 SET SWITCH = 2 (COS+, NO SIN/COS EXCHANGE)
CD 0,P10VER2 *
BL **14 * IF X GREATER THAN PI/2
LA 0,1 * SET X = PI-X
L0DR 0,0 * AND SET SWITCH =1 (COS-)
AD 0,PI *
CD 0,P10VER4 * IF X GREATER THAN PI/4
BL **16 * SET X = PI/4 + PI/4 -X
L0CR 0,0 * AND SET SWITCH = (SIN/COS EXCHANGE)
L0DR 0,0 *
AD 0,P10VER4 *
AD 0,P10VER4 *
LER 2,0
AU 2,*X'45000000' 4500000X 1ST HEX DIGIT OF X = INDEX
STE 2,68(13)
L 1,68(13) 45000001
SLL 1,3 81 =CONSTANT INDEX I=0,1,2,...,12
LDR 2,0 X TO FRO AND FRO
MDR 2,*2 X*X =X2
LDR 4,*2 X2
MD 4,S5(1)
AD 4,S4(1)
MDR 4,*2
AD 4,S3(1)
MDR 4,*2
AD 4,S2(1)
MDR 4,*2
AD 4,S1(1)
MDR 0,*4 SIN(X) VIA 9TH DEGREE ODD POLYNOMIAL
LDR 4,2
CE 4,=E*25
BL ++18
MD 2,C6-64(1)
AD 2,C5(1)
MDR 2,4
B ++8
MD 2,C5(1)
AD 2,C4(1)
MDR 2,4
AD 2,C3(1)
MDR 2,4
AD 2,C2(1)
MDR 2,4
AD 2,C1(1)
LTR 0,0
BP ++12
LCR 0,0
LDR 4,2
LDR 2,0
LDR 0,4
BCT 0,**6
LDR 2,2
LTD 6,6
BCT 10,14
BCT 0,0
BR 14
USING *,15
ENTRY DCOS,DCOSX
DCOS L 1,0(1)
LD 0,0(1)
LDR 0,0
AD 0,P10VER4
AD 0,P10VER4
LA 15,DSINX
BR 15
USING *,15
DCOSX LDR 0,0
AD 0,P10VER4
AD 0,P10VER4
LA 15,DSINX
BR 15
ENTRY DSIN,DSINX
DSIN L 1,0(1)
LD 0,0(1)
LA 15,12(15)
USING *,15
DSINX LDR 6,0
LPDR 0,0
CD 0,TWOPI
BL ++20
LDR 2,0
DD 2,TWOPI
AW 2,ZERO14
MD 2,TWOPI

X2
X2 VS 1/4
8TH DEGREE COS IF X LESS THAN 1/2
10TH DEGREE COS IF X GREATER THAN 1/2

COS(X) VIA 8 OR 10 DEGREE POLYNOMIAL
TEST SWITCH FOR SIGN
NO EXCHANGE IF +
OTHERWISE SET SWITCH +1 OR +2
AND
EXCHANGE
SIN AND COS

* IF SWITCH = 1
SET COSINE -
* IF SIGN OF SIN +
* EXIT
OTHERWISE SET SIGN -
AND EXIT

USING *,15
ENTRY DCOS,DCOSX
DCOS L 1,0(1)
LD 0,0(1)
LDR 0,0
AD 0,P10VER4
AD 0,P10VER4
LA 15,DSINX
BR 15
USING *,15
DCOSX LDR 0,0
AD 0,P10VER4
AD 0,P10VER4
LA 15,DSINX
BR 15
ENTRY DSIN,DSINX
DSIN L 1,0(1)
LD 0,0(1)
LA 15,12(15)
USING *,15
DSINX LDR 6,0
LPDR 0,0
CD 0,TWOPI
BL ++20
LDR 2,0
DD 2,TWOPI
AW 2,ZERO14
MD 2,TWOPI

34
SDR 0,2
CD 0,PI
BL **+10
LCDR 6,6
SD 0,PI
CD 0,PIOVER2
BL **+10
LCDR 0,0
AD 0,PI
CD 0,PIOVER4
BNL COSS
LER 2,0
AU 2,=X'45000000'
STE 2,68(13)
L 1,68(13)
SLL 1,3
LDR 2,0
MDR 2,2
LDR 4,2
MD 4,S5(1)
AD 4,S4(1)
MDR 4,2
AD 4,S3(1)
MDR 4,2
AD 4,S2(1)
MDR 4,2
AD 4,S1(1)
MDR 0,4
LTDR 6,6
BCR 10,14
LCDR 0,0
BR 14

COSS
LCDR 0,0
AD 0,PIOVER4
AD 0,PIOVER4
LER 2,0
AU 2,=X'45000000'
STE 2,68(13)
L 1,68(13)
SLL 1,3
MDR 0,0
LDR 2,0
CE 2,=E'25'
BL **+18
MD 0,C6-64(1)
AD 0,C5(1)
MDR 0,2
B **+8
MD 0,C5(1)
AD 0,C4(1)
MDR 0,2
AD 0,C3(1)
MDR 0,2
AD 0,C2(1)
MDR 0,2
<table>
<thead>
<tr>
<th>DC</th>
<th>\textbf{SINE COEFFICIENTS}</th>
</tr>
</thead>
</table>

**S1**
- \texttt{DC X'40FFFFFFFFFFFFFA' 0.0}
- \texttt{DC X'40FFFFFFFFFFFFF37' 0.6250000000D-01}
- \texttt{DC X'40FFFFFFFFFFFFFD7C' 0.1250000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFE8D8' 0.1875000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFC96A' 0.2500000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFF49BA' 0.3125000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFCFB23' 0.3750000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFO157' 0.4375000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFC5C90' 0.5000000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFF68C98' 0.5625000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFFFE63DA15' 0.6250000000D 00}
- \texttt{DC X'40FFFFFFFFFFFFCLE8C81' 0.6875000000D 00}
- \texttt{DC X'40FFFFFFFFFFFF85A960' 0.7500000000D 00}

**S2**
- \texttt{DC X'C02AAAAAAA713F' 0.0}
- \texttt{DC X'C02AAAAAAA931F5' 0.6250000000D-01}
- \texttt{DC X'C02AAAAAAA8FD27' 0.1250000000D 00}
- \texttt{DC X'C02AAAAAAA2D215' 0.1875000000D 00}
- \texttt{DC X'C02AAAAAAA9E6858' 0.2500000000D 00}
- \texttt{DC X'C02AAAAAA8CDDDA' 0.3125000000D 00}
- \texttt{DC X'C02AAAAAA4CBBA7' 0.3750000000D 00}
- \texttt{DC X'C02AAAAAA947FA52' 0.4375000000D 00}
- \texttt{DC X'C02AAAAA71E4DEB' 0.5000000000D 00}
- \texttt{DC X'C02AAAAA826B13' 0.5625000000D 00}
- \texttt{DC X'C02AAAA98025048' 0.6250000000D 00}
- \texttt{DC X'C02AAAAA84FD178E' 0.6875000000D 00}
- \texttt{DC X'C02AAAAA69F049B' 0.7500000000D 00}

**S3**
- \texttt{DC X'3F222221FB15B5CF' 0.0}
- \texttt{DC X'3F2222212DC75F04' 0.6250000000D-01}
- \texttt{DC X'3F22222122B8A09A' 0.1250000000D 00}
- \texttt{DC X'3F22222222D5BEFl' 0.1875000000D 00}
- \texttt{DC X'3F2222221106E925' 0.2500000000D 00}
- \texttt{DC X'3F22222203229637' 0.3125000000D 00}
- \texttt{DC X'3F222221DF3FF490' 0.3750000000D 00}
- \texttt{DC X'3F2222215C81CD5' 0.4375000000D 00}
- \texttt{DC X'3F222220934E1941' 0.5000000000D 00}
- \texttt{DC X'3F222221F3A705DF' 0.5625000000D 00}
- \texttt{DC X'3F22221CBB9904EF' 0.6250000000D 00}
- \texttt{DC X'3F222218FE37316B' 0.6875000000D 00}
- \texttt{DC X'3F2222146F31054' 0.7500000000D 00}

**S4**
- \texttt{DC X'BDCFBE6DA833F92D' 0.0}
- \texttt{DC X'BDD00B956DAEA4B' 0.6250000000D-01}
- \texttt{DC X'BDD00C4AC2G8225A' 0.1250000000D 00}
- \texttt{DC X'BDD00C1538ED130' 0.1875000000D 00}
| DC | \( \times' \text{BDD00C4960797744}' \) | 0.25000000000D 00 |
| DC | \( \times' \text{BDD00C081B0853F7}' \) | 0.31250000000D 00 |
| DC | \( \times' \text{BDD00B3087649F44}' \) | 0.37500000000D 00 |
| DC | \( \times' \text{BDD0099428EEBB20}' \) | 0.43750000000D 00 |
| DC | \( \times' \text{BDD00789E59C8C04}' \) | 0.50000000000D 00 |
| DC | \( \times' \text{BDD004B8BBF3436D}' \) | 0.56250000000D 00 |
| DC | \( \times' \text{BDD000816BF3313}' \) | 0.62500000000D 00 |
| DC | \( \times' \text{BDCFFFFB46C616A714}' \) | 0.68750000000D 00 |
| DC | \( \times' \text{BDF55C0D99F68D}' \) | 0.75000000000D 00 |
| \text{S5} | \( \times' \text{BDD0019B0975}' \) | 0.0 |
| DC | \( \times' \text{3C274D02016B0880}' \) | 0.62500000000D-01 |
| DC | \( \times' \text{3C2DC3E3A8A969C91}' \) | 0.12500000000D 00 |
| DC | \( \times' \text{3C2DE2C5F202B50E}' \) | 0.18750000000D 00 |
| DC | \( \times' \text{3C2E01B43E4997D5}' \) | 0.25000000000D 00 |
| DC | \( \times' \text{3C2DF9F7766C5869}' \) | 0.31250000000D 00 |
| DC | \( \times' \text{3C2E450BC081B71}' \) | 0.37500000000D 00 |
| DC | \( \times' \text{3C2DC2608E20BA18}' \) | 0.43750000000D 00 |
| DC | \( \times' \text{3C2DA291A6506887}' \) | 0.50000000000D 00 |
| DC | \( \times' \text{3C2D76F6C3453E649}' \) | 0.56250000000D 00 |
| DC | \( \times' \text{3C2D545C68C1566F}' \) | 0.62500000000D 00 |
| DC | \( \times' \text{3C2D284AE4A50DCB}' \) | 0.68750000000D 00 |
| DC | \( \times' \text{3C2D0066FD7D092D}' \) | 0.75000000000D 00 |

**COSINE COEFFICIENTS**

| DC | \( \times' \text{411000000000000D}' \) | 0.0 |
| DC | \( \times' \text{411000000000000D}' \) | 0.62500000000D-01 |
| DC | \( \times' \text{411000000000000D}' \) | 0.12500000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFFF6B5}' \) | 0.18750000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFFO2CC}' \) | 0.25000000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFF9D8B0}' \) | 0.31250000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFFB4AF3}' \) | 0.37500000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFF66CA3C}' \) | 0.43750000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFFBF8694}' \) | 0.50000000000D 00 |
| DC | \( \times' \text{411000000002779F}' \) | 0.56250000000D 00 |
| DC | \( \times' \text{41100000000484D}' \) | 0.62500000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFF0ACD4}' \) | 0.68750000000D 00 |
| DC | \( \times' \text{40FFFFFFFFFFF328278}' \) | 0.75000000000D 00 |

| DC | \( \times' \text{C08000000000008FC}' \) | 0.0 |
| DC | \( \times' \text{C080000000001F56}' \) | 0.62500000000D-01 |
| DC | \( \times' \text{C08000000000A56E}' \) | 0.12500000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFFF2422C}' \) | 0.18750000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFFC1841D}' \) | 0.25000000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFFE2E9A8}' \) | 0.31250000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFFBA034A}' \) | 0.37500000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFF24D4282}' \) | 0.43750000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFFC7615}' \) | 0.50000000000D 00 |
| DC | \( \times' \text{C080000002240B38}' \) | 0.56250000000D 00 |
| DC | \( \times' \text{C0800000006D49C3}' \) | 0.62500000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFF1A5797}' \) | 0.68750000000D 00 |
| DC | \( \times' \text{C07FFFFFFFFFF8B5A64}' \) | 0.75000000000D 00 |

| DC | \( \times' \text{3FAAAAAAACFAD91B}' \) | 0.0 |
| DC | \( \times' \text{3FAAAAAAAB4F60A}' \) | 0.62500000000D-01 |
| DC | \( \times' \text{3FAAAAAACBF2FE5}' \) | 0.12500000000D 00 |
| DC | \( \times' \text{3FAAAAAA9BD824DF}' \) | 0.18750000000D 00 |
| DC | \( \times' \text{3FAAAAAA487F9F90}' \) | 0.25000000000D 00 |
| DC | \( \times' \text{3FAAAAAA985B06FCA}' \) | 0.31250000000D 00 |
| C4   | 0.3750000000D 00 | DC   | 3FAAAAAA756C2D236' |
|      | 0.4375000000D 00 |      | 3FAAAAAA2085E052B' |
|      | 0.5000000000D 00 |      | 3FAAAAAA8A34E183E' |
|      | 0.5625000000D 00 |      | 3FAAAAAAB665E57D' |
|      | 0.6250000000D 00 |      | 3FAAAAAAABBFB8614' |
|      | 0.6875000000D 00 |      | 3FAAAAAA5544824F' |
|      | 0.7500000000D 00 |      | 3FAAAAAA8E16081B' |
|      | 0.5250000000D-01 |      | B508A3D232D3' |
|      | 0.1250000000D 00 |      | BE5B05B13C670B8D' |
|      | 0.1875000000D 00 |      | BE5B059A0ECCD046' |
|      | 0.2500000000D 00 |      | BE5B0563A45C8E91' |
|      | 0.3125000000D 00 |      | BE5B05118160D0F42' |
|      | 0.3750000000D 00 |      | BE5B046D420B0F0B' |
|      | 0.4375000000D 00 |      | BE5B0375D64C08AC' |
|      | 0.5000000000D 00 |      | BE5B053B60D50E39' |
|      | 0.5625000000D 00 |      | BE5B05C9F9CE1EFA5' |
|      | 0.6250000000D 00 |      | BE5B05AF4705E015' |
|      | 0.6875000000D 00 |      | BE5B059FF606D358' |
|      | 0.7500000000D 00 |      | BE5B0576C21575A' |

| C5   | 0.6250000000D-01 | DC   | 3D1A0381E558ED90' |
|      | 0.1250000000D 00 |      | 3D19FD902FF6A01A' |
|      | 0.1875000000D 00 |      | 3D19F150857FB484' |
|      | 0.2500000000D 00 |      | 3D19E3CB68016FA1' |
|      | 0.3125000000D 00 |      | 3D19D6896D105CAD' |
|      | 0.3750000000D 00 |      | 3D19C48042D34407' |
|      | 0.4375000000D 00 |      | 3D19B076C277FE83' |
|      | 0.5000000000D 00 |      | 3D19F45FAA3F5386' |
|      | 0.5625000000D 00 |      | 3D19A040A69823422' |
|      | 0.6250000000D 00 |      | 3D19A0107B2439E18' |
|      | 0.6875000000D 00 |      | 3D19F654702218' |
|      | 0.7500000000D 00 |      | 3D19FD9A88121E8' |

| C6   | 0.5000000000D 00 | DC   | BB4010A69645256D' |
|      | 0.5625000000D 00 |      | BB4ABF0291131076' |
|      | 0.6250000000D 00 |      | BB4AF9D94FC6AC452' |
|      | 0.6875000000D 00 |      | BB487121A5139E61' |
|      | 0.7500000000D 00 |      | BB479FD89AA019C2' |

| END  | =X'45000000' =E'1.25' |