ON OPTIMIZING THE TREATMENT OF EXCHANGE PERTURBATIONS

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ABSTRACT

In a practical treatment of exchange perturbation problems one seeks
the best wavefunction of the form \( \psi(1) = A(\phi_0 + \phi_1) \) where \( A \) is a symmetry
projection operator, \( \phi_0 \) is the eigenfunction of the unperturbed Hamiltonian, \( H_0 \), and \( \phi_1 \) is the solution to a first order perturbation
equation of the form \( (H_0 - E_0)\phi_1 + (V - E_1)\phi_0 = (1 - A)F_1 \). Most previous
treatments correspond to assuming that \( F_1 = \sum_{\nu,j} \nu \sum_{\nu} \alpha_j + \nu \beta_j V \phi_0 \).
Here \( \nu \) labels the irreducible representation, \( j \) labels the row of
the representation. The choice of the constants \( \nu \alpha_j \) and \( \nu \beta_j \) depends

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upon the particular method. In this paper the function $F_1$ is determined so that the resulting $\Psi(1)$ is equal to the exact wavefunction or any given function having the symmetry of $A$. It is hoped that this analysis will suggest desirable functional forms for $F_1$ to use in practical problems. Attempts to determine $\phi_1$ by optimizing the sum of the first and second order energies were not successful.
There are a great many different theories of exchange perturbations (for reviews, see Refs. (1)-(6)) and most of them would give the exact energy and wavefunction if the treatment were carried to an infinite order. However, in practice, the calculations are sufficiently difficult that the wavefunction is usually truncated after the first order (for examples, see Refs, (7) and (8)). Unfortunately the different methods give different values for the second order energy \( E_2 \) and different values for \( E(1) \), the expectation value of the Hamiltonian corresponding to the zeroth plus first order wavefunction. In the present paper it is shown that the zeroth plus first order wavefunction obtained by optimizing the basic equation which is used in most exchange perturbation treatments is the exact wavefunction for the perturbed system and \( E(1) \) is the exact energy! On the other hand, there is no unique value of \( E_2 \) since it can be made arbitrarily large or small. We hope that the equation which we use for determining the optimum first order function will serve as the basis for improving the techniques which can be used to solve practical exchange perturbation problems.

The basic difficulty inherent in exchange perturbation problems is that the symmetry group * corresponding to the perturbed Hamiltonian

*If the Hamiltonian is spin-free, then the energy eigenfunctions (which we call wavefunctions in the present paper) are really the spin-free components of the complete wavefunction. The construction of projection operators to be used in the calculation of the spin-free components of a complete wavefunction is discussed in Refs. (9)-(14).
\[ H = H_0 + V \] is different from the symmetry group corresponding to the unperturbed Hamiltonian \( H_0 \). Thus, if \( A \) is the idempotent projection operator corresponding to a particular irreducible representation (or to a particular row of this irreducible representation) of the symmetry group of the perturbed system, then \( A \) commutes with \( H \) but not with \( H_0 \). This leads to the relation

\[ A H_0 - H_0 A = V A - A V \]  \( \text{(1)} \)

The right-hand-side of Eq. (1) appears to be first order whereas the left-hand-side appears to be zeroth order. Thus, the concept of order is not clearly defined by the "power of \( V \" as in the ordinary Rayleigh-Schrödinger perturbation theory.

Exchange perturbation treatments start with a zeroth order Schrödinger equation

\[ (H_0 - E_0)\phi_0 = 0 \]  \( \text{(2)} \)

Here \( E_0 \) is taken to be the zeroth order energy, but \( \psi_0 = A \phi_0 \), rather than \( \phi_0 \) itself, is generally taken to be the zeroth order wavefunction. We seek approximations to the exact energy \( E \) and the exact wavefunction \( \psi = A \psi \) of the perturbed system. In most exchange perturbation treatments, each order of the perturbed wavefunction has the symmetry of \( \psi \).

Thus, we seek a first order function \( \phi_1 \) such that the first order wavefunction is \( \psi_1 = A \phi_1 \). The wavefunction correct through the first order is then

\[ \psi(1) = A(\phi_0 + \phi_1) \]  \( \text{(3)} \)

and

\[ E(1) = \frac{\langle \phi_0 + \phi_1 | A(H_0 + V)A | \phi_0 + \phi_1 \rangle}{\langle \phi_0 + \phi_1 | A | \phi_0 + \phi_1 \rangle} \]  \( \text{(4)} \)
Three observations:

(a) If \( \phi_1 \) is replaced by \( \phi_1' = \phi_1 + (1-A)S \), where \( S \) is an arbitrary function, then \( E(1) \) remains unchanged. This is because \( \phi_1 \) occurs in Eq. (3) only in the form \( A\phi_1 \).

(b) If \( V \) is replaced by either \( V' = V + (1-A)T \) or \( V'' = V + T(1-A) \), where \( T \) is an arbitrary function or operator, then \( E(1) \) remains unchanged. Similarly, \( H_0 \) can be replaced by either \( H_0 - (1-A)T \) or \( H_0 = T(1-A) \).

(c) If \( \phi_1 \) is varied without constraints, the optimum (or stationary) value of \( E(1) \) is obtained when \( \phi_1 \) satisfies the equation \( (H - E(1))A(\phi_0 + \phi_1) = 0 \). In this case it is clear that \( \psi(1) = \psi \) and \( E(1) = E \). Such a procedure corresponds to the usual variational method at fixed charge and spin quantum number in \( 1 \).

In the Appendix, Eq. (3) is rewritten in an equivalent form and expanded into terms which correspond to the apparent first, second, ... orders. It is then shown that the optimum expression for \( \phi_1 \) cannot be obtained by varying \( \phi_1 \) so as to make the sum of the first and second order energies stationary. **This does not necessarily mean that only the component of \( V \) which has the symmetry of \( A \) determines the value of \( E(1) \). For example, if \( A = (1/2)(1-P) \), then it is the \((1-A)\) component of \( V \) which enters into Eq. (3) since \( AVA\phi \) is equal to the product of \((1-A)V \) times \( A\phi \).**

Except that it must satisfy the same boundary and continuity conditions as are imposed upon acceptable wavefunctions.
Most Rayleigh-Schrodinger type treatments are based upon rewriting the perturbed Schrodinger equation in the form

$$(\mathbf{H} - E)\mathbf{A}\psi = A[(\mathbf{H}_0 - E_0) + (V - E_1) - E_2 - \ldots](\phi_0 + \phi_1 + \ldots) = 0$$

This leads to a first order equation

$$(\mathbf{H}_0 - E_0)\phi_1 + (V - E_1)\phi_0 = (1-A)F_1$$

where $F_1$ is an arbitrary function and $E_1$ is the constant which makes Eq. (5) mathematically consistent,

$$E_1 = \frac{\langle \phi_0 | V\phi_0 - (1-A)F_1 \rangle}{\langle \phi_0 | \phi_0 \rangle}$$

The various perturbation schemes differ in their choice of $F_1$. In most of the schemes that have been proposed, $F_1$ is taken to have the form

$$F_1 = \sum_{\nu, \alpha, j} \nu_{\alpha, j}(\nu_{\alpha, j} + \nu_{\beta, j})\phi_0$$

Here the $\nu$ labels the irreducible representation and $j$ labels the row within that representation. The particular choice of the constants...-

†The various choices of $F_1$ and the corresponding higher order functions $F_2, F_3, \ldots$ which lead to the various perturbation schemes proposed in the literature are discussed in detail in Ref. (15). This paper also presents detailed calculations for $H_2$ comparing the results for the different theories.

††Ref. (16) gives the values of $\nu_{\alpha, j}$ and $\nu_{\beta, j}$ corresponding to various proposed treatments. It gives a thorough discussion of the symmetry problems involved in exchange perturbations. Also, it presents accurate calculations for $H_2$ comparing the results for the different theories.

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\( \alpha_j \) and \( \beta_j \) varies from method to method. Expediency is the only justification for limiting the choice of \( F_1 \) in this manner. In the present paper, we seek to optimize \( F_1 \).

Actually, we can show that if \( \Psi(1) \) is an arbitrary given function having the symmetry \( \Psi(1) = A \Psi(1) \) and \( E_1 \) is an arbitrary given value, then it is always possible to find a function \( F_1 \) such that Eq. (5) is satisfied by the first order function

\[
\Psi_1 = \Phi_0 + \Psi(1) + (1-A)G
\]  

(8)

Note that Eq. (8) is consistent with the requirement that \( \Psi(1) = A(\Phi_0 + \Phi_1) \) so that \( \Psi(1) \) can be considered to be the perturbed wavefunction truncated after the first order. In Eq. (8), \( G \) is any function such that

\[
AF_1 = 0
\]  

(9)

If \( AF_1 = 0 \), then Eq. (5) (making use of Eq. (8)) gives the identity

\[
F_1 = (V-E_1)\Phi_0 + (H_0-E_0)(\Psi(1) + (1-A)G)
\]  

(10)

Making use of Eq. (1), the statement that \( AF_1 = 0 \) can be written in the form

\[
A \Phi_0 (1-A)G = A X
\]  

(11)

where

\[
X = (V-E_1)\Phi_0 + (H_0-E_0)\Psi(1)
\]  

(12)

As will be shown below, for a two particle system (or one containing only gerade and ungerade symmetries), Eq. (11) completely determines the solution of \( F_1 \) and \( \Phi_0 \), and the physical wavefunction is
(1-A)G. For a perturbed system with three or more particles (or having more than two symmetry elements), it is always possible to find a function (1-A)G such that Eq. (11) is satisfied, but the solution often is not unique. For such cases, \( A_F \) and \( A_{\phi} \) are still uniquely determined by the specification of \( \psi(1) \) and \( E_1 \), but the functions \( F_1 \) and \( \phi_1 \) are not uniquely specified.

The optimum function \( F_1 \) would be determined by Eqs. (10)-(12) in which \( \psi(1) \) is taken to be the exact perturbed wavefunction \( \psi \). In this case the \( E(1) \) of Eq. (3) becomes the exact energy \( E \).

**EXAMPLE:** The \( S_2 \) Permutation Group

The simplest physical examples of exchange perturbations are the interactions of a hydrogen atom with either another hydrogen atom or else a proton. In either case, the symmetry group of the perturbed system is the \( S_2 \) permutation group having the elements \( 1 \) and \( P \). This group has two one dimensional irreducible representations, gerade (g) and ungerade (u). The corresponding projection operators are:

\[
\bar{g} = (1/2)(1+P) \quad \text{and} \quad \bar{u} = (1/2)(1-P).
\]

It is convenient to let the super-script \( g \) or \( u \) designate the symmetry component of a function, for example: \( gV = \bar{g}V \). For this \( S_2 \) group, we must distinguish two cases:

1) The state under consideration has gerade symmetry so that \( A \equiv \bar{g}A \).

Since \( V = gV + uV \) and since the product \( gVgG \) is ungerade and \( uVuG \) is gerade, Eq. (11) has the unique solution,

\[
(1-\bar{g})G = (1/\bar{u})\bar{X} \quad \text{(13)}
\]

2) The state under consideration has ungerade symmetry so that \( A \equiv \bar{u}A \).

Similarly, since \( gVgG \) is gerade and \( uVuG \) is ungerade, Eq. (11) has the unique solution.
\[(1-U)G = (1/\nu)Ux\]  
(14)

If \(\Psi(1)\) is a given function and \(E_1\) is a given value then \(F_1\) is uniquely given by Eq. (10) and \(\phi_1\) is uniquely given by Eq. (8).

**EXAMPLE:** The \(S_3\) Permutation Group

The exchange of three like particles, as for example in the inter- 
action of \(H_2\) with \(H_1\), corresponds to the \(S_3\) symmetry group having 
six symmetry elements. Its three irreducible representations correspond 
to the Young diagrams \(\begin{array}{c} \begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \end{array}\), \(\begin{array}{c} \begin{array}{c} \cdot \\ \cdot \\ \\ \cdot \end{array} \end{array}\), and \(\begin{array}{c} \begin{array}{c} \cdot \\ \cdot \end{array} \end{array}\) (which we 
refer to as 1, 2, and 3 respectively). The projectors for the one 
dimensional representations 1 and 3 are

\[1^A = (1/6) (1 + \nu_{12} + \nu_{13} + \nu_{23} + \nu_{123} + \nu_{132})\]  
(15)

\[3^A = (1/6) (1 - \nu_{12} - \nu_{13} - \nu_{23} + \nu_{123} + \nu_{132})\]  
(16)

We include the projector \(1^A\) for the sake of completeness even though 
a spatial wavefunction which transforms as \(1^A\) is not Pauli-allowed.

The two dimensional representation 2 has the two projectors

\[2_{11} = (1/6) (2 + 2\nu_{12} - \nu_{13} - \nu_{23} - \nu_{123} - \nu_{132})\]  
(17)

\[2_{22} = (1/6) (2 - 2\nu_{12} + \nu_{13} + \nu_{23} - \nu_{123} - \nu_{132})\]  
(18)

and the two shift operators

\[\nu_1 \cdot \nu_2 \cdot \nu_3 \cdot \nu_4 \cdot \nu_5 \cdot \nu_6 \]

Here we are using the convention \(\nu_{12} = \delta_{12} \phi_{12}\). Note that 
in Goscinski and Lowdin's Ref. (13) notation, the rows and columns are 
interchanged.
\[ 2A_{12} = (1/12)^{1/2}(p_{13} - p_{23} + p_{123} - p_{132}) \]  
\[ 2A_{21} = (1/12)^{1/2}(p_{13} - p_{23} - p_{123} + p_{132}) \]  

The projector corresponding to the character of 2 is, of course,
\[ 2A = 2A_{11} + 2A_{22} = (1/3)(2 - p_{123} - p_{132}) \]  

After reducing the representation of VG to its irreducible components (and using the notation \( 2A_{i,j} V = 2V_{i,j} \)), the four symmetry components of Eq. (11) are

\[ 1_X = 1/2[2G_{11} 2V_{11} + 2G_{21} 2V_{21} + 2G_{22} 2V_{22} + 2G_{12} 2V_{12}] + 3G 2V \] (22)

\[ 2_X_{11} = 1_G 2V_{11} + 1/2[2G_{22} 2V_{22} - 2G_{12} 2V_{12}] + 2G_{22} 3V + 3G 2V_{22} \] (23)

\[ 2_X_{22} = 1_G 2V_{22} + 1/2[2G_{11} 2V_{22} + 2G_{21} 2V_{12}] + 2G_{11} 3V + 3G 2V_{11} \] (24)

\[ 3_X = 1_G 3V + 1/2[2G_{11} 2V_{22} - 2G_{21} 2V_{12} + 2G_{22} 2V_{11} - 2G_{12} 2V_{21}] \] (25)

In addition, we obtain the equation corresponding to the character projector \( 2A \)
\[ 2_X = 1_G 2V + 3G 2V \] (26)

For each of the symmetry cases Eq. (22), or (23) or (24) or (25), provides one condition on the four independent components* of \( G (1_G, 2G_{11}, 2G_{22}, \) and \( 3_G) \).

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* Note that \( 2G_{21} = 2A_{21} 2G_{11} \) and \( 2G_{12} = 2A_{12} 2G_{22} \) so that \( 2G_{21} \) and \( 2G_{12} \) can be determined from a knowledge of \( 2G_{11} \) and \( 2G_{22} \) respectively.

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Hence \( G \) and \( (1-A)G \) can be chosen in an infinite number of ways.

On the other hand, if we used the character projector to express the symmetry of the two dimensional representation, we would obtain from Eq. (26) the unique solution for \( (1 - 2A)G \), corresponding to wavefunctions which have the 2 symmetry,

\[
(1 - 2A)G = 2X / 2V
\]  

(27)

Thus, some simplification in the determination of \( (1-A)G \) and in the equation (5) for the determination of \( \phi_1 \) may result from the use of the character projector \( 2A \). However, the wavefunction for the perturbed system has the symmetry corresponding to a particular row of the irreducible representation. Thus, \( \psi(1) = 2A_{11}(k\psi_0 + \phi) \) or \( \psi(1) = 2A_{22}(k\psi_0 + \phi) \). As far as the equations for \( 1X \) and \( 3X \) are concerned, no simplification results from trying to use the \( 2A \) projector.

For wavefunctions having either the 1 or the 3 symmetry, a particular solution for \( (1-A)G \) results from choosing \( 2G_{11} \) and \( 2G_{22} \) to be zero (hence \( 2G_{2} = 2G_{12} = 0 \)). In this case, Eqs. (22) and (25) give

\[
(1 - 1A)G = 1X / 3V \quad \text{and} \quad (1 - 3A)G = 3X / 3V
\]  

(28)

We believe that our analysis has shed a bit of light on the structure of the exchange perturbation equations. It remains to be seen how this analysis will be used to develop better functions \( F_1 \) which will make the practical solution of exchange perturbation problems easier or more accurate.
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APPENDIX

An Attempt to Determine $\phi_1$ by Optimizing $E_1 + E_2$

It is not easy to determine the various orders of perturbation energy from Eq. (3). For example, the apparent zeroth order energy would seem to be

$$E_0' = \frac{\langle \phi_0 \mid A_0 A \phi_0 \rangle}{\langle \phi_0 \mid A \phi_0 \rangle}$$  (A1)

instead of $E_0$. This difficulty disappears if we rewrite Eq. (3) in the form

$$E(1) = \frac{\langle \phi_0 + \phi_1 \mid (A_0 + A_0) + (A_V + V_A) \mid \phi_0 + \phi_1 + \phi_1 \rangle}{2\langle \phi_0 + \phi_1 \mid A \mid \phi_0 + \phi_1 \rangle}$$  (A2)

Here we are expressing the first order function $\phi_1$ as the sum of its two components $A_0 \phi_1$ and $(1-A)\phi_1$, which can be varied separately. In Eq. (3), the zeroth order operator $(A_0 + A_0)$ and the first order operator $(A_V + V_A)$ are still Hermitian. However, we would have obtained the same conclusions if we had used the non-Hermitian zeroth and first order operators $A_0$ and $A$ (or equivalently, $A_0 A$ and $V_A$). Our conclusion is that we cannot obtain an optimum first order function $\phi_1$ by varying the sum of the first and second order energies.
For convenience, let us normalize $\phi_0$ so that

$$\langle \phi_0 | \hat{\mathbf{A}} | \phi_0 \rangle = 1$$

Expanding the right hand side of Eq. (A2) and grouping the terms according to their apparent order of perturbation, and letting c.c. denote the conjugate complex we obtain:

$$E_1 = E_1^{\text{HL}} + B_1$$

$$E_2 = E_1' + D_2 - \frac{1}{2} B_1 d$$

Here $E_1^{\text{HL}}$ is the usual Heitler-London first order energy

$$E_1^{\text{HL}} = \frac{1}{2} \langle \phi_0 | \mathbf{A} \mathbf{V} + \mathbf{V} \mathbf{A} | \phi_0 \rangle$$

The additional first order energy term is

$$B_1 = \frac{1}{2} \left[ \langle \mathbf{A} \phi_1 + (1 - \mathbf{A}) \phi_0 | (\mathbf{H}_0 - \mathbf{E}_0) \mathbf{A} | \phi_0 \rangle + \text{c.c.} \right]$$

The second order energy terms are

$$E'_2 = \langle \mathbf{A} \phi_1 | \mathbf{H}_0 - \mathbf{E}_0 | \mathbf{A} \phi_1 \rangle$$

$$+ \frac{1}{2} \left[ \langle (1 + \mathbf{A}) \phi_0 | \mathbf{V} - E_1^{\text{HL}} | \mathbf{A} \phi_1 \rangle + \text{c.c.} \right]$$

$$D_2 = \frac{1}{2} \left[ \langle (1 - \mathbf{A}) \phi_0 | (\mathbf{H}_0 - \mathbf{E}_0) \mathbf{A} \phi_1 + (\mathbf{V} - E_1^{\text{HL}}) \mathbf{A} \phi_0 \rangle + \text{c.c.} \right]$$

and the third term is $-\frac{1}{2} B_1 d$ where

$$d = \langle \phi_0 | \mathbf{A} \phi_1 \rangle + \text{c.c.}$$
Although the function \((1-A)\phi_1\) does not affect \(E(1)\), by varying \((1-A)\phi_1\) values of \(E_1\) and \(E_2\) can be made arbitrarily large or small. This follows from the fact that the expressions for \(E_1\) and \(E_2\) are linear in \((1-A)\phi_1\).

We might still expect to obtain a good expression for \(\phi_1\) by varying \(A\phi_1\) (while holding \((1-A)\phi_1\) constant) so as to make the sum of \(E_1 + E_2\) stationary. In order for the resulting equation to be mathematically consistent, \(B_1\) must be equal to zero or \(E_1 = E^{HL}_1\). The resulting equation for the first order function \(\phi_1 = A\phi_1 + (1-A)\phi_1\) (where the \(A\phi_1\) has been optimized while \((1-A)\phi_1\) has been held fixed) is then

\[
A[(H_0 - E_0)\phi_1 + (V - E^{HL}_1)\phi_0] + (H_0 - E_0)A(\phi_1 + (1-d)\phi_0) + (V - E^{HL}_1)A\phi_0 = 0
\]

If now Eq. (A10) is multiplied by \((1-A)\) and we make use of Eq. (1), we obtain an equation of the form

\[(1-A)(H_0 - E_0)AY = 0\]  \hspace{1cm} (A11)

where \(Y = \phi_1 - d\phi_0\). Again, making use of Eq. (1), Eq. (A11) can be expressed in the form

\[(AV - VA)AY = 0\]  \hspace{1cm} (A12)

However, the basic premise in perturbation theory is that \(V\) does not commute with \(A\), so Eq. (A12) can be true only if \(AY = 0\). Thus,

\[A\phi_1 = d A\phi_0\]  \hspace{1cm} (A13)
Multiplying Eq. (A13) by $\phi_0^*$, integrating and remembering the definition of $d$, we see that $d = 0$ and so $A\phi_1 = 0$. Thus, we conclude that Eq. (A10) does not have a (useful) solution. As a result, we cannot obtain an optimum first order function $\phi_1$ by making use of Eqs. (A3) and (A4) for the first and second order energies.

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