RANGE AND RANGE RATE ERROR ANALYSIS
FOR THE
ATS-F/NIMBUS-E TDRS EXPERIMENT

by
C. Filippi

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THE Magnavox COMPANY
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advanced systems analysis office
This report presents a system design evaluation study of the proposed ATS-F/NIMBUS-E Tracking and Data Relay Satellite Experiment, as well as performance analyses of specific problem areas of interest to NASA/GSFC. The tracking signal processing in the ATS-F/NIMBUS-E system is analyzed to verify the system concept feasibility, and a range and range rate error analysis is performed to illustrate the accuracy limitations of a tracking network employing a relay satellite. The extraction and processing of range rate data is studied by comparing the error performance capabilities of destructive- and nondestructive-count extractors, both at the raw and smooth data stages. The ionospheric effects in the extraction of range and range rate signals based on GRARR system principles are analyzed.
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1. GENERAL INTRODUCTION

A basic feature of the tracking and data relay satellite (TDRS) system is that it permits continuous communication between user spacecrafts and selected earth stations, thus reducing the complexity of the ground network required for communication and tracking purposes. If the inherent advantages of a space relay station are to be fully exploited, the tracking and data relay operations should not compromise performance attainable with an all-ground tracking network, at least to any major extent. In fact a successful tracking of the user spacecrafts by the TDRS system can but improve the user position location when a joint orbiting and ground tracking mode is considered, but a poor tracking performance by the orbiting mode will hinder the TDRS system concept applications.

In order to both evaluate the TDRS system concept feasibility and compare its tracking and data relay performance to that attainable with a ground network, NASA/GSFC intends to conduct an experiment utilizing the ATS-F and NIMBUS-E spacecrafts as the relay satellite and user spacecraft respectively, along with their corresponding ground tracking networks. The basic experiment configuration is summarized in Fig. 1: the ATS-F relays test and tracking signals to NIMBUS-E, and receives from the latter data and retransmitted signals for ground relay purposes. The relayed signals received by the ATSR ground stations can then be compared to those directly received from NIMBUS-E by its ground network, so as to evaluate the communication and tracking performance of the relay satellite system.

The primary modes being considered for the ATS-F/NIMBUS-E experiment involve non-simultaneous transmission of data and tracking signals, and the tracking mode represents the high priority issue from study program considerations at this stage. The signal generation and processing for the tracking
mode may be summarized with the aid of Fig. 2 as follows:

(a) Generation and transmission of a 6 GHz PM sidetone ranging signal from the ATSR ground station to the ATS-F spacecraft.

(b) Coherent translation of the 6 GHz signal received by ATS-F so as to generate a 2 GHz PM signal to be relayed to the NIMBUS-E spacecraft.

(c) Incoherent translation of the 2 GHz PM signal received at NIMBUS-E to a subcarrier frequency (1.4, 2.4 or 3.2 GHz) and generation of an S-band PM/PM signal to be retransmitted back to the ATS-F spacecraft or to the NIMBUS ground receiver. The S-band carriers are derived from the NIMBUS-E local oscillator, and the spacecraft thus operates as a GRARR transponder.

(d) Incoherent translation of the S-band signal received by ATS-F so as to relay a 4 GHz PM/PM signal to the ATSR ground receiver. The heterodyning operation employs a scaled coherent replica of the 6 GHz uplink received carrier as the mixing reference.

(e) Acquisition and tracking of the 4 GHz signal received at the ATSR ground station, plus extraction and data processing of the range and range rate signals following GRARR receiver principles.

The ATS-F/NIMBUS-E experiment and TDRS system performance objectives introduce many new issues never considered in past studies of the ATS and NIMBUS systems. With reference to the ATS system, the available ATSR ground receivers have been designed to handle a PM sidetone ranging signal and not a GRARR-like PM/PM ranging signal, since the ATS ranging principles are based on a 5 MHz high-frequency tone for range rate purposes rather than a GRARR subcarrier. The need for a modified ATSR receiver processing is thus evident, and it is of
Fig. 2  Tracking Signal Processing of the ATS-F/NIMBUS-E System
interest to introduce minimum-complexity modifications to maximize interfacing and reduce cost without compromising system performance.

With reference to both the ATS and NIMBUS systems, their tracking performance exhibited a nil error contribution from retransmitted noise effects, since downlink noise predominated over retransmitted uplink noise in a simple ground-satellite link. However, the ATS-F/NIMBUS-E system introduced a satellite-satellite link, and retransmitted noises accompanying the downlink noise in the ATS-F/G.S. link or the NIMBUS-E/STADAN RCVR link need no longer be secondary effects and must be accounted for in the receiver design, error analysis and tracking performance evaluation. Moreover, some of these noises do not represent additive effects and require a more complex analysis, such as the ATS-F/NIMBUS-E link noise which phase-modulates the NIMBUS-E signal carrier in the PM/PM signal generation.

On this basis, a study program was developed by NASA/GSFC and MAGNAVOX/ASAO to evaluate the ATS-F/NIMBUS-E experiment and its TORS system performance objectives, and the following tasks were stated to reflect NASA's interests and high-priority issues. A close contact with NASA was maintained throughout the program to update the analysis to reflect any changes in link parameters, orbital conditions, power budgets, or operational blocks of the system.

(a) To perform an evaluation study of the proposed ATS/NIMBUS experiment for the TDRS, investigating its conceptual feasibility and conducting a performance analysis to ensure adequate signal acquisition, tracking and demodulation operations.
(b) To provide a detailed analysis of the ground receiver (ATSR) modification, specifying any critical functions or potential problems, exhibiting existing design compromises and recommending solutions compatible with the overall system concept, signal processing principles and minimum cost considerations.

(c) To perform a complete error analysis on the range and range rate data extracted under the pertinent link parameters, orbital conditions and channel specifications, and recommend any operational or data processing modification that will improve the tracking performance under the TDRS configuration.

(d) To provide whatever technical assistance may be required and study any specific problems that may arise during the course of the program, including assistance in the generation of testing procedures and in test data interpretation.

While maintaining the overall system concept and transponder operations, NASA/GSFC introduced two important system redesign considerations during the course of the program. The ATS-F/NIMBUS-E link frequency was changed from 1800 MHz to 2062.85 MHz to comply with existing frequency revisions, which altered the downlink carrier/subcarrier frequency ratio and required a corresponding ground receiver modification so as to extract the proper doppler signal. Also, in order to avoid a complete redesign of the ATS-F phased-locked transponder so as to match with the aforesaid change, a beacon signal phase-coherent to the uplink carrier is to be transmitted along with the latter in the G.S./ATS-F link. The beacon signal frequency will be 12.15 MHz higher than the PM carrier, and its level should be substantially lower than the PM carrier.
so as not to compromise the ranging signal power budget, yet sufficient to provide for self-acquisition and lock maintenance at the ATS-F loop.

The study program performed for the original 1800 MHz and present 2062.85 MHz system concepts involve analogous considerations, except for the inclusion of the beacon signal implications in the latter, plus yield commensurate results. Hence we will concentrate here on the present 2062.85 MHz system concept, and omit a full repetition of the analogous block diagrams, system analysis and performance evaluation germane to the 1800 MHz system concept, in accordance with the NASA/GSFC suggestions during the preparation of this report.
2. ATS-F/NIMBUS-E TDRS SYSTEM ANALYSIS

In this chapter we present a performance evaluation and error analysis of the ATS-F/NIMBUS-E tracking system. The signal processing specifications presented in Fig. 2 inherently restrict the doppler extraction potential of the system, hence we first establish such potential regardless of any given receiver realization. After the doppler extraction capabilities are thus identified, we proceed to analyze the specific ATSR receiver configuration under consideration, so that we can indeed evaluate the merits of the realization by noting to what extent it fully exploits the intrinsic doppler extraction potential of the system. We then present a logical analysis and discussion of the effects of both oscillator and thermal noise effects on system performance. We identify the additional phase noise contributions and SNR degradation effects directly caused by the TDRS system concept and otherwise absent in an all-ground based tracking network, and conclude with a range and range rate error analysis to illustrate the accuracy and error performance capabilities of the system.

2.1 Doppler Extraction Considerations

The signal processing specified for the ATS-F and NIMBUS-E transponders can be used to establish the doppler extraction potential inherent in the ATS-F/NIMBUS-E experiment, i.e., the doppler signal that can be extracted by an ideal ground receiver that processes the downlink signal in an optimum way. The motivation towards establishing this potential is that the pseudo-coherent ATS-F translation (uplink coherent, downlink incoherent) and the GRARR-like NIMBUS-E processing result in an asymmetrical treatment of the doppler effects in the
G.S./ATS-F vs the ATS-F/NIMBUS-E links, and in fact eliminate the possible extraction of a coherent doppler signal proportional to the round-trip doppler effect. It is important to be aware of the doppler reproduction capabilities and limitations conditioned on the experiments by the specified transponder operations in order to properly evaluate any proposed receiver configuration and doppler extraction scheme, i.e., we must know how much we can expect from an ideal receiver so as to evaluate any particular realization.

To this effect, we assume constant doppler shifts of different magnitudes (and perhaps sign) in both the G.S./ATS-F and ATS-F/NIMBUS-E links. Since all transmission or reference frequencies involved in the system are in principle coherent to the G.S. master oscillator, the NIMBUS-E local oscillator, or a weighted sum of the two oscillators, we need only specify the doppler effects on these two oscillations or any scaled multiples of their frequencies. On this basis, we denote by $f_T = 6$ GHz the uplink ground carrier and by $f_o = 37.55$ MHz the NIMBUS-E oscillator, and respectively represent their one-way doppler effects by unprimed $D_T$ or $D_o$ in the G.S./ATS-F link and by primed $D'_T$ or $D'_o$ in the ATS-F/NIMBUS-E link.

The doppler signal propagation is illustrated in Fig. 3, assuming a 2.4 MHz downlink subcarrier is employed. The $(\pm)$ symbol appearing in the ATS-F side-stepping of the downlink signal is intended to imply that any one of the two signs (conversion procedures) may be used with the proper choice of L-value. While the NIMBUS-E conversion for subcarrier generation is specified from existing S-band GRARR transponders, the downlink ATS-F translation remains open at present insofar as the NIMBUS-E/ATS-F carrier may be added or subtracted to generate the ATS-F/G.S. carrier. The interest in distinguishing between the two options is that the ground receiver processing must be adopted accordingly
G.S. XMTR

\[ f_T \rightarrow \text{to ATS-F} \]

ATS-F PROCESSING OF G.S. SIGNAL

\[ f_T + D_T \rightarrow \text{PLL} \rightarrow \frac{1}{K}(f_T + D_T) \rightarrow \text{to NIMBUS-E} \]

NIMBUS-E PROCESSING OF ATS-F SIGNAL

\[ \frac{1}{K}(f_T + D_T + D'_T) \rightarrow X \rightarrow \text{PM} \rightarrow Nf_0 \text{ carrier} \rightarrow \text{to ATS-F} \]

\[ Mf_0 \rightarrow \frac{1}{K}(f_T + D_T + D'_T) \text{ subcarrier} \]

ATS-F PROCESSING OF NIMBUS-E SIGNAL

\[ N(f_0 + D'_0) \text{ carrier} \]

\[ M(f_0 + D'_0) - \frac{1}{K}(f_T + D_T + 2D'_T) \text{ subcarrier} \]

\[ L(f_T + D_T) \pm N(f_0 + D'_0) \text{ carrier} \]

\[ L(f_T + 2D_T) \pm N(f_0 + D'_0 + D''_0) \text{ carrier} \]

G.S. RCVR

\[ M(f_0 + D'_0 + D''_0) - \frac{1}{K}(f_T + 2D_T + 2D'_T) \text{ subcarrier} \]

\[ 2LM_0 + \frac{2}{K}(D_T + D'_T) \]

Fig. 3  Doppler Signal Processing
to subtract or add the carrier and subcarrier doppler effects when compensating for the incoherent NIMBUS-E oscillator effects and extracting a coherent doppler signal.

It should be noted that even though the NIMBUS-E oscillator compensation is indeed feasible, the ground receiver will not be capable of extracting a doppler signal proportional to the round-trip doppler $2(D_T + D_T')$. We must accept instead a weighted sum of the doppler effects $D_T$ and $D_T'$ of the two links involved, with unequal weighting coefficients. The doppler signal will be proportional to the term $\left(\frac{KLM}{N}\pm 1\right)(2D_T) \pm 2D_T'$ (sign depending on the aforesaid option) and the different weighting coefficients must be accounted for in the orbital determination procedure. The possible use of these available values as an approximation to the round-trip doppler $\pm(2D_T + 2D_T')$ is a valid question since the relation $D_T << D_T'$ will be usually satisfied (except under negligible $D_T'$ conditions). The approximation involves a bias error term of $\pm \frac{KLM}{N}(2D_T)$ for the two options, and this is about 15 Hz and 56 Hz maximum assuming $(D_T)_{\text{max}} = 20$ Hz, which correspond to a maximum bias of 0.4 m/s and 1.4 m/s in the round-trip range rate over the entire G.S./ATS-F/NIMBUS-E path. The ultimate position-location effects when using the approximation rather than the different weighting coefficients depend on the orbital determination procedure.

Once the pseudo-coherent ATS-F and GRARR-like NIMBUS-E processings are specified as initial conditions, the exact reproduction of the round-trip doppler $2D_T + 2D_T'$ would require additional signaling. For example, the downlink transmission by ATS-F to ground of a pilot signal coherent to the uplink carrier received by ATS-F from ground will suffice, as shown in Fig. 4. It is of interest to note that the use of a coherent NIMBUS-E transponder, along with the pseudo-coherent ATS-F, will not suffice and a pilot signal insertion would still be required, as illustrated in Figs. 5(a) and 5(b).
ATS-F retransmission to G.S.

\[ L(f_T+D_T) \pm N(f_o+D'_o) \] carrier

\[ [M(f_o+D'_o)-\frac{1}{K}(f_T+D_T+D'_T)] \] subcarrier

G.S. RCVR processing

\[ L'(f_T+2D_T) \] pilot

\[ \frac{N}{MK} \left[ \frac{KLM}{N} (2D_T) - (2D_T+2D_T') \right] \]

doppler from G.S. RCVR of Fig. 3

Fig. 4 Round-Trip Doppler Reproduction via Pilot Tone Insertion
ATS-F processing of G.S. signal

from G.S. \( f_T + D_T \) \( \xrightarrow{\text{PLL}} \) \( \frac{1}{K} (f_T + D_T) \) to NIMBUS-E

NIMBUS-E processing of ATS-F signal

from ATS-F \( \frac{1}{K} (f_T + D_T + D_T') \) \( \xrightarrow{\text{PLL}} \) \( \frac{J}{K} (f_T + D_T + D_T') \) to ATS-F

ATS-F processing of NIMBUS-E signal

(a) from NIMBUS-E \( \frac{J}{K} (f_T + D_T + 2D_T') \) \( \xrightarrow{\text{adder}} \) \( \frac{J+I}{K} (f_T + D_T) + \frac{J}{K} (2D_T) \) to G.S.

(b) from NIMBUS-E \( \frac{J}{K} (f_T + D_T + 2D_T') \) \( \xrightarrow{\text{adder}} \) SIGNAL ADDER \( \frac{J+I}{K} (f_T + D_T) + \frac{J}{K} (2D_T') \) carrier \( \xrightarrow{\text{adder}} \) \( \frac{H}{K} (f_T + D_T) \) pilot to G.S.

(Figure continues on next page)
G.S. RCVR

(a) from ATS-F

\[ \frac{J+I}{K} (f_T + 2D_T) + \frac{J}{K} (2D'_T) \]

PLL

\[ \frac{J}{K} [(2D_T + 2D'_T) + \frac{I}{J}(2D_T)] \text{ doppler data} \]

(b) from ATS-F

\[ \frac{J+I}{K} (f_T + 2D_T) + \frac{J}{K} (2D'_T) \text{ carrier} \]

\[ \frac{H}{K} (f_T + 2D_T) \text{ pilot} \]

PLL

\[ \frac{J}{K} (2D_T + 2D'_T) \text{ doppler data} \]

Fig. 5 Doppler Extraction using a Coherent NIMBUS-E Transponder
2.2 The Modified ATSR Doppler Extractor

In this section we characterize and evaluate the signal processing techniques being at present considered as the ATSR modification for doppler extraction purposes. The original proposed diagrams are included here as Figs. 6-8 for reference purposes, and their basic processing can be summarized as shown in Figs. 9-11 to guide the doppler signal extraction analysis. The following notation is used in the figures:

\[ \omega_x = \text{nominal x-MHz frequency} \]

\[ \theta_x^{(G)} = \text{phase instabilities of ground master oscillator scaled to x-MHz, with the time dependence omitted for simplicity} \]

\[ \theta_x^{(2.4)} = \text{phase instabilities of ground crystal oscillator at the downlink 2.4 MHz subcarrier frequency, with the time dependence omitted for simplicity} \]

\[ \theta_c \text{ and } \theta_{sc} = \text{phase fluctuations of the received downlink carrier and subcarrier signals, with the time dependence omitted for simplicity.} \]

\[ \theta_{v,c} \text{ and } \theta_{v,sc} = \text{phase fluctuations of the receiver carrier and subcarrier loop VCO's, with the time dependence omitted for simplicity.} \]

\[ \Delta \theta_c \text{ and } \Delta \theta_{sc} = \text{phase locking errors of the receiver carrier and subcarrier loops, with the time dependence omitted for simplicity.} \]

It is important to notice that the carrier VCO phase \( \theta_{v,c} \) ideally tracks the carrier signal phase \( \theta_c \), while (minus) the subcarrier VCO phase \( -\theta_{v,sc} \) ideally tracks the subcarrier signal phase \( \theta_{sc} \), as made evident by the carrier locking error \( \Delta \theta_c = \theta_c - \theta_{v,c} \) and the subcarrier locking error \( \Delta \theta_{sc} = \theta_{sc} + \theta_{v,sc} \) expressions in the absence of oscillator instabilities.

The sign reversal in the subcarrier VCO is introduced when mixing the signals (4) and (5) of Fig. 10. The net effect on the doppler signal extracted can
Figure 7  SUBCARRIER RECEIVER/RANGE TONE DEMODULATION FUNCTION BLOCK DIAGRAM
Figure 8  DOPPLER EXTRACTOR FUNCTIONAL BLOCK DIAGRAM
\( E_c \sin[\omega_{3953} t + \theta_c] \pm E_{sc} \sin[(\omega_{3953} \pm \omega_{2.4}) t + \theta_c \pm \theta_{sc}] \)

(2) \( \cos[(\omega_{3953} - \omega_{70}) t + \theta_{3953}^{(G)} - \theta_{70}^{(G)}] \)

(3) \( E_c' \sin[\omega_{70} t + \theta_c - \theta_{3953}^{(G)} + \theta_{70}^{(G)}] \pm E_{sc}' \sin[(\omega_{70} \pm \omega_{2.4}) t + \theta_c - \theta_{3953}^{(G)} + \theta_{70}^{(G)} \pm \theta_{sc}] \)

(4) \( \cos[\omega_{70} t + \theta_{v,c}] \)

(5) \( \cos[\omega_{11} t + \theta_{11}^{(G)}] \)

(6) \( \cos[\omega_{81} t + \theta_{v,c} + \theta_{11}^{(G)}] \)

(7) \( E_c \sin[\omega_{11} t + \theta_{v} - \theta_{c} + \theta_{11}^{(G)} + \theta_{3953}^{(G)} - \theta_{70}^{(G)}] \)

(8) \( \sin[\theta_c - \theta_{3953}^{(G)} + \theta_{70}^{(G)} - \theta_{v,c}] = \sin \Delta \theta_c \)

---

**Fig. 9** Carrier Loop Processing
(1) $E_c' \sin[\omega_70 t + \theta \phi_70 (G) + \theta (G)]$
$\pm E_s' \sin[(\omega_70 + \omega_2.4) t + \theta \phi_{3953} + \theta (G) + \theta (G) + \theta_s c]$

(2) $\cos[\omega_70 t + \theta \phi_{v,c}]$

(3) $E_s'' \sin[\omega_{2.4} t + \theta \phi_{s,c} + \theta_s c] = 2E_s'' \cos \theta \phi_c \sin[\omega_{2.4} t + \theta \phi_{s,c}]$

(4) $\cos[\omega_{42.4} t + \theta (G) + \theta (x)]$

(5) $\cos[\omega_{30} t + \theta \phi_{v,sc}]$

(6) $\cos[\omega_{12.4} t - \theta \phi_{v,sc} + \theta (G) + \theta (x)]$

(7) $E_s'' \cos \theta \phi_c \sin[\omega_{10} t - \theta \phi_{v,sc} - \theta \phi_{s,c} + \theta (G) + \theta (x)]$

(8) $\cos[\omega_{10} t + \theta (G)]$

(9) $\sin[\theta \phi_{s,c} - \theta (G) - \theta (x) + \theta \phi_{v,sc}] = \sin \theta \phi_{s,c}$

Fig. 10 Subcarrier Loop Processing
Fig. 11 Doppler Signal Extraction

\begin{align*}
(1) & \quad \cos[\omega_{70}t+\theta_v,c] \\
(2) & \quad \cos[\frac{11}{12}\omega_{70}t+\frac{11}{12}\theta_v,c] \\
(3) & \quad \cos[\omega_{12.4}t-\theta_v,sc+\theta_{40}^{(G)}+\theta_{2.4}^{(x)}] \\
(4) & \quad \cos[\omega_{2.4}t+\theta_{2.4}^{(x)}] \\
(5) & \quad \cos[\omega_{10}t-\theta_v,sc+\theta_{40}^{(G)}] \\
(6) & \quad \cos[(\frac{11}{12}\omega_{70}-\omega_{10.5})t+\frac{11}{12}\theta_{70}^{(G)}-\theta_{10.5}^{(G)}] \\
(7) & \quad \cos[(\frac{11}{12}\omega_{70}-\omega_{0.5})t-\theta_v,sc+\frac{11}{12}\theta_{70}^{(G)}+\theta_{29.5}^{(G)}] \\
(8) & \quad \cos[\omega_{0.5}t+\frac{11}{12}\theta_v,sc+\theta_v,sc-\frac{11}{12}\theta_{70}^{(G)}-\theta_{29.5}^{(G)}] \\
\end{align*}
be established from (8) of Fig. 11: such signal has a phase \( \frac{11}{12} \theta_v, c^{+\theta}_v, sc \) in the absence of oscillator instability effects, which ideally corresponds to \( \frac{11}{12} \theta_s, c^{-\theta}_sc \) in the absence of locking errors. Hence, the doppler extractor subtracts the weighted carrier and subcarrier doppler effects, by first reversing the subcarrier phase at the subcarrier VCO and then adding the weighted carrier and subcarrier VCO phases in the doppler signal extraction.

We can now refer back to Fig. 3 and consider the two possible side-stepping procedures for the ATS-F relay of the NIMBUS-E signal to the ATSR receiver. In the case where the downlink carrier is generated via \( L_f^+ + N_f^0 \), the carrier and subcarrier doppler will exhibit the same sign in the term contributed by the NIMBUS-E oscillator, and thus the interest is to have the ATSR receiver subtract the weighted carrier and subcarrier doppler so as to compensate for the incoherent NIMBUS-E oscillator effects. Conversely, if the downlink is generated via \( L_f^- - N_f^0 \), the carrier and subcarrier doppler contributions from the NIMBUS-E oscillator will have opposite signs, and the interest is to add their weighted reproductions by the ATSR receiver to provide the desired compensation.

It is evident that the ATSR receiver will perform the desired compensation of the NIMBUS-E oscillator effects if the downlink carrier is generated via \( L_f^+ + N_f^0 \) at the ATS-F transponder, but not if the \( L_f^- - N_f^0 \) procedure is used in which case the doppler signal extracted will include doppler effects of the NIMBUS-E oscillator. In summary, the ATSR receiver modification must be compatible with the ATS-F side-stepping procedure for the downlink signal, and the proposed modification assumes the NIMBUS-E/ATS-F carrier frequency is added to that of the ATS-F reference in the heterodyning operation. This approach has been recently suggested in the literature (Sabelhaus et al,
ATS-F and G Communications Subsystem, Proc. IEEE, Feb. 1971, p 206), and under such condition the ATSR receiver will properly cancel the incoherent NIMBUS-E oscillator effects, and moreover will match the doppler extraction potential inherent in the ATS-F/NIMBUS-E system, which was discussed in the previous section.

2.3 Effects of Retransmitted and Local Oscillator Noise

The ranging signal processing for the ATS-F/NIMBUS-E TORS experiment is based on a quasi-coherent ATS-F transponder operation (uplink coherent, downlink incoherent), as shown in Figs. 2 and 3. In particular, the uplink signal extracted by the ATS-F loop is used to derive the mixing references involved in the side-stepping of the NIMBUS-E downlink signal for ground retransmission purposes. As a result, the downlink carrier received by the ATSR station will: (a) exhibit not only the phase instabilities of the NIMBUS-E oscillator at 2253 MHz, but also those contributed by the side-stepping reference and whose origin was the G.S./ATS-F uplink signal and hence the ground oscillator, (b) have a frequency higher than that received by a ground station directly linked to the NIMBUS-E spacecraft (namely 3953 vs 2253 MHz), so that a higher r-f local reference is required in the ATSR receiver for i-f conversion purposes, and larger local-oscillator phase instabilities will be involved. These extra oscillator instabilities of (a) and (b) above are a direct consequence of the relay satellite introduced in-between the ground station and the user spacecraft, and their error performance implications must be identified for a proper comparative evaluation of a TDRS tracking system vs an all-ground tracking system.
The ATS-F transponder operation is summarized in Fig. 12(a), with the understanding that the 25 MHz VCO is phase-locked to the beacon signal rather than the carrier signal. The discussion that follows will concentrate on the middle set of design frequencies tabulated below the figure, and analogous considerations exist for the other sets. The NIMBUS-E transponder operation is then summarized in Fig. 12(b) for the case of a 2.4 MHz subcarrier channel. The downlink carrier/subcarrier contributions from the NIMBUS-E local oscillator can be verified to now exhibit a frequency ratio of \( M/N = (56-1)/60 = 11/12 \), rather than the \( 48/60 = 12/15 \) ratio existing in the original system employing an 1800 MHz ATS-F/NIMBUS-E link.

The basic signal processing for the round-trip propagation in the G.S./ATS-F/NIMBUS-E link is now summarized in Table 1, where the ranging sidetones have been omitted for simplicity. In order to keep track of oscillator frequency and phase coherence, propagation effects and phased-locked filtering, the following notation rules will be used:

(a) The nominal operational frequency in MHz appears as a subscript to \( \omega \); e.g., \( \omega_{6150} \) represents a 6150 MHz oscillation.

(b) The phase fluctuations caused by oscillator instabilities or doppler effects are denoted by \( \theta \), and the time-dependence is omitted for simplicity. A subscript is used to scale the fluctuations to a given frequency using the same notation as in (a) above.

(c) A superscript (G) or (N) is used to identify the origin of the oscillator phase fluctuations as the ground (G) or NIMBUS-E (N) oscillator; e.g., \( \theta_{2253}^{(N)} \) is the phase of the NIMBUS-E oscillator scaled to 2253 MHz.
Fig. 12 ATS-F and NIMBUS-E Transponder Processings
TABLE 1

(1) \( (GS)_{XMT} = A_c \sin[\omega_{6137.85} t + \theta_{6137.85}] + A_b \sin[\omega_{6150} t + \theta_{6150}] \)

(2) \( (ATS-F)_{RCV} = A_c' \sin[\omega_{6137.85} t + \theta_{6137.85}] + A_b' \sin[\omega_{6150} t + \theta_{6150}] \)

(3) \( (ATS-F)_{VCO} = \cos[\omega_{25} t + \theta_{25}] \)

(4) \( (ATS-F)_{IF} = A'' \sin[\omega_{137.85} t + \theta_{137.85} + \Delta \theta_{6000}] + A'' \sin[\omega_{150} t + \theta_{150} + \Delta \theta_{6000}] \)

(5) \( (ATS-F)_{XMT} = B_c \sin[\omega_{2062.85} t + \theta_{2062.85} + \Delta \theta_{4075}] + B_b \sin[\omega_{2075} t + \theta_{2075} + \Delta \theta_{4075}] \)

(6) \( (NIM-E)_{RCV} = B_c' \sin[\omega_{2062.85} t + \theta_{2062.85} + \Delta \theta_{4075}] + B_b' \sin[\omega_{2075} t + \theta_{2075} + \Delta \theta_{4075}] \)

(7) \( (NIM-E)_{L0} = \sin[\omega_{37.55} t + \theta_{37.55}] \)

(8) \( (NIM-E)_{IF_1} = B'' \cos[\omega_{39.95} t + \theta_{2102.8} \pm \theta_{2062.85} - \Delta \theta_{4075}] \)

(9) \( (NIM-E)_{IF_2} = B_c' \cos[\omega_{2.4} t + \theta_{2065.25} - \theta_{2062.85} - \Delta \theta_{4075}] \)

(10) \( (NIM-E)_{XMT} = C_c \sin[\omega_{2253} t + \theta_{2253}] \pm C \sin[\omega_{2253+2.4} t + \theta_{2253} \pm \theta_{2065.25} - \theta_{2062.85} - \Delta \theta_{4075}] \)

(11) \( (ATS-F)_{RCV} = C_c' \sin[\omega_{2253} t + \theta_{2253}] \pm C' \sin[\omega_{2253+2.4} t + \theta_{2253} \pm \theta_{2065.25} - \theta_{2062.85} - \Delta \theta_{4075}] \)

(12) \( (ATS-F)_{IF} = C'' \sin[\omega_{153} t + \theta_{2253} \pm 1200] \pm C'' \sin[\omega_{153+2.4} t + \theta_{2253} \pm 1200] \)

(13) \( (ATS-F)_{XMT} = D_c \sin[\omega_{3953} t + \theta_{3953}] \pm D \sin[\omega_{3953+2.4} t + \theta_{3953} \pm 1700] \)

(14) \( (GS)_{RCV} = D_c' \sin[\omega_{3953} t + \theta_{3953} \pm 1700] \pm D' \sin[\omega_{3953+2.4} t + \theta_{3953} \pm 1700] \)
(d) A subscript 1, 2, 3 or 4 is attached to \((G)\) or \((N)\) to identify a propagation in the G.S./ATS-F link \((1)\), the ATS-F/NIMBUS-E link \((2)\), the NIMBUS-E/ATS-F link \((3)\) or the ATS-F/G.S. link \((4)\); e.g., \(\theta_{2253}\) is the phase of the NIMBUS-E local oscillator at 2253 MHz after (one-way) propagating through the NIMBUS-E/ATS-F/G.S. path.

(e) A hat is attached to the integer subscripts in (d) above to identify PLL estimates of the signal phases; e.g., \(\theta_{G}^{6150}\) is the phase of the ground oscillator at 6150 MHz after (one-way) propagating through the G.S./ATS-F path and extracted by a PLL filter.

For simplicity, open-loop filters are assumed to ideally reproduce the input phase fluctuations of their respective signals and group delay effects are neglected. Any other considerations such as the joint interrogation of the NIMBUS-E spacecraft by the ATS-F and STADAN-network channels, and the resultant joint PM of the NIMBUS-E carrier by these two signals, are not included here.

The transmitted and received RF signals in the G.S./ATS-F link are given by \((1)\) and \((2)\); the primed symbol on the received signal amplitude reflecting the net link attenuation, and the subscript 1 on the G reflecting the change in phase due to the propagation. The ATS-F PLL then tracks the beacon so that the VCO phase is a reproduction of the received beacon phase referred to 25 MHz; the VCO signal \((3)\) reflecting this phase estimation via the \(\wedge\) symbol. Of course, the VCO cannot distinguish between estimating the received beacon phase vs the received carrier phase, since these two signals are coherently derived from the same ground oscillator, and hence the VCO signal can be used to coherently transpond the carrier signal.
To this effect, the received signal (2) is first down-converted to the IF signal (4), and then up-converted to the RF signal (5) for NIMBUS-E transmission purposes. If a phase error

\[ \Delta \theta_{6150} = \theta_{6150} - \theta_{6150} \]

is introduced in the PLL estimate of the received signal phase, the IF carrier signal will exhibit a phase

\[ \theta_{6137.85} - 240^0_{25} = \theta_{137.85} + \theta_{6000} - \theta_{6000} \]

which represents a phase error of

\[ \Delta \theta_{6000} = \theta_{6000} - \theta_{6000} = \frac{6000}{6150} \Delta \theta_{6150} \]

or \( \frac{6000}{6150} \) times the loop locking error. The same phase error can be verified to be present in the IF beacon signal, relative to its nominal phase of \( \theta_{150} \). Similarly, the RF signals transmitted to NIMBUS-E can be verified to exhibit a phase error of

\[ \Delta \theta_{4075} = \theta_{4075} - \theta_{4075} = \frac{4075}{6150} \Delta \theta_{6150} \]

or \( \frac{4075}{6150} \) times the loop locking error, relative to the nominal phases of \( \theta_{2062.85} \) and \( \theta_{2075} \) of the retransmitted carrier and beacon signals.

The RF signals (6) received by NIMBUS-E are down-converted to the 2.4 MHz subcarrier via two heterodyning operations (8) and (9), representing a net mixing of the received signal with a higher 55 \( \times \) 37.55 = 2065.25 MHz reference derived from the NIMBUS-E local oscillator (7). The low beacon
signal component must be verified to not generate any important undesired IF components accompanying the subcarrier signal. The 2.4 MHz subcarrier then phase-modulates the 2253 MHz derived from the local oscillator, producing the carrier and subcarrier sidebands given in (10).

The ATS-F receives the NIMBUS-E signal (11) and side-steps it for G.S. transmission purposes, using references derived from the 25 MHz VCO and hence transferring the (scaled) loop locking errors to the carrier and sidebands. The net mixing with a 1700 MHz reference adds a phase \( \theta_{1700} = \theta_{1700} - \Delta \theta_{1700} \) thus introducing a phase error of \( \Delta \theta_{1700} \) or \( \frac{1700}{6} \times 1700 \) times the loop locking error into the signal components as shown in (13). The signal (14) is finally received at ground.

The proposed ATSR receiver modification has already been presented as Figs. 6-8, and their basic processing has been summarized in Figs. 9-11, where the carrier loop, subcarrier loop and doppler extractor operations are formulated. The carrier and subcarrier phases are given by

\[
\begin{align*}
\theta_c &= \theta_{2253} + \theta_{1700} - \Delta \theta_{1700} \\
\theta_{sc} &= \theta_{2065.25} - \theta_{2062.85} - \Delta \theta_{4075}
\end{align*}
\]

as previously derived in Table 1. The carrier-loop VCO phase \( \hat{\theta}_{v,c} \) estimates \( \theta_c - \delta_{3953} + \delta_{70} \), while the subcarrier-loop VCO phase \( \hat{\theta}_{v,sc} \) estimates \( -\theta_{30} + \delta_{29.4} \), and the resultant doppler signal phase is given by \( \theta_D = \frac{11}{12} \hat{\theta}_{v,c} + \hat{\theta}_{v,sc} - \frac{11}{12} \theta_{70} - \delta_{29.5} \). If we neglect oscillator instability effects, we then have \( \hat{\theta}_{v,c} = \hat{\theta}_c, \hat{\theta}_{v,sc} = -\hat{\theta}_{sc} \) and \( \theta_D = \frac{11}{12} \hat{\theta}_{v,c} + \hat{\theta}_{v,sc} = \frac{11}{12} \hat{\theta}_c - \hat{\theta}_{sc} \) (where the hats imply estimates) thus subtracting the weighted
carrier and subcarrier phase estimates, hence it provides the desired doppler compensation under ideal tracking conditions and the doppler signal fed to the range rate extractor is then given by $\frac{11}{12} D_{1700}^{(G)14} + D_{2062.85}^{(G)1234}$, which agrees with Fig. 3 using $L = \frac{1700}{6150}$.

The contribution of short-term oscillator instabilities may be investigated by treating the components of $\theta_c$ and $\theta_{sc}$ as caused by such phase fluctuations. If the doppler effects on these phase instabilities are neglected, the received carrier phase instabilities are due to the NIMBUS-E oscillator effects at 2253 MHz plus the ground oscillator effects at 1700 MHz estimated by the ATS-F loop. In turn, the received subcarrier phase instabilities are due to the NIMBUS-E oscillator effects at 2065.25 MHz, the ground oscillator effects at 2062.85 MHz, and the phase error in reproducing the 6150 MHz ground oscillator at the ATS-F loop but scaled by a factor.

It is of interest to compare these effects to those existing in the case where the NIMBUS-E tracking is done via an all-ground network without any ATS-F relay satellite involved. In the latter case, the received carrier would not exhibit the 1700 MHz ground-oscillator estimate contributions, which is introduced in the ATS-F relay of the NIMBUS-E signal, and the received subcarrier would not exhibit the $\frac{4075}{6150}$-scaled locking error to the 6150 MHz ground-oscillator, which is introduced in the ATS-F relay of the ground signal to NIMBUS-E.

The receiver carrier loop is wider than the ATS-F loop and the retransmitted phase instability estimate $\theta_{1700}^{(G)14}$ will be essentially reproduced by the former, which also filters the phase instabilities in the terms $\theta_{2253}^{(N)34}$. 
and \( \theta^{(G)}_{3953} \) within the carrier loop passband. The phase process \( \theta^{(N)}_{34} \) is uncorrelated to the others, and the phase process \( \theta^{(G)}_{1700} \) is not fully correlated to \( \theta^{(G)}_{70} \) due to the ATS-F loop filtering and the propagation delays. The situation is summarized in Fig.13, which presents the equivalent phase processing operations. In the case of an all-ground tracking network, the carrier loop would still filter the NIMBUS-E oscillator process at 2253 MHz plus a ground-oscillator process at 2253-70 MHz (rather than 3953-70 MHz), with no retransmitted, ground-oscillator, estimate contribution. The basic distinction between the all-ground and relay-satellite tracking networks is illustrated in Table 2, where \( T' \) is the NIMBUS-E/G.S. link delay in an all-ground operation. Indeed, in the case of no locking errors or propagation delays we have \( \hat{\theta}^{(G)}_{1700}(t-T_1-T_4) = \theta^{(G)}_{1700}(t) \), and no distinction would exist between the carrier loop inputs (the difference between \( T_3+T_4 \) vs \( T' \) in the NIMBUS-E contribution is irrelevant since the \( \theta^{(N)} \) process is assumed stationary and is independent to the \( \theta^{(G)} \) process).

The differential effect in Table 2 is thus the presence of the \( \theta^{(G)}_{1700}(t) - \hat{\theta}^{(G)}_{1700}(t-T_1-T_4) \) term in the relay-satellite network, vs its zero value in the all-ground network.

**TABLE 2**

<table>
<thead>
<tr>
<th></th>
<th>Relay-Satellite</th>
<th>All-Ground</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) NIMBUS-E Osc.</td>
<td>( \theta^{(N)}_{2253}(t-T_3-T_4) )</td>
<td>( \theta^{(N)}_{2253}(t-T_2') )</td>
</tr>
<tr>
<td>(2) G.S. RCVR RF Osc.</td>
<td>( \theta^{(G)}<em>{2253}(t)+\theta^{(G)}</em>{1700}(t)-\theta^{(G)}_{70}(t) )</td>
<td>( \theta^{(G)}<em>{2253}(t)-\theta^{(G)}</em>{70}(t) )</td>
</tr>
<tr>
<td>(3) G.S. Retransmit. Osc.</td>
<td>( \hat{\theta}^{(G)}_{1700}(t-T_1-T_4) )</td>
<td>none</td>
</tr>
<tr>
<td>(4) Carrier Loop input</td>
<td>(1)-(2)+(3)</td>
<td>(1)-(2)</td>
</tr>
</tbody>
</table>
Fig. 13  Summary of Phase Processing Propagation
The ATS-F estimation error at 1700 MHz should be nil since the loop locking error at 6150 MHz should itself be small; i.e., the loop estimates the ground oscillator phase at 6150 MHz but scales the estimate by $\frac{1700}{6150}$ for side-stepping the downlink carrier thus reducing locking errors by a proportionate amount. The predominant distinction in Table 2 has then to do with the difference between the propagation time and the effective correlation time of the ground-oscillator phase instability process (the effective width of its autocorrelation function). If this difference is small the differential effect is also small and the relay-satellite and all-ground operations will essentially exhibit the same instability process at the carrier loop input.

In the worst case of total uncorrelation between the local and retransmitted phase process at 1700 MHz, the relay-satellite case compares to the all-ground case as follows insofar as the carrier loop is concerned: (a) both have an input process $\theta^{(N)}_{2253}(t)$ independent of the others, (b) the relay-satellite case has an input process $\theta^{(G)}_{3883}(t)$ while the all-ground case has only $\theta^{(G)}_{2183}(t)$, (c) the relay-satellite case has an extra independent process $\theta_{1700}(t)$ (for mean-square evaluation purposes) with spectral density identical to the $\theta^{(G)}_{1700}(t)$ process (excluding the ATS-F loop effect) or a filtered version of such a process (including the loop effect). The net result will be an increase in mean-square carrier loop locking errors and carrier VCO phase instabilities in the relay-satellite case, but the differential effect should not be critical since the 1700 MHz ground-oscillator contribution should be secondary to the 2253 MHz NIMBUS-E oscillator contribution and $\frac{3883}{2183} < 2$. 
With reference to the subcarrier loop input, the basic distinction between the two cases is the presence of the $\Delta \hat{\theta}_{4075}^{(G)}(t - \Sigma T_i)$ term in the relay-satellite case, which should be a small effect since the ATS-F loop locking error at 6150 MHz should itself be small. The NIMBUS-E and ground-oscillator processes $\hat{\theta}^{(N)}_{2065.25}(t-T_3-T_4)$ and $\hat{\theta}^{(G)}_{2062.85}(t-\Sigma T_i)$ are independent, so that their mean-square effects are additive and should predominate over the scaled locking error contribution.

With reference to the doppler signal extraction in the relay-satellite mode, the following phase process contributions will be fed to the range rate extractor (see Figs. 9-11):

Carrier loop VCO estimate
scaled by 11/12

\[
\frac{11}{12} \left[ \hat{\theta}^{(N)}_{2253}(t-T_3-T_4) + \hat{\theta}^{(G)}_{1700}(t-T_1-T_4) - \hat{\theta}^{(G)}_{3953}(t) + \hat{\theta}^{(G)}_{2.4}(t) \right]
\]

Subcarrier loop VCO estimate

\[
- \left[ \hat{\theta}^{(N)}_{2065.25}(t-T_3-T_4) - \hat{\theta}^{(G)}_{2062.85}(t-\Sigma T_i) - \Delta \hat{\theta}^{(G)}_{4075}(t-\Sigma T_i) \right] + \left[ \hat{\theta}^{(G)}_{2.4}(t) + \hat{\theta}^{(G)}_{29.5}(t) \right]
\]

Local reference processes

\[
- \left[ \frac{11}{12} \hat{\theta}^{(G)}_{70}(t) + \hat{\theta}^{(G)}_{29.5}(t) \right]
\]

If we assume identical carrier and subcarrier loop designs their estimates of the NIMBUS-E oscillator phase process will be proportionate and the NIMBUS-E terms cancel. Similarly, their estimates of the local (not retransmitted) ground-oscillator terms will be proportionate, and the contribution from these terms plus the local reference processes is
whereas the contribution from the retransmitted ground-oscillator processes is

\[ -\frac{11}{12} \hat{\theta}_{3953}(t) + \Delta \hat{\theta}_{70}(t) + \left[ \hat{\theta}_{0.5}(t) - \Delta \hat{\theta}_{30}(t) \right] + \hat{\theta}_{2.4}(t) \]

An equivalent all-ground operation would only differ from the previous formulation in that the 1700 MHz term would be \( \frac{11}{12} \hat{\theta}_{1700}(t) \) instead, hence fully correlated to the local reference (not retransmitted) processes and cancelling some of the \( \hat{\theta}_{3953}(t) \) reducing it to a \( \hat{\theta}_{2253}(t) \) contribution as happened in the carrier loop. If we assume nil propagation time relative to the effective correlation time of the filtered processes, and neglect locking errors, the phase process fed to the range rate extractor is of course identical for both relay-satellite and all-ground modes and given by \( \hat{\theta}(t) + \left[ \hat{\theta}(t) - \hat{\theta}(t) \right] \). If we assume the propagation times much larger than the effective correlation times, the predominant terms in the relay-satellite mode are \( -\frac{11}{12} \hat{\theta}_{3953}(t) \) while the predominant terms in the all-ground mode are \( -\frac{11}{12} \hat{\theta}_{2253}(t) \) and are uncorrelated. Under worst conditions for the relay-satellite mode, its second-bracket terms will be fully correlated and represent a 1.7 increase relative to the corresponding term in the all-ground mode, whereas the first-bracket term always represents a 1.7 increase relative to its image under any conditions. Hence, this worst relay-satellite conditions will increase the oscillator phase noise fed to the range rate extractor by 1.7 relative to the all-ground mode, and thus increase the range rate error due to additive oscillator noise by the same factor. The assumption of identical carrier and subcarrier loop designs
(used to cancel NIMBUS-E oscillator noise effects) is reasonable since third-order Mallinckrodt loop designs are available in the existing ATSR-system carrier loop and the GRARR-system subcarrier loops, and this same design is at present being considered for the ATSR modification.

2.4 Effects of Uplink and Downlink Thermal Noise

In the previous section we discussed the increase in oscillator phase noise errors directly caused by the relay satellite involved in the TDRS configuration, and found retransmitted oscillator noise to be an important contributor to these effects. An analogous problem exists with thermal noise which causes signal level degradation and extra additive noise as a direct consequence of retransmitted uplink noise effects. In past studies of tracking systems based on an all-ground network, such effects have been properly neglected due to the high SNR's existing in the uplink, but the presence of a satellite-user link (ATS-F/NIMBUS-E) in the TDRS configuration represents a low SNR uplink that must be accounted for. The limiter at NIMBUS-E may cause signal suppression effects and the resultant PM/PM downlink signal will exhibit not only attenuated signal components but retransmitted PM noise as well.

The nominal/worst overall signal-to-noise density levels are summarized in Table 3, along with the corresponding level reduction in the signal components due to the PM and PM/PM operations. The downlink signal levels include the effects of intermodulation terms present in the uplink signal as noted, but no limiter suppression effects are included in the table. The computation details are summarized in Appendix 1 for reference purposes.
TABLE 3

(A) Nominal/Worst Signal-to-Noise Density Levels

<table>
<thead>
<tr>
<th></th>
<th>G.S./ATS-F</th>
<th>ATS-F/NIMBUS-E</th>
<th>NIMBUS-E/ATS-F</th>
<th>ATS-F/G.S.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>97.3/62.1</td>
<td>63.7/57.7</td>
<td>59.8/53.0</td>
<td>87.5/55.0</td>
</tr>
</tbody>
</table>

(B) Uplink Signal Component Levels in dB Down from Available Power Due to PM Effect

<table>
<thead>
<tr>
<th></th>
<th>2 tones at 1.2 rad each</th>
<th>2 tones at 1.2 and 0.5 rad</th>
<th>1 tone at 1.2 rad</th>
<th>0.5 rad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carrier</td>
<td>-6.93</td>
<td>-4.02</td>
<td>-3.46</td>
<td>-0.55</td>
</tr>
<tr>
<td>Major Tone Sideband</td>
<td>-9.51</td>
<td>-6.60</td>
<td>-6.05</td>
<td>--</td>
</tr>
<tr>
<td>Minor Tone Sideband</td>
<td>-9.51</td>
<td>-15.78</td>
<td>--</td>
<td>-12.31</td>
</tr>
</tbody>
</table>

(C) Downlink Signal Component Levels in dB Down from Available Power Due to PM/PM Effect (no limiter suppression)

<table>
<thead>
<tr>
<th></th>
<th>2 UL tones at 1.2 rad each</th>
<th>2 UL tones at 1.2 and 0.5 rad</th>
<th>1 UL tone at 1.2 rad</th>
<th>0.5 rad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subcarrier Index</td>
<td>0.676</td>
<td>0.945</td>
<td>1.007</td>
<td>1.408</td>
</tr>
<tr>
<td>Major Tone Index</td>
<td>0.502</td>
<td>0.702</td>
<td>0.747</td>
<td>--</td>
</tr>
<tr>
<td>Minor Tone Index</td>
<td>0.502</td>
<td>0.244</td>
<td>--</td>
<td>0.363</td>
</tr>
<tr>
<td>Carrier</td>
<td>-4.87</td>
<td>-4.52</td>
<td>-4.86</td>
<td>-5.59</td>
</tr>
<tr>
<td>Subcarrier Sideband</td>
<td>-13.79</td>
<td>-9.96</td>
<td>-9.67</td>
<td>-5.87</td>
</tr>
<tr>
<td>Major Tone Sideband</td>
<td>-16.63</td>
<td>-13.08</td>
<td>-12.73</td>
<td>--</td>
</tr>
<tr>
<td>Minor Tone Sideband</td>
<td>-16.63</td>
<td>-22.66</td>
<td>--</td>
<td>-20.22</td>
</tr>
</tbody>
</table>

Note: The effect of uplink IM terms is relevant only in the first column and has been included (an extra 1.65 dB reduction from the no IM case).

In the discussion that follows we restrict ourselves to the 1.2-radian, single-tone, modulation case which characterizes the high-accuracy tracking mode of interest after the acquisition and ambiguity-resolving operations have occurred. It should be noted that the overall SNR in a 40 MHz IF bandwidth is 21.3 dB nominal and -13.9 dB worst (the carrier and tone-sidebands
being respectively 3.5 and 6.1 dB down from both of these values, and the beacon signal much lower when all signals are present), so that a narrower effective IF bandwidth must be realized in the beacon loop if threshold effects are to be disregarded.

2.4.1 ATS-F Transponder Processing

We now consider the acquisition and tracking behavior of the PLL extracting the beacon signal at the ATS-F transponder. We shall attempt a single mode design where the same loop parameters are used for acquisition and tracking operations, assuming above-threshold conditions exist.

We assume a conventional 2nd-order loop design with finite gain $K$ sec$^{-1}$, damping factor $\zeta$ and resonance frequency $\omega_r$ rad/sec, so that the (two-sided) equivalent noise bandwidth is given by $B_n = \frac{1+4\zeta^2}{4\zeta} \omega_r$ Hz for $K >> \frac{\omega_r}{2\zeta}$, and we can use $B_n$ (in Hz) = $\omega_r$ (in rad/sec) for $0.5 \leq \zeta \leq 0.7$ and $K >> \omega_r$ which represent conventional designs.

With reference to doppler and thermal noise effects only, the design compromise involved is specified by the following relations:

\[
pull-in\,range = \frac{0.7}{\pi} \sqrt{KB_n} \text{ Hz}
\]

\[
pull-in\,time\,for\,D-Hz\,doppler\,shift = \begin{cases} \frac{40D^2}{B_n^3} \text{ sec for } B_n < D \\ \frac{10}{B_n} \text{ sec for } B_n > D \end{cases}
\]

steady-state error to D-Hz doppler shift = \frac{360D}{K} \text{ degrees}

steady-state error to D-Hz/sec doppler rate = \frac{360D}{B_n^2} \left(1 + \frac{B_n^2t}{K}\right) \text{ degrees}

rms thermal noise error = \frac{40.5}{\sqrt{(SNR)B_n}} \text{ degrees}
A maximum one-way doppler shift $D = 20$ Hz and doppler rate $\dot{D} = 0.02$ Hz/s referred to a 6000 MHz frequency correspond to the specified maximum relative velocity of 1 m/s and acceleration of 1 mm/s in the G.S./ATS-F link. The doppler shift error can then be maintained below 1° for $K>7500$ sec$^{-1}$ which is reasonable. The doppler rate error will be evaluated neglecting the time-dependent term as usual (the constant doppler rate duration thus assumed to be short enough relative to the tracking time). The design compromise between dynamic response (fast acquisition, small doppler rate error) and noise rejection (small thermal noise error) is shown in Table 4 for nominal/worst beacon SNR conditions. The beacon signal is assumed to be at a level 20 dB below the carrier signal in the tabulated values, and the limiter effect will be discussed later.

The use of noise bandwidths of 10 to 100 Hz yield a nil dynamic error and maintain the acquisition time in the 16 to 0.1 sec duration. A 20 dB loop SNR corresponds to an rms noise error of 4.05 degrees so that even the wider loop design and worst SNR condition yields thermal noise errors below 5 degrees rms. It should be noted that the worst conditions cited refer to operation in the extreme sidelobes of the horn antenna pattern, and hence cover other possibilities such as beam-edge operation under boresight pointing conditions. Finally, a loop gain $K > \frac{8000}{B_n}$ sec$^{-1}$ will suffice from pull-in range considerations, hence secondary to the loop gain requirement for doppler shift tracking purposes.
TABLE 4

<table>
<thead>
<tr>
<th>$B_n$ (Hz)</th>
<th>Tacq(s)</th>
<th>Doppler Rate Error (deg peak)</th>
<th>Nom/Wst (SNR) $B_n$ for 1-tone (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16000</td>
<td>10</td>
<td>73.8/38.6</td>
</tr>
<tr>
<td>2</td>
<td>2000</td>
<td>2.5</td>
<td>70.8/35.6</td>
</tr>
<tr>
<td>5</td>
<td>128</td>
<td>0.4</td>
<td>66.8/31.6</td>
</tr>
<tr>
<td>10</td>
<td>16</td>
<td>0.1</td>
<td>63.8/28.6</td>
</tr>
<tr>
<td>20</td>
<td>0.5-2</td>
<td>nil</td>
<td>60.8/25.6</td>
</tr>
<tr>
<td>50</td>
<td>0.2</td>
<td>nil</td>
<td>56.8/21.6</td>
</tr>
<tr>
<td>100</td>
<td>0.1</td>
<td>nil</td>
<td>53.8/18.6</td>
</tr>
</tbody>
</table>

The pre-detection limiter maintains a constant signal level at the loop input in the absence of noise. When noise is present, the composite signal fed to the loop exhibits level variations dependent on the input SNR and governed by the limiter, but not commensurable with the inverse-range-squared variation that would exist otherwise. The limiter causes a smooth variation in the loop gain and bandwidth, since these parameters depend on the signal level through the phase detector sensitivity. As the limiter input SNR increases from its minimum value at maximum range, the signal level at the loop input exhibits a smoother increase due to the limiter effect, and the loop gain and bandwidth exhibit correspondingly smooth increases from their design values at minimum SNR conditions. The doppler dynamic and oscillator noise errors become smaller relative to those pertaining to minimum-SNR design parameters, and the rms noise error does not exceed its minimum-SNR counterpart since the bandwidth expansion is compensated by the signal level increase.
The previous discussion has essentially assumed that the power density spectrum of the additive noise maintains its flat characteristic, and the limiter effect is restricted to signal-to-noise suppression without any major modification in the spectral shape. The actual output spectra of a limiter whose input is a (modulated or unmodulated) signal in noise consists of various spectral terms representative of signal, noise, signal x signal, signal x noise and noise x noise contributions, and the post-limiting filter restricts the problem to the consideration of only those spectral terms present in the desired signal passband. A flat output noise spectrum will be the predominant term arising from a flat input noise spectrum, and the remaining signal x signal, signal x noise and noise x noise contributions in the passband can be verified to be secondary effects for the high SNR conditions prevailing in the G.S./ATS-F link.

However, the limiter can cause some beacon signal suppression when the carrier signal is present since the former is transmitted at a much lower level. For the case of two additive independent signals in noise, Jones ("Hard Limiting of Two Signals in Random Noise," IEEE Trans. on I.T., Jan. 1963, p 34) shows that if one signal predominates over the other (S_1 >> S_2) and the noise (S_{NR} >> 1), then the parameters

\[ L_j = \frac{S_{jNR}}{(S_{jNR})_{in}}, \quad K_2 = \frac{(S_2/S_1)_{out}}{(S_2/S_1)_{in}} = \frac{L_2}{L_1} \]

specifying the power transfer of the limiter are given by \( L_1 = 2, L_2 = 0.5 \) and \( K_2 = 0.25 \); i.e., the larger signal SNR increases by 3 dB at the expense of the smaller signal SNR. The presence of modulation on the stronger (carrier) signal in our application may be accounted by superposing the noise and modulation effects separately (i.e., reducing the carrier and ranging tone sideband by the conventional Bessel-function terms) as a first approximation.
A more detailed analysis becomes very complex not mainly because of the modulation, but because of the correlation between the phase angles of the carrier, modulation and beacon signals, since they are all derived from the same ground oscillator and exhibit proportionate doppler effects on the G.S./ATS-F link. The presence of such correlation introduces otherwise-absent cross-terms in the limiter spectral analyses available up to date (which exploit statistical averaging over independent phase angles) and the problem is yet to be solved. At the present state of the program it seems more appropriate to assume the beacon signal at some arbitrary low level (say, 15-20 dB) below the carrier at the limiter output and verify it has a negligible effect on system performance, since the distinction between say 10-20 dB below the transmitted carrier is academic from transmitter power demand considerations.

It should be mentioned that a signaling procedure under consideration consists on first sending only the beacon signal with no carrier present, in which case a large beacon SNR prevails during the acquisition process. The carrier with modulation is then introduced after the beacon is acquired, and the SNR level of the latter is reduced so that the previous comments then apply. It can thus be concluded that acquisition and tracking can be accommodated with a single design in the ATS-F loop from doppler and thermal noise effects provided above-threshold input SNR's are provided by proper IF filtering. With reference to oscillator noise locking errors, Sydnor et al ("Frequency Stability Requirements for Space Communications and Tracking Systems," Proc. IEEE, Feb. 1966, p 231) cite 0.1-10° rms errors depending on the oscillator quality for $B_n=10-100$ Hz and S-band loops, so these bandwidths should be adequate in our C-band application.
Some of the thermal phase noise present in the VCO signal will be retransmitted to NIMBUS-E when generating the S-band carrier signal, and also to the ATSR station in the heterodyning of the downlink carrier received from NIMBUS-E. The phase noise retransmitted to NIMBUS-E consists of the ATS-F loop noise error scaled by \( \frac{4075}{6150} \) (e.g., see Fig. 13) and will be induced on the subcarrier signal modulating the downlink carrier. It will eventually be filtered and extracted along with the subcarrier phase information by the ATSR receiver subcarrier loop and it will appear as filtered phase noise on the range rate signal. The phase noise added when heterodyning the downlink carrier consists of the ATS-F loop noise error scaled by \( \frac{1700}{6150} \) and will be induced in the downlink carrier. It is filtered and extracted by the ATSR carrier loop, and \( \frac{11}{12} \)-scaled to contribute phase noise to the range rate signal. These two phase noise contributions have opposite signs since the doppler extractor yields \( \frac{11}{12} \) \( e_{c}^{-\theta_{sc}} \), and may be to some extent correlated depending on the propagation delays and the carrier/subcarrier loop designs. In any case, they will be secondary effects to other downlink noise contributions yet to be discussed, namely downlink thermal noise will predominate over retransmitted uplink noise effects so the former will govern the phase noise contribution to the range rate signal. There is no contribution of retransmitted noise to the range signal since such noise is added to the downlink carrier and subcarrier as phase noise, and the range tone only suffers amplitude attenuation by the cosine of the locking errors in the carrier and subcarrier loops.

2.4.2 NIMBUS-E Transponder Processing

The signal received at NIMBUS-E is heterodyned to the subcarrier IF with the spacecraft local oscillator and then used to generate the downlink
PM/PM signal based on GRARR principles. The small S-band beacon signal relayed along with the PM carrier signal is assumed to be filtered in the subcarrier generation process. The G.S./ATS-F uplink additive noise around the ranging signal is also retransmitted towards NIMBUS-E, but is negligible compared to the ATS-F/NIMBUS-E uplink additive noise (97.3/62.1 vs 63.7/57.7 dB-Hz nominal/worst). The phase noise appearing on the PM carrier and originating in the ATS-F beacon loop will be processed as phase modulation and will not govern the IF SNR at NIMBUS-E. Such SNR is thus mainly due to the ATS-F/NIMBUS-E link noise and corresponds to 6.3 dB nominal and 0.3 dB worst assuming a 550 kHz IF bandwidth.

The signal + noise is bandpass-limited at NIMBUS-E and the resultant output phase-modulates the NIMBUS local oscillator to generate the PM/PM signal. The main aspects to be considered are: (a) the signal suppression by noise and the corresponding change in signal mod index from its nominal value, (b) the variation in the noise spectral density shape at the limiter output relative to the input, (c) the resultant effects in the output spectrum of the phase modulator. These issues have been properly neglected in past analyses of GRARR-transponder processing where no relay satellite existed between the ground station and user spacecraft, which permitted high SNR and nil retransmitted noise conditions. However, these effects must now be investigated in relay applications where a moderate-to-low input SNR prevails at the user.

The problem has been studied by T. J. Grenchik (Analysis of Signal and Noise Turnaround in the GRARR Transponder, NASA/GSFC X-551-69-323, Aug. 1969) under the following specifications: (a) the input signal is unmodulated,
(b) the noise spectral density has a Gaussian characteristic at the limiter input, (c) this characteristic is preserved at the limiter output and the limiter effect is restricted to signal vs noise level suppression as established from a constant-output-power requirement. Computer solutions were developed under these conditions, and they illustrate that the carrier level at the phase modulator output remains essentially constant with the limiter input SNR (the rationale being that either the signal or the noise, whichever prevails, drives the modulator) for a nominal mod index of 1.5, while the subcarrier output level only remains constant at high input SNR conditions and exhibits a nearly linear reduction with decreasing, low, input SNR. The noise spectral density at the demodulator output exhibits spectral spread relative to the input at low SNR conditions, but no closed-form expression is available.

With reference to signal-to-noise densities, the carrier exhibits a nearly linear reduction in this parameter with decreasing, high input SNR, but settles down to a constant at low input SNR's. The subcarrier exhibits an essentially linear reduction in this parameter with decreasing input SNR, regardless of the value of the latter. It should be understood that these signal-to-noise densities are evaluated at the carrier and subcarrier frequencies and represent significant parameters only if the receiver tracking loops at the ground station are narrow enough, so that only the flat portion of the noise spectral density around the carrier and subcarrier frequencies need be considered in the loop error analysis.

The assumption of the Gaussian characteristic for the noise spectral density at the limiter input may be more reasonable than a rectangular spectrum from BPF sharp-cutoff considerations, but it suffers from its fast decay off
the center-frequency and an excessive Gaussian-filter width may be required to maintain a flat passband around the center for signal filtering in the presence of doppler shifts. For example, a Butterworth characteristic would be more advantageous since it can provide both a flat central passband and non-ideal sharp cutoff.

However, the Gaussian shape for the noise spectral density at the limiter output may still represent a realistic approximation. Such spectrum is known to consist of noise x noise and signal x noise, and signal x signal spectral conditions which are derived by the successive spectral convolution terms characteristic of the limiter effect. Under high input SNR conditions, the input noise spectrum is the predominant term and the limiter preserves the noise spectral density shape (this was the case for the ATS-F spacecraft limiter), but for moderate-to-low SNR's various terms need be considered and the limiter alters the noise spectral density shape. In particular, if the input noise density is assumed to be rectangular, the convolutions will successively yield rectangular, triangular, parabolic-decay, etc., patterns which are scaled and added (and filtered) to generate the noise spectral density at the limiter output. The resultant shape will have bell-like features, and a Gaussian approximation may indeed be realistic (and the BPF's need not be assumed to be Gaussian but rectangular or sharp-cutoff Butterworth will suffice).

The case where a range tone phase-modulates the subcarrier has also been formulated by the author in unpublished work and his computer simulation results for the case of interest are shown in Fig. 14. The subcarrier curve represents the algebraic sum of the two sideband powers, and the range tone
carrier signal power, $S_c$

subcarrier signal power $S_{SC}$

Range tone signal power $S_{RT}$

$B_{XPDR} = 1.5 \quad B_{RT} = 1.2$

**Fig. 14** Limiter-Phase Modulator Suppression Effect
curve represents the algebraic sum of the four sideband powers. Note that the results of Appendix 1 hold for high SNR conditions: the carrier is 4.9 dB down, each subcarrier sideband is 9.7 dB down (hence their sum is 6.7 dB down) and each tone sideband is 12.7 dB down (hence their sum is 6.7 dB down), which agrees with the curves. The same qualitative results previously observed in the absence of modulating tone still hold, namely an essentially constant carrier and a linear reduction in sideband power level with decreasing input SNR beyond 0 dB. At the 6.3/0.3 dB nominal/worst input SNR of interest, the curves indicate reductions of 0/0, 1/3 and 1/3 dB for the carrier, subcarrier and tone power respectively. This also corresponds to a 1/3 dB reduction per sideband at the 6.3/0.3 dB nominal/worst input SNR, as illustrated in Fig. 15.

2.4.3 Downlink Signal-to-Noise Characteristics

The signal-to-noise density in the NIMBUS-E/ATS-F downlink is 59.8/53.0 dB-Hz nominal/worst. If the uplink noise (G.S./ATS-F/NIMBUS-E link) effects are temporarily ignored, then the downlink signal components will exhibit the signal-to-noise density levels shown in Table 5 below.

<table>
<thead>
<tr>
<th></th>
<th>Carrier</th>
<th>Subcarrier Sideband</th>
<th>Tone Sideband</th>
</tr>
</thead>
<tbody>
<tr>
<td>PM/PM effect</td>
<td>-4.9/-4.9</td>
<td>-9.7/-9.7</td>
<td>-12.7/-12.7 dB nominal/worst</td>
</tr>
<tr>
<td>Limiter effect</td>
<td>0/0</td>
<td>-1/-3</td>
<td>-1/-3 dB nominal/worst</td>
</tr>
<tr>
<td>Net effect</td>
<td>-4.9/-4.9</td>
<td>-10.7/-12.7</td>
<td>-13.7/-15.7 dB nominal/worst</td>
</tr>
<tr>
<td>Component-to-Noise density</td>
<td>54.9/48.1</td>
<td>49.1/40.3</td>
<td>46.1/37.3 dB nominal/worst</td>
</tr>
</tbody>
</table>
(a) Case of \((\text{SNR})_\text{in} = \infty\)

- Carrier: -4.9 dB down
- Subcarrier: -9.7 dB down
- Tone: -12.7 dB down

\[\text{carrier: } -4.9 \text{ dB down} \]
\[\text{subcarrier: } -9.7 + 3 = -6.7 \text{ dB down} \]
\[\text{tone: } -12.7 + 6 = -6.7 \text{ dB down} \]

(b) Case of \((\text{SNR})_\text{in} = 6.3 \text{ dB}\)

- Carrier: -4.9 dB down
- Subcarrier: -10.7 dB down
- Tone: -13.7 dB down

\[\text{carrier: } -4.9 \text{ dB down} \]
\[\text{subcarrier: } -10.7 + 3 = -7.7 \text{ dB down} \]
\[\text{tone: } -13.7 + 6 = -7.7 \text{ dB down} \]

(c) Case of \((\text{SNR})_\text{in} = 0.3 \text{ dB}\)

- Carrier: -4.9 dB down
- Subcarrier: -12.7 dB down
- Tone: -15.7 dB down

\[\text{carrier: } -4.9 \text{ dB down} \]
\[\text{subcarrier: } -12.7 + 3 = -9.7 \text{ dB down} \]
\[\text{tone: } -15.7 + 6 = -9.7 \text{ dB down} \]

**FIGURE 15** SIDEBAND VS SUM OF SIDEBAND DEGRADATION
In turn, the signal-to-noise density in the ATS-F/G.S. downlink is 87.5/55.0 dB-Hz nominal/worst. The ATS-F side-steps the NIMBUS-E signal so that the output signal sidebands will exhibit the same level reduction tabulated above as the net effect, and it is thus evident that the NIMBUS-E/ATS-F downlink noise predominates over the ATS-F/G.S. downlink noise contribution.

Of course, we must also account for the retransmitted uplink noise which has various contributions. The G.S./ATS-F uplink noise around the carrier retransmitted to NIMBUS-E has already been neglected relative to the ATS-F/NIMBUS-E uplink noise. The latter appears as an additive signal phase-modulating the NIMBUS-E downlink carrier and its spectral characteristics at the NIMBUS-E output signal do not have a compact formulation. If we use the limiter suppression results cited in the last section as a guideline (based on Gaussian noise at the limiter output) plus assume that the presence of the tone modulation only reduces the signal component levels but preserves the noise power density spectral shape at the modulator output, then the retransmitted subcarrier would exhibit a signal-to-retransmitted-noise density of the order of 65/55 dB-Hz (nominal/worst) per sideband. If we compare these values to the 49.1/40.3 dB-Hz due to the NIMBUS-E/ATS-F downlink noise shown in Table 5, then the latter also predominates over the ATS-F/NIMBUS-E retransmitted uplink noise. It should be acknowledged that the retransmitted uplink noise sidebands will eventually fold coherently at the ground receiver, whereas the downlink additive noise sidebands fold incoherently and do not introduce an extra 3 dB degradation, but this amount will not alter the aforesaid conclusion of NIMBUS-E/ATS-F downlink noise predominance over ATS-F/NIMBUS-E retransmitted uplink noise. In summary,
the uplink additive noise is neglected when compared to the downlink additive noise, but its signal-suppression effects at the NIMBUS-E transponder are included (see Table 5) to evaluate downlink signal levels.

Finally, we consider the retransmitted phase noise which originates in the ATS-F loop and appears both as subcarrier phase noise at the NIMBUS-E output and as carrier phase noise at the ATS-F output; (both effects induced in the side-stepping operations). Both these phase noises represent scaled replicas of the ATS-F loop noise error (see section 2.4.1) yet they may be uncorrelated due to the round-trip propagation delay in the ATS-F/NIMBUS-E baseline (e.g., see Fig. 13). The G.S./ATS-F uplink signal-to-noise density level is 97.3/62.1 dB-Hz nominal/worst so that if we assume the beacon signal to be 20 dB below the carrier signal then the beacon signal-to-noise density will be 73.8/38.6 dB-Hz nominal/worst. The carrier phase noise is the ATS-F loop noise scaled by 1700/6150 which essentially increases the previous values by $(3.6)^2=10.8$ dB to 84.6/49.4 dB-Hz nominal/worst. The subcarrier phase noise is the ATS-F loop noise scaled by 4075/6150 which essentially increases the previous values by $(1.5)^2=3.4$ dB to 77.2/42.0 dB-Hz nominal/worst. Hence the retransmitted carrier and subcarrier phase noise is negligible relative to the downlink additive noise under nominal conditions, and becomes commensurable under worst conditions for a beacon signal 20 dB below the carrier component at the ATS-F limiter output. Of course, we can have a larger beacon level relative to the carrier (say 15 dB down) without any real transmitter power demands, and thus we can make the beacon-loop retransmitted phase noise to be also negligible under the worst conditions. It should be noted when selecting such a level that the downlink additive noise
will be reduced (NSR-wise) by 3 dB due to folding effects at the receiver loops while the retransmitted phase noise does not exhibit this effect.

In summary, the NIMBUS-E/ATS-F downlink additive noise predominates over all other additive noise effects and can be made to predominate over all retransmitted phase noise effects without requiring a large beacon signal relative to the carrier component. Even a beacon 20 dB below the carrier will suffice under nominal conditions and will yield only commensurable retransmitted noise effects (relative to the aforesaid additive noise) under worst conditions. Such worst conditions refer to a G.S./ATS-F link 30 dB below nominal and 20 dB of those are due to operation in the extreme sidelobes of the horn pattern during ATS-F slewing maneuvers (see ATS-F/NIMBUS-E Data Relay Experiment, Technical Summary, NASA/GSFC DRI-0000, Revision B, Feb. 3, 1969). On this basis, we will concentrate on the NIMBUS-E/ATS-F downlink additive noise as the main contributor to thermal noise effects and error performance limitations. It should be emphasized that the NIMBUS-E/ATS-F downlink component-to-noise density is obtained by including the signal suppression caused by the low SNR at NIMBUS-E (see Table 5). Hence, the worst-condition dB-Hz is a superposition of two worst conditions, namely worst ATS-F/NIMBUS-E uplink signal-to-noise density (to yield maximum uplink signal suppression) and worst NIMBUS-E/ATS-F downlink signal-to-noise density (to yield maximum downlink noise addition).

2.5 Range and Range Rate Error Analysis

The 3953 MHz PM/PM signal received by the ATSR station is converted with a local reference coherent to the transmitted signals to yield the 70 MHz IF signal fed to the carrier and subcarrier loops. The received
carrier and subcarrier doppler shifts are given by (see Fig. 3)

\[ D_c = 2 \frac{L D_T + N(D_o + D'_o)}{6150} \quad D'_T = 50 \text{ kHz max} \]

\[ D_{sc} = \frac{M(D_o + D'_0)}{K} \left(D_T + D'_T\right) = \left(\frac{2065.25}{6150} - 2 \frac{2062.85}{6150}\right)D'_T = -45 \text{ kHz max} \]

where the approximation neglects the G.S./ATS-F link doppler. Hence, loop gains of \( 2 \times 10^6 \) (seconds)\(^{-1} \) or higher will maintain the locking error within 10°. The gain requirements vary in inverse proportion to the error magnitude so even tighter bounds still represent state-of-the-art design. The maximum doppler rate conditions are in turn \( \dot{D}_c = 75 \text{ Hz/s} \) and \( \dot{D}_{sc} = -65 \text{ Hz/s max} \), so that (two-sided) loop noise bandwidths of 55 Hz or wider will maintain the locking error within 10°, assuming a conventional second-order loop design for simplicity. A tighter bound will be accompanied by a square-root proportionate increase in bandwidth; e.g., a 1° error requires 175 Hz. In what follows we will assume a loop noise bandwidth of 200 Hz, which is provided in the present ATSR receiver design.

The loop SNR's and noise locking errors corresponding to a 200 Hz noise bandwidth can now be obtained by assuming the (signal-suppressed) NIMBUS-E/ATS-F link SNR conditions, using the values in Table 5 and the results of the last section:

\[ (\text{SNR})_C = 31.9/25.1 \text{ dB nominal/worst} \]
\[ (\text{SNR})_{sc} = 29.1/20.3 \text{ dB nominal/worst} \]
\[ \sigma_c = 1.03/2.25 \text{ deg rms nominal/worst} \]
\[ \sigma_{sc} = 1.45/3.91 \text{ deg rms nominal/worst} \]

where the 3 dB SNR improvement due to the subcarrier-sideband folding effect has been included.
These errors are adequate for locking purposes and their corresponding range rate accuracy limitations must now be established. The doppler signal is extracted by combining the $\frac{11}{12}$-scaled carrier and the subcarrier VCO signals, and the loop phase noise will thus be fed to the range rate extractor. The latter is based on cycle-counting principles for average doppler and range rate measurement, and the phase noise causes zero-crossing time uncertainties that can be directly formulated as doppler and range rate resolution limitations.

The rms error contribution of phase noise in a doppler measurement based on cycle-counting techniques is given by $\sigma_D = \frac{\sigma_{\Delta \phi}}{2\pi T}$ Hz, where $\sigma_{\Delta \phi}$ is the rms phase-error difference in radians at the beginning and end of the counting interval of T-sec duration. The phase error contribution due to thermal noise is the amount of filtered phase noise present in $\frac{11}{12} \theta_{V,C} + \theta_{V,SC}$ (see Fig. 11). Hence, denoting the loop phase noise processes by $\phi_{n,c}$ and $\phi_{n,sc}$, then the phase-error difference in question is given by

$$\Delta \phi(t,T) = \frac{11}{12} \left[ \phi_{n,c}(t-T) - \phi_{n,c}(t) \right] + \left[ \phi_{n,sc}(t-T) - \phi_{n,sc}(t) \right]$$

and since these processes are statistically independent when the NIMBUS-E/ATS-F link noise predominates, the mean-square phase-error difference (assuming zero-mean stationary processes) is

$$\sigma^2_{\Delta \phi} = 2\left(\frac{11}{12}\right)^2 [R_C(o) - R_C(T)] + [R_{SC}(o) - R_{SC}(T)] \text{rad}^2$$

where the $R(\cdot)$ are the loop noise autocorrelation functions, so that $\sigma^2_C = R_C(o)$ and $\sigma^2_{SC} = R_{SC}(o)$ are the mean-square loop noise errors. If the effective correlation time of the processes are small relative to the measurement
duration so that \( R_C(T) - R_{sc}(T) = 0 \), then this expression reduces to

\[
\sigma_{\Delta \phi}^2 \approx 2 \left[ \left( \frac{11}{12} \right)^2 \sigma_C^2 + \sigma_{sc}^2 \right] = \frac{(11/12)^2}{(SNR)_C} + \frac{1}{(SNR)_{sc}} \text{ rad}^2
\]

The counting time \( T \) corresponds to data rates of either 0.1, 1, 2, 4 or 8 samples per sec (neglecting dead zones if destructive cycle-counting is used), and large \( B_n T \) values occur for loop bandwidths \( B_{n,c} = B_{n,sc} = 200 \text{ Hz} \) so the simple expression can be used. The rms doppler error is then

\[
\sigma_D = \frac{1}{2\pi} \left[ \frac{(11/12)^2}{(SNR)_B} + \frac{1}{(SNR)_{sc}} \right]^{1/2} \text{ Hz}
\]

The range rate error itself requires some careful interpretation under the circumstances. If the doppler being measured could be viewed as a two-way doppler on a certain reference frequency \( f_x \), then the rms range rate error would be given by \( \sigma_R = \frac{c}{2f_x} \sigma_D \), where \( c \) is the velocity of light. This would be the situation if the weighting coefficients for the two baseline dopplers were the same, in which case we would measure a doppler \( 2D_x = v(2D_T + 2D_L) \) and \( f_x = vf_T \) would be used to determine range rate errors. However we cannot specify such a frequency \( f_x \) in the present system since the weighting coefficients differ; the most we can do is to acknowledge that usually \( D_T \ll D^L_T \) so that \( 2D_x = \frac{2063}{6150} (2D_T) \) and the doppler being measured can be viewed as approximately the two-way doppler on \( f_x = \frac{2063}{6150} f_T = 2063 \text{ MHz} \). The effects of this approximation were discussed before and their ultimate position-location implications depend on the orbital determination procedures to be employed in the ATS-F/NIMBUS-E experiment.

The subcarrier loop SNR is lower than the carrier loop SNR, and hence the rms doppler error is essentially given by \( \sigma_D = \frac{1}{2\pi} (SNR)_{sc}^{1/2} \text{ Hz} \). If we
assume the 200 Hz loop bandwidth design along with the signal and noise levels of Table 5, the rms doppler error is \( \sigma_D = \frac{0.035}{2\pi T} \) Hz nominal and \( \sigma_D = \frac{0.097}{2\pi T} \) Hz worst. If we use the 2063 MHz range rate interpretation discussed above, the corresponding rms range rate error due to noise is \( \sigma_R = \frac{12}{165} \sigma_D \) m/s. For example, a non-destructive cycle-counting procedure would yield the following error performance due to thermal noise.

<table>
<thead>
<tr>
<th>( B_n,sc ) (Hz)</th>
<th>( T ) (sec)</th>
<th>( \sigma_D ) (Hz, nominal/worst)</th>
<th>( \sigma_R ) (m/s, nominal/worst)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>10</td>
<td>0.00056/0.00154</td>
<td>0.000041/0.000112</td>
</tr>
<tr>
<td>200</td>
<td>1</td>
<td>0.0056/0.0154</td>
<td>0.00041/0.00112</td>
</tr>
<tr>
<td>200</td>
<td>0.5</td>
<td>0.0112/0.0308</td>
<td>0.00082/0.00224</td>
</tr>
<tr>
<td>200</td>
<td>0.25</td>
<td>0.0224/0.0616</td>
<td>0.00164/0.00448</td>
</tr>
<tr>
<td>200</td>
<td>0.125</td>
<td>0.0448/0.1232</td>
<td>0.00328/0.00896</td>
</tr>
</tbody>
</table>

With reference to the range tone loop, the maximum two-way doppler rate on the 100 kHz tone is about \( 7 \times 10^{-3} \) Hz/s so that if we assume the existing loop noise bandwidth options of 1, 5 or 25 Hz, then the peak dynamic errors tabulated below will result. The rms thermal noise error can be obtained by using these bandwidths along with the results of Table 5, and including a 6 dB SNR improvement due to the tone-sideband folding effects. The results are also tabulated below and the compromise between dynamic and noise errors can be interpreted in range error terms by noting that a 1° tracking error at 100 kHz corresponds to 8.3 m in the two-way propagation.

<table>
<thead>
<tr>
<th>( B_n )</th>
<th>peak dynamic error (°)</th>
<th>tone loop SNR (dB)</th>
<th>rms noise error (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( B_n = 1 ) Hz</td>
<td>2.52° (20.92m)</td>
<td>52.1/43.3 dB</td>
<td>0.10/0.28° (0.83/2.32m)</td>
</tr>
<tr>
<td>( B_n = 5 ) Hz</td>
<td>0.10° (0.83m)</td>
<td>45.1/36.3 dB</td>
<td>0.22/0.62° (1.83/5.15m)</td>
</tr>
<tr>
<td>( B_n = 25 ) Hz</td>
<td>0.004° (0.033m)</td>
<td>38.1/29.3 dB</td>
<td>0.50/1.38° (4.15/11.45m)</td>
</tr>
</tbody>
</table>
2.6 **Summary of Results and Conclusions.**

The tracking signal processing operations of the ATS-F/NIMBUS-E system are characterized by a coherent ATS-F transponder in the forward link, an incoherent GRARR-like transponder at NIMBUS-E, and an incoherent side-stepping ATS-F transponder in the return link. These specifications inherently restrict the doppler extraction potential of an ideal receiver, and we first established such potential as a guideline to evaluate any specific receiver configuration. In the absence of other signaling schemes (such as pilot signal insertion for resolution of the G.S./ATS-F link doppler), the ideal doppler signal extractor cannot reproduce a doppler proportional to the round-trip doppler effect on any given reference frequency, but must be satisfied with extracting a linear combination of the doppler effects in the two system baselines (G.S./ATS-F and ATS-F/NIMBUS-E) with known but different weighting coefficients. The choice between taking this effect into account in an orbital determination procedure vs assuming nil G.S./ATS-F doppler as a first approximation requires position and velocity error analysis of orbital determination programs beyond the scope of range and range rate error resolution.

The specific ATSR receiver modification under consideration was then analyzed and shown to indeed match the doppler extraction potential of the ideal receiver, provided that the heterodyning operations at ATS-F, NIMBUS-E and the ATSR receiver r-f result in the proper carrier and subcarrier doppler reversal conditions implicitly assumed and exploited in the modified ATSR receiver processing. The existing NIMBUS-E and ATSR receiver r-f conversions are set, so the emphasis is to assure that the proper number of carrier doppler reversals are introduced by the ATS-F transponder in its forward and
return relay operations. This was ascertained by investigating the ATS-F transponder configuration under development, which has recently been documented in the literature.

The presence of a relay satellite in the link introduces retransmitted noise considerations that must be accounted for. With reference to oscillator phase noise effects, the uplink carrier retransmitted to NIMBUS-E will include the beacon phase noise filtered by the ATS-F loop and transferred (with scale factors) into the mixing references used to derive the S-band carrier. Similarly, the downlink NIMBUS-E carrier will receive filtered and scaled beacon phase noise when being side-stepped into the downlink C-band carrier in the return link. Moreover, the ATSR receiver will now require a larger r-f reference (C-band vs S-band) for i-f conversion purposes, with a corresponding increase in local oscillator phase noise contribution. All these considerations are absent in an all-ground tracking system with no relay satellite involved, and hence they must be properly investigated and formulated to identify any possible error degradation factors directly caused by the insertion of a relay satellite in the tracking system network.

On this basis, the oscillator phase noise propagation throughout the system was analyzed and compared to an equivalent all-ground tracking network by-passing the relay satellite. Insofar as the receiver carrier loop is concerned, the relay satellite tracking mode exhibited an increase in loop locking errors and VCO phase instabilities but not to a critical extent due to the NIMBUS-E S-band oscillator contribution being present in any case. With reference to the subcarrier loop, there was essentially no distinction, because the beacon signal extracted at ATS-F should be coherent to the carrier signal under proper ATS-F loop locking conditions, and only scaled ATS-F
locking errors would distinguish the relay-satellite and all-ground modes. With reference to the doppler signal and range rate extraction, the basic distinction is that (a) the relay-satellite network requires a larger r-f reference representing a $\frac{3953}{2253} = 1.7$ increase in local oscillator phase noise effects; (b) the relay-satellite system exhibits an $\frac{11}{12}$-scaled 1700-MHz retransmitted phase noise added to the 2062 MHz phase noise present in the doppler signal (the first term being absent in all-ground system), and these two effects may or may not be correlated since they involve different propagation times but at most represent a $\frac{2062+1700}{2062} = 1.7$ increase in received oscillator phase noise. Hence, a worst condition would exhibit a net 1.7 increase in (local plus received) oscillator phase noise effects relative to an all-ground tracking network.

An analogous problem exists with thermal noise effects since the relay-satellite network not only introduces potentially low SNR links in the ATS-F/NIMBUS-E baseline, which can cause signal suppression and pertinent retransmitted phase noise at the NIMBUS-E limiter/phase-modulator stages, but also adds thermal phase noise to the downlink relayed signal since the side-stepping references are derived from the loop VCO which tracks the uplink beacon and the latter has a signal-to-noise density level much lower than that of the uplink carrier. In summary, the issues involve the four uplink and downlink additive noises around the carrier, the uplink additive noise around the beacon which becomes thermal phase noise in the ATS-F loop and is thus retransmitted to NIMBUS-E and to the ATSR station in the relay operations, the signal suppression at NIMBUS-E due to the low SNR ATS-F/NIMBUS-E link, and the retransmitted non-thermal phase noise at the output of the NIMBUS-E phase modulator.
The thermal noise considerations discussed above were analyzed in detail and shown to conclude on a predominating NIMBUS-E/ATS-F downlink additive noise as the main contributor to noise error performance, but including the signal-suppression effects caused by the ATS-F/NIMBUS-E uplink additive noise in the signal-to-noise density level evaluations. The ATSR receiver loop designs were verified to yield tracking bandwidth requirements compatible to existing realizations, and range and range rate error accuracy limitations were evaluated and shown to result in acceptable values when compared to analogous data characteristic of all-ground tracking networks.
APPENDIX 2.1

Case: 2 simultaneous tones @ 1.2 rad each

Uplink Carrier $\sum_{n=0}^{N} J_0^2(1.2) = -6.93 \, \text{dB}$

Uplink Tone Sideband $\sum_{n=0}^{N} J_1(1.2)J_0(1.2) = -9.51 \, \text{dB}$

Downlink Subcarrier Mod Index $1.5 \sum_{n=0}^{N} J_0^2(1.2) = 0.676$

Downlink Tone-Sideband Mod Index $1.5 \sum_{n=0}^{N} J_1(1.2)J_0(1.2) = 0.502$

Downlink Carrier $J_0(0.676)J_0^4(0.502) = -3.22 \, \text{dB}$

Downlink Subcarrier Sideband $J_1(0.676)J_0^4(0.502) = -12.14 \, \text{dB}$

Downlink Tone Sideband $J_0(0.676)J_1(0.502)J_0^3(0.502) = -14.98 \, \text{dB}$

Note: The effect of uplink IM terms has been neglected in the computations above, yet they are passed by the wideband (40 MHz) ATS-F repeater and may also be passed by the NIMBUS-E filtering preceding the PM operation. The net effect of these terms is to add an extra 1.65 dB reduction as shown below:

\[
\begin{align*}
J_1(1.2)J_1(1.2) & = -12.10 \, \text{dB} & 0.372 & 4 \text{ components} \\
J_2(1.2)J_0(1.2) & = -19.42 \, \text{dB} & 0.160 & 4 \text{ components} \\
J_2(1.2)J_1(1.2) & = -22.05 \, \text{dB} & 0.119 & 8 \text{ components} \\
J_2(1.2)J_2(1.2) & = -31.92 \, \text{dB} & 0.038 & 4 \text{ components} \\
J_3(1.2)J_0(1.2) & = -33.12 \, \text{dB} & 0.033 & 4 \text{ components} \\
\end{align*}
\]

\[
\begin{align*}
J_0(0.372) & = -0.30 \, \text{dB} & (x4 = -1.20 \, \text{dB}) & -1.63 \, \text{dB by} \\
J_0(0.160) & = -0.045 \, \text{dB} & (x4 = -0.18 \, \text{dB}) & \text{those within} \\
J_0(0.119) & = -0.031 \, \text{dB} & (x8 = -0.25 \, \text{dB}) & \text{20 dB of tone} \\
J_0(0.038) & = -0.0034 \, \text{dB} & (x4 = -0.014 \, \text{dB}) & \\
J_0(0.033) & = -0.0026 \, \text{dB} & (x4 = -0.010 \, \text{dB}) & -1.65 \, \text{dB}
\end{align*}
\]
Case: 2 simultaneous tones @ 1.2 and 0.5 rad respec.

Uplink Carrier
\[ J_0(1.2)J_0(0.5) = -4.02 \text{ dB} \]

Uplink Tone Sideband
\[ J_1(1.2)J_0(0.5) = -6.60 \text{ dB} \]
\[ J_1(0.5)J_0(1.2) = -15.78 \text{ dB} \]

Downlink Subcarrier Mod Index
\[ 1.5 J_0(1.2)J_0(0.5) = 0.945 \]

Downlink Tone-Sideband Mod Index
\[ 1.5 J_1(1.2)J_0(0.5) = 0.702 \]
\[ 1.5 J_1(0.5)J_0(1.2) = 0.244 \]

Downlink Carrier
\[ J_0(0.945)J_0^2(0.702) = -4.52 \text{ dB} \]

Downlink Subcarrier Sideband
\[ J_1(0.945)J_0^2(0.702) = -9.96 \text{ dB} \]

Downlink Tone Sideband
\[ J_0(0.945)J_1(0.702)J_0^2(0.244) = -13.08 \text{ dB} \]
\[ J_0(0.945)J_0^2(0.702)J_1(0.244)J_0(0.244) = -22.66 \text{ dB} \]

Note: The effect of uplink IM terms is nil as shown below:

\[ x \times 1.5 \]

\[ J_1(1.2)J_1(0.5) = -18.36 \text{ dB} \]
\[ J_2(1.2)J_0(0.5) = -16.51 \text{ dB} \]
\[ J_2(0.5)J_0(1.2) = -33.75 \text{ dB} \]
\[ J_3(1.2)J_0(0.5) = -30.21 \text{ dB} \]
\[ J_3(0.5)J_0(1.2) = -55.30 \text{ dB} \]
\[ J_2(1.2)J_1(0.5) = -28.27 \text{ dB} \]
\[ J_2(0.5)J_1(1.2) = -36.34 \text{ dB} \]
\[ J_2(1.2)J_2(0.5) = -46.25 \text{ dB} \]
\[ J_0(0.224) = -0.11 \text{ dB} \]
\[ J_0(0.181) = -0.071 \text{ dB} \]
\[ J_0(0.057) = -0.0078 \text{ dB} \]
\[ J_0(0.046) = -0.0052 \text{ dB} \]
\[ J_0(0.031) = -0.0017 \text{ dB} \]
\[ J_0(0.181) = -0.071 \text{ dB} \]
\[ J_0(0.057) = -0.0078 \text{ dB} \]
\[ J_0(0.046) = -0.0052 \text{ dB} \]
\[ J_0(0.031) = -0.0017 \text{ dB} \]
Case: **2 seq. tones @ 1.2 or 0.5 rad**

<table>
<thead>
<tr>
<th>Description</th>
<th>Formula</th>
<th>Power Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uplink Carrier</td>
<td>$J_0(1.2) = -3.46 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$J_0(0.5) = -0.55 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>Uplink Tone Sideband</td>
<td>$J_1(1.2) = -6.05 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$J_1(0.5) = -12.31 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>Downlink Subcarrier Mod Index</td>
<td>$1.5 J_0(1.2) = 1.007$</td>
<td>$1.5 J_0(0.5) = 1.408$</td>
</tr>
<tr>
<td>Downlink Tone-Sideband Mod Index</td>
<td>$1.5 J_1(1.2) = 0.747$</td>
<td>$1.5 J_1(0.5) = 0.363$</td>
</tr>
<tr>
<td>Downlink Carrier</td>
<td>$J_0(1.007)J_0^2(0.747) = -4.86 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$J_0(1.408)J_0^2(0.363) = -5.59 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>Downlink Subcarrier Sideband</td>
<td>$J_1(1.007)J_0^2(0.747) = -9.67 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$J_1(1.408)J_0^2(0.363) = -5.87 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>Downlink Tone Sideband</td>
<td>$J_0(1.007)J_1(0.747)J_0(0.747) = -12.73 \text{ dB}$</td>
<td>$J_0(1.408)J_1(0.363)J_0(0.363) = -20.22 \text{ dB}$</td>
</tr>
</tbody>
</table>

**Note:** The effect of uplink IM terms is nil as shown below:

<table>
<thead>
<tr>
<th>Formula</th>
<th>Power Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_2(1.2) = -15.96 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>$J_3(1.2) = -29.66 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>$1.5 J_2(1.2) = 0.239$</td>
<td></td>
</tr>
<tr>
<td>$1.5 J_3(1.2) = 0.0579$</td>
<td></td>
</tr>
<tr>
<td>$J_0(0.239) = -0.125 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>$J_0(0.0579) = -0.007 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>$-0.132 \text{ dB}$</td>
<td></td>
</tr>
<tr>
<td>$x 2$</td>
<td></td>
</tr>
<tr>
<td>$-0.264 \text{ dB}$</td>
<td></td>
</tr>
</tbody>
</table>

**Note:** The effect of uplink IM terms is nil as shown below:
3. RANGE RATE EXTRACTION AND DATA PROCESSING

The subject of this chapter is the extraction of range rate data in two-way coherent doppler systems, where the round-trip doppler effect \( f_d(t) \) exhibited by a sinusoidal signal of nominal frequency \( f_s \) is used to establish range rate according to the relation \( \dot{r}(t) = \left( \frac{c}{2 f_s} \right) f_d(t) \), where \( c \) is the velocity of light. We restrict ourselves to range rate extractors based on average-doppler principles, i.e., average range rate samples \( \dot{r}_k \) are produced at discrete times \( kT_0 \) separated by doppler measurement blocks of \( T_0 \) sec duration, where average-doppler effects are determined employing cycle-counting techniques. The discrete "raw" range rate samples thus produced are to be used in an existing orbital determination procedure, after being smoothed by a data processor.

In particular, we will be concerned with two specific realizations of range rate extractors based on cycle-counting techniques. They both yield the average doppler \( \bar{f}_d \) corresponding to a raw range-rate sample as the solution of the equation \( N = T(f_b + \bar{f}_d) \), where \( N \) is the number of bias (\( f_b \)) plus doppler cycles that occur in a time interval \( T \). The average doppler is thus determined by fixing one parameter from the pair \((N,T)\) and measuring the other, the actual choice of fixed vs variable parameter essentially specifying the two extractor realizations of interest.

We assume a fixed data rate of \( T_0^{-1} \) raw range rate samples per second at the extractor output. In the fixed-\( T \) case, this only means that \( N_k \) bias plus doppler cycles are measured over successive \((k=1,2..)\) blocks of \( T_0 \)-seconds duration per block without any loss of data. However, in the fixed-\( N \) case the parameter \( T_k \) may vary with the average doppler from measurement to
measurement, so that the actual cycle-counting time involved in a single measurement block only represents a fraction $T_k/T_0 < 1$ of the block duration, with corresponding dead zones of perhaps variable duration $U_k = T_0 - T_k$ where the doppler information is not being used. The terms nondestructive-count and destructive-count are respectively used to refer to these extractors based on these data loss characteristics, and this designation will be employed in the sections that follow.

3.1 The Destructive vs Non-Destructive Extractor Realizations

The destructive-count procedure is based on measuring the time duration $T_k$ of a fixed number of bias-plus-doppler cycles in time units quantized to $1/f_r$ seconds as shown in Fig. 16(a), where $f_r$ is some high frequency reference. The counter $C_{BD}$ receives the bias-plus-doppler signal and develops start and stop pulses at the 1st and $(N+1)$-th zero-crossings, which are used to trigger the reference counter $C_R$. The number $M_k$ of reference cycles that occur in-between these trigger pulses measures their time separation in the aforesaid units, and the average doppler and range rate measurements are then given by

$$
\bar{r}_k = \frac{c}{2f_s} \bar{f}_{d,k} = \frac{c}{2f_s} \left( \frac{N}{T_k} - f_b \right) = \frac{c}{2f_s} \left( \frac{Nf_r}{M_k} - f_b \right)
$$

(3.1)

The presence of undesired random phase fluctuations accompanying the bias-plus-doppler signal advances or retards the zero-crossings (assuming high SNR conditions such that cycle slippage effects can be neglected), thus increasing or decreasing the reference count and introducing an error in the average doppler and range rate measurement. If we assume a stationary, zero-mean, phase noise process $\phi(t)$ then the time error introduced is given by
3-3

**Fig. 16(a) Destructive-Count Range Rate Extraction**

**Fig. 16(b) Nondestructive-Count Range Rate Extraction**
\[(\Delta T)_k = \frac{(\Delta \phi)_k}{2\pi (f_b + f_{d,k})} \quad (3.2)\]

where \((\Delta \phi)_k\) is the phase noise difference between the \((N+1)\)-th and 1st zero-crossings of the \(k\)-th measurement block. The first-order error contributions to the measured parameters are then given by

\[ (\Delta \bar{r})_k = \frac{c}{2f_s} (\Delta \bar{f}_{d,k}) = -\frac{c}{4\pi f_s T_k} (\Delta \phi)_k \quad (3.3) \]

so that the rms range rate error expressed in terms of the phase noise correlation function is given by

\[ \sigma_{\bar{r},k}^2 = \frac{c}{2\sqrt{2\pi f_s T_k}} [R\phi(0)-R\phi(T_k)]^2 \quad (3.4) \]

In turn, the nondestructive-count procedure is based on the non-real time measurement of the number of bias-plus-doppler cycles \(N_k\) that occur in a fixed interval \(T_0\), as shown in Fig. 16(b). Note the count \(N'_k\) of the counter \(C_{BD}\) does not span the entire \(T_0\) seconds since the cycle fractions that occur before the 1st zero-crossing and after the \((N'_k+1)\)-th one are not accounted for, so another counter \(C_{R}\) is used to measure these fractions in time units quantized to \(1/f_r\) seconds. The net number of bias-plus-doppler cycles that occur during the \(T_0\)-sec duration is then

\[ N_k = N'_k + 1 + \frac{f_b + f_{d,k}}{f_r} (M_k - M_{k+1}) \quad (3.5) \]

which illustrates the non-real time processing involved, since the \(M_{k+1}\) count is used to determine the \(N_k\) count and we must wait for the \((N'_k+1)\)-th zero crossing to resolve the time elapsed between the \(N'_k\)-th zero crossing and the end of the \(k\)-th measurement block. Even though the cycle fractions represented by the reference counts \(M_k\) and \(M_{k+1}\) are small relative to the integer
count \( N'_k \), they cannot be neglected from error analysis considerations as will be shown.

Note that (3.5) is dependent on the unknown doppler \( \bar{f}_{d,k} \) and hence does not yield \( N_R \) explicitly. We must equate this expression to

\[ N_k = T_0(f'_b + \bar{f}_{d,k}) \]

thus developing an implicit equation to be resolved for \( \bar{f}_{d,k} \) in terms of the measured quantities \( N'_k, M_k, M_{k+1} \). The solution of this equation yields the average range rate and doppler measured in the \( k \)-th block as

\[ \bar{r}_k = \frac{c}{2 f_s} \bar{f}_{d,k} = \frac{c}{2 f_s} \left[ \frac{(N'_{k+1}) f_r}{T_0 + (M_{k+1} - M_k)} - f_b \right] \quad (3.6) \]

It should be acknowledged that the implicit equation in question is based on the assumption of a constant doppler per measurement block, but it should also be emphasized that this assumption is intrinsic in the usage of cycle-counting realizations for range rate extraction. The model of a quantized doppler \( \bar{f}_{d,k} \) per measurement block is compatible with such extractor realizations and any inquiries on this issue must proceed by questioning the usage of average doppler and average range rate measurements in the first place.

The presence of accompanying phase noise in the input signal will cause errors not only in the reference counts \( M_k \) and \( M_{k+1} \) by advancing or retarding zero-crossings, but can also increase or decrease the bias-plus-doppler count \( N'_k \) by inserting or deleting zero-crossings into or out-of the \( k \)-th measurement block. Assuming high SNR conditions, the errors \( \Delta N'_k \) will be limited to 0, ±1, ±2, and the relations between \( \Delta N'_k, \Delta M_k, \Delta M_{k+1} \) and \( \Delta \phi_k \) are derived in Appendix 3.1 for all possible cases. The first-order dependence of \( \Delta \bar{r}_k \) on \( \Delta N'_k, \Delta M_k, \Delta M_{k+1} \) can be obtained from (3.6)
and the substitution of any of the cases tabulated in Fig. 3.2 yields

\[
(\Delta r)_k = \frac{c}{2fs} (\Delta r_{d,k}) = - \frac{c}{4\pi fsT_o} (\Delta \phi)_k
\]

(3.7)

and

\[
\sigma_{r,k} = \frac{c}{2\sqrt{2\pi fsT_o}} \left[ R_\phi(o) - R_\phi(T_o) \right]^{1/2}
\]

(3.8)

which are expressions analogous to (3.3) and (3.4).

It should be emphasized that even though \( T_o f_r \gg M_{k+1} - M_k \) may represent a valid approximation tantamount to neglecting the fractional counts and eliminating the need for a reference counter, their exclusion will yield different answers as the results of Appendix 3.1 indicate; i.e., the existing dependence between \( \Delta N_k \) and \( \Delta M_{k+1} - \Delta M_k \) will not be accounted, and the expression for \( (\Delta r)_k \) will become ambiguous since it will be based solely on \( \Delta N_k \) and the latter is multiple-valued (0, ±1, ±2). Hence, the realization of the non-destructive-count extractor must include the fractional correction procedure and non real-time processing required if a proper error comparison with the destructive-count extractor is to be performed. Under such conditions, the two extractor realizations will then only differ in the dead-zone effect previously mentioned \( (T_k \ vs \ T_o) \) insofar as the error analysis is concerned.

3.2 Raw Range Rate Error Analysis

The rms error in the raw range rate samples obtained from a single-measurement block has been shown to be given by (where \( \sigma^2_\phi = R_\phi(o) \)):

\[
\sigma_{\bar{r},k} = \begin{cases} 
\frac{c_\phi}{2\sqrt{2\pi fsT_k}} \left[ 1 - \frac{R_\phi(T_k)}{R_\phi(o)} \right]^{1/2} & \text{(destructive)} \\
\frac{c_\phi}{2\sqrt{2\pi fsT_o}} \left[ 1 - \frac{R_\phi(T_o)}{R_\phi(o)} \right]^{1/2} & \text{(non-destructive)}
\end{cases}
\]

(3.9)
The non-destructive error can thus be computed upon knowledge of 
$R_\phi(t)$, but the destructive error is also a function of the doppler $f_{d,k}$ which determines the parameter $T_k$. In the special case of $|R_\phi(T_k)| < R_\phi(o)$ for all $T_k$ and $T_0$, then the rms errors are independent of the autocorrelation function, and the destructive error exceeds the nondestructive error by $T_0/T_k > 1$.

A possible approach to eliminate the destructive error dependence on the parameter $T_k$ is to statistically average the error by assuming a stationary (k-independent) prior probability law for the quantized doppler $f_{d,k}$ and determining its induced law on $T_k$. The presence of the $R_\phi(T_k)$ term could make difficult the computations and simple general results are restricted to the case of $|R_\phi(T_k)| < R_\phi(o)$. For example, a uniform p.d.f. over $\pm f_D$ Hz induces an inverse-square p.d.f. on $T_k$, namely

$$p_{T_k}(t) = \frac{N}{2f_D t^2}, \quad \frac{N}{f_b+f_D} \leq t \leq \frac{N}{f_b-f_D}$$

(3.10)

and the mean of the destructive rms error is then given by

$$E(\sigma_\epsilon_k) = \int \frac{c_\sigma_\phi}{2\sqrt{2\pi}f_s t} p_{T_k}(t) dt = \frac{c_\sigma_\phi}{2\sqrt{2\pi}f_s (N/f_b)}$$

(3.11)

if the assumption $|R_\phi(T_k)| < R_\phi(o)$ is satisfied. Note that if we assume the bias $f_b$ large relative to the doppler such that $T_k \approx N/f_b$ for all $k$, the destructive rms error would also be given by the last result in (3.11) without any prior assignment and error expectation involved, but assuming $|R_\phi(N/f_b)| < R_\phi(o)$. 


We next consider a sequence of measurement blocks and the correlation effects between successive measurements. The covariance of the i-th and j-th raw range rate sample errors is given by

\[
E[(\Delta \bar{r}_i)(\Delta \bar{r}_j)] = \frac{C^2}{(4\pi f_s)^2T_i T_j} E[(\Delta \phi)_i(\Delta \phi)_j]
\]

\[
= \begin{cases} 
R_\phi[(j-i)T_0+T_j-T_i] + R_\phi[(j-i)T_0] - R_\phi[(j-i)T_0+T_j] - R_\phi[(j-i)T_0-T_i] & \text{(destructive)} \\
2R_\phi[(j-i)T_0] - R_\phi[(j-i+1)T_0] - R_\phi[(j-i-1)T_0] & \text{(non-destructive)}
\end{cases}
\]

(3.12)

If we assume that the duration \(T_0\) of a measurement block is large relative to the effective correlation time of the phase noise such that \(R_\phi(\tau) \approx 0\) for \(\tau > T_0\), then only adjacent measurement blocks will exhibit error correlation effects and the following correlation coefficients result

\[
\rho_{ij}^{\text{(destructive)}} = \begin{cases} 
1 & \text{for } i=j \\
-\frac{R_\phi(U_k) - R_\phi(T_0+T_k+1-T_k)}{2R_\phi(o)} & \text{for } |i-j|=1 \text{ and } k=\min(i,j) \\
0 & \text{for } |i-j| > 2
\end{cases}
\]

(3.13a)

\[
\rho_{ij}^{\text{(nondestructive)}} = \begin{cases} 
1 & \text{for } i=j \\
\frac{1}{2} & \text{for } |i-j|=1 \\
0 & \text{for } |i-j| > 2
\end{cases}
\]

(3.13b)

where \(U_k = T_0 - T_k\) is the dead zone of the destructive extractor.

The destructive extractor expression can be further simplified by neglecting the \(R_\phi(T_0+T_k+1-T_k)\) by assuming the average doppler does not vary significantly over adjacent measurement blocks. Under such conditions, the
distinction between the two results above is a variable correlation coefficient of \( -\frac{1}{2} \frac{R_{\phi}(U_k)}{R_{\phi}(o)} \) for the destructive extractor vs a constant \(-\frac{1}{2}\) for the nondestructive extractor.

In order to compare the two extractors at the raw sample stage on the basis of successive measurements, we consider the arithmetic mean

\[
\bar{R} = \frac{1}{n} \sum_{k=1}^{n} R_k
\]

of \( n \) consecutive samples, with corresponding error \( \Delta \bar{R} = \frac{1}{n} \sum_{k=1}^{N} (\Delta R)_k \) and rms error (assuming adjacent-measurement correlation only):

\[
\sigma = \frac{c_{\phi}}{2\sqrt{2n\pi f_s}} \left[ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{T_k^2} + \frac{2}{n} \sum_{k=1}^{n-1} \frac{\rho_{k,k+1}}{T_k T_{k+1}} \right]^{\frac{1}{2}}
\]

(destructive)

\[
\approx \left\{ \begin{array}{ll}
\frac{c_{\phi}}{2\sqrt{2n\pi f_s}} \left[ \frac{1}{n} \sum_{k=1}^{n} \frac{1}{T_k^2} \right]^{\frac{1}{2}} & \text{(non-destructive)} \\
\frac{c_{\phi}}{2\sqrt{2n\pi f_s}} \left[ \frac{1}{n} \sum_{k=1}^{n} \frac{R_{\phi}(U_k)}{T_k T_{k+1}} \right]^{\frac{1}{2}} & \text{(destructive)}
\end{array} \right.
\]

(3.14)

The operation of taking the arithmetic mean is usually meaningful only when the doppler is essentially constant over consecutive measurements, and we are interested in sample averaging for error reduction purposes. Under this condition, we can assume \( T_k = T \leq T_o \) in the destructive case which reduces the above expressions to

\[
\sigma = \left\{ \begin{array}{ll}
\frac{c_{\phi}}{2\sqrt{2n\pi f_s T}} \left[ 1 + 2\rho \left( \frac{n-1}{n} \right) \right]^{\frac{1}{2}} & \text{(destructive)} \\
\frac{c_{\phi}}{2\sqrt{2n\pi f_s T_o}} \left[ \frac{1}{n} \right]^{\frac{1}{2}} & \text{(non-destructive)}
\end{array} \right.
\]

(3.15)

where \( \rho = -R_{\phi}(U)/2\sigma_\phi \) and \( U = T_o - T \) in the destructive case. The \( 1/\sqrt{n} \) appearing
as a common multiplier in front of the brackets is the conventional rms error reduction factor characteristic of independent-samples averaging.

In the nondestructive case the \(-\frac{1}{2}\) correlation effect always contributes an extra \(1/\sqrt{n}\) factor as shown by the bracket term in (3.15), which is not matched by the nondestructive expression since \(\left|R_\phi(U)\right| \leq \sigma_\phi\) implies \(|\rho| \leq \frac{1}{2}\) and \(1+2\rho\left(\frac{n-1}{n}\right) \geq \frac{1}{n}\). The \((T_0/T)\) single-measurement, rms-error, advantage of the nondestructive extractor is still present in the multiple-measurement plus arithmetic averaging results, and is enhanced by a \(\sqrt{n} \left[1+2\rho\left(\frac{n-1}{n}\right)\right]\frac{1}{2}\) factor contributed by error correlation effects. In the special case of large dead zones such that \(\left|R_\phi(U)\right| \ll R_\phi(0)\) and \(\rho = 0\) in the destructive case, the nondestructive improvement will be the aforesaid \(\sqrt{n}\) factor contributed by the \(\rho = -\frac{1}{2}\) correlation effects.

3.3 Dead-Zone Effects on the ATS-F/NIMBUS-E System

We consider the case where the phase noise process derived from the receiver loops represents the thermal noise accompanying the input signal, which becomes phase noise when filtered by the tracking loops. In applications involving incoherent transponders the doppler signal is obtained by combining the received carrier doppler with received subcarrier or pilot-tone doppler, so as to compensate for the incoherent oscillator effects. The thermal phase noise process is then composed of two independent additive processes, and its power spectral density is given by

\[
S_\phi(\omega) = (k_1)^2 \frac{\phi_1}{E_1} \left|H_1(j\omega)\right|^2 + (k_2)^2 \frac{\phi_2}{E_2} \left|H_2(j\omega)\right|^2
\]

(3.16)

where \(\phi_i/E_i\) is the corresponding phase-noise power spectral density at the loop input, \(H_i(j\omega)\) is the loop transfer function, and \(k_i\) is the frequency scaling factor involved in the doppler signal extraction. For GRARR-like
transponder applications, the two contributions above refer to carrier and subcarrier loop signals and we shall consider the specific case of identical carrier and subcarrier loop designs which is common in practice, in which case the expression above reduces to

\[ S_\phi(\omega) = \phi_{eq} |H(j\omega)|^2 \] (3.17)

In order to concentrate on the ATS-F/NIMBUS-E TDRS system we shall assume the loop transfer functions to be

\[ H(s) = \frac{9\omega_0}{4} \frac{(s + \frac{\omega_0}{3})^2}{(s + \frac{\omega_0}{4})(s + \omega_0)^2} \] (3.18)

which represents the existing carrier loop design in the ATSR system and the expected subcarrier loop design for the modified ATSR receiver. The evaluation of the autocorrelation function \( R_\phi(\tau) \) as the inverse transform of \( S_\phi(\omega) \) is complicated because of the magnitude-square operation involved, but a measure of the correlation time (the effective width of the correlation function) can be obtained from

\[ \tau_c = \left[ \frac{\int \tau^2 R_\phi(\tau) d\tau}{\int R_\phi(\tau) d\tau} \right]^{\frac{1}{2}} = \left[ \frac{S_\phi''(0)}{S_\phi'(0)} \right]^{\frac{1}{2}} \] (3.19)

and the derivation details are summarized in Table 6. The effective correlation time is shown there to be \( \tau_c = \frac{6\sqrt{6}}{\omega_0} \) for the loop in question, and this result can be expressed in terms of the (two-sided) loop noise bandwidth

\[ B_n = \int_{-\infty}^{\infty} \frac{|H(j\omega)|^2 d\omega}{2\pi} = \frac{297}{200} \omega_0 \text{ Hz} \] (3.20)
TABLE 6

\[ S_\phi(\omega) = \phi_{eq} |H(\omega)|^2 = \phi_{eq} H(\omega)H(-\omega) \]

\[ S'_\phi(\omega) = \phi_{eq} [H'(\omega)H(-\omega)+H(\omega)H'(-\omega)] \]

\[ S''_\phi(\omega) = \phi_{eq} [H''(\omega)H(-\omega)+2H'(\omega)H'(-\omega)+H(\omega)H''(-\omega)] \]

\[ = -\phi_{eq} [H''(s)H(-s)+2H'(s)H'(-s)+H(s)H''(-s)] \], \( s = j\omega \)

\[ H(s) = \frac{9\omega_o}{4} \frac{N(s)}{D(s)} \]

\[ H'(s) = \frac{9\omega_o}{4} \frac{N'(s)D(s)-D'(s)N(s)}{D^2(s)} \]

\[ H''(s) = \frac{9\omega_o}{4} \frac{[N''(s)D(s)-D''(s)N(s)][D^2(s)]-[N'(s)D(s)-D'(s)N(s)][2D(s)D'(s)]}{D^4(s)} \]

\[ N(s) = s^2 + \frac{2}{3} \omega_o s + \frac{\omega_o^2}{9} \]
\[ D(s) = s^3 + \frac{9}{4} \omega_o s^2 + \frac{3}{2} \omega_o^2 s + \frac{\omega_o^3}{4} \]

\[ N'(s) = 2s + \frac{2}{3} \omega_o \]
\[ D'(s) = 3s^2 + \frac{9}{2} \omega_o s + 3\omega_o^2 \]

\[ N''(s) = 2 \]
\[ D''(s) = 6s + \frac{9}{2} \omega_o \]

\[ H(\omega) = 1, \quad H'(\omega) = -\frac{6}{\omega_o}, \quad H''(\omega) = \frac{144}{\omega_o^2} \]

\[ H(-\omega) = 1, \quad H'(-\omega) = \frac{6}{\omega_o}, \quad H''(-\omega) = \frac{144}{\omega_o^2} \]

\[ S_\phi(\omega) = \phi_{eq}, \quad S''_\phi(\omega) = -\frac{216}{\omega_o^2} \phi_{eq}, \quad \tau_{sc} = \frac{S''_\phi(\omega)}{S_\phi(\omega)} = \frac{216}{\omega_o^2} \]
which yields

\[ \tau_c = \frac{6\sqrt{6}}{\omega_0} = \frac{9\sqrt{6}}{B_n} \text{ sec} \quad (3.21) \]

If we assume maximum two-way doppler conditions of ±100 kHz accompanying the 500 kHz bias frequency as characteristic of the ATS-F/NIMBUS-E experiment, the maximum and minimum counting times for a destructive-count extractor employing the S-band GRARR system data rates are now tabulated along with the corresponding minimum and maximum dead zones.

<table>
<thead>
<tr>
<th>( T_0 (s) )</th>
<th>( N(\text{cycles}) )</th>
<th>( T_{\text{max}} (s) )</th>
<th>( T_{\text{min}} (s) )</th>
<th>( U_{\text{min}} (s) )</th>
<th>( U_{\text{max}} (s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3,133,956</td>
<td>7.8</td>
<td>5.2</td>
<td>2.2</td>
<td>4.8</td>
</tr>
<tr>
<td>1</td>
<td>229,263</td>
<td>0.57</td>
<td>0.38</td>
<td>0.43</td>
<td>0.62</td>
</tr>
<tr>
<td>0.5</td>
<td>131,007</td>
<td>0.33</td>
<td>0.22</td>
<td>0.17</td>
<td>0.28</td>
</tr>
<tr>
<td>0.25</td>
<td>65,503</td>
<td>0.16</td>
<td>0.11</td>
<td>0.09</td>
<td>0.14</td>
</tr>
<tr>
<td>0.125</td>
<td>32,751</td>
<td>0.082</td>
<td>0.055</td>
<td>0.043</td>
<td>0.070</td>
</tr>
</tbody>
</table>

Hence, if we assume \( B_n = 200 \text{ Hz} \) (see chapter 2) the effective correlation time is 0.11 sec so that the dead zones are 3.9 to 5.6 times larger for a 1-second measurement block duration and the assumption of small \( R(U) \) values seems reasonable as a first approximation. A more detailed analysis would require the inversion of \( S_\phi(\omega) \) for an exact evaluation of the correlation function, rather than using rms correlation times as in the discussion above.

### 3.4 The Least-Squares Polynomial Fit

The data processor of interest for smoothing the raw data samples is based on fitting a \((K-1)\)-th order polynomial of the form \( \hat{r}_i = \sum_{j=0}^{K-1} a_j i^j \) to the \( i \)-th raw sample \( r_i \), where \( i = 1,2,...,n \) and the bar denoting the average.
nature of the raw range rate sample is dropped for notation simplicity.
The coefficients \( a_j \) do not vary with \( i \) and are chosen so as to minimize
the sum of squares of the \( n \) residuals, i.e.,
\[
\sum_{i=1}^{n} (\hat{r}_i - \tilde{r}_i)^2 = \sum_{i=1}^{n} (\tilde{r}_i - \sum_{j=0}^{K-1} a_j i^j)^2 = \text{to be minimized by choice of } a_j \text{'s}
\]
which leads to the set of equations
\[
\sum_{j=0}^{K-1} a_j [\sum_{i=1}^{n} i^{j+v}] = \sum_{i=1}^{n} i^v y_i; \quad v = 0, 1, \ldots, K-1
\]
whose solution represents the optimum choice of \( a_j \text{'s} \). The explicit evaluation
of the latter is not necessary since their optimum choice can be shown to
imply a polynomial fit given by
\[
\hat{r}_i = \sum_{j=1}^{n} b_j (i) \tilde{r}_j = -\sum_{j=1}^{n} \frac{\det B_j(i)}{\det B} \tilde{r}_j; \quad i=1, 2, \ldots, n
\]
where \( B_j(i) \) and \( B \) are rectangular matrices specified by
\[
B_j(i) = \begin{bmatrix}
0 & 1 & j & j^{K-1} \\
1 & A_0 & A_1 & A_{K-1} \\
i & A_1 & A_2 & A_K \\
i^2 & A_2 & A_3 & A_{K+1} \\
\vdots & \vdots & \vdots & \vdots \\
i^{K-1} & A_{K-1} & A_K & A_{2K-2}
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
A_0 & A_1 & A_{K-1} \\
A_1 & A_2 & A_K \\
\vdots & \vdots & \vdots \\
A_{K-1} & A_K & A_{2K-2}
\end{bmatrix}, \quad A_K = \sum_{i=1}^{n} i^K
\]
In other words, the net effect of using optimum \( a_j \) values in the
polynomial fit according to a least-sum-of-squares criterion results in a
non real-time, time-varying, discrete filter processing the raw data samples;
i.e., the raw sample $r_i$ is estimated by the smooth sample $\hat{r}_i$, and the latter is a linear combination of past, present and future raw samples $r_j$ with the weighting coefficients $b_j(i)$ dependent on the raw sample $r_i$ being estimated.

The least-sum of squares is a limited figure of merit to measure the smoothed data quality in the presence of random disturbances. If we assume the raw measurements to consist of a true value plus a random error term contributed by a zero-mean stationary process, the variance of the $i$-th smooth sample is given by

$$
\sigma^2_{r_i} = \left\{ \sum_{j=1}^{n} [b_j(i)]^2 + 2 \rho_1 \sum_{j=1}^{n-1} b_j(i) b_{j+1}(i) \right\} \sigma^2_e + 2 \rho_\nu \sum_{j=1}^{n-\nu} b_j(i) b_{j+\nu}(i) \right\} \sigma^2_e
$$

where $\rho_\nu$ is the correlation coefficient between the random error samples at the raw stage, and $\sigma^2_e$ is the error process variance also at the raw stage. Note that the direct application of these results to the two extractors under consideration is in general only valid for the nondestructive case where $\sigma^2_e$ is given by Eq. (3.8), since in the destructive case the error process variance is not constant but depends on the sample number as evidenced by Eq. (3.4) and the $k$-dependence of $T_k$ and $R_\phi(T_k)$. Hence any further discussion on this matter with reference to the destructive case must assume essentially constant doppler conditions over successive samples to remove the $k$-dependence and permit the usage of Eq. (3.26).

* B. Kruger, Effects of Correlated Noise with Applications to Apollo Tracking Problems, NASA/GSFC, TND-4121, Feb. 1968.
Even if the raw sample error process is stationary (which it is in the nondestructive case and may be assumed in the destructive case under constant doppler conditions), the smooth sample variance will vary with the sample number \( j_i \) in Eq. 3.26 as a direct consequence of the time-varying nature of the raw sample estimator. Hence it is inappropriate to compare the smooth sample variance (nonstationary) to the raw sample variance in any case. In order to bypass this last restriction, Kruger uses the arithmetic mean of the smooth-sample variances as a figure of merit, namely

\[
\sigma_{\text{r}}^2 = \frac{1}{n} \sum_{i=1}^{n} \sigma_{r_i}^2
\]  

(3.27)

and then proceeds to compare this figure with \( \sigma_{e}^2 \) (the stationary raw sample variance in his error model) and develop nondestructive/destructive improvement factors at the smooth sample stage. It should thus be emphasized that any results thus obtained must assume essentially constant doppler over successive measurement blocks, since otherwise the raw-sample error process does not fit the stationary model assumed.

Moreover, the figure of merit given by Eq. (3.27) must itself be questioned, since the arithmetic mean of the variances indeed overcomes the nonstationary limitation of the smooth-sample error process, but does not take into account the correlation existing between the error samples at the smooth stage. The correlation coefficient appearing in Eq. (3.26) only exhibits the raw-sample correlation effect on each smooth sample variance, but no smooth-sample correlation considerations are accounted for in Eq. (3.27). If we consider the arithmetic mean of the smooth-sample errors (not of their variance) then our interest lies in the variance of a sum which differs from the sum of the variances given in (3.27) when the random variables (the smooth
sample errors) in question are correlated. It should be noted that such correlation will exist at the smooth sample stage since it already exists at the raw sample stage, and each smooth sample is derived using all the raw samples available as shown in Eq. (3.24).

It is evident that a more proper figure of merit is the variance of the arithmetic mean of the smooth-sample errors, and that of course this still leaves open the issue of the destructive raw samples having a non-stationary error process in general, which would limit any results on such extractor to constant doppler conditions. If we assume raw sample errors $\varepsilon_j$ with stationary variance $\sigma^2_{\varepsilon_j} = \sigma^2$ as before, the smooth sample error is given by (see Eq. 3.24)

$$\hat{e}_i = \sum_{j=1}^{n} b_j(i) \varepsilon_j$$

(3.28)

so that the arithmetic mean of the smooth-sample errors is then

$$\hat{e} = \frac{1}{n} \sum_{i=1}^{n} \hat{e}_i = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} b_j(i) \varepsilon_j$$

(3.29)

and its variance is given by

$$\sigma^2_e = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{i'=1}^{n} \sum_{j=1}^{n} \sum_{j'=1}^{n} b_j(i)b_{j'}(i') \rho_{jj'} \sigma^2_{\varepsilon_j} \sigma^2_{\varepsilon_{j'}}$$

(3.30)

The net effect of using the sum of the variances instead as the figure of merit is to neglect the contribution of the $i \neq i'$ terms in the above sum. The direct evaluation of Eq. (3.30) by substitution of the $b_j(i)$ given by Eqs. (3.24) and (3.25) is too complicated; it is better to go back and substitute such coefficients in Eq. (3.29) and study the results. The arithmetic mean of the smooth-sample errors then becomes
\[ 
\hat{\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i = -\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\det B_j(i)}{\det B} \varepsilon_j 
\]

since

\[
\sum_{i=1}^{n} \det B_j(i) = \sum_{i=1}^{n} \frac{\det B_j(i)}{\det B} = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_j 
\]

In summary, the random variable represented by the arithmetic mean of the smooth-sample errors is identical to the arithmetic mean of the raw-sample errors, so that their variances will be equal (in fact, all their statistics) and the data processor in question does not introduce any average-error improvements beyond those already existing at the raw sample stage. It should
be emphasized that the improvements cited in the aforesaid reference were derived using the sum of the smooth-sample variances (rather than the variance of the sum) as a figure of merit, and such choice is inappropriate since it neglects smooth-sample correlation effects and hence does not reflect the true characteristics of the error data, as evidenced by the fact that no improvement exists when the performance index is modified to account for such correlation effects.

3.5 Summary of Results and Conclusions

The error performance characteristics of destructive vs nondestructive range rate extractor realizations has been analyzed, both from single and multiple measurement considerations to account for error correlation effects, and both at the raw and smooth sample stage by assuming a least-squares polynomial-fit data processor. The implementation of both realizations was presented, and it was shown that the nondestructive-count procedure requires the measurement of the cycle fractions that occur just before and after the end of the measurement block if a proper error comparison between the two realizations is to be performed. It is tempting to neglect these fractional counts as small relative to the integer counts of bias-plus-doppler cycles, since it would eliminate the reference counter and the non real-time processing in the nondestructive extractor realization.

The rms range rate error caused by a stationary phase noise process accompanying the signal is given by Eq. (3.9) for a single-measurement, and such error is noted to be non-stationary for the destructive extractor as evidenced by the k-dependence. If we assume large counting times relative to the correlation time of the phase noise process, the rms error improvement
of the nondestructive extractor is a factor of $T_o/T_k \geq 1$ and also nonstationary unless essentially constant doppler can be assumed over successive measurements. If we assume a stationary uniform prior distribution for the doppler and consider the expected rms error in the destructive case, thus by-passing the k-dependence, the nondestructive improvement factor is then $T_o f_b/N$ but this must be then interpreted as an average effect.

The analysis of raw sample averaging towards an error reduction under constant doppler conditions must account for correlation effects over successive measurement blocks. If we assume adjacent-block correlation only, the error correlation coefficient is given by Eq. (3.13) and may be approximated by $-\frac{1}{2} \frac{R_\phi(U_k)}{R_\phi(o)}$ for the destructive extractor vs an exact $-\frac{1}{2}$ value for the nondestructive extractor, where $U_k$ is the dead zone duration. If we assume the phase noise correlation time to be small relative to the dead zone, then the destructive extractor will have no correlation over successive measurements and the rms error reduction by averaging will be limited to the conventional $1/\sqrt{n}$ factor for such extractor, whereas the $-\frac{1}{2}$ correlation exhibited by the nondestructive extractor will induce an extra $1/\sqrt{n}$ factor as evidenced by Eq. (3.15). For the third-order loops under consideration in the ATS-F/NIMBUS-E system, the rms correlation time of the thermal phase noise is $9/\sqrt{5}/B_n = 0.11s$ for $B_n = 200$ Hz while the dead zone durations range from 0.43 to 0.62 s for a GRARR-like destructive extractor operating at a 1 per second data rate, so that small correlation effects if any are expected for such extractor and the previous comparison is indeed representative.

The smooth samples were assumed as generated by a least-squares, polynomial-fit, data processor of interest at NASA/GSFC. Each smooth sample is a nonstationary linear combination of past and future raw samples, so
that error correlation effects on the latter are induced on the smooth samples obtained. It is important to account for such induced correlation when analyzing the smooth-sample errors, and hence to use a performance index that properly reflects such effects. The previous analyses of the problem employed the arithmetic mean of the variances of the smooth-sample errors as the figure of merit, and thus ignored the smooth-sample error correlation which can be accounted for by instead using the variance of the arithmetic mean of the errors as the figure of merit. The distinction is rather important since the previous analyses derived error improvement factors in favor of the nondestructive extractor (relative to the destructive case) as introduced by the data processor itself, but based on the inappropriate performance index cited above. However, we showed that the random variable represented by the arithmetic sum of the sample errors is exactly preserved in value at the input (raw samples) and output (smooth samples) of the data processor in question, so that all their statistics are identical and no error improvement factor is introduced by the processor beyond that already existing at the raw sample stage.
APPENDIX 3.1

The purpose of this section is to illustrate the relations involved in the single-measurement, nondestructive, range rate error of Eq. (3.7). The pertinent cases are presented in Fig. 17, which shows the three possible transitions (due to phase noise) that can happen at a zero-crossing occurring near the beginning or the end of a measurement block. In case (a), the noise causes a zero-crossing (hence a cycle) to be missed in the k-th interval count and an extra one counted in the (k+1)-th interval, in case (b) the converse occurs, and in case (c) no count is lost or gained in either of the two blocks. The net effect may be summarized as follows:

Case (a): $\Delta N'_k = +1$, $\Delta \delta_k = \frac{\Delta \phi}{2\pi} \frac{1}{f_b + f_d}$, $\Delta M_k = \frac{\Delta \phi}{2\pi} \frac{-1}{f_d}$

Case (b): $\Delta N'_k = -1$, $\Delta \mu_{k-1} = \frac{\Delta \phi}{2\pi} \frac{-1}{f_b + f_d}$, $\Delta M_k = \frac{\Delta \phi}{2\pi} \frac{1}{f_d}$

Case (c): $\Delta N'_k = 0$, $\Delta \delta_k = \frac{\Delta \phi}{2\pi} \frac{1}{f_b + f_d}$, $\Delta M_k = \frac{\Delta \phi}{2\pi} \frac{1}{f_d}$

Notice that these cases refer to either the beginning or the end of a counting block, so that we have the following nine possibilities:

<table>
<thead>
<tr>
<th>Transition at $kT_o$</th>
<th>$\Delta N'_k$</th>
<th>$\Delta M_k$</th>
<th>$\Delta M_{k+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$kT_o$</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_b + f_d}$</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_d}$</td>
<td>$(\Delta \phi \frac{2\pi}{1} + 1)$ $\frac{f_r}{f_d}$</td>
</tr>
<tr>
<td>fwd</td>
<td>0</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_b + f_d}$</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_d}$</td>
</tr>
<tr>
<td>fwd</td>
<td>2</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_b + f_d}$</td>
<td>$(\Delta \phi \frac{2\pi}{1} + 1)$ $\frac{f_r}{f_d}$</td>
</tr>
<tr>
<td>fwd</td>
<td>1</td>
<td>$(\Delta \phi \frac{2\pi}{1} - 1)$ $\frac{f_r}{f_b + f_d}$</td>
<td>$(\Delta \phi \frac{2\pi}{1} + 1)$ $\frac{f_r}{f_d}$</td>
</tr>
<tr>
<td>Action</td>
<td>Action</td>
<td>( \Delta N_k' )</td>
<td>( \Delta M_k )</td>
</tr>
<tr>
<td>--------</td>
<td>--------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td>bwd</td>
<td>fwd</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
<td>( \frac{1}{2 \pi} (\Delta \phi - 1) )</td>
</tr>
<tr>
<td>bwd</td>
<td>bwd</td>
<td>0</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
</tr>
<tr>
<td>bwd</td>
<td>none</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
</tr>
<tr>
<td>none</td>
<td>fwd</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
</tr>
<tr>
<td>none</td>
<td>bwd</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
</tr>
<tr>
<td>none</td>
<td>none</td>
<td>0</td>
<td>( \frac{1}{2 \pi} (\Delta \phi + 1) )</td>
</tr>
</tbody>
</table>

Hence, in general we have that \( \Delta N_k' = f I_1 \), \( \Delta M_k = (\frac{\Delta \phi}{2 \pi} - I_2) \) \( \frac{f r}{f b + f d} \), \( \Delta M_{k+1} = (\frac{\Delta \phi}{2 \pi} - I_3) \) \( \frac{f r}{f b + f d} \), where the I's represent the possible choices of integers illustrated by the table above. Using this notation, the range rate error may be written as

\[
(\Delta \tilde{r})_k = -\frac{c}{4\pi f s T_0} \left[ (\Delta \phi)_k - 2\pi(I_1 - I_2 + I_3) \right] = -\frac{c}{4\pi f s T_0} (\Delta \phi)_k
\]

where we have used the fact that \( I_1 - I_2 + I_3 = 0 \) as can be verified by the table above.

The fact that the same answer results for all cases is not surprising when reflecting that the measurement operation supplements the count \( N_k' \) with the measurements of \( \delta_k \) and \( \nu_k \), so that all data intervals are accounted for and any information gained or lost in the \( N_k' \) count via a zero-crossing transition is excluded or included in the \( \delta_k \) or \( \nu_k \) measurements. This argument could have been used to only analyze the simple no-transition case corresponding to \( \Delta N_k = 0 \) (last entry in table) and then claim the result valid in general, though one can question whether a-posteriori obvious results are indeed a-priori obvious.
Fig. 17 Effects of Noise on 1st or Last Zero-Crossing
4. IONOSPHERIC EFFECTS IN THE GRARR VHF RANGING SYSTEM

The problem of interest is to evaluate the time delay correction terms that are induced in the GRARR VHF range and range rate signals as a consequence of propagation through the ionosphere. The use of group and phase delay principles is a well-established approach to analyze the effects of the medium on sinusoidal fields, but the direct use of existing results on modulated field applications requires further analytical support. To this effect, we first summarize the group and phase delay concepts and their wave propagation interpretation, and then consider the effect of the ionosphere medium ionization (but neglecting electron collisions and the earth's magnetic field as usual) on the analysis. The distinction between unmodulated vs modulated field derivations is rigorously compared to illustrate the constraints that will permit the use of conventional unmodulated-field results in modulated-field applications. In particular, the case of a ranging tone PM'ed on a sinusoidal carrier is analyzed in detail and two solution approaches are presented for narrowband PM conditions: one based on phase/group delays and one based on phase delays only without engaging in group concepts. The results are then applied to the GRARR VHF ranging system and ionospheric time-delay expressions are derived for the range and range rate signals extracted at the receiver. The results are then interpreted, and a logical short procedure that bypasses the longer derivation is presented and illustrated to reproduce the results.
4.1 Group and Phase Delays

The analysis of bandpass signals and their processing is often more conveniently done by exploiting their complex envelope representation. The relation between a real bandpass signal \( v(t) \) and its complex envelope \( u(t) \) is given by

\[
v(t) = \text{Re}\{u(t)e^{j\omega_c t}\}
\]

where \( \omega_c \) is the carrier angular frequency. The complex envelope \( u(t) \) specifies the joint AM-PM modulating signal since the polar form \( u(t) = a(t)e^{j\theta(t)} \) yields

\[
v(t) = a(t) \cos[\omega_c t + \theta(t)]
\]

so that the magnitude and phase functions of the complex signal \( u(t) \) indeed correspond to the amplitude \( a(t) \) and phase \( \theta(t) \) modulations on the carrier.

There is a one-to-one correspondence between a bandpass signal \( v(t) \) and its complex envelope \( u(t) \) provided the latter is lowpass relative to the carrier frequency, and such correspondence permits the analysis of bandpass signals to be effected in terms of their complex envelopes. For example, the passage of a bandpass signal \( v_1(t) \), with complex envelope \( u_1(t) \) and carrier \( \omega_c \), through a bandpass filter with impulse response \( h(t) \) and center frequency \( \omega_c \) will yield an output bandpass signal \( v_0(t) \) with complex envelope

\[
u_0(t) = \frac{1}{2} u_1(t) \otimes g(t)
\]

and carrier \( \omega_c \), where \( g(t) \) is the complex envelope of \( h(t) \) and \( \otimes \) is the convolution operator.
In particular, consider a bandpass filter $H(\omega) = e^{-j\phi(\omega)}$ with a unity magnitude spectrum and a phase spectrum $\phi(\omega)$ that can be expanded as a power series

$$\phi(\omega) = \sum_{i=0}^{\infty} \frac{K_i}{i!} (\omega - \omega_c)^i, \quad \omega > 0$$  \hfill (4.4)

around the carrier frequency. If we can neglect the nonlinear terms in such expansion, then the complex envelope of the filter can be shown to be

$$g(t) = 2e^{-jK_0\delta(t-K_1)}$$  \hfill (4.5)

so that the complex envelope of the output is given by

$$u_0(t) = \frac{1}{2} u_1(t) \otimes 2e^{-jK_0\delta(t-K_1)} = e^{-jK_0\delta(t-K_1)} u_1(t-K_1)$$  \hfill (4.6)

and the output bandpass signal is then

$$v_o(t) = \text{Re}(u_0(t)e^{j\omega_c t}) = \text{Re}(u_1(t-K_1)e^{j(\omega_c t-K_0)})$$

$$= a(t-K_1)\cos[(\omega_c t-K_0) + \theta(t-K_1)]$$  \hfill (4.7)

In summary, assuming the narrowband approximations pertinent to the complex envelope representation of bandpass signals, and also assuming a linear phase spectrum for the bandpass filter centered at the carrier frequency, then the unmodulated carrier exhibits a "phase delay"

$$t_{ph} = \frac{K_0}{\omega_c} = \frac{\phi(\omega_c)}{\omega_c}$$  \hfill (4.8a)

while the modulating signals exhibit a "group delay"

$$t_{gr} = K_1 = \phi'(\omega_c)$$  \hfill (4.8b)
and these two terms are related by

\[ t_{gr}(\omega) = \frac{d}{d\omega} \left[ \omega t_{ph}(\omega) \right] = t_{ph}(\omega) + \omega \dot{t}_{ph}(\omega) \quad (4.9) \]

### 4.2 Wave Propagation Fundamentals

We first consider the solution of Maxwell's equation in a medium of permittivity \( \varepsilon_0 \), permeability \( \mu_0 \), and conductivity \( \sigma \). If we assume a plane wave with electric field vector in the x-direction, magnetic field vector in the y-direction and Poynting vector in the z-direction, then the curl \( \nabla \times E \) and curl \( \nabla \times H \) equations simplify to

\[
\frac{\partial E(x,t)}{\partial x} = - \frac{\partial B(x,t)}{\partial t}, \quad - \frac{1}{\mu_0} \frac{\partial B(x,t)}{\partial x} = \sigma E(x,t) + \varepsilon_0 \frac{\partial E(x,t)}{\partial t} \quad (4.10)
\]

so that the time and frequency domain forms of the wave equation are

\[
\frac{\partial^2 E(x,t)}{\partial x^2} = \sigma \mu_0 \frac{\partial E(x,t)}{\partial t} + \varepsilon_0 \frac{\partial^2 E(x,t)}{\partial t^2} \quad (4.11a)
\]

\[
\frac{\partial^2 E(x,\omega)}{\partial x^2} = j\omega \sigma \mu_0 E(x,\omega) + (j\omega)^2 \varepsilon_0 E(x,\omega) \quad (4.11b)
\]

and the frequency domain solution is given by

\[
\frac{E(x,\omega)}{E(0,\omega)} = e^{-\alpha(\omega)x} e^{-j\beta(\omega)x} \quad (4.12)
\]

where

\[
\alpha(\omega) = \omega \left[ \frac{\sigma}{2} \left( \sqrt{1 + \frac{\sigma^2}{\varepsilon_0 \mu_0 \omega^2}} - 1 \right) \right]^{1/2} \quad \longrightarrow 0 \quad \text{if} \quad \sigma \longrightarrow 0 \quad (4.12a)
\]

\[
\beta(\omega) = \omega \left[ \frac{\sigma}{2} \left( \sqrt{1 + \frac{\sigma^2}{\varepsilon_0 \mu_0 \omega^2}} + 1 \right) \right]^{1/2} \quad \longrightarrow \omega/\sqrt{\mu_0 \varepsilon_0} \quad \text{if} \quad \sigma \longrightarrow 0 \quad (4.12b)
\]

In the case of \( \sigma = 0 \) so that no frequency-dependent attenuation exists, the transfer function \( e^{-j\beta(\omega)x} \) is analogous to the filter effect considered in the previous section for fixed \( x \), with \( \phi(\omega) \) now given by \( \beta(\omega)x \) and being
dependent on the propagation distance \( x \). In particular, the phase and group delay expressions now become

\[
t_{\text{ph}} = \frac{x\beta(\omega_c)}{\omega_c}, \quad t_{\text{gr}} = \frac{x\beta'(\omega_c)}{\beta'(\omega_c)} \quad (4.13)
\]

and the fact that the equivalent filter interpretation yields input-output expressions corresponding to a propagation distance \( x \) then permits us to also introduce phase and group velocities as

\[
v_{\text{ph}} = \frac{x}{t_{\text{ph}}} = \frac{\omega_c}{\beta(\omega_c)}, \quad v_{\text{gr}} = \frac{x}{t_{\text{gr}}} = \frac{1}{\beta'(\omega_c)} \quad (4.14)
\]

which respectively represent the speed at which the carrier and modulating signals propagate through the medium.

Notice that the carrier/modulation interpretation of these time delays and velocities requires a linear \( \beta(\omega) \), since otherwise the results of the last section are not applicable (in particular, the expression for \( g(t) \) is not valid, and such expression is responsible for yielding an output modulation identical to the input modulation except for a delay).

In our development we have assumed \( \mu \) and \( \xi \) to be constants independent of frequency \( \omega \) or distance \( x \), and then \( \beta(\omega) \) is indeed linear in \( \omega \) with slope \( \sqrt{\mu_0 \xi_0} \) and zero intercept, so that \( t_{\text{ph}} = t_{\text{gr}} = x \sqrt{\mu_0 \xi_0} = x/c \) and \( v_{\text{ph}} = v_{\text{gr}} = 1/\sqrt{\mu_0 \xi_0} = c \) in such case.

### 4.3 Ionospheric Wave Propagation

We next consider the effect of medium ionization, though neglecting electron collisions and the earth's magnetic field. The electronic motion in the presence of a plane wave with an electric field \( E_c \cos \omega_c t \) satisfies
the force equation

\[ m \frac{dv}{dt} = q E_c \cos \omega_c t \]  

(4.15)

where \( m \) and \( q \) are the electron mass and charge respectively, with the ion mass having been neglected relative to the electron mass. The electron velocity obtained as the solution of the equation is the used to specify convection and displacement current-densities as

\[ i_c = N q v = \frac{N q^2}{m \omega_c} E_c \sin \omega_c t \]  

(4.16a)

\[ i_d = \varepsilon_0 \frac{\partial E}{\partial t} = -\varepsilon_0 \omega_c E_c \sin \omega_c t \]  

(4.16b)

where \( N \) is the electron content of the medium (assumed first to be constant).

The total current-density is then given by

\[ i_d + i_c = -\varepsilon_0 \omega_c (1 - \frac{N q^2}{m \varepsilon_0 \omega_c^2})(E_c \sin \omega_c t) = -\varepsilon_0 \omega_c (1 - \frac{\omega_p^2}{\omega_c^2})(E_c \sin \omega_c t) \]  

(4.17)

where \( \omega_p = (\frac{N q^2}{m \varepsilon_0})^{\frac{1}{2}} \) is the critical (plasma) frequency above which ionospheric reflections disappear. The expression (4.17) illustrates the effective reduction of the permittivity, dielectric constant and refractive index to

\[ n = \sqrt{\frac{\varepsilon_c}{\varepsilon_0}} = \sqrt{1 - \frac{\omega_p^2}{\omega_c^2}} \]  

(4.18)

and for the special case of sinusoidal excitation and constant electron content then the equivalent phase spectrum in the transfer function \( e^{-j\beta(\omega)x} \) of (4.12) would be given by

\[ x_\beta(\omega) = x_\omega \sqrt{\frac{\varepsilon_0 \omega_c}{\mu_0}} = x \sqrt{\frac{\varepsilon_0}{\mu_0}} \sqrt{\frac{\omega^2 - \omega_p^2}{\omega_c^2}} = \frac{x}{c} \sqrt{\frac{\omega^2 - \omega_p^2}{\omega_c^2}} \]  

(4.19)
which is nonlinear in \( \omega \). The power series expansion of this expression about \( \omega = \omega_c \) is given by

\[
x_{\beta}(\omega) = x[K_0 + K_1(\omega - \omega_c) + \frac{K_2}{2}(\omega - \omega_c)^2 \ldots]
\]

(4.20)

where

\[
K_0 = \beta(\omega_c) = \frac{\omega_c}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right) \approx \frac{\omega_c}{c} \left(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}\right) \text{ for } \omega_p^2 \ll \omega_c^2.
\]

\[
K_1 = \beta'(\omega_c) = \frac{1}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right)^{-\frac{1}{2}} \approx \frac{1}{c} \left(1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}\right) \text{ for } \omega_p^2 \ll \omega_c^2.
\]

\[
K_2 = \beta''(\omega_c) = -\frac{\omega_c^2}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right)^{-\frac{3}{2}} \approx -\frac{\omega_c^2}{c} \left(1 + \frac{3}{2} \frac{\omega_p^2}{\omega_c^2}\right) \text{ for } \omega_p^2 \ll \omega_c^2.
\]

(4.20a)

so that a linear approximation for \( x_{\beta}(\omega) \) is indeed valid for \( \omega_p^2 \ll \omega_c^2 \). The phase delay and velocity are then given by

\[
t_{\text{ph}} = \frac{x_{\beta}(\omega_c)}{\omega_c} = \frac{x}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right) \approx \frac{x}{c} \left(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}\right)
\]

(4.21a)

\[
v_{\text{ph}} = \frac{\omega_c}{\beta'(\omega_c)} = c(1 - \frac{\omega_p^2}{\omega_c^2}) \approx c(1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2})
\]

(4.21b)

and the corresponding group delay and velocity would be given by

\[
t_{\text{gr}} = x_{\beta'}(\omega_c) = \frac{x}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right)^{-\frac{1}{2}} \approx \frac{x}{c} \left(1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}\right)
\]

(4.22a)

\[
v_{\text{gr}} = \frac{1}{\beta'(\omega_c)} = c(1 - \frac{\omega_p^2}{\omega_c^2}) \approx c(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_c^2})
\]

(4.22b)

However, it must be understood that the use of the group delay and velocity as derived above in modulated field applications is not only conditioned on a constant electron content \( N \) independent of distance \( x \),
but also on the assumption that the presence of modulation does not essentially affect the refractive index expression previously derived using an unmodulated sinusoidal field in the force equation. For example, it should be noted that the results of the two previous sections did not involve any unmodulated wave assumption inherently linked to the derivation, since the results of section 1 are based on filter input/output transfer functions hence valid regardless of the input spectrum, and the equivalent transfer function $e^{-\alpha(\omega)x}e^{-j\beta(\omega)x}$ of (4.12) in section 2 was obtained by solving the equation in the frequency domain employing transform relations also applicable regardless of the excitation. Hence the group vs phase delay or velocity notions induced by the developments in these two sections remain well-defined for modulated sinusoidal excitations.

On the other hand, the effective permittivity $\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\omega_0^2}{\omega^2}\right)$ of (4.18) in section 3 was obtained not via transform techniques, but from the solution to the force equation for an unmodulated sinusoidal excitation. As a result, only phase delay or velocity notions are meaningful while the corresponding group concepts may not be so since the same $\varepsilon(\omega)$ need not result when a modulated sinusoid is used as the excitation in the derivation in question. It can be noted that such an excitation in the force equation $m \frac{dv}{dt} = qE(t)$ will in general yield convection and displacement current densities that no longer differ simply by the $1 - \frac{\omega_0^2}{\omega^2}$ factor previously obtained, and hence the effective dielectric constant may not exhibit this dependence for an arbitrary modulating function, and in fact the dependence will be governed by the modulation time-variation itself.
A possible solution method could be based on quasi-static solutions; i.e., treating the modulation as a constant in the force equation thus restricting all results to specific time-duration constraints and ultimately verifying the latter hold for the propagation times of interest in a particular application. A more direct solution method is to assume the particular modulation of interest and analyzing the resultant force equation with the modulated sinusoidal excitation, towards the derivation of the specific $e(\omega)$ dependence that results. This last approach is advantageous in multitone PM applications since the modulated excitation appearing in the force equation can then be written as a sum of unmodulated sinusoids using the conventional Bessel expansion procedure, and the differentiation and integration operations involved in the derivation of the displacement and convection current-densities can be performed component-wise on the series.

In particular, for the case of a single-tone PM the force equation reads

$$m \frac{dv}{dt} = qE_c \cos[\omega_c t + \delta \sin \omega_m t] = qE_c \sum_K J_K(\delta) \cos(\omega_c + K\omega_m)t$$

(4.23)

so that

$$v = \frac{qE_c}{m\omega_c} \sum_K \frac{J_K(\delta)}{K\omega_m} \frac{\sin(\omega_c + K\omega_m)t}{1 + \frac{K\omega_m}{\omega_c}}$$

(4.24a)

$$i_c = \frac{Nq^2E_c}{m\omega_c} \sum_K \frac{J_K(\delta)}{K\omega_m} \frac{\sin(\omega_c + K\omega_m)t}{1 + \frac{K\omega_m}{\omega_c}}$$

(4.24b)
\[ i_d = -\varepsilon_0 \omega_c E_c \sum_K J_K(\delta) \cdot \left(1 + \frac{K_{\omega m}}{\omega_c}\right) \sin(\omega_c + K_{\omega m})t \]  

(4.24c)

\[ i_d + i_c = -\varepsilon_0 \omega_c E_c \sum_K J_K(\delta) \left[\left(1 + \frac{K_{\omega m}}{\omega_c}\right) - \frac{Nq^2}{mc^2} \left(1 + \frac{K_{\omega m}}{\omega_c}\right)^{-1}\right] \sin(\omega_c + K_{\omega m})t \]  

(4.24d)

We can now follow two possible approaches: (a) We assume sufficiently narrowband PM and recognize that whenever \(1 + \frac{K_{\omega m}}{\omega_c} \approx 1\) does not hold in (4.24d) such term is multiplied by \(J_K(\delta) = 0\) so that we can rewrite the total current density as

\[ i_d + i_c \approx -\varepsilon_0 \omega_c \left(1 - \frac{Nq^2}{mc^2} \right) E_c \sum_K J_K(\delta) \sin(\omega_c + K_{\omega m})t \]

\[ = -\varepsilon_0 \omega_c \left(1 - \frac{\omega_P^2}{\omega_c^2}\right) E_c \sin(\omega_c t + \delta \sin(\omega_m t)) \]  

(4.25)

which shows that the same \(\varepsilon(\omega)\) of (4.18) results in the presence or absence of our specific modulation, so that group delay and velocity concepts are indeed applicable under the narrowband PM conditions.

(b) We consider the modulated signal as a series of sinusoids with related amplitudes and phases, and use only phase delay and velocity notions component-wise without engaging in group concepts. The total current-density can be exactly written as

\[ i_d + i_c = -\varepsilon_0 \sum_K J_K(\delta) \cdot (\omega_c + K_{\omega m}) \left[1 - \frac{Nq^2}{mc^2 (\omega_c + K_{\omega m})^2}\right] \sin(\omega_c + K_{\omega m})t \]  

(4.26)

so that the individual component phase delay and velocity are

\[ t_{ph}(\omega_c + K_{\omega m}) = \frac{\omega_P^2}{c} \left[1 - \left(\frac{\omega_P}{\omega_c + K_{\omega m}}\right)^2\right]^{\frac{1}{2}} \]  

(4.27a)
\[ v_{ph}(\omega_{c+k\omega_m}) = c \left[ 1 - \frac{\omega_p^2}{(\omega_{c+k\omega_m})^2} \right]^{-\frac{3}{2}} \]  

(4.27b)

and we can then take as many components as dictated by the modulation index and study their propagation delay \( t_{ph}(\omega_{c+k\omega_m}) \) in a given application. Under narrowband PM conditions, the ultimate results derived by working only with the phase delays of carrier and sideband components should of course agree with the first approach based on carrier phase delay and modulation group delay techniques.

Finally, the case where the electron content \( N \) and the parameter \( \omega_p \) are not constant in the link but vary along the propagation path can be handled by breaking down the path into subpaths \( \Delta r_i \) of essentially constant \( \omega_{p,i} \) parameter and adding the phase shifts or time delays of the subpaths. This procedure leads to an equivalent plasma frequency term

\[ \omega_{p,eq}^2 = \lim_{\Delta r_i \to 0} \frac{1}{R} \sum_i \omega_{p,i}^2 \Delta r_i = \frac{q^2}{m \epsilon_0} \int \frac{N \cdot dr}{j \omega c} = \frac{q^2 N_{eq}}{m \epsilon_0} \]  

(4.28)

where \( R = \int dr \) is the net propagation distance involved. In what follows we will not distinguish between \( \omega_p^2 \) and \( \omega_{p,eq}^2 \) for simplicity of notation, but it should be understood that the latter is of course implied if the application requires it.

4.4 Application to Ranging Systems

In order to illustrate the application of the results, we first consider a simple one-way link with a single-tone, narrowband-PM, ranging signal. The transmitter signal is thus effectively represented by a carrier plus a pair of antisymmetrical PM sidebands, and the receiver processing
consists of the coherent extraction of the carrier followed by product
demodulation of the ranging tone. The transmitted and received r-f signals
are thus given by

\[
\text{XMTD: } \cos[\omega_c t + \theta_c] + \cos[(\omega_c + \omega_m) t + (\theta_c + \theta_m)] \tag{4.29a}
\]

\[
\text{RCVD: } \cos[\omega_c (t - T_c) + \theta_c] + \cos[\omega_c (t - T_{sb}) + (\theta_c + \theta_m)] \tag{4.29b}
\]

where the amplitudes of the components have been normalized for simplicity,
and where the propagation delays of the carrier and sideband components are
respectively given by

\[
T_c = \frac{R}{c} \left(1 - \frac{\omega_p^2}{\omega_c^2}\right)^{\frac{1}{2}} \tag{4.30a}
\]

\[
T_{sb} = \frac{R}{c} \left(1 - \frac{\omega_p^2}{(\omega_c + \omega_m)^2}\right)^{\frac{1}{2}} \tag{4.30b}
\]

The receiver then extracts a quadrature replica of the received

\[
\sin[\omega_c (t - T_c) + \theta_c] \tag{4.31}
\]

carrier as

and the product demodulation of the sideband components yields

\[
\pm \sin[\omega_m t + \theta_m - (\omega_c + \omega_m) T_{sb} + \omega_c T_c] = \sin[\omega_m t + \theta_m - (\omega_c + \omega_m) T_{sb} + \omega_c T_c] \tag{4.32}
\]

so that the carrier and tone phase shifts due to propagation delays are
respectively given by (for \(\omega_p^2 < \omega_c^2\)):  

\[
-\omega_c T_c = -\frac{R}{c} \omega_c \left(1 - \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}\right) \tag{4.33a}
\]
\[
\bar{f}(\omega_c + \omega_m)T_{sb} + \omega_c T_c \approx - \frac{R}{c} \omega_m \left( 1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2} \right) \approx - \frac{R}{c} \omega_m \left( 1 + \frac{1}{2} \frac{\omega_D^2}{\omega_c^2} \right) (4.33b)
\]

and these results identify the decomposition into a free-space nominal term plus a correction term caused by the ionosphere. It can be noted that if these phase delays are referred to a common frequency by normalizing by the ratio \( \omega_m / \omega_c \) so that we are essentially looking at time delays, then the correction terms present in the carrier and tone will be of the same magnitude but opposite signs.

We next reproduce the results by working with group and phase delay concepts. The carrier phase delay is obtained from (4.21a) as

\[
T_c = \frac{R}{c} \left( 1 - \frac{\omega_p^2}{\omega_c^2} \right) \approx \frac{R}{c} \left( 1 - \frac{1}{2} \frac{\omega_p^2}{\omega_c^2} \right) (4.34a)
\]

and the modulation group delay is obtained from (4.22a) as

\[
T_m = \frac{R}{c} \left( 1 - \frac{\omega_p^2}{\omega_c^2} \right) \approx \frac{R}{c} \left( 1 + \frac{1}{2} \frac{\omega_D^2}{\omega_c^2} \right) (4.34b)
\]

which agrees with the previous results of (4.33a) and (4.33b).

4.5 Application to the GRARR VHF Ranging System

We next consider the two-way GRARR link of interest as illustrated in Figure 18. and we refer to the corresponding expressions characterizing the signal processing and presented in Table 7. The uplink carrier \( \omega_T \) and modulating tone \( \omega_s \) specify the uplink signal carrier and major sidebands of \( \textcircled{1} \) and \( \textcircled{2} \), at the ground transmitter and transponder receiver respectively. The GRARR transponder processing is specified by \( \textcircled{3} \) and \( \textcircled{4} \), and the ground received signal components are given in \( \textcircled{5} \). The change in
FIGURE 18 GRARR VHF SYSTEM PROCESSING
(1) \[ \cos(\omega t + \theta_T) \pm \cos((\omega_T \pm \omega_S)t + (\theta_T \pm \theta_S)) \]

(2) \[ \cos(\omega(t - T_1) + \theta_T) \pm \cos((\omega_T \pm \omega_S)(t - T_2) + (\theta_T \pm \theta_S)) \]

\[ T_1 = \frac{R}{c} \left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad T_2 = \frac{R}{c} \left(1 - \frac{v^2}{(\omega_T \pm \omega_S)^2}\right)^{1/2} \]

(3) \[ \cos((\omega - M_0 \omega_0)t + (\theta_T - M_0 \theta_0 - \omega_T T_1)) \pm \cos((\omega_T \pm \omega_S - M_0 \omega_0)t + (\theta_T \pm \theta_S - M_0 \theta_0 - (\omega_T \pm \omega_S)T_2) \pm) \]

(4) \[ \cos(N_0 t + \theta_0) \]

\[ \pm \cos([N_0 \pm (\omega_T - M_0 \omega_0)]t + [N_0 \pm (\theta_T - M_0 \theta_0 - \omega_T T_1)]) \]

\[ \pm \cos([N_0 \pm (\omega_T + \omega_S - M_0 \omega_0)]t + [N_0 \pm (\theta_T + \theta_S - M_0 \theta_0 - (\omega_T + \omega_S)T_2) \pm]) \]

\[ \pm \cos([N_0 \pm (\omega_T - \omega_S - M_0 \omega_0)]t + [N_0 \pm (\theta_T - \theta_S - M_0 \theta_0 - (\omega_T - \omega_S)T_2 -)]) \]
\[\cos(N_0(t-T_3)+\theta_0)\]

\[\pm \cos([N_0^\pm(\omega - M_0)](t-T_4^\pm)+[N_0^\pm(\theta_1-M_0)-(\omega - T_1)])\]

\[\pm \cos([N_0^\pm(\omega + \omega - M_0)](t-T_5^\pm)+[N_0^\pm(\theta + \theta - M_0)-(\omega + \omega - T_2^\pm)])\]

\[\mp \cos([N_0^\pm(\omega - \omega - M_0)](t-T_6^\pm)+[N_0^\pm(\theta - \theta - M_0)-(\omega - \omega - T_2^-)])\]

\[T_3 = \frac{R}{c} \left(1 - \frac{\omega_2^2}{(N_0^2)^2}\right), \quad T_4^\pm = \frac{R}{c} \left(1 - \frac{\omega_2^2}{[N_0^\pm(\omega - M_0)]^2}\right)\]

\[T_5^\pm = \frac{R}{c} \left(1 - \frac{\omega_2^2}{[N_0^\pm(\omega - \omega - M_0)]^2}\right), \quad T_6^\pm = \frac{R}{c} \left(1 - \frac{\omega_2^2}{[N_0^\pm(\omega - \omega - M_0)]^2}\right)\]

or

\[\cos(\omega c^\pm \theta c - \omega c T_c)\]

\[\pm \cos[(\omega c^\pm m_1) t + (\theta_1^\pm m_1) - (\omega c^\pm m_1) T_{s1}^\pm]\]

\[\pm \cos[(\omega c^\pm m_2) t + (\theta_2^\pm m_2) - (\omega c^\pm m_2) T_{s2}^\pm]\]

\[\mp \cos[(\omega c^\pm m_3) t + (\theta_3^\pm m_3) - (\omega c^\pm m_3) T_{s3}^\pm]\]

\[\omega_c = N_0 \omega_0, \quad \theta_c = N_0 \theta_0\]

\[\omega_{m1} = \omega - M_0, \quad \theta_{m1} = \theta - M_0 - \omega - T_1\]

\[\omega_{m2} = \omega + \omega - M_0, \quad \theta_{m2} = \theta + \theta - M_0 - (\omega + \omega - T_2^+)\]

\[\omega_{m3} = \omega - \omega - M_0, \quad \theta_{m3} = \theta - \theta - M_0 - (\omega - \omega - T_2^-)\]

\[T_c = T_3, \quad T_{s1} = T_4, \quad T_{s2} = T_5, \quad T_{s3} = T_6\]
\[ \sin(\omega_c t) = \sin[(\omega_c - \omega_0) t + (\theta_c - \theta_0 - \omega c T_c)] \]

\[ \sin[\omega_0 t + \theta_0] \]

\[ \pm \sin[(\omega_0 - \omega_1) t + \theta_0 - \omega_1 T_c + (\omega c - \omega_1) T_{s_1}] \]

\[ \pm \sin[(\omega_0 - \omega_2) t + \theta_0 - \omega_2 T_c + (\omega c - \omega_2) T_{s_2}] \]

\[ \pm \sin[(\omega_0 - \omega_3) t + \theta_0 - \omega_3 T_c + (\omega c - \omega_3) T_{s_3}] \]

\[ \sin[M(\omega_c - \omega_5) t + M(\theta_c - \theta_5)] \]

\[ \sin[\omega_m t + \theta_m] \pm \omega c T_c + (\omega c - \omega_m) T_{s_1} \]

\[ \pm \sin[\omega_m t + \theta_m] \pm \omega c T_c + (\omega c - \omega_m) T_{s_2} \]

\[ - \sin[\omega_m t + \theta_m] \pm \omega c T_c + (\omega c - \omega_m) T_{s_3} \]

or

\[ \sin[\omega_m t + \theta_m] - \frac{R}{c} \omega_m (1 + \frac{\omega_p^2}{\omega_c^2}) \approx \sin[\omega_m t + \theta_m - \frac{R}{c} \omega_m (1 + \frac{\omega_2^2}{\omega_c^2})] \]

or

\[ \sin[\omega_m t + \theta_m] - \frac{R}{c} \omega_m (1 + \frac{\omega_p^2}{\omega_c^2}) \approx \sin[\omega_m t + \theta_m - \frac{R}{c} \omega_m (1 + \frac{\omega_2^2}{\omega_c^2})] \]

or

\[ \sin[\omega_m t + \theta_m] - \frac{R}{c} \omega_m (1 + \frac{\omega_p^2}{\omega_c^2}) \approx \sin[\omega_m t + \theta_m - \frac{R}{c} \omega_m (1 + \frac{\omega_2^2}{\omega_c^2})] \]

or

\[ \sin[\omega_m t + \theta_m] - \frac{R}{c} \omega_m (1 + \frac{\omega_p^2}{\omega_c^2}) \approx \sin[\omega_m t + \theta_m - \frac{R}{c} \omega_m (1 + \frac{\omega_2^2}{\omega_c^2})] \]
$$\cos[\omega_V' t + \theta_V'] = \cos[(\omega_{10} + \omega_{sc}) t + \theta_{10} + \theta_{sc} - \omega_1 T_1 - \frac{R}{c} \omega_{sc} (1 + \frac{1}{2} \frac{\omega^2}{\omega_c^2})]$$

$$\sin[\omega_S t + \theta_S \pm \omega_T T_1 \pm \frac{R}{c} \omega_{sc} (1 + \frac{1}{2} \frac{\omega^2}{\omega_c^2}) \mp (\omega_1 \pm \omega_S) T_2 \mp \frac{R}{c} (\omega_{sc} \pm \omega_s) (1 + \frac{1}{2} \frac{\omega^2}{\omega_c^2})]$$

$$\sin[((\omega_T + \omega_{g7}) - M(\omega_v - \omega_5)] t + [(\theta_T + \theta_{g7}) - M(\theta_v - \theta_5)]$$

$$\sin[(\omega_V' - (\omega_T + \omega_{g7}) + M(\omega_v - \omega_5)] t + [(\theta_V' - (\theta_T + \theta_{g7}) + M(\theta_v - \theta_5)]$$
notation shown in 5 identifies the downlink carrier by \( \omega_c \) and the uplink
modulating sidebands (subcarrier plus uplink tone sidebands) by \( \omega_{m_i}, i=1 \) to 3,
with corresponding downlink delays \( T_c \) and \( T_{sbi} \).

The assumption of a carrier loop VCO signal given by \( \sin(\omega_v t + \theta_v) \) and
zero locking errors in the carrier loop then permits us to write the fre-
quency and phase lock conditions as

\[
N\omega_v - \omega_c - \omega_{60} = 0 \quad (4.35a)
\]

\[
N\theta_v - \theta_c - \theta_{60} + \omega_c T_c = 0 \quad (4.35b)
\]

by following the carrier component from the input 5 to the phase detector
output. The input 7 and output 9 expressions for the wideband phase
detector are then specified, and the relation

\[
\frac{1}{2} \omega_c T_c \frac{\omega_c}{\omega_{m_i}} \frac{\omega_{m_i}}{\omega_c} \frac{1}{2} \omega_c^2 = - \frac{R}{c} \omega_{m_i} (1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2})^2 - \frac{R}{c} \omega_{m_i} (1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}) \frac{1}{2} \omega_c^2 \quad (4.36)
\]

used in 9 is merely a replica of the one-way link example presented in the
previous section (indeed, the subcarrier and uplink tone-sideband extraction
from the downlink carrier signal is representative of that example).

The change in notation shown in 9 then identifies the downlink sub-
carrier by \( \omega_{sc} \) and the tone sidebands by \( \omega_{sc} + \omega_s \). The assumption of a
subcarrier loop VCO signal given by \( \cos(\omega_v t + \theta_v) \) and zero locking errors in
the subcarrier loop yields the frequency and phase lock conditions as

\[
\omega_v' - \omega_{10} - \omega_{sc} = 0 \quad (4.37a)
\]

\[
\theta_v' - \theta_{10} - \theta_{sc} + \omega_T T_1 + \frac{R}{c} \omega_{sc} (1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2}) = 0 \quad (4.37b)
\]
by following the subcarrier signal from the input $9$ to the phase detector output $11$. The expression for the latter is then specified and the substitution of $\omega_c, \omega_{sc}, T_1$ and $T_2^+$ then yields the extracted tone as

$$
\sin \left\{ \omega_s t + \theta_s - \frac{R}{c} \left[ 2\omega_s + \frac{1}{2} \frac{\omega_p^2}{\omega_T} \cdot \frac{2 \omega_s \omega_{sc}}{\omega_T^2 - \left( \frac{\omega_{sc}}{\omega_T} \right)^2} \right] - \frac{1}{2} \omega_s \omega_p^2 \left( \frac{1}{\omega_T^2} - \frac{1}{N^2 \omega_o^2} \right) \right\}
$$

$$
\approx \sin \left\{ \omega_s t + \theta_s - 2 \frac{R}{c} \omega_s \left[ 1 + \frac{1}{4} \frac{\omega_p^2}{\omega_T^2} \left( \frac{1}{\omega_T^2} + \frac{1}{N^2 \omega_o^2} \right) \right] \right\} \quad (4.38)
$$

which clearly illustrates the free-space and ionospheric correction terms present in the range signal.

In turn, the range rate signal extraction can be followed from $6$ and $10$ up to $14$ to show the given expression. The net frequency and phase of the range rate signal $14$ can be solved using

$$
\omega_v' = \omega_{10} + \omega_{sc} = \omega_{10} + \omega_T - M\omega_o \quad (4.39a)
$$

$$
\omega_v = \frac{1}{N} \left( \omega_c + \omega_{60} \right) = \omega_o + \omega_{12} \quad (4.39b)
$$

and

$$
\theta_v' = \theta_{10} + \theta_{sc} - \frac{R}{c} \omega_{sc} \left( 1 + \frac{1}{2} \frac{\omega_p^2}{\omega_c^2} \right) = \theta_{10} + \theta_T - M\theta_o - \frac{R}{c} \omega_T \left( 1 - \frac{1}{2} \frac{\omega_p^2}{\omega_T^2} \right)
$$

$$
- \frac{R}{c} \left( \omega_T - M\omega_o \right) \left( 1 + \frac{1}{2} \frac{\omega_p^2}{N^2 \omega_o^2} \right) \quad (4.40a)
$$

$$
\theta_v = \frac{1}{N} \left( \theta_c + \theta_{60} - \omega_c T \right) = \theta_o + \theta_{12} - \frac{R}{c} \omega_o \left( 1 - \frac{1}{2} \frac{\omega_p^2}{N^2 \omega_o^2} \right) \quad (4.40b)
$$

to get the results

$$
\omega_{\text{net}} = \omega_{10} - \omega_{9.97} = \omega_{0.03} \quad (30 \text{ kHz}) \quad (4.41a)
$$
\[ \theta_{\text{net}} = \theta_{0.03} - 2 \frac{R}{c} \omega_T \left[ 1 - \frac{1}{4} \omega_p^2 \left( \frac{1}{\omega_T} - \frac{1}{N^2 \omega_o^2} + \frac{2M \omega_o / \omega_T}{N^2 \omega_o^2} \right) \right] \]

\[ \theta_{\text{net}} = \theta_{0.03} - 2 \frac{R}{c} \omega_T \left[ 1 - \frac{1}{4} \omega_p^2 \left( \frac{1}{\omega_T} + \frac{1}{N^2 \omega_o^2} - \frac{2(\omega_T - M \omega_o)}{\omega_T (N^2 \omega_o^2)} \right) \right] \] (4.41b)

which illustrates the free-space and ionospheric correction terms present besides the 30 kHz oscillator phase.

A comparison between the range and range rate signal time delays (phase delays normalized to a common frequency) shows that they exhibit the same free-space term but differ in the ionospheric correction term as follows:

\[ \text{range signal time delay} \quad 2 \frac{R}{c} \left[ 1 + \frac{1}{4} \omega_p^2 \left( \frac{1}{\omega_T^2} + \frac{1}{N^2 \omega_o^2} \right) \right] \] (4.42a)

\[ \text{range rate signal time delay} \quad 2 \frac{R}{c} \left[ 1 - \frac{1}{4} \omega_p^2 \left( \frac{1}{\omega_T^2} + \frac{1}{N^2 \omega_o^2} - \frac{2(\omega_T - M \omega_o)}{\omega_T (N^2 \omega_o^2)} \right) \right] \] (4.42b)

where

\[ \omega_p^2 = \frac{q^2 \eta_{\text{eq}}}{m \epsilon_o} \approx 3200 \quad \eta_{\text{eq}} \approx 4\pi^2 \left( 81 \right) \eta_{\text{eq}} \] (4.43a)

\[ N_{\text{eq}} = \frac{1}{R} \int_R N \, dr \] (4.43b)

It should be noted that while the range signal correction term is always positive, the range rate signal correction term will be negative provided that the ratio of downlink/uplink carrier frequencies satisfies

\[ \left( \frac{\omega_c}{\omega_T} \right)^2 = \left( \frac{N \omega_o}{\omega_T} \right)^2 > 1 - \frac{2M \omega_o}{\omega_T} = 1 - 2 \frac{M}{N} \left( \frac{\omega_c}{\omega_T} \right) \] (4.44)
This can be checked to be the case in our system so that the ionospheric time-delay effects of the range and range rate signals will indeed exhibit opposite polarities.

4.6 Discussion of the GRARR System Results

The time delay results exhibited in (4.41a) by the range signal can be easily identified as the sum of the uplink and downlink group delays respectively evaluated using the uplink and downlink carrier frequencies; i.e.,

\[
\text{range signal } = \frac{R}{c} \left(1 + \frac{\omega_D^2}{\omega_U^2}\right) + \frac{R}{c} \left(1 + \frac{\omega_D^2}{\omega_C^2}\right)
\]

However, it is not immediately clear how the time delay results exhibited in (4.41b) by the range rate signal can be interpreted or even obtained with little effort. We next present a simple logical rationale that jointly exploits the phase/group delay concepts along with the intrinsic doppler signal extraction principles characteristic of the GRARR system, to interpret and reproduce the aforesaid results. We start by noting that the range rate signal is obtained in the GRARR VHF system by adding the received subcarrier signal to the M/N-weighted received carrier signal (the generation of the bias frequency being of course academic to the discussion). The subcarrier is generated from the ground and spacecraft oscillators according
to \( \omega_{sc} = \omega_T - M\omega_o \), while the weighted carrier is obtained from the spacecraft oscillator according to \( \frac{M}{N\omega_c} = \frac{M}{N(N\omega_o)} = M\omega_o \).

We now consider each of the individual contributions just mentioned.

The \( \omega_T \) term representing the ground oscillator goes uplink as a carrier thus introducing an uplink phase delay

\[
\frac{R}{c} \left( 1 - \frac{\omega_T^2}{\omega_c^2} \right) \approx \frac{R}{c} \left( 1 - \frac{\omega_T^2}{\omega_c^2} \right)
\]

(4.46a)

and returns as modulation with a corresponding downlink group delay

\[
\frac{R}{c} \left( 1 + \frac{\omega_T^2}{\omega_c^2} \right) \approx \frac{R}{c} \left( 1 + \frac{\omega_T^2}{\omega_c^2} \right)
\]

(4.46b)

In turn, the \(-M\omega_o\) term contribution of the spacecraft oscillator to the downlink subcarrier also returns as modulation thus specifying a group delay that when normalized to the \( \omega_T \) frequency (for time rather than phase comparison) becomes

\[
- \frac{R}{c} \cdot \frac{M\omega_o}{\omega_T} \left( 1 + \frac{\omega_T^2}{\omega_c^2} \right) \approx - \frac{R}{c} \cdot \frac{M\omega_o}{\omega_T} \left( 1 + \frac{\omega_T^2}{\omega_c^2} \right)
\]

(4.46c)

and finally the \(+M\omega_o\) term contribution of the spacecraft oscillator via the weighted downlink carrier specifies a phase delay that when normalized to the \( \omega_T \) frequency becomes

\[
+ \frac{R}{c} \cdot \frac{M\omega_o}{\omega_T} \left( 1 - \frac{\omega_T^2}{\omega_c^2} \right) \approx + \frac{R}{c} \cdot \frac{M\omega_o}{\omega_T} \left( 1 - \frac{\omega_T^2}{\omega_c^2} \right)
\]

(4.46d)

The net delay in the range rate signal is obtained by superposing the phase shifts (whether caused by phase or group delays) of these four contributions and normalizing to a common frequency for time delay evaluation. The normalization has already been introduced in the process above so we need only add the four time delays to get.
range rate signal \( R_{\text{rate}} = \frac{R}{c} (1 - \frac{\omega_1^2}{\omega_0^2}) + \frac{R}{c} (1 + \frac{\omega_2^2}{\omega_0^2}) - \frac{R}{c} \frac{M_0}{\omega_T} (1 - \frac{\omega_2^2}{\omega_0^2}) + \frac{R}{c} \frac{M_0}{\omega_T} (1 + \frac{\omega_2^2}{\omega_0^2})
\)

time delay
\[
\approx 2 \frac{R}{c} - \frac{1}{2} \left( \frac{R}{c} \omega_0^2 \left[ \frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} + \frac{2M_0/\omega_T}{\omega_0^2} \right] \right)
\]
\[
= 2 \frac{R}{c} \left[ 1 - \frac{1}{4} \left( \frac{1}{\omega_1^2} - \frac{1}{\omega_2^2} + \frac{2M_0/\omega_T}{\omega_0^2} \right) \right], \quad \omega_c = N_0\omega_0
\]

which is the desired result in (4.42b).

4.7 Case of Different Uplink vs Downlink Paths

We finally extend the results for the case where the propagation path followed by the uplink signal through the ionosphere differs from that followed by the downlink signal. The net effect can be introduced in the time delays \( T_1 \) to \( T_6 \) as follows:

\[
T_1 = \frac{R_1}{c} \left( 1 - \frac{\omega_1^2}{\omega_0^2} \right), \quad T_2 = \frac{R_1}{c} \left( 1 - \frac{\omega_1^2}{\omega_0^2} \right),
\]
\[
T_3 = \frac{R_2}{c} \left( 1 - \frac{\omega_2^2}{\omega_0^2} \right), \quad T_4 = \frac{R_2}{c} \left( 1 - \frac{\omega_2^2}{\omega_0^2} \right),
\]
\[
T_5 = \frac{R_2}{c} \left( 1 - \frac{\omega_2^2}{\omega_0^2} \right), \quad T_6 = \frac{R_2}{c} \left( 1 - \frac{\omega_2^2}{\omega_0^2} \right)
\]

in order to distinguish \( R_1 \) vs \( R_2 \) and \( \omega_1 \) vs \( \omega_2 \). The resultant changes in the formulation of section 5 can be traced to conclude that the extracted range and range rate signals are now given by
range signal = \sin\left(\omega_s t + \theta_s - \left(\frac{R_1+R_2}{c}\right)\omega_s - \frac{1}{2} \left[\left(\frac{R_1}{c}\right) \frac{\omega p_1^2}{\omega_T^2} + \left(\frac{R_2}{c}\right) \frac{\omega p_2^2}{\omega_T^2}\right]\right)

\text{(4.49a)}

range rate
\text{signal} = \sin(\omega_{0.03} t + \theta_{0.03} - \left(\frac{R_1+R_2}{c}\right)\omega_T^t - \frac{1}{2} \left[\left(\frac{R_1}{c}\right) \frac{\omega p_1^2}{\omega_T^2} - \left(\frac{R_2}{c}\right) \frac{\omega p_2^2 - 2M}{\omega_T}\right]\right)

\text{(4.49b)}

where \omega_c = N\omega_o as before. The corresponding time delays are thus given by

\text{range signal} = \frac{R_1+R_2}{c} + \frac{1}{2} \left[\left(\frac{R_1}{c}\right) \frac{\omega p_1^2}{\omega_T^2} + \left(\frac{R_2}{c}\right) \frac{\omega p_2^2}{\omega_T^2}\right]

\text{(4.50a)}

\text{range rate} \text{ time-delay} = \frac{R_1+R_2}{c} - \frac{1}{2} \left[\left(\frac{R_1}{c}\right) \frac{\omega p_1^2}{\omega_T^2} - \left(\frac{R_2}{c}\right) \frac{\omega p_2^2 - 2M}{\omega_T}\right]

\text{(4.50b)}

and these results could also have been obtained by the short procedure presented in section 6.

4.8 Summary and Conclusions

A conventional approach in the analysis of ionospheric time delay effects consists of replacing the free-space permittivity by an equivalent frequency-dependent term derived from the resultant displacement and convection current-densities generated by the force equation in an ionized medium. The dielectric constant or refractive index thus obtained is then applied to specify phase and group delay notions to be used to follow the signal propagation in a given application. However, the group notions imply a modulated field, and a portion of the derivation cited above is explicitly limited to an unmodulated field. In particular, the specific
frequency-dependence obtained may not be preserved in modulated field applications, and the corresponding group delay results induced by it may not be applicable.

On this basis, we have first carefully distinguished the parts of the derivation that remain valid regardless of the excitation from those strictly limited to unmodulated field excitations, and then analyzed the latter part to observe how the results are modified for our specific range-tone PM application. As evidenced by (4.17) vs (4.24d), the unmodulated-field frequency-dependence given by (4.18) will no longer hold regardless of the mod index and the condition $J_k(\delta) \cdot \left(1 \pm \frac{K \omega_m}{\omega_c} \right)^{\pm 1} \approx J_k(\delta)$ or zero is required for group delay notions based on the conventional expression of (4.18) to be meaningful. This condition is indeed characteristic of narrow-band PM applications and hence a group delay approach is valid.

As an alternate approach we can consider our particular PM signal as a sum of weighted sinusoids and then work in terms of sideband phase delays without engaging in group notions at all. The different phase delays of the spectral components of the PM signal can be followed in a given system, and the time delays of any signal reproduced by the system processing can be obtained from the net phase shift contribution of different sideband components. The method is illustrated in (4.29)-(4.33) for a simple one-way link, and then applied to the more complex two-way GRARR VHF link, which involves a PM/PM downlink signal and incoherent transponder processing.

The sideband phase delays are followed throughout the system up to the range and range rate signals extracted, and the resultant time
delays of (4.42) are obtained. While the range signal delay is easily identified as the sum of two (uplink plus downlink) group delays, the range rate signal delay does not exhibit an obvious decomposition. We then presented a novel procedure that links group/phase delay notions with the intrinsic doppler extraction principles characteristic of the GRARR system, and reproduced the range rate signal delay with a minimum effort and without dealing with the specific signal processing blocks of a particular receiver realization, for the rationale exploits basic GRARR signal extraction requirements that must be satisfied regardless of the structure used to this effect. The case of different uplink and downlink paths is finally presented as a direct extension of the previous results.