The Application of Signal Detection Theory to Optics

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ABSTRACT

Research on three topics is described in this report: (1) The role of measurements of noncommuting quantum observables in detection of signals and estimation of signal parameters by quantum receivers. (2) The restoration of images focused on a photosensitive surface when the data are the numbers of photoelectrons ejected from various parts of the surface. (3) The detection of an image formed on a photosensitive surface in the presence of background illumination when the data are as in (2).
Communication theorists search for the highest efficiency attainable in communicating through a given channel. By comparing existing or contemplated systems with the ideal system that would achieve that highest efficiency, they can estimate how much room there is for improvement and assess the effort required. A basic component of any communication system is the receiver, whose function is to decide periodically which signals have been sent through the channel, and an important factor in the efficiency of the system is the reliability of those decisions. To evaluate that reliability is a problem in detection theory.

In a typical communication system, one of an alphabet of $M$ symbols is dispatched every $T$ seconds by transmitting a uniquely associated signal. The receiver must decide, every $T$ seconds, which of the $M$ signals is present at its input. The reliability of these decisions is measured—in a negative way—by the average probability of error, or more generally, when different costs are attached to different types of errors, by an average cost of operation or Bayes cost. Detection theory seeks lower bounds to these error probabilities or Bayes costs. The bounds depend on the nature of the signals and of the

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random noise that corrupts them at the input to the receiver. When the signals have low enough frequencies that they and their associated noise background can be treated by classical physics, ordinary probability theory and the statistical decision theory derived from it suffice to determine the optimum receiver and its reliability. When the signals lie in the optical domain, as in a communication system transmitting laser pulses, the analysis of reception and decision must take account of the laws of quantum mechanics.

In order to determine the lower bounds on error probability or Bayes cost, we consider an ideal receiver, which for optical signals can be thought of as an aperture behind which is a lossless enclosure or cavity, initially empty. During each signaling interval of $T$ seconds' duration, the electromagnetic fields—signal and background light—are allowed to enter the cavity, after which it is closed, and the field inside is measured in order to obtain data on the basis of which to decide which signal is present. The hypothesis that the $k$-th signal is present is called $H_k$, and there are $M$ of them, $H_1, \ldots, H_M$.

The incident field excites the normal modes of the receiver cavity, which are subject to the laws of quantum mechanics and must be described in terms of the possible quantum mechanical states to which they might be excited. The internal field of the receiver is associated with a grand Hilbert space in which a pure state of the entire field will be represented by a state vector $|\psi\rangle$. When a known signal, say the $k$-th, has arrived at the aperture, the field will not, however, be in a pure state; because of random background light also incident, it will be in a statistical mixture of states described by a density operator $\rho_k$, $k = 1, \ldots, M$. The receiver must decide which of the density operators $\rho_1, \ldots, \rho_k, \ldots, \rho_M$ represents the field at hand, and it must do so with minimum average error probability or Bayes cost.
The aperture field can be decomposed into an orthonormal set of two-dimensional spatial modes. When as usually the background light is distributed uniformly over a somewhat wider cone of directions than encompasses the rays from the signal source, these aperture modes will be statistically independent. Each aperture mode excites a particular set of normal modes of the cavity field, and that set behaves much like the normal modes of a lossless transmission line, matched to an antenna or signal generator. It is convenient to talk about such a transmission-line receiver, particularly when discussing the effect of amplification. A generator at one end represents the antenna, with its radiation impedance $Z_g$. The line is assumed long enough to contain the entire signal, and the decision among the various hypotheses can be based on measurements of the modes of the line, which behave like harmonic oscillators. Classically, we can process simultaneous measurements of the mode amplitudes in such a way that for detecting coherent signals in white Gaussian noise, the same minimum error probability is attained as is calculated by the ordinary analysis involving the temporal form of the input signal and noise. It is because simultaneous observations of spatial field modes are easier to treat in quantum-mechanics than those made in temporal succession that our ideal receiver is constructed in this fashion.

Just as the ordinary analysis leads to filters matched to the signals to be detected, so in the transmission line one forms new "matched" modes that are linear combinations of the line modes appropriate to the several signals. Each such matched mode also behaves like a harmonic oscillator, and when—as we shall postulate—the noise is Gaussian, the density operator for the matched mode has a Gaussian $P$-representation. In an a-m communication system in which all the signals have the same form, but differ in amplitude and phase, only a single such matched mode needs to be considered, and we shall mostly illustrate our discussion...
The density operator under hypothesis $H_k$ will then have the form

$$\rho_k = (\pi N)^{-1} \int \exp[-|a - \mu_k|^2/N] |a\rangle\langle a| \, d^2a, \quad k = 1, \ldots, M, \quad (1)$$

where $N$ is the mean number of noise photons in the matched mode and is given by the Planck formula

$$N = (e^x - 1)^{-1}, \quad x = \hbar \omega / K T,$$

in which $\omega$ is the angular frequency of the mode, $K$ is Boltzmann's constant, and $T$ is the effective absolute temperature of the background. Here $\mu_k$ is the complex mode amplitude for the $k$-th possible signal; $|\mu_k|^2 = N_{sk}$ is the average number of signal photons supplied.1

How shall the decision among the $M$ density operators be made? The $M$ hypotheses form a set of logical alternatives, only one of which can be true, and von Neumann showed long ago that a decision among them is equivalent to measuring $M$ commuting projection operators $\Pi_j$, forming a resolution of the identity

$$\sum_{j=1}^{M} \Pi_j = \mathbb{1}, \quad \Pi_j \Pi_k = \Pi_j \delta_{kj}, \quad (2)$$

where $\mathbb{1}$ is the identity operator in our grand Hilbert space.2

The probability of choosing hypothesis $H_j$ when $H_k$ is true is $\text{Tr}(\rho_k \Pi_j)$, and if the cost of such a decision is $C_{jk}$ and the prior probability of hypothesis $H_k$ is $\zeta_k$, the average cost of operating our decision system is

$$\bar{C} = \sum_{k=1}^{M} \sum_{j=1}^{M} \zeta_k C_{jk} \text{Tr}(\rho_k \Pi_j). \quad (3)$$

This is to be minimized by picking the operators $\Pi_j$ under the constraints of eq. (2). How to do so when the density operators $\rho_k$ do not commute is as yet unknown, except for binary decisions ($M = 2$).1,3
For \( C_{jk} \equiv 1, j \neq k, C_{jj} = 0, \xi_j \equiv 1/M \), we get \( \bar{C} \) equal to the average probability of error

\[
P_e = 1 - \frac{1}{M} \sum_{k=1}^{M} \text{Tr}(\rho_k \Pi_k).
\]

H. Yuen has found that a necessary condition on the projection operators \( \Pi_k \) is then

\[
\sum_{j=1}^{M} \Pi_j \rho_j = \sum_{j=1}^{M} \rho_j \Pi_j,
\]

but it is unclear how this can be used to find the optimum projection operators \( \Pi_j \).

Here is an alternative formulation of the problem. Suppose we measure an operator with a continuum of eigenstates \(|\beta\rangle\) that are complete and orthonormal in the sense that

\[
\langle \beta | \beta' \rangle = \delta(\beta - \beta'), \quad \int |\beta\rangle \langle \beta| \, d^n\beta = 1;
\]

\( \beta \) may be a multidimensional parameter, and \( d^n\beta \) is the volume element in its space of values. If after the measurement the system is left in the state \(|\beta\rangle\), we choose hypothesis \( H_j \) with a probability \( \pi_j(\beta) \), where

\[
\sum_{j=1}^{M} \pi_j(\beta) = 1.
\]

The probability density of the parameter \( \beta \) under hypothesis \( H_k \) is \( \langle \beta | \rho_k | \beta \rangle = p_k(\beta) \), and the average Bayes cost is

\[
\bar{C} = \sum_{k=1}^{M} \sum_{j=1}^{M} \xi_k C_{jk} \int \pi_j(\beta) \, p_k(\beta) \, d^n\beta.
\]

The minimization of \( \bar{C} \) is now the same as in classical decision theory. Compute the posterior risk \( R_j(\beta) \) of hypothesis \( H_j \), given the outcome \( \beta \) of the measurement,
and choose that hypothesis for which \( \mathcal{R}_j(\beta) \) is smallest. The optimum projection operators are then

\[
\Pi_i = \int_{R_i} |\beta\rangle \langle \beta| \, d^n\beta = \int_{\pi_i(\beta)} |\beta\rangle \langle \beta| \, d^n\beta, \tag{9}
\]

where \( R_i \) is the range of values of \( \beta \) for which \( \mathcal{R}_i(\beta) < \mathcal{R}_j(\beta), \forall j \neq i \). There are many orthonormal sets \(|\beta\rangle\) for which \( C \) equals the minimum \( C_{\text{min}} \); all yield the same set of commuting projection operators \( \{\Pi_i\} \), but no one knows how to find them.

It has been suggested that we should admit to our formulation procedures that in effect cast the system into one of an overcomplete set of states \(|\beta\rangle\); that is, although still \( \int |\beta\rangle \langle \beta| \, d^n\beta = 1 \), the kets \(|\beta\rangle\) are not orthonormal.\(^6\) One defines an ideal measurement as one in which the multidimensional p.d.f. of the outcome is still given by \( \langle \beta|\rho_k|\beta\rangle \) under hypothesis \( H_k \).

As we are mainly concerned with the harmonic oscillator, the favorite overcomplete set is that of the coherent states \(|\beta\rangle\) made famous by R. Glauber.\(^7\) Here \( \beta \) is a complex number, \( \beta = \beta_x + i\beta_y \), and \(|\beta\rangle\) is a right eigenstate of the non-Hermitian annihilation operator

\[
b = \frac{(\omega q + ip)}{\sqrt{2\hbar \omega}}, \tag{10}
\]

where \( q \) and \( p \) are the coordinate and momentum operators and \( \omega \) is the frequency of the oscillator. When \( \rho_k \) is Gaussian, as in eq. (1), the joint p.d.f. of \( \beta_x \) and \( \beta_y \) after an ideal measurement is also Gaussian,

\[
\langle \beta|\rho_k|\beta\rangle = p_k(b_x, b_y) = \frac{1}{\pi(N + 1)} \exp[-|\beta - \mu_k|^2/(N + 1)]. \tag{11}
\]

The procedure resembles an approximate measurement of noncommuting observables \( q \) and \( p \) on the same oscillator.
Gordon and Louisell have shown that in order to carry out such a procedure on our receiver or oscillator, which we shall now designate as the "system" S, it will be necessary to bring up an auxiliary system, or "apparatus" A, which interacts with S and is later decoupled. The interaction is such that at a time $t_f$ afterward we can measure commuting observables on A alone—or possibly on both S and A—and deduce from the outcomes of these measurements the values of $\beta$ in such a way that $\langle \beta | \rho_k | \beta \rangle$ indeed represents their joint p.d.f. under hypothesis $H_k$.

When commuting projection operators are measured on S and A in order to decide among the M hypotheses, the minimum Bayes cost is independent of the time at which the measurement is made. This independence arises from the unitarity of the transformation that takes the density operators $\rho_k(t)$ for one time into those for a subsequent time. A set of projection operators $\Pi_k(t)$ optimum for measurements at one time $t$ can be converted into an optimum set for the other time by applying the same unitary transformation. The minimum Bayes cost, depending only on the traces $\text{Tr}(\rho_k \Pi_j)$, is invariant.

One might just as well, therefore, make one's measurements at a time $t_0$ before the system S and the apparatus A have interacted. At that time, the density operator $\rho_k^{(S+A)}(t_0)$ must factor into independent parts for system S and apparatus A. The density operator $\rho^{(A)}(t_0)$ for the apparatus A, which now has no information about the state of the system S, must be independent of which hypothesis $H_k$ is true. Hence

$$\rho_k^{(S+A)}(t_0) = \rho_k^{(S)}(t_0) \rho^{(A)}(t_0).$$

One would think that the optimum measurements would now be made on S and A separately, and that those on A could give no information that would reduce the probabilities of making errors in decisions about S. Thus the optimum
projection operators $\Pi_j^{(S+A)}$ should factor into a part $\Pi_j^{(S)}$ referring to the system $S$ and the identity operator $1_A$ for the apparatus $A$. The Bayes cost would now be given by eq. (3) with reference only to system $S$, and it would be minimized by properly choosing the commuting projection operators $\Pi_j^{(S)}$ on the Hilbert space for $S$ alone. The minimum Bayes cost $\bar{C}(t_0)$ would equal $\bar{C}(t_f)$. In brief, the apparatus $A$, before its interaction with the system $S$, can furnish only data irrelevant to our decision.

In classical decision theory, it is easy to show that measurements on $A$ cannot lead to a lower Bayes cost than what can be attained by measurements on $S$ alone. Quantum-mechanically, we must show that

$$\bar{C} = \text{Tr} \rho^{(A)} \sum_{j=1}^{M} \sum_{k=1}^{M} \xi_j \ C_{jk} \rho_{k}^{(S)} \Pi_j^{(S+A)}$$

(13)

is minimized by $\Pi_j^{(S+A)} = \Pi_j^{(S)} 1_A$, and this appears very hard to do for $M > 2$; the proof may be just as difficult as finding the minimizing operators $\Pi_j$ in the first place.

There are three cases in which we know that measurements of commuting projection operators on the original system $S$ alone will suffice to attain the minimum Bayes cost. The first of these is the binary decision, $M = 2$. Multiplying the two density operators $\rho_1^{(S)}$ and $\rho_2^{(S)}$ by a common factor $\rho^{(A)}$ for the density operator of an auxiliary apparatus does not change the decision strategy or the minimum attainable Bayes cost.

The second case is the choice among $M$ pure states, $\rho_k = |\psi_k\rangle \langle \psi_k|$. This would represent a receiver with vanishing background interference, and with signals producing coherent states $|\psi_k\rangle$, $k = 1, 2, \ldots, M$, in the matched mode of a transmission-line receiver. The decision scheme can then be restricted to the $M$-dimensional subspace spanned by the vectors $|\psi_k\rangle$, and it is necessary to
measure $M$ commuting projection operators

$$\Pi_j = |\beta_j\rangle\langle\beta_j|, \quad j = 1, \ldots, M, \quad \sum_{j=1}^{M} \Pi_j = 1,$$

(14)

where the $|\beta_j\rangle$ are an orthonormal set of state vectors spanning the same subspace. A finite complete set of unit vectors, i.e., one satisfying a completeness relation $\sum_{j=1}^{M} |\beta_j\rangle\langle\beta_j| = I$, is also orthonormal.\(^5\)

The third case in which an auxiliary apparatus would be unnecessary is the choice among $M$ commuting density operators $\rho_k$, $k = 1, \ldots, M$. Consider, for instance, a system transmitting incoherent light pulses of $M$ different energies for the $M$ different symbols in which messages are written. If they excite a single mode to different mean photon numbers $N_{s1}, N_{s2}, \ldots N_{sM}$, the best receiver simply counts the number $n$ of photons in the mode and decides on the basis of likelihood ratios formed from its distributions

$$P_k(n) = (1 - v_k)^n v_k, \quad v_k = (N_{sk} + N)/(N_{sk} + N + 1),$$

$$k = 1, 2, \ldots, M,$$

(15)

where $N$ is the mean number of background-induced photons in the mode.\(^1,3\)

It is to be expected that extending one's measurements to an auxiliary system, coupled somehow to the receiver, but ignorant of what state the receiver is in, would inevitably introduce additional noise because of the quantum uncertainties about the state of the auxiliary system, and the minimum Bayes costs should in general be attained by measuring commuting operators on the receiver itself. However, a general proof going beyond the three cases I have just mentioned seems to be unknown.

Admitting an auxiliary apparatus $A$ gives one the freedom to choose not only the $M$ commuting projection operators $\Pi_j^{(S+A)}$ on the combined system, but also the density operator $\rho^{(A)}$ of the apparatus, and indeed the structure of
the apparatus as well, in order to minimize the Bayes cost. There are then two possibilities: either the optimum set of projection operators \( \prod_j^{(S+A)} \) and the resulting minimum cost are independent of the density operator \( \rho^{(A)} \) of the apparatus, or they are not. The only way in which the minimum cost could be completely independent of the state of the apparatus would appear to be for the optimum projection operators to factor as \( \prod_j^{(S)} \prod_j^A \).

The alternative is that a given set \( \rho_k^{(S)} \) of possible density operators for the system S under the \( M \) hypotheses \( H_k \) determines a unique state or mixture \( \rho^{(A)} \) in the apparatus and indeed, because the apparatus itself is arbitrary, in an outside world that knows nothing about which hypothesis is true. This density operator \( \rho^{(A)} \) is the one that, when combined with the \( \rho_k^{(S)} \), allows the choice of projection operators \( \prod_j^{(S+A)} \) leading to an absolute minimum Bayes cost. It is difficult to see by what mathematics such a most favorable density operator \( \rho^{(A)} \) could be calculated.

New facets appear on this question of the role of an auxiliary measuring apparatus A when the optimum estimation of signal parameters is studied. In an "analog" communication system the task of the receiver is not to decide among discrete hypotheses, but to estimate one or more parameters of the received signals—parameters whose values convey information from the source. Parameter estimation is also required in an optical receiver such as a telescope that must fix the location of an object, or in a spectrometer that measures the wavelength of a spectral line. In a laser radar it is necessary to measure the time of arrival and sometimes the Doppler shift of a coherent pulse of light reflected from the target. The unknown parameters \( (\theta_1, \theta_2, \ldots, \theta_m) \), which we combine into a vector \( \theta \), are parameters of the density operator \( \rho = \rho(\theta) \) of the field in the receiver.

Quantum estimation theory seeks strategies for producing estimates
the parameters by measurements on the field of the receiver. As in classical estimation theory, the costs of errors in these estimates are specified by a cost function \( C(\hat{\theta}, \theta) \), of which the squared error is the most commonly adopted,

\[
C(\hat{\theta}, \theta) = \sum_{i=1}^{m} c_i (\hat{\theta}_i - \theta_i)^2,
\]

the \( c_i \)'s being certain positive weights. In addition, the relative frequencies with which certain values of the parameters occur are embodied in a prior p.d.f. \( z(\theta) \). We now define a set of estimating operators \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \), that when measured on the receiver ("the system") yield numbers \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \) as estimates of the parameters. By substitution into the cost function \( C(\hat{\theta}, \theta) \) we can then determine a cost operator \( C(\hat{\theta}, \theta) \), provided the estimating operators \( \hat{\theta}_1, \ldots, \hat{\theta}_m \) commute; if they do not, there may be some difficulty in defining this operator \( C(\hat{\theta}, \theta) \). The average cost of error, or Bayes cost, is then

\[
\bar{C} = \int \text{Tr}[C(\hat{\theta}, \theta) \rho(\theta)] z(\theta) \, d^m \theta,
\]

and the optimum receiver uses such estimating operators \( \hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_m \) that \( \bar{C} \) is minimum. For estimation of a single parameter \( \theta \) with the quadratic cost function \( (\hat{\theta} - \theta)^2 \), S. Personick has worked out the optimum estimating operator \( \hat{\theta} \). How to find the optimum estimators when the density operators \( \rho(\theta) \) do not commute for different sets of values of \( m \) parameters \( \theta \) is in general unknown when \( m > 1 \). When they do commute, classical estimation theory can be applied.

Parameter estimation can be considered a continuous version of multiple hypothesis testing. Indeed, if all we ask is whether the parameters lie in one of a finite number of finite ranges, the two are identical. What we said about reducing minimum Bayes costs by bringing in an auxiliary apparatus ought to
apply to parameter estimation as well.

As in classical estimation theory, lower bounds on mean-square errors of unbiased estimates of the parameters \( \Theta = (\theta_1, \ldots, \theta_m) \) of a density operator \( \rho(\Theta) \) can be set by means of a version of the Cramér-Rao inequality. We define a covariance matrix \( B \) whose elements are

\[
B_{ij} = \frac{1}{2} \text{Tr} \rho[(\hat{\theta}_i - \theta_i)(\hat{\theta}_j - \theta_j) + (\hat{\theta}_j - \theta_j)(\hat{\theta}_i - \theta_i)] \tag{18}
\]

and another matrix \( A \) whose elements are

\[
A_{ij} = \frac{1}{2} \text{Tr} \rho(L_i L_j + L_j L_i), \tag{19}
\]

where \( L_i \) is a symmetrized logarithmic derivative of the density operator \( \rho \), defined by

\[
\frac{\partial \rho}{\partial \theta_i} = \frac{1}{2} (\rho L_i + L_i \rho). \tag{20}
\]

The Cramér-Rao inequality can then be stated in two equivalent ways. If \( Z \) is a row vector of real variables \( (z_1, z_2, \ldots, z_m) \), the inequality

\[
Z B^{-1} Z \preceq Z A Z \tag{21}
\]

states that the concentration ellipsoid \( \tilde{Z} B^{-1} \tilde{Z} = m + 2 \) of the estimates lies outside the ellipsoid given by \( \tilde{Z} A \tilde{Z} = m + 2 \). Alternatively,

\[
\tilde{Z} B \tilde{Z} \succeq \tilde{Z} A^{-1} \tilde{Z} \tag{22}
\]

yields lower bounds on certain combinations of the variances and covariances of the estimates; for instance

\[
\text{Var} \hat{\theta}_i = \text{Tr} \rho(\hat{\theta}_i - \theta_i)^2 \succeq (A^{-1})_{ii}, \tag{23}
\]

where \((A^{-1})_{ii}\) is the \( i \)-th diagonal element of the matrix \( A \).
As with the minimum Bayes cost of a decision strategy, the Cramér-Rao bound is independent of the time of measurement because of the unitarity of the transformation that carries a quantum-mechanical system from one time to another. If one contemplates estimating parameters by means of ideal measurements as defined by Gordon and Louisell, one couples that auxiliary apparatus $A$ to the system $S$ and measures on $S + A$ commuting observables $\tilde{\theta}_i(t_f)$ at a later time $t_f$ after $S$ and $A$ have interacted. The Cramér-Rao minimum, however, will be the same as if the measurements are made, with appropriately modified operators $\tilde{\theta}_i(t_0)$, at a time $t_0$ before $S$ and $A$ interact. The density operator $\rho(\theta)$ then takes the form

$$\rho(\theta) = \rho^{(S)}(\theta) \rho^{(A)}$$  \hspace{1cm} (24)

where $\rho^{(A)}$ is independent of the parameters $\theta$. The s.l.d. operators must be obtained from eq. (21), which now becomes

$$\frac{\partial \rho^{(S)}}{\partial \theta_1} \rho^{(A)} = \frac{1}{2} \left[ \rho^{(S)}(\theta) \rho^{(A)} L_1 + L_1 \rho^{(S)}(\theta) \rho^{(A)} \right].$$  \hspace{1cm} (25)

As $\rho^{(A)}$ commutes with $\rho^{(S)}(\theta)$, the solution of this equation is

$$L_1 = L_1^{(S)} \mathbb{1}_A,$$  \hspace{1cm} (26)

where $L_1^{(S)}$ satisfies eq. (20) with $\rho = \rho^{(S)}(\theta)$. Thus the Cramér-Rao lower bounds based on measurements of commuting estimators $\tilde{\theta}_i$, $i = 1, \ldots, m$, cannot be undercut by measuring commuting operators in a larger Hilbert space than that of the system $S$ alone, nor by making ideal measurements on $S$ as contemplated by Gordon and Louisell.\(^8\)

In my opinion, the estimating operators $\tilde{\theta}_i$ must commute in order for mean-square errors and Bayes costs to be sensibly defined. The Hilbert space on which they operate may be larger than that needed by the excited receiver.
modes (our system S), but the state of the apparatus represented by the remaining factor of the product space cannot depend on the unknown parameters $\theta$. What we have shown is that the Cramér-Rao lower bounds cannot be reduced by bringing in an apparatus $A$ and so enlarging the Hilbert space. Whether those bounds can be attained by measurements restricted to $S$ is uncertain; we know of cases where they doubtless cannot, for to do so would require measuring noncommuting operators on $S$.

An important example is the estimation of the components $\mu_x$ and $\mu_y$ of the complex amplitude $\mu$ of a coherent signal in a matched mode of our ideal receiver. When the noise is Gaussian, as we have seen, the density operator of the mode is

$$
\rho(\mu_x, \mu_y) = \rho(\mu) = \left( \frac{1}{\pi N} \right)^{\frac{1}{2}} \exp\left(-\frac{|\alpha - \mu|^2}{N}\right) |\alpha\rangle\langle\alpha| d^2\alpha. \quad (27)
$$

If the phase of the signal is known, we can take it as zero, put $\mu_y = 0$, and estimate only $\mu_x$. Then an efficient estimator of $\mu_x$, that is, one attaining the minimum mean-square error specified by the Cramér-Rao inequality, is $\frac{1}{2} (a + a^+)$, where $a$ is the annihilation operator for the mode, and

$$
\text{Tr} \rho(\hat{\mu}_x - \mu_x)^2 = \frac{1}{2} (N + \frac{1}{2}). \quad (28)
$$

If both $\mu_x$ and $\mu_y$ are unknown and must be estimated, the Cramér-Rao inequality yields the same lower bound for the mean-square error in each, but the efficient estimators $\hat{\mu}_x = \frac{1}{2} (a + a^+)$ and $\hat{\mu}_y = \frac{1}{2} i(a - a^+)$ that it produces do not commute and cannot be measured on the same system.

An ingenious variant of the Cramér-Rao inequality that fits this problem has been discovered by Yuen and Lax. In its simplest form it involves a pair of estimanda, which we take as $\mu_x$ and $\mu_y$, and their estimating operators $\hat{\mu}_x$ and $\hat{\mu}_y$. The estimanda are combined into a complex parameter $\mu = \mu_x + i\mu_y$, the
estimators into a complex operator \( \bar{u} = \bar{u}_x + i\bar{u}_y \). The new inequality is

\[
\text{Tr}[\rho (\bar{u} - \mu)(\bar{u}^* - \mu^*)] \geq (\text{Tr} \rho \Lambda^+ \Lambda)^{-1},
\]

(29)

where \( \Lambda \) is a complex logarithmic derivative operator defined by

\[
\frac{\partial \rho}{\partial \mu} \Lambda \rho.
\]

(30)

Equality is attained when the operator \( \bar{u} \) is proportional to \( \Lambda^+ \). For the density operator in eq. (27),

\[
\Lambda = a^+/(N + 1),
\]

(31)

whereupon

\[
\text{Tr} \rho (\bar{u} - \mu)(\bar{u}^* - \mu^*) = \text{Tr} \rho (\bar{u}_x^* - i\bar{u}_y^*)(\bar{u}_x - i\bar{u}_y) \geq N + 1
\]

(32)

Now if the estimators \( \bar{u}_x \) and \( \bar{u}_y \) commute, we find

\[
\text{Tr} \rho (\bar{u}_x^2 + \bar{u}_y^2) = \text{Var} \bar{u}_x + \text{Var} \bar{u}_y \geq N + 1.
\]

(33)

Thus when the estimating operators commute and can be measured on the same system, the sum of the mean-square errors in the components of the complex mode amplitude of the signal cannot be made less than \( N + 1 \). There are two ways in which this lower bound can be attained. One is by amplifying the mode by such a large gain that its field can be treated by classical physics. If this is done in such a way as to introduce minimum additional noise, the sum of the mean-square errors is just \( N + 1 \). Alternatively, as S. Personick has shown, we can adjoin an "apparatus" \( A \) consisting of a separate oscillator in the ground state \( |0_A\rangle \), taking the density operator of the combined system \( S + A \) as

\[
\rho^{(S+A)} = \rho(\mu) |0_A\rangle \langle 0_A|
\]

(34)
and the estimating operator as \( \bar{p} = a_S + a_A^+ \), where \( a_S \) and \( a_A \) are photon annihilation operators for the receiver mode \( S \) and the apparatus \( A \), respectively.\(^{12}\)

If, on the other hand, we were to put \( p = a_S \), the estimators \( \bar{p}_x \) and \( \bar{p}_y \) would not commute, and the new form of the Cramér-Rao inequality would yield

\[
\text{Tr} \rho \left( \bar{p}_x'^2 + \bar{p}_y'^2 + i \left( \bar{p}_y' \bar{p}_x' - \bar{p}_x' \bar{p}_y' \right) \right) = \\
\text{Tr} \rho \left( \bar{p}_x'^2 + \bar{p}_y'^2 \right) + \frac{1}{2} \geq N + 1,
\]

as given also by the first version.

This example places the question of the role of an auxiliary system before us in particularly clear terms. No way is known to estimate both \( \mu_x \) and \( \mu_y \) by measuring commuting operators \( \bar{p}_x \) and \( \bar{p}_y \) on the system \( S \) alone and to attain at the same time the lower bound \( N + 1 \) on the sum of their mean-square errors. Why should an apparatus containing no information about the state of the system \( S \) allow one to estimate parameters of \( S \) with minimum Bayes cost? How can one determine from an arbitrary density operator \( \rho^{(S)}(\theta) \) the density operator \( \rho^{(A)} \) that, when the apparatus \( A \) is coupled with \( S \), permits estimation of the parameters \( \theta \) of \( S \) with minimum Bayes cost, or minimum total mean-square error? The answers to these questions would undoubtedly contribute to our understanding of quantum detection as well, and perhaps even to elucidating certain aspects of quantum theory of measurement.
References


11. H. H. Yuen, M. Lax, "Multiple Parameter Quantum Estimation with Applications to Radar and Analog Communication", IEEE Information Theory Symposium,
Asilomar, California, January 31, 1972.

An image is formed on a photoelectric surface divided like a mosaic into a number of small spots of area $s$, which emit photoelectrons when the light impinges on them. The number $n_i$ of photoelectrons from the $i$-th spot has a Poisson distribution with mean value

$$\mathbb{E}(n_i) = \alpha s I(x_i), \quad (1)$$

where $I(x)$ is the image illuminance, $\alpha$ is a constant proportional to the quantum efficiency of the surface, and $x_i$ is the 2-vector of coordinates at the center of the $i$-th spot. The random variables $n_i$ are assumed statistically independent, as when the product WT of the bandwidth of the light and the observation time is very large, $WT \gg 1$.

The illuminance $I(x)$ is given in terms of that of the true or geometrical image $J(x)$ by the equation

$$I(x) = \int K(x - y) J(y) \, d^2y, \quad (2)$$

where $K(u)$ is the point-spread function, so normalized that

$$\int K(x) \, d^2x = 1. \quad (3)$$

The true image is given by

$$J(x) = I_0 + j(x), \quad (4)$$

where $j(x)$ is a mean-zero Gaussian random process with covariance

$$\mathbb{E}[j(x_1) j(x_2)] = \varphi(x_1 - x_2), \quad (5)$$
and $I_0$ is the mean illuminance. We wish to estimate the values of $j(x)$ at sampling points $x_i$, $j(x_i) = \xi_i$, from the data $n_i$, putting

$$I(x_i) - I_0 = \eta_i = \sum_j K_{ij} \xi_j,$$

$$K_{ij} = s K(x_i - x_j) = K_{i-j}, \quad \sum_p K_p = 1.$$  

(6)

(7)

The covariance matrix of the $\xi_i$'s is

$$\varphi_{ij} = \varphi_{i-j} = E[\xi_i \xi_j] = \varphi(x_i - x_j).$$  

(8)

We have shown\(^1\) that the maximum-likelihood estimates of the true image samples $\xi_i$ are obtained by solving the equations

$$\xi_m = \sum_i K_{mi} \left[ \frac{n_i}{I_0 + \eta_i} - \alpha s \right],$$

$$K_{mi} = \sum_j \varphi_{mj} K_{ij},$$  

(9)

(10)

with the $n_i$ given by eq. (6).

In order to test this scheme, a one-dimensional version was tried out by means of a digital computer. The $M$ true image samples $\xi_i$ were generated by means of a first-order autoregressive process,

$$\xi_i = r \xi_{i-1} + z_i, \quad i = 0, 1, \ldots, M-1,$$  

(11)

where $|r| \leq 1$ and the $z_i$ are independent Gaussian random variables with mean zero and variance $\sigma^2$. The covariance matrix of the true image was then

$$\varphi_{ij} = \varphi_{|i-j|} = \sigma^2 |i-j|.$$  

(12)

In order to use fast Fourier transform routines, the rows of all Toeplitz

matrices $K, K'$, and $\varphi$ were assumed periodic with period $M$, as were the true image \{\xi_m\} and the distorted image \{\eta_m\}. The distorting matrix $K$ had elements

$$K_{ij} = K_{i-j} = a \text{sinc}^2[2\rho(i-j)], \quad (13)$$

corresponding to the point-spread function of a slit aperture; $a$ is a normalization constant enabling eq. (7) to hold. The numbers $n_i$ of photoelectrons, which were the data from which the estimates $\hat{\xi}_i$ of the images were to be determined, were generated by a random-number routine in such a way that $n_i$ had a Poisson distribution with mean value $\alpha(I_0 + n_i)$. The noisy image is then given by $n_i/\alpha$ as before any processing.

The nonlinear equations (9) were solved by Newton's method, the starting point $\{\hat{\xi}_i\}'$ of which was the true image as estimated by the method of least squares applied to the data $n_i$, \[ \xi_i' = \sum_j L_{i-j} \left( \frac{n_j}{\alpha \xi} - I_0 \right), \quad (14) \]

where the coefficients $L_p$ represented the Wiener filter and were given by \[ \sum_m L_{i-m} \psi_{mj} = K_{ij}', \quad (15) \]

\[ \psi_{mj} = \sum_{k,n} K_{mk} K_{jn} \varphi_{kn} + (I_0/\alpha) \delta_{mj}, \quad (16) \]
a set of equations that could be solved once for all by Fourier transforms. The discrete Fourier transform $\{\Lambda_p\}$ of the sequence $\{L_p\}$ is given by \[ \Lambda_p = k_p \Phi_p \left[ |k_p|^2 \Phi_p + (I_0/\alpha) \right]^{-1}, \quad (17) \]
in terms of the Fourier transforms $\{k_p\}$ and $\{\Phi_p\}$ of the sequences $\{K_p\}$ and $\{\varphi_p\}$ given by eqs. (7), (8), (12), and (13). The convolution expressed by eq. (14) could also be carried out by Fourier transforms in terms of $\{\Lambda_p\}$. 

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In Newton's method the differences

\[ f_m = \xi_m - \sum_i K_{ni} \left( \frac{n_i}{\lambda_0 + n_i} - \alpha s \right) \]  

are made as small as possible by correcting each set \( \xi_m \) by an amount \( \delta \xi_m \) determined by solving the equations

\[ \sum_p \frac{\partial f_m}{\partial \xi_p} \delta \xi_p = -f_m, \]  

where

\[ \frac{\partial f_m}{\partial \xi_p} = \delta_{mp} + \sum_i \frac{K_{ni} K_{in}}{(\lambda_0 + n_i)^2}; \]  

the new trial values are then \( \xi_m + \delta \xi_m \). Only two or three iterations, each of which required a matrix inversion as in eq. (19), were necessary.

It was found that the maximum-likelihood estimates \( \hat{\xi}_m \) found by this method differed little from those obtained by the method of least squares, as in eq. (14). The following table lists the parameters of the various trials, made with different contrasts and signal-to-noise ratios. As a measure of the improvement of the image we have used the sum of squares of the deviations of the estimates \( \hat{\xi}_i \) from the true image values \( \xi_i \), expressed as a percentage of the norm \( \sum_i \xi_i^2 \) of the true image.
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Reliability of the Likelihood Ratio Detector for a Photoelectric Image

by

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ABSTRACT

The reliability of the likelihood ratio detector for a photoelectric image has been approximated by an asymptotic expansion in descending powers of the decision level determined from the detection criterion. Furthermore the likelihood ratio detector can be approximated by the threshold detector for a weak signal. To investigate the performance of those detectors, a Gaussian image has been assumed, and numerical calculations have been carried out at various signal-to-noise ratios.
I. Introduction

In modern observational astronomy, it is often a practice to use an image tube or a phototube as a sensor to detect a star for the purpose of navigation or observation. In a binary optical communication system, a phototube is used as the sensor to detect a bit "1" when an optical signal is transmitted. In most applications, noise caused by the fluctuation of the light source through a turbulent medium or by the background radiation, or noise inherent in the instrument, is present and thus limits the performance of the sensor. It is therefore practical and necessary in describing the detectability and information capacity of the sensor to consider the presence of the noise. In this paper only the statistical noise caused by the background radiation and photoelectric emission is considered.

The likelihood-ratio detector has been discussed previously by Reiffen and Sherman\(^1\) and Helstrom\(^2\). The detection statistic was based on the likelihood ratio, and its characteristic function was derived\(^2\). Because of the complexity of the characteristic function, the probability density function (p.d.f.), which is the inverse Fourier transform of the characteristic function, was approximated by a Gaussian function from the Gram-Charlier expansion. The false-alarm probability and the detection probability, a pair of probabilities defined as the reliability of the detector, were then expressed by error functions. However, when the signal-to-noise ratio is large, the k\(^{th}\)-order cumulant of the logarithm of the characteristic function increases as k becomes larger. Therefore a new form of approximation must be used. For large values of the statistic the p.d.f. can be approximated\(^3\) by the classical saddle-point method, that is, by taking the integration path of steepest descent that passes through the saddle point. The probabilities that the statistic exceeds a large decision
level $g_0$ can then be computed by numerical integration or summation. On the other hand, the reliability can be expressed directly by an integral in terms of the characteristic function. This integral has not only one saddle point, but also a singularity at the origin. For a very large decision level $g_0$, one portion of the integral lies near the singularity, and the other portion comes from the integral by taking the integration path through the saddle point. However, for moderate values of the decision level, the interaction between the saddle point and the singularity must be also considered. Rice developed a general method for expanding a saddle-point integral in a uniform asymptotic series. The reliability of the detector will then be approximated by the asymptotic expansion in descending powers of the decision level $g_0$ as shown in eq. (15).

When the signal is very weak, the likelihood-ratio detector can be approximated by the threshold detector. To investigate the performance of those detectors, the Gaussian charge distribution of the star image on the surface has been assumed. Standard signals have been used, and the numerical calculations of the reliability for both detectors have been carried out.
II. Statistic from the Likelihood Ratio Detector

In general, the receptor of the sensor is taken to be the photosensitive surface of the image tube. This surface is then divided like a mosaic into a large number $M$ of small equal areas $\{\delta A_k\}$. The observed data $\{n\}$ are the set $n_1, n_2, \ldots, n_k, \ldots$, where $n_k$ is the number of charges observed from the $k$th area $\delta A_k$ of the surface during the time interval $\tau$. There are two hypotheses, $H_0$ and $H_1$, that will be used for the system. Hypothesis $H_0$ is the statement that there is no star, but only noise is present; and hypothesis $H_1$ is the statement declaring the presence of a star.

The likelihood ratio $\Lambda(n)$ has been derived \(^2\) and is expressed as

$$\Lambda(n) = \frac{p\{n_1, n_2, \ldots, n_M|H_1\}}{p\{n_1, n_2, \ldots, n_M|H_0\}}$$

$$= \prod_{k=1}^{M} \left[1 + \mu_s(x_k)/\mu_0(x_k)\right]^{n_k} \exp[-\tau \mu_s(x_k) \delta A_k],$$  \hspace{1cm} (1)

where $\mu_s(x_k) = I_s G(x_k)$ is the count rate per unit area due to the star intensity $I_s$ weighted by $G(x_k)$ and

$$\mu_0(x_k) = I_b G_b$$

is the count rate per unit area due to the background intensity $I_b$ weighted by $G_b$, which is usually a constant independent of the location $x_k$ of the area $\delta A_k$. $p\{n_1, n_2, \ldots, n_M|H_j\}$ is the joint conditional probability of obtaining the data $\{n_1, n_2, \ldots, n_M\}$ under hypothesis $H_j$ ($j = 0$ or $1$). $p\{n_1, n_2, \ldots, n_M|H_j\}$ is a Poisson probability function. Then

$$g = \sum_k n_k \ln[1 + \mu_s(x_k)/\mu_0(x_k)] = \sum_k n_k \ln H(x_k)$$  \hspace{1cm} (2)
is the optimum statistic obtained from the logarithm of the likelihood ratio $\Lambda(n)$, where

$$H(x_k) = 1 + \left[ \mu_s(x_k)/\mu_0(x_k) \right] = 1 + \left( I_s/I_b G_b \right) G(x_k). \quad (3)$$

The characteristic function $h_j(-i\omega)$ of the statistic $g$ is then

$$h_j(-i\omega) = \mathbb{E}[\exp(i\omega)g | H_j]$$

$$= \prod_{k=1}^{M} \mathbb{E}\{\exp(i\omega N_k \ln H(x_k) - i\omega) | H_j\}$$

$$= \exp \left\{ \tau I_b G_b \sum_k [H(x_k)]^j \{[H(x_k)]^{i\omega} - 1\} \delta A_k \right\}. \quad (4)$$

If the areas $\{\delta A\}$ are very small, the characteristic function becomes

$$h_j(-i\omega) = \exp \left\{ \tau I_b G_b \int_A [H(x)]^j \{[H(x)]^{i\omega} - 1\} \, dx \right\}, \quad (5)$$

with $A$ the area of the receptor. The mean and variance of the statistic $g$ under hypothesis $H_j$ can be obtained from eq. (5) as

$$\mathbb{E}[g | H_j] = \tau I_b G_b \int_A [H(x)]^j \ln(H(x)) \, dx$$

$$\text{Var}[g | H_j] = \tau I_b G_b \int_A [H(x)]^j \ln^2(H(x)) \, dx, \quad j = 0 \text{ or } 1. \quad (6)$$
III. Detection Reliability from the Asymptotic Expansion for Saddle-Point Integrals

Since the statistic $g$ is non-negative and its probability density function (p.d.f.) is integrable for all $g$, the Laplace transform of the conditional p.d.f. $p_j(g)$ is interchangeable with its characteristic function; that is, the Laplace transform of $p_j(g)$ can be written as

$$h_j(s) = E[e^{-sg}|H_j]$$

$$= \exp \left\{ T b G_b \int_{A} [H(x_k)]^j \{[H(x_k)]^{-s} - 1 \} dx \right\}, \quad j = 0 \text{ or } 1,$$

(7)

where $s = a + i \omega$ is complex for $a > 0$. The false-alarm probability $Q_0$ and the detection probability $Q_d$ are defined as

$$Q_0 = \Pr\{g > g_0|H_0\} = \int_{g_0}^{\infty} p_0(g) \, dg,$$

$$Q_d = \Pr\{g > g_0|H_1\} = \int_{g_0}^{\infty} p_1(g) \, dg,$$

(8)

where $g_0$ is the decision level of the detector.

The Laplace transform of $Q_0$ is

$$\mathcal{L}[Q_0] = \mathcal{L}[1 - \int_{0}^{\infty} u(g_0 - g) \, p_0(g) \, dg] = [1 - h_0(s)]/s,$$

where $u(x) = 1$ for $x > 0$, and $u(x) = 0$ for $x \leq 0$. Similarly,

$$\mathcal{L}[Q_d] = [1 - h_1(s)]/s.$$

Thus eq. (8) can be rewritten in terms of the Laplace transform $h_j(s)$, that is,

$$Q_0 = \mathcal{L}^{-1} \mathcal{L}[Q_0]$$
\[ Q_d = \mathcal{L}^{-1} \mathcal{L}[Q_d] \]
\[ = 1 - \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{1}{s} \exp\left[ g_0 \, \phi_0(s) \right] ds, \]  
\[ \phi_0(s) = \left( \ln \frac{h_j(s)}{g_0} \right) \ln \frac{h_j(s)}{g_0} + s \]
\[ = (T^1_b G_b / g_0) \int [H(x)]^j \{ [H(x)]^{-s} - 1 \} \, dx + s \]  
\[ \text{with } \alpha > 0 \text{ and} \]
\[ \phi_j(s) = (1/g_0) \ln h_j(s) + s \]
\[ = (\tau I^1_b G_b / g_0) \int \int_A [H(x)]^j \{ [H(x)]^{-s} - 1 \} \, dx + s \]  
is the complex phase of the integral.

Because of the difficulty of obtaining the p.d.f.'s \( p_j(g) \), eq. (9) will be used for the calculation of the reliability. In general, the integral in eq. (9) with large parameter \( g_0 \) can be approximated by an asymptotic expansion that is a power series in \( g_0^{-1} \). The probability \( Q_0 \) or \( Q_d \) will then be determined by adding up to that term in the series after which the terms begin to increase. The coefficients of the series can be determined by making use of the classical saddle-point expansions about each individual saddle point \( s_i \), which is real and satisfies the equation \( \frac{d}{ds} \phi_j(s) \big|_{s=s_i} = 0 \). For the complex phase given in eq. (10), there is only one saddle point \( s_j \) under hypothesis \( H_j \), and it can be determined from the equation
\[ g_0 = \tau I^1_b G_b \int \int_A [H(x)]^j \ln H(x) \, dx, \quad j = 0 \text{ or } 1. \]  
In the case where \( g_0 \) is extremely large and the saddle point is far from the origin, eq. (9) can be approximated by the classical saddle-point method, that
is, by taking the path of the integral through the saddle point $s_j$, which satisfies eq. (11). If the origin is also a singular point, one portion of the path will lie close to the origin. Thus eq. (9) can be approximated as

\[ Q_0 = 1 - I_0 \quad s_j > 0 \]
\[ Q_d = 1 - I_1 \]
\[ Q_0 = -I_0 \quad s_j < 0 \]
\[ Q_d = -I_1 \]

(12)

where

\[
I_j = \frac{1}{2\pi i} \int_{L_j} \frac{1}{s} \exp[g_0 \phi_j(s)] \, ds
\]

\[
\approx \frac{\exp[g_0 \phi_j(s_j)]}{[2\pi g_0 \phi_j''(s_j)]^{1/2}} \sum_{k=0}^{\infty} \left[-2/g_0 \phi_j''(s_j)\right]^k s_j^{-1}
\]

\[
\times \sum_{n=0}^{2k} (-s_j)^{-2k+n} \sum_{l=0}^{n} A_{l,n} (1/2)_{l+k}, \quad j = 0 \text{ or } 1
\]

(12a)

is the asymptotic expansion by taking the integral path $L_j$ through the saddle point $s_j$ where

\[
A_{l,n} = \begin{cases} 
0 & \text{for } n < l \text{ or } n = 0, \ n \geq 1 \\
1 & \text{for } l = n = 0,
\end{cases}
\]

\[
A_{l+1,n+1} = -\frac{2}{n+1} \sum_{m=1}^{n-l+1} \frac{m \phi_j''(s_j)}{(m+2)!} \phi_j''(s_j)^k A_{l,n-m+1}
\]

(12b)

and

\[
\left( \frac{1}{2} \right)_m = \frac{1}{2} \left( \frac{1}{2} + 1 \right) \left( \frac{1}{2} + 2 \right) \ldots \left( \frac{1}{2} + m - 1 \right), \quad \left( \frac{1}{2} \right)_0 = 1.
\]

$A_{l,n}$ are the coefficients which can be calculated by the recurrence relation through the $n$\textsuperscript{th} derivatives of the complex phase $\phi_j(s_j)$ as shown.
The $n$th derivatives $\phi_j(s)$ for $n \geq 2$ can be expressed as

$$\phi_j^n(s) = (-1)^n I_b C_b \int_A [H(x)]^{j-s} \ln^n[H(x)] \, dx.$$  

(13)

However, when the decision level $g_0$, though large, is not extremely large and the saddle point is not far away from the origin, the contributions from taking the integral path through the saddle point and around the singularity at $s = 0$ cannot be treated separately. Therefore, a new asymptotic expansion must be used instead of eq. (12). Since eq. (11) gives only one saddle point, the asymptotic expansion can be derived by considering the integral of the kind with one saddle point and a simple pole at the origin. The complex phase $\phi_j(s)$ is analytic in the neighborhood of the origin and behaves much like a second-order polynomial in $s$. This suggests changing of variables of the integration from $s$ to $z$ where

$$\varphi_j(z) = z^2 - 2z z_j = \varphi_j(s), \quad j = 0 \text{ or } 1$$  

(13)

is the new complex phase in the $z$ plane, to be chosen with the saddle point $z_j$ corresponding to $s_j$ in $s$-plane, that is, $\varphi_j(z_j) = \varphi_j(s_j) = -z_j^2$. In order to make $z_j$ and $s_j$ have the same sign, the new saddle point $z_j$, which is also real, can be expressed in terms of $\phi_j(z_j)$ as

$$z_j = \text{sign}(s_j)(-\phi_j(s_j))^{1/2},$$

where

$$\phi_j(s_j) < 0$$

and

$$\text{sign}(x) = 1 \text{ for } x > 0, \quad \text{sign}(x) = -1 \text{ for } x < 0.$$

The probabilities from eq. (9) can be approximated by the new asymptotic expansion as

$$Q_0 = 1 - q_0(g_0) - I_0'$$

$$Q_d = 1 - q_1(g_0) - I_1'$$

(15)
where

\[ q_j(g_0) = \frac{1}{2\pi i} \int_{z>0} z^{-1} \exp[g_0 \varphi_j(z)] \, dz \]

\[
= \begin{cases} 
1 - \text{erfc}((-2g_0 \phi_j(s_j))^{1/2}), & s_j < 0 \\
\text{erfc}((-2g_0 \phi_j(s_j))^{1/2}), & s_j > 0,
\end{cases} \quad j = 0 \text{ or } 1 \tag{15a}
\]

is the integration by taking the path at the right-hand side of the z plane with the given complex phase \( \varphi_j(z) \). It can be proved that \( q_j(g_0) \) is a function of error functions where

\[ \text{erfc}(y) = (2\pi)^{-1/2} \int_y^\infty \exp(-a^2/2) \, da \]

and

\[
I_j' \sim \frac{\exp[g_0 \phi_j(s_j)]}{[2\pi g_0 \phi_j^{(2)}(s_j)]^{1/2}} \sum_{k=0}^{\infty} \left\{ \left[ -\frac{2}{g_0} \phi_j^{(2)}(s_j) \right]^k s_j^{-1} \right\} \sum_{n=0}^{2k} (-s_j)^{-2k+n} \sum_{\ell=0}^{n} A_{\ell n} (1/2)^{\ell+k} - \text{sign}(s_j) \cdot (1/2)^k \left[ -\phi_j^{(2)}(s_j) / 2 \phi_j(s_j) \right]^{1/2} \left[ g_0 \phi_j(s_j) \right]^{-k} \]

\tag{15b}

is the asymptotic expansion in power series in \( g_0^{-1} \) with the coefficients obtained by making use of the classical saddle-point method, known as the "Ursell method." \( A_{\ell n} \) is the same coefficient in terms of the \( n \)-th derivatives \( \phi_j^{(n)}(s_j) \) as given in eq. (12b).

When the decision level \( g_0 \) is very large and the saddle point \( s_j \) is far from the origin, the function \( q_j(g_0) \) itself in eq. (15a) can be approximated by the classical saddle-point method in the z plane with an approximation form.
which is replaceable by the second power series from eq. (15b). Thus eq. (15) reduces to eq. (12). In other words, for large decision levels $g_0$, the tails of the distribution of the statistic $g$ can be approximated asymptotically by either eq. (15) or eq. (12). However as the decision level $g_0$ approaches the mean value $E[g | H_j]$ under hypothesis $H_j$ ($j = 0$ or 1), the saddle point $s_j$ will move toward the origin. Therefore, eq. (15) is the asymptotic approximate form one should use.

As an example we take a circular Gaussian image

$$G(x) = \frac{1}{2\pi \sigma^2} \exp[-(x_1^2 + x_2^2)/2\sigma^2]$$  \hspace{1cm} (16)

to be detected against a uniform background. $\sigma^2$ is a constant. Thus from eq. (3)

$$H(x) = 1 + \frac{I_s}{I_b G_b} \frac{1}{2\pi \sigma^2} \exp[-(x_1^2 + x_2^2)/2\sigma^2]$$

$$= 1 + D \nu(x), \hspace{1cm} (17)$$

where $D = I_s/(2\pi \sigma^2 I_b G_b)$ is defined as the signal-to-noise ratio and

$$\nu(x) = \exp[-(x_1^2 + x_2^2)/2\sigma^2].$$

By substituting eq. (17) into eq. (7) and changing the variables, the Laplace transform $h_j(s)$ can be simplified as

$$h_j(s) = \exp \left\{ N_0 \int_0^1 (1 + Dy)^{-s} - 1 \right\} y^{-1} dy \right\}. \hspace{1cm} (18)$$

where $N_0 = 2\pi \sigma^2 I_b G_b \tau$ is the average number of charges observed during the time interval $\tau$ over the area $2\pi \sigma^2$ under hypothesis $H_0$. Similarly the function $\phi_j(s)$ from eq. (10) becomes

$$\phi_j(s) = g_0^{-1} N_0 \int_0^1 (1 + Dy)^{-s} - 1 \right\} y^{-1} dy + s. \hspace{1cm} (19)$$
The saddle point $s_j$ is obtained by solving the equation $\phi_j^{(1)}(s_j) = 0$ or

$$g_0 = N_0 \int_0^1 (1 + Dy)^{j-s} \ln(1 + Dy) y^{-1} \, dy.$$  \hfill (20)

The $n^{th}$ derivative of $\phi_j(s)$ for $n \geq 2$ becomes

$$\phi_j^{n}(s) = (-1)^n g_0^{n-1} N_0 \int_0^1 (1 + Dy)^{j-s} \ln^n(1 + Dy) y^{-1} \, dy,$$  \hfill (21)

$j = 0 \text{ or } 1$.

The mean values of $g$ under $H_j$ are

$$E[g|H_0] = N_0 \int_0^1 \ln(1 + Dy) y^{-1} \, dy$$

$$E[g|H_1] = N_0 \int_0^1 (1 + Dy) \ln(1 + Dy) y^{-1} \, dy.$$  \hfill (22)

The reliability of the likelihood-ratio detector defined by eq. (8) can be calculated by using eq. (15) when the parameters $N_0$ and $D$ are given. The saddle point $s_j$, the complex function $\phi_j(s)$, and the $n^{th}$ derivative of $\phi_j(s)$ under hypothesis $H_j$ can be calculated from eqs.(19), (20), and (21), once the decision level $g_0$ has been determined from the detection criterion. If the cost functions and the prior probability of the hypothesis $H_j$ ($j = 0 \text{ or } 1$) are given, the decision level $A_0$ on the likelihood ratio from eq. (1) can be calculated according to the Bayes strategy and

$$g_0 = \ln A_0 + N_S,$$  \hfill (23)

where $N_S$ is the average total number of charges observed from the receptor due to the star intensity $I_s$. If the prior probability of the hypothesis $H_j$ is unknown, then the minimax criterion can be used where the decision level can be obtained by maximizing the average cost. For the Neyman-Pearson criterion, a false-alarm probability $Q_0$ will be preassigned, and the decision level $g_0$ can
be obtained from eq. (15) by numerical iteration, as by Newton's method. The system will declare the hypothesis $H_j$ after comparing the statistic $g$ with the calculated $g_0$, that is

$$
\begin{align*}
H_0 & \quad \text{if} \quad g \leq g_0 \\
H_1 & \quad \text{if} \quad g > g_0 \\
\end{align*}
$$

(24)

The detection probability $Q_d$ can be calculated again from eq. (15). To investigate the performance of the detector numerical examples will be given.
IV. Threshold Detector

When the signal is very weak, that is, the signal-to-noise ratio \( D \) is much less than 1, the likelihood detector from eq. (2) can be approximated by

\[
g = \sum_k n_k \left[ \frac{\mu_s(x_k)}{\mu_0(x_k)} \right] = \sum_k n_k \left( \frac{I_s}{I_b G_b} \right) G(x_k).
\]  

(25)

Since \( I_s, I_b, \) and \( G_b \) are all constant, for a circular Gaussian image the statistic \( g \) can be simply represented by

\[
g = \sum_k n_k v(x_k).
\]  

(26)

The Laplace transform of the p.d.f. \( p_j(g) \) is

\[
h_j(s) = E\{e^{-sg} | H_j \}
\]

\[
= \prod_k \exp \left\{ \tau I_b G_b \sum_k (1 + D v(x_k))^j [\exp(-s v(x_k)) - 1] \delta A_k \right\}.
\]  

(27)

For many small \( \delta A_k \), eq. (27) becomes

\[
h_j(s) = \exp \left\{ \tau I_b G_b \int_A (1 + D v(x))^j [\exp(-s v(x)) - 1] \, dx \right\}
\]

\[
= \exp \left\{ N_0 \int_0^1 (1 + Dy)^j [\exp(-sy) - 1] \, dy \right\},
\]  

\( j = 0 \) or 1.  

(28)

The mean value and variance of the threshold statistic \( g \) under \( H_j \) are given as

\[
E[g | H_j] = N_0 \int_0^1 (1 + Dy)^j \, dy = N_0(1 + jD/2)
\]

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\[
\text{Var}[g_{ij}] = N_0 \int_0^1 (1 + Dy)^j y \, dy = N_0 (\frac{1}{2} + JD/3),
\]
\[j = 0 \text{ or } 1. \tag{29}\]

The complex phase from eq. (10) becomes
\[
\phi_j(s) = g_0^{-1} N_0 \int_0^1 (1 + Dy)^j \left[\exp(-sy) - 1\right] y^{-1} \, dy + s. \tag{30}\]

The saddle point \(s_j\) is obtained by solving the equation \(\phi_j^{(1)}(s_j) = 0\) or
\[
g_0 = N_0 \int_0^1 (1 + Dy)^j \exp(-sy) \, dy
= s^{-1} N_0 \left[1 - \exp(-s) + JD[s^{-1} - (1 + s^{-1}) \exp(-s)]\right]. \tag{31}\]

Eq. (31) has only one single saddle point for positive \(g\).

The \(n^{\text{th}}\) derivative of \(\phi_j(s)\) for \(n \geq 2\) are given by
\[
\phi_j^{(n)}(s) = (-1)^n g_0^{-1} N_0 \int_0^1 (1 + Dy)^j y^{n-1} \exp(-sy) \, dy
= (-1)^n g_0^{-1} N_0 [f_{n-1}(s) + JD f_n(s)], \quad j = 0 \text{ or } 1, \tag{32}\]

where
\[
f_n(s) = \int_0^1 y^n \exp(-sy) \, dy
= n! s^{-n-1} - s^{-1} \exp(-s) \sum_{k=0}^{n} \frac{[n!/(n-k)!]}{s^k}. \tag{33}\]

Again when the total background average count \(N_0\) from the area \(2\pi a^2\) during the interval \(\tau\) and the signal-to-noise ratio \(D\) are given, the reliability of the threshold detector can be calculated by substituting eqs. (30), (31), and (32) into eq. (15). The decision level \(g_0\) is obtained according to the detection strategy as discussed in III.
V. Numerical Example and Discussion

To investigate the performance of the optimum detector and the threshold detector, a program has been written by using equations (15), (15a), and (15b). Some numerical calculations of the reliability have been carried out at various signal-to-noise ratios. The typical average number of background charges $N_0$ is assumed to be 5.

1. The reliability of the optimum detector versus the decision level $g_0$ is plotted in Fig. 1 at two different signal-to-noise ratios $D = 1$ and $D = 6$.

2. The detection probability $Q_d$ of the optimum detector versus the signal-to-noise ratio $D$ with the false-alarm probability $Q_0$ fixed at $10^{-3}$ and $10^{-4}$ is plotted in Fig. 2.

3. The false-alarm probability $Q_0$ of the threshold detector versus the decision level $g_0$ is plotted in Fig. 3 where the detection probabilities $Q_d$ are also shown at $D = 0.1, 1, 2, 3, 4$, and $6$.

4. The detection probabilities $Q_d$ for both optimum and threshold detectors are plotted versus the signal-to-noise ratio $D$ at fixed $Q_0 = 10^{-3}$ in Fig. 4.

From the numerical examples, it is clear that the detection probability $Q_d$ of the likelihood detector or the threshold detector increases as the signal-to-noise ratio $D$ becomes larger at fixed false-alarm probability $Q_0$ and parameter $N_0$ (Figures 1, 2, and 3). Since the distribution of the threshold statistic is independent of the signal strength under hypothesis $H_0$, the advantage of this detector at weak signals is that the decision level $g_0$ is fixed for a preassigned $Q_0$, whereas the decision level $g_0$ of the likelihood detector must be readjusted for each different value of $D$. At strong signals, however, the likelihood detector is superior because of the higher detection probability $Q_d$ (Fig. 4).
When the Neyman-Pearson strategy is used, the false-alarm probability $Q_0$ is preassigned. The decision level $g_0$ is calculated from eq. (20) or eq. (31), where the saddle point $s_0$ must be obtained by using eq. (15) with a simple numerical iteration method. In other words if we let $s_N$ be the saddle point that gives a false-alarm probability $Q_N$ less than the preassigned value $Q_0$, and if we let $s_P$ be the saddle point that gives a false-alarm probability $Q_P$ greater than the preassigned value $Q_0$, the trial value $s_0^T$ of the saddle point can be determined by the equation

$$s_0^T = s_N + \frac{s_P - s_N}{Q_P - Q_N} (Q_0 - Q_N).$$  

(34)

Then we substitute $s_0^T$ into eq. (15), calculate the false-alarm probability $Q_0^T$, and compare it with the preassigned value $Q_0$. If $Q_0^T$ is greater than $Q_0$, the values of $s_P$ and $Q_P$ will be replaced by the values of $s_0^T$ and $Q_0^T$, and a new trial value $s_0^T$ will be obtained by using equation (34). If $Q_0^T$ is less than $Q_0$, the values of $s_N$ and $Q_N$ will be replaced by the values $s_0^T$ and $Q_0^T$, and the new trial value $s_0^T$ will again be calculated by the same equation. Repeat the procedure until the calculated value $Q_0^T$ approaches the preassigned value $Q_0$ within a tolerable error. The detection probability $Q_d$ is thereupon computed by using eq. (15).

To calculate the $n^{th}$ derivatives of the complex phase $\phi_j^{(n)}(s)$ and the decision level $g_0$, Simpson's rule can be used for the numerical integration. The number of points used in integration procedure depends on the relative error the system requires. More points can be used when higher accuracy is required. Also when the saddle point $s_j$ ($j = 0$ or 1) is small, the value in the parenthesis of the eq. (15b) may be the difference of two large values. Double precision may therefore be required to reduce the computation error.
The approximation of the reliability by the asymptotic expansion is based on the assumption that \( g_0 \) is reasonably large. The values of the probabilities are obtained by adding up the terms in the asymptotic series as long as they continue to decrease. The accuracy of the approximation is determined by the smallest terms or the last term added to the sum. When the signal is weak, in order to maintain large enough \( g_0 \), longer observation time can be used. Alternatively, if the higher-order cumulants of the logarithm of the characteristic function are much less than the variance, the statistic can be assumed to be normally distributed, and the reliability can be computed from error functions. Nevertheless, when the decision level \( g_0 \) is large enough, the reliability can be approximated by the asymptotic expansion in descending power series of \( g_0 \) for all values of the parameters \( N_0 \) and \( D \).

When the image on the photosensitive surface is translated into a time-varying signal by a scanning technique, the likelihood ratio detector is constructed by dividing the observation time into \( M \) intervals, as has been discussed by Reiffen and Sherman. The charges observed in each interval will be weighted by a time function \( G(t) \) instead of the function \( G(x) \) in space, and the reliability of the detector should be calculated by the same asymptotic expansion form as shown in eq. (15). Furthermore, when a circular uniform image is assumed, the weight function \( G(x) \) will be a constant and independent of the location. Thus the statistic \( g \) will now be Poisson-distributed. When the Neyman-Pearson criterion is used, the decision level \( g_0 \) can be obtained by using randomization, and the reliability of the detector can be expressed by the equations

\[
Q_0 = \sum_{g > g_0} N_B^g \exp(-N_B)/g! + f N_B^g_0 \exp(-N_B)/g_0!
\]

\[
Q_d = \sum_{g > g_0} (N_B + N_S)^g \exp(-N_B - N_S)/g! + f (N_B + N_S)^g_0 \exp(-N_B - N_S)/g_0!
\]

(35)
where $N_B$ or $N_S$ is the total average number of charges observed from the image due to the background radiation or signal alone. The decision level $g_0$ is the largest integer such that the preassigned probability $Q_0 > \sum_{g=0}^{g_0} \frac{N_B^g \exp(-N_B)}{g!}$, and it can be easily determined when $N_B$ is given. Here $f$ is the probability for the system to choose $H_1$ when $g = g_0$ and can be determined from the expression for $Q_0$ after $g_0$ is calculated with fixed $Q_0$. Thus the detection probability $Q_d$ can be calculated for given $N_B$, $N_S$, and $g_0$. 
References


Figure 1

Observed interval $T$
$N_0 = 5$ (counts)
Optimum detector

$Q_0 = 10^{-4}$
$Q_0 = 10^{-3}$

Observed interval $\tau$
$N_0 = 5$ (counts)

Figure 2
Observed interval $T$

$N_0 = 5$ (counts)

$Q_0 = 10^{-3}$

Figure 4

Detection Probability $Q_d$ (%)