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ABSTRACT

Given a linear time invariant multivariable system $\dot{\underline{x}}=\hat{A}\underline{x} + \hat{B}\underline{u}$, $\underline{y}=\hat{C}\underline{x}$ with m inputs and p outputs, Davison [2] has shown that p closed loop poles of the system can be preassigned arbitrarily using constant gain output feedback provided $[\hat{A},\hat{B}]$ is controllable. This paper shows that if $m \neq p$, $[\hat{A},\hat{B},\hat{C}]$ controllable & observable, Rank $\hat{B}=m$ and Rank $\hat{C}=p$, then max (m,p) poles of the system can be assigned arbitrarily using constant gain output feedback. This result is particularly useful in designing systems for high integrity in the event of transducer failure [3].

A NOTE ON POLE ASSIGNMENT

Introduction

Consider a linear time-invariant multivariable system

$$\begin{aligned} \dot{\underline{x}} &= \hat{A}\underline{x} + \hat{B}\underline{u} \\ y &= \hat{C}x \end{aligned} \qquad (1)$$

Where $\underline{x}$ is an n vector of states,

$\underline{u}$ is an m vector of inputs and

$\underline{y}$ is a p vector of outputs.

The problem of pole assignment i.e. the placement of eigenvalues of the closed loop system $\dot{\underline{x}} = (\hat{A} + \hat{B}K\hat{C})\underline{x}$ where K is a constant gain matrix has been studied by Retallack [1], Davison [2] and others. It has been shown by Davison that if $(\hat{A},\hat{B})$ is controllable, and if Rank $\hat{C}$ = p, then p poles of the system can be arbitrarily placed using output feedback. This correspondence shows that given $(\hat{A},\hat{B},\hat{C})$ controllable and observable, Rank $\hat{B}$ = m and Rank $\hat{C}$ = p, then at least max (m,p) poles can be arbitrarily placed using output feedback. In certain cases, as shown in the example, more than max (m,p) poles can be arbitrarily placed.

Theorem

Given the system (1) with $m \neq p$, Rank $\hat{B} = m \leq n$, Rank $\hat{C} = p \leq n$, then a linear feedback of the output $\underline{u} = K\underline{y}$, where K is a (mxp) constant gain matrix, can always be found such that max (m,p) eigenvalues of the closed loop system can be made to take preassigned values.

Proof

Let $(\lambda_1, \lambda_2, ---, \lambda_n)$ and $(\rho_1, \rho_2, ---, \rho_n)$ be the eigenvalues of the open-loop and closed-loop system respectively.

We have

open loop characteristic polynomial = $|sI-\hat{A}| = (s-\lambda_1)(s-\lambda_2) --- (s-\lambda_n)$ (2)

and closed loop characteristic polynimial = $|sI-\hat{A}+\hat{B}K\hat{C}| = (s-\rho_1)(s-\rho_2) ---)s-\rho_n)$

Then (3)

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-A|} = \det [I + \hat{B}K\hat{C}\ (sI-\hat{A})^{-1}]. \qquad (4)$$

Choosing $K = fd^T$ where f is a mx1 (column) vector and d^T is a 1xp (row) vector, and using the identity det $[I + MN] = \det [I + NM]$, equation (4) becomes

$$\begin{aligned} \frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-A|} &= 1 + d^T\hat{C}\ (sI-\hat{A})^{-1}\hat{B}f \\ &= 1 + d^T\hat{C}T(sI-T^{-1}\hat{A}T)^{-1}\ T^{-1}\hat{B}f \qquad (5) \\ &= 1 + d^TC\ (s\ -A)^{-1}\ Bf \end{aligned}$$

where $C = \hat{C}T$, $A = T^{-1}\hat{A}T$, $B = T^{-1}\hat{B}$ and T is a nxn nonsingular matrix.

Now, equation (5) gives

$$\frac{|sI-\hat{A}+\hat{B}K\hat{C}|}{|sI-A|} = 1 + \sum_{i=1}^{n} \frac{\alpha_1}{(s-\lambda_i)} \qquad (6)$$

The value of α_1 depends on the closed loop eigenvalues $(\rho_1, ---, \rho_n)$.

From (5) and (6),

$$d^TC\ (sI-A)^{-1}\ Bf = \sum_{i=1}^{n} \frac{\alpha_1}{(s-\lambda_i)} \qquad (7)$$

For simplicity it is assumed that A has distinct eigenvalues. However, the following analysis can easily be modified when A has multiple eigenvalues.

Choosing T as a modal matrix equation (7) becomes

$$d^TC(sI-\Lambda)^{-1}\ Bf = \sum_{i=1}^{n} \frac{\alpha_i}{(s-\lambda_i)} \qquad (8)$$

where $\Lambda = \text{diag.}\ (\lambda_1,\lambda_2,---,\lambda_n)$.

Let c^i be the i^{th} column of C and b_i be the i^{th} row of B. Then,

$$\alpha_i = d^T c^i b_i f - \qquad i = 1,2,---, n. \qquad (9)$$

Case (i)

Let $p > m$ i.e. more outputs than inputs. Choose f_i such that $b_i f = \delta_i \neq 0$ $i = 1,2,---n$. This can always be done since $b_i \neq 0$, for controllability.

Hence, $d^T c^i = \alpha_i/\delta_i \qquad i = 1, ---, n. \qquad (10)$

This gives $C^T d = \underline{\alpha} \qquad (11)$

where $\underline{\alpha} = \text{col}\ [\alpha_1/\delta_1, \alpha_2/\delta_2, ---, \alpha_n/\delta_n]$.

Now, let C_p be the matrix made of the p independent rows of C^T and $\underline{\alpha}_p$ the corresponding subset of $\underline{\alpha}$. Then,

$$d = C_p^{-1}\ \underline{\alpha}_p \qquad (12)$$

Thus $(d_1, d_2, ---, d_p)$ can be chosen corresponding to the p desired pole locations. Once this is done the remaining (n-p) poles are fixed automatically.

Case (ii)

Let $m > p$ i.e. more inputs than outputs.

Choose d_i such that $d^T c^i = \gamma_i \neq 0 \qquad i = 1, 2, ---, n.$

This can always be done since $c^i \neq 0$, for observability.

Hence, $b_i f = \alpha_i/\gamma_i \qquad i = 1, ---, n. \qquad (13)$

This gives $Bf = \underline{\alpha} \qquad (14)$

where $\underline{\alpha} = \text{col}\ [\alpha_1/\delta_1, \alpha_2/\gamma_2, - - -, \alpha_n/\gamma_n]$

Since the rank of B is m, there are m independent rows of B, B_m, such that

$$B_m f = \underline{\alpha}_m$$

where $\underline{\alpha}_m$ is the corresponding subset of $\underline{\alpha}$.

$$f = B_m^{-1}\ \underline{\alpha}_m \qquad (15)$$

Thus $(f_1, f_2, \cdots f_m)$ can be chosen corresponding to the m desired pole locations and the remaining (n-m) poles are located automatically. From case (i) and case (ii) it is evident that at least max (m,p) poles of the system can be assigned arbitrarily.

<u>Example 1</u>:

$$\underline{\dot{x}} = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underline{u}$$

$$\underline{y} = |0 \quad 1 \quad 0| \; \underline{x}.$$

According to Davison one pole can be arbitrarily assigned. However, according to the theorem stated in this paper all the three poles can be arbitrarily assigned, i.e. m = 3.

The open-loop poles are at 1, -2 and 3. Let the closed-loop poles be at -1, -3 and -4.

If the modal matrix T and its inverse T^{-1} are given by

$$T = \begin{bmatrix} -1 & 11 & 1 \\ 1 & 1 & 1 \\ 1 & -14 & 1 \end{bmatrix} \text{ and } \quad T^{-1} = \frac{1}{30} \begin{bmatrix} -15 & 25 & -10 \\ 0 & 2 & -2 \\ 15 & 3 & 12 \end{bmatrix} .$$

Then the transformed system equations become

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{x} + \frac{1}{30} \begin{bmatrix} -15 & -10 & 25 \\ 0 & -2 & 2 \\ 15 & 12 & 3 \end{bmatrix} \underline{u}$$

We have, open-loop characteristic polynomial = $(s-1)(s+2)(s+3) = s^3-2s^2-5s+6$.

and closed-loop characteristic polynomial = $(s+1)(s+3)(s+4) = s^3+8s^2+19s+12$.

Now, from (6)

$$\frac{s^3+8s^2+19s+12}{s^3-2s^2-5s+6} = 1 + \frac{10s^2+24s+6}{(s-1)(s+2)(s-3)}$$

$$= 1 - \frac{40/6}{s-1} - \frac{2/15}{s+2} + \frac{168/10}{s-3}$$

This gives $\alpha_1 = -40/6$, $\alpha_2 = -2/15$ and $\alpha_3 = 168/10$.

K is given by $K = fd^T = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} d$.

and choosing $d = 1$, we get the equations

$$15f_1 + 10f_2 + 25f_3 = 200$$

$$15f_1 \div 12f_2 + 3f_3 = 504$$

$$f_2 - f_3 = 2$$

solving these equations gives $K = \begin{bmatrix} 22 \\ 12 \\ 10 \end{bmatrix}$. With this choice of K the closed loop poles are located at -1, -3 and -4.

<u>Example 2</u>:

$$\underline{\dot{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \underline{u}$$

$$\underline{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \underline{x}$$

Let the desired closed loop poles be -1, -2 and -3. It is easily seen that $\alpha_1 = 12, \alpha_2 = -60$ and $\alpha_3 = 60$. $K = fd^T = [f_1, f_2] \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and we have the

equations

$$d_1 f_1 = 12$$

$$d_2 f_2 = -60$$

$$d_2(f_1 + f_2) = 60.$$

Making $d_1 = 1$, we get $d_2 = 10$, $f_1 = 12$ and $f_2 = -6.0$.

Hence,

$$K = \begin{bmatrix} 12 & 120 \\ -6.0 & -60 \end{bmatrix}$$

places all the three closed loop poles at the desired locations. The theorem, however, assures at least two poles can be assigned arbitrarily.

Conclusion

It has been shown that for a controllable, observable system at least max (m,p) poles can be assigned arbitrarily. In certain cases more than max (m,p) poles can be assigned. This result is particularly useful in designing systems for high integrity in the event of transducer failure. Due to failure there may be loss of inputs and/or outputs in the system. In such an event this design takes advantage of the unequal number of inputs and outputs so as to assure no loss in pole assignability.

References

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