ADAPTIVE CONTROL OF STOCHASTIC LINEAR SYSTEMS WITH UNKNOWN PARAMETERS

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by

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ABSTRACT

This thesis considers the problem of optimal control of linear discrete-time stochastic dynamical system with unknown and, possibly, stochastically varying parameters on the basis of noisy measurements. It is desired to minimize the expected value of a quadratic cost functional. Since the simultaneous estimation of the state and plant parameters is a nonlinear filtering problem, the extended Kalman filter algorithm is used. The open-loop feedback optimal control technique is investigated as a computationally feasible solution to the adaptive stochastic control problem. The open-loop feedback optimal control system adaptive gains depend on the current and future uncertainty of the parameters estimation. Thus, the standard Separation Theorem does not hold in this problem. Suboptimal control system in which Separation Theorem is arbitrarily enforced is also considered. The identifier is the same as that of the open-loop feedback optimal control system. Several qualitative and asymptotic properties of the open-loop feedback optimal control and the enforced separation scheme are discussed. Simulation results via Monte Carlo method show that, in terms of the performance measure, for stable systems the open-loop feedback optimal control system is slightly better than the enforced separation scheme, while for unstable systems the latter scheme is far better.
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CHAPTER 1

INTRODUCTION

1.1 Brief Historical Review

The theory of stochastic optimal control of linear systems with known dynamics with respect to quadratic performance criterion is fairly well developed [1],[2],[3],[4]. The uncertainties arise from the plant and observation disturbances and initial states of the system. The statistical laws of these uncertainties are assumed to be known. A class of such problems has been considered in discrete time by Joseph and Tou [5] and Gunckel and Franklin [6], and in continuous time by Wonham [7]. Under fairly general assumptions of the Gaussian noise structure, the Separation Theorem or the Certainty-Equivalence Principle [1],[2] is constructive in studying the optimal control problem of purely stochastic systems. The optimal closed-loop stochastic control can be obtained by combining the solution to two separate problems — optimal linear estimation of the states and optimal feedback control of the corresponding deterministic system. The results have been extended to more general performance criteria by Striebel [8] and Wonham [9].

However, in many practical control systems, the dynamics of the system are not completely known. Such problems occur in a variety of engineering designs of aerospace and process control systems. In the most general setting, the plant and observation parameters, the various noises, the initial conditions and/or description of the inputs are imperfectly known. We assume that we know the structure of the dynamics. In our
approach it is assumed that all the uncertainties are random processes with known statistics.

The control of linear systems with unknown plant dynamics has to be parameter adaptive. An adaptive control system should have an identifier that generates the estimates of the states, the system parameters (adaptive filtering) and their levels of uncertainty. Based on the identification of the plant, the adaptive controller makes a decision followed by modification or activation.

If the plant is imperfectly known because of random time-varying parameters, then the initial identification, decision, and modification procedures must be done continuously. This constant self-organization of the system is characteristic of all adaptive systems. The controller should reflect (1) the initial uncertainty about the system and the desire to minimize that uncertainty; (2) the dependence upon current estimates and the confidence one can attach to these values, and (3) the ultimate and basic objective to minimize the cost functional. Thus the control input must be used for the identification of parameters and for attaining the desired system response. The "dual" nature of adaptive control is clearly emphasized in this philosophy of adaptive systems. The feedback estimation controller subsystems approach implies that the Separation Theorem will not hold in adaptive systems. We shall find that the adaptive control gains depend upon the parameter estimation accuracy.

The problem of controlling a system with imperfectly known parameters operating in a stochastic environment has been considered by several people. An optimal solution of this problem may be obtained based on Feldbaum's "dual" control approach [10]. Practical on-line
computation of the optimal closed-loop control is not currently feasible, however, due to computer limitations. Rigorous close approximations to the optimal solution such as the "parameter-adaptive self-organizing" approach proposed by Stein and Saridis [24] contain computational algorithms too complicated for practical implementation. Therefore, different suboptimal but practical solution methods were proposed in the literature. Lee [30] has suggested an arbitrary separation of the parameter identification, state estimation, and control. Identification algorithms based on stochastic approximation [39] or maximum likelihood [40] preclude the use of dynamic feedback control depending on the current estimates of the unknown parameters. Farison et al [23] have considered a suboptimal closed-loop adaptive control scheme for unknown systems with perfect measurements using conditional quadratic cost function to force separation of identification and control. Florentin [22] and Murphy [25] considered the stochastic optimal control of linear systems with unknown but constant gain in discrete time by formulating the identification as a linear estimation problem and approximating the predictor equations and, using suboptimal approach, reduced it to a two-point-boundary value problem. Gorman and Zaborszky [26] considered the problem in continuous time and arrived at results similar to Murphy's [25]. Assuming separation, Saridis and Lobbia [27] considered an online stochastic approximation algorithm for parameter identification and showed that the per-interval feedback controller gave better performance than the overall optimal feedback controller. Schmidt [28] worked out a linear perturbation-controller for systems with unknown parameters by finding the optimal open-loop control minimizing the cost functional which was
expanded in a power series. Aoki [4], Spang [11], Bar-Shalom and Sivan [12], Dreyfus [13],[14], and Curry [15] all used the optimal open-loop feedback approach for linear stochastic discrete-time systems to obtain a suboptimal closed-loop control. Tse and Athans [16],[29] showed that the use of this design concept leads to a stochastic control system that is adaptive and computationally feasible for on-line implementation.

In this thesis, the open-loop feedback optimal technique is used to consider the problem of adaptive control of linear discrete-time systems whose poles and zeros are unknown based on inaccurate measurements, with respect to a quadratic cost functional. The unknown parameter values may be time-varying and random. The disturbances in the state and measurement equations are assumed to be additive, white Gaussian stationary noise sequences. All the uncertainties are assumed to have known statistical laws.

The simultaneous estimation of state and plant parameters is a nonlinear filtering problem. Since the truly optimal nonlinear estimator cannot be implemented exactly with current digital computers, one is forced to use a suboptimal estimation algorithm such as the extended Kalman filter [17],[18],[19],[20]. The approximate expressions for the conditional means and error covariance matrix are summarized in Section 2.3.

1.2 Structure of the Thesis

The structure of the thesis is as follows. In Chapter 2 we describe the adaptive stochastic control problem under consideration and give the statistical assumptions. We discuss the philosophy of control
based on the open-loop feedback doctrine. We state the solution to the resulting deterministic optimal control problem. We define all the variables and summarize the equations of the open-loop feedback optimal control algorithm. In Chapter 3 we shall first modify the original cost functional into a deterministic cost functional. The original problem is thus reformulated as a completely deterministic optimal control problem. Appendix A contains the details of the reformulation. We then derive the optimal open-loop control law via dynamic programming. We shall derive the "conditional open-loop optimal cost-to-go" to describe the performance of the optimal open-loop control sequence. We then interpret the derived solution in a feedback sense.

In Chapter 4 we present the results on the uniqueness and existence of the open-loop feedback optimal control. We shall consider the asymptotic behavior of the identifier and the overall adaptive control system as the time index $k \to \infty$. We define the enforced separation scheme, in which the actual parameter values are replaced by their current estimates, and discuss its asymptotic properties. In Chapter 5 we discuss the open-loop feedback optimal approach and the qualitative properties of the results obtained. The O.L.F.O. equations given in Chapter 2 and derived in Chapter 3 are given further heuristic interpretations.

In Chapter 6 we present the simulation results on first-order linear time-invariant (stable and unstable) single-input single-output systems. The actual plant parameter values are, thus, unknown constants. Simulation results will compare the response of the resultant stochastic control system when 1) the actual parameters are known 2) O.L.F.O. design is employed, and 3) enforced separation is employed. The identifier used
in 2) and 3) is the extended Kalman filter and reduces to the optimal filter in 1). In Chapter 7 we discuss the results of the simulation studies in light of the general qualitative properties given in Chapters 4 and 5. We consider, finally, the computational aspects of the two sub-optimal closed-loop control systems. In Chapter 8 we summarize the theoretical and simulation results on the adaptive control systems. Further research in this area are also discussed.

1.3 Contribution of the Thesis

This thesis extends the previous work by Tse and Athans [29] to a larger class of practical control problems, which involve imperfectly known system dynamics as well as input gains. We solve this nonlinear stochastic control problem using the open-loop feedback optimal control via dynamic programming. The main analytical result is the development of the open-loop feedback optimal control structure and equations. The precise variation of the open-loop feedback optimal control adaptive gains as a function of the future expected uncertainty of the parameters is derived. Application of the OLFO adaptive gain plus correction term control to the N-stage state error plus control effort stochastic linear regulator problem with unknown but constant parameters is compared with the design in which the Separation Theorem is arbitrarily enforced [23] in terms of the performance measure. The system resulting from the arbitrary use of the current estimates for the actual (but unknown) parameters is much simpler to implement since the propagation of the covariance matrices is not needed. It was expected that the enforced separation scheme would be worse than that obtained by OLFO design.
However, simulation results via Monte Carlo method showed that for stable systems the OLFO control system is slightly better than the enforced separation, while for unstable systems, the latter design is far better.
2.1 Problem Statement

In this section we shall state the problem of interest — the control of discrete-time linear stochastic dynamical system with unknown parameters based on noisy observations of its output. The dimension of the system is assumed to be known, but the pole and zero locations may not be completely known and they may vary in a stochastic manner. Basically then, we assume a canonical structure for the state-space model of the system, but the actual plant parameter values are not completely specified. The particular class of problems that we shall examine has both the plant time constants and input gain vector imperfectly known. The performance criterion is chosen so as to minimize the expectation of a quadratic form in the state and control variables over a fixed interval of time.

Suppose we have a discrete-time n-dimensional linear dynamical system, with an imperfectly known initial state with noise disturbances entering the plant equation and the output measurement, governed by the following vector stochastic difference equations (integer k is the time index)

Plant: \( x(k+1) = A(k)x(k) + b(k)u(k) + \xi(k) \) \( \quad k=0,1,\ldots,N-1 \) (2.1.1)

Measurement: \( z(k) = Cx(k) + \theta(k) \) \( \quad k=0,1,\ldots,N \) (2.1.2)

where we assume that \( x(k), b(k) \) and \( \xi(k) \in \mathbb{R}^n \), \( z(k) \) and \( \theta(k) \in \mathbb{R}^m \), \( C \) is a known constant \( m \times n \) matrix, and \( u(k) \) is an unconstrained scalar input.

Furthermore, we assume that the actual (but unknown) \( n \times n \) plant matrix
\( \mathbf{A}(k) \) has the following canonical representation or is convertible to this equivalent system by a linear nonsingular transformation. [30]

\[
\mathbf{A}(k) = \begin{bmatrix}
I_{n-1} & \cdot & \cdot & \cdot \\
\cdot & I_{n-1} & \cdot & \cdot \\
\cdot & \cdot & I_{n-1} & \cdot \\
\cdot & \cdot & \cdot & I_{n-1}
\end{bmatrix}, \quad \mathbf{a}(k) = \begin{bmatrix}
a_1(k) \\
a_2(k) \\
\vdots \\
a_n(k)
\end{bmatrix} \in \mathbb{R}^n
\]

(2.1.3)

The matrix \( \mathbf{A}(k) \) is called a companion matrix. The unknown plant parameter vector \( \mathbf{a}(k) \) and gain vector \( \mathbf{b}(k) \) are assumed to satisfy the difference equations

\[
\mathbf{a}(k+1) = \mathbf{a}(k) + \mathbf{\delta}(k)
\]

(2.1.4)

\[
\mathbf{b}(k+1) = \mathbf{b}(k) + \mathbf{\gamma}(k)
\]

(2.1.5)

respectively, where \( \mathbf{\delta}(k), \mathbf{\gamma}(k) \in \mathbb{R}^n \). We shall assume also that the triplet \( [\mathbf{A}(k), \mathbf{b}(k), \mathbf{c}] \) of Eqs. (2.1.1) and (2.1.2) represents a completely controllable and completely observable system [31].

We shall assume that the stationary random sequences \( \{\mathbf{x}(\cdot), \mathbf{\xi}(\cdot), \mathbf{\delta}(\cdot), \mathbf{\gamma}(\cdot)\} \) are white Gaussian, zero-mean, and uncorrelated with each other and with the Gaussian random vectors \( \{\mathbf{x}(0), \mathbf{a}(0), \mathbf{b}(0)\} \). The statistical laws of the underlying random vectors are assumed known for all \( k \). The symbol \( N[\mathbf{m}, \mathbf{\Sigma}] \) denotes normal distribution with mean equals to \( \mathbf{m} \) and finite covariance matrix equals to \( \mathbf{\Sigma} \). Thus, let

\[
\mathbf{x}(0) \sim N[\mathbf{x}_0, \mathbf{\Sigma}_{\mathbf{x}_0}]
\]

(2.1.6)

\[
\mathbf{a}(0) \sim N[\mathbf{a}_0, \mathbf{\Sigma}_{\mathbf{a}_0}]
\]

(2.1.7)

\[
\mathbf{b}(0) \sim N[\mathbf{b}_0, \mathbf{\Sigma}_{\mathbf{b}_0}]
\]

(2.1.8)
where the covariance matrices have the properties

\[
\begin{align*}
\Sigma_{x_0} &= \Sigma_{x_0}', > 0 \\
\Sigma_{a_0} &= \Sigma_{a_0}', > 0 \\
\Sigma_{b_0} &= \Sigma_{b_0}', > 0 \\
\Xi(k) &= \Xi'(k) > 0 \\
\Theta(k) &= \Theta'(k) > 0 \\
\Delta(k) &= \Delta'(k) > 0 \\
\Gamma(k) &= \Gamma'(k) > 0
\end{align*}
\]

Thus, additive discrete white plant driving noise $\xi(k)$, observation noise $\theta(k)$, and parameter noises $\delta(k)$ and $\gamma(k)$ are used to model the uncertainties in the state evolution $x(k)$, the measurements $z(k)$ and the parameter vectors $a(k)$ and $b(k)$, respectively.

We are also given a quadratic cost functional $J(u)$ which is given by

\[
J(u) = \frac{1}{2} x'(N)Q(N)x(N) + \frac{1}{2} \sum_{k=0}^{N-1} \{ x'(k)Q(k)x(k) + r(k)u^2(k) \} 
\]

with $N$ fixed and finite. The objective of the problem is to find the optimal control sequence that minimizes in some sense Eq. (2.1.20). We will want to make use of all the available information in the computation of the optimal control sequence. Thus, the control input at each time $k$
will, in general, be a function of the measurements and all a priori information up to and including time k. We are seeking, therefore, a physically realizable control.

Since the system Sl is operating in a stochastic environment, both the state trajectory \( \hat{x}(\cdot) \) and the input \( u(\cdot) \) are random sequences. A suitable performance criterion to choose then is the scalar real-valued cost functional

\[
\bar{J}(u) = \frac{1}{2} \mathbb{E}\{x'(N)Q(N)x(N) + \sum_{k=0}^{N-1} x'(k)Q(k)x(k) + r(k)u^2(k)\} \tag{2.1.21}
\]

where \( \mathbb{E} \) denotes expected value. The expectation is taken over all the underlying random processes, \( a(0), b(0), x(0), \xi(\cdot), \theta(\cdot), \delta(\cdot), \) and \( \gamma(\cdot) \). Thus, given the noise-corrupted unknown linear dynamic system, the objective is to find the admissible control sequence \( U(0,N-1) \triangleq \{u(j)\}_{j=0}^{N-1} \) which performs best "on the average" such that it minimizes the performance measure \( \bar{J}(u) \) of Eq. (2.1.21) subject to the system equations (2.1.1) and (2.1.2).

Physically, the optimum control sequence will "on the average" drive the state \( x(k) \) of the system Sl to zero without excessive expenditure of control energy. We shall assume in Eq. (2.1.21) that \( Q(\cdot) \) is a positive semidefinite symmetric matrix and \( r(\cdot) \) is a positive scalar, and that the initial and final times are fixed and finite \( (N<\infty) \).

To complete the problem statement we must now define what we mean by admissible control sequence. For a stochastic optimal control problem it is very important to specify precisely that data or information pattern which is available for determining the control input.
Depending on what measurements the computation of the control sequence $U(0,N-1)$ is based on exactly, different formulations of the optimal stochastic control problem are possible. In the general case when the parameters are not independent random variables, one cannot obtain explicit solutions to the optimal closed-loop controller, so that one has to resort to a reasonable and feasible controller. We shall, therefore, restrict our consideration to the deterministic open-loop controls, and derive the open-loop feedback optimal controller.

The open-loop feedback optimal control policy can be interpreted as follows. At time $k$ ($k=0,1,\ldots,N-1$) we are to control a system based on measurements taken up to time $k$; we assume that no observations will be made in the future. Under this assumption one generates an optimal control sequence $\{u^O(j|k)\}_{j=k}^{N-1}$ according to the open-loop policy. Only the first element of this sequence $u^*(k) \Delta = u^O(k|k)$ is actually used. The applied control changes the probabilistic information provided by the estimator at time $k+1$. At time $k+1$, an additional measurement becomes available. The next control input $u^*(k+1)$ is again computed according to the open-loop policy, but based on all the information at time $k+1$. Thus, we shall recompute the open-loop optimal deterministic control after new information becomes available at each time instant. This technique applies in general, and includes the case when the parameters are random. It turns out that this open-loop control can be expressed as a function of the state and parameter statistics; hence, the name open-loop feedback optimal (O.L.F.O.) is used. The form of the open-loop feedback controller is shown in Fig. 2.1. The control at each time $k$ is an explicit
Fig. 2.1 Block Diagram of the Open-Loop Feedback Control System
function of state and parameter estimates and their level of uncertainty. The controller is to be designed from a deterministic system, once the estimator and the predictor are obtained. We shall see that the open-loop feedback optimal control sequence is in some sense "adaptive".

2.2 Definitions of Variables

Let the present time be indexed by $k$. Denote the accumulative observation statistic, that is, the (random) observation outputs or the available information at time $k$ by

$$Z_k \triangleq \{z(0), z(1), \ldots, z(k)\} \quad (2.2.1)$$

Let us also assume that the optimal control sequence $U^*(0,k-1) = \{u(j)\}_{j=0}^{k-1}$ has been applied to the system. Let us then define for $j > k$ the conditional expectations

$$\hat{x}(j|k) \triangleq E\{x(j)|Z_k\} \quad (2.2.2)$$

$$\hat{a}(j|k) \triangleq E\{a(j)|Z_k\} \quad (2.2.3)$$

$$\hat{b}(j|k) \triangleq E\{b(j)|Z_k\} \quad (2.2.4)$$

the error vectors

$$e_x(j|k) \triangleq \hat{x}(j|k) - x(j) \quad (2.2.5)$$

$$e_a(j|k) \triangleq \hat{a}(j|k) - a(j) \quad (2.2.6)$$

$$e_b(j|k) \triangleq \hat{b}(j|k) - b(j) \quad (2.2.7)$$
and the conditional error covariance matrix

\[ \Sigma(j|k) \triangleq E\{ e'(j|k): e'(j|k): e'(j|k)| z_k \} \]  

(2.2.8)

where \( \Sigma(j|k) \) can be decomposed into its submatrices:

\[ \Sigma(j|k) = \begin{bmatrix} \Sigma_{xx}(j|k): \Sigma_{xa}(j|k): \Sigma_{xb}(j|k) \\ \Sigma_{ax}(j|k): \Sigma_{aa}(j|k): \Sigma_{ab}(j|k) \\ \Sigma_{bx}(j|k): \Sigma_{ba}(j|k): \Sigma_{bb}(j|k) \end{bmatrix} \]  

(2.2.9)

Let us also define for \( k \leq j < N - 1 \) the \( 3n \times 3n \) Jacobian matrix of the nonlinear augmented state vector system

\[ S2: \begin{bmatrix} x(i+1) \\ a(i+1) \\ b(i+1) \end{bmatrix} = \tilde{A}(i,a(i),u^*(i)) \begin{bmatrix} x(i) \\ a(i) \\ b(i) \end{bmatrix} + \begin{bmatrix} \xi(i) \\ \delta(i) \\ \gamma(i) \end{bmatrix} \]  

(2.2.10)

where

\[ \tilde{A}(i,a(i),u^*(i)) \triangleq \begin{bmatrix} A(i): 0: u^*(i)I_n \\ 0: I_n: 0 \\ 0: 0: I_n \end{bmatrix} \]  

(2.2.11)

and \( A(i) \) is given by Eq. (2.1.3)
Hence, the $3n \times 3n$ Jacobian matrix $\hat{F}(i, \hat{a}(i|i), \hat{x}(i|i), u^*(i))$ of Eq. (2.2.10) is given by

$$
\hat{F}(i, \hat{a}(i|i), \hat{x}(i|i), u^*(i)) \triangleq \begin{bmatrix}
\hat{A}(i|i) & \hat{X}(i|i) & u^*(i)I_n \\
\vdots & \ddots & \vdots \\
0 & \ddots & 0 \\
0 & \ddots & I_n \\
\end{bmatrix}
$$

(2.2.12)

where

$$
\hat{A}(i|i) \triangleq \begin{bmatrix}
0 & I_{n-1} \\
\vdots & \ddots \\
0 & 0_{n-1} \\
\end{bmatrix}
$$

(2.2.13)

and

$$
\hat{X}(i|i) \triangleq \begin{bmatrix}
0 & 0_{n-1} \\
\vdots & \ddots \\
0 & 0_{n-1} \\
\end{bmatrix}
$$

(2.2.14)

2.3 Structure of the Open-Loop Feedback Optimal Control

We shall state in this section the solution to the deterministic optimal control problem to be formulated in Section 3.2. The detailed derivation via dynamic programming is given in Section 3.3. The symbol $u^O(j|k)$ denotes the optimal open-loop control conditioned on the observations up to and including time $k$, and is given for $j \geq k$ below. It should be stressed that $\hat{x}(j|k)$ and $\Sigma(j|k)$ are not the exact conditional means and error covariance matrix.
where the \( n(2n+1) \times n(2n+1) \) symmetric matrix \( \bar{K}(j+1|k) \) is the unique solution of the nonlinear matrix Riccati difference equation for \( k + 1 \leq j \leq N-1 \)

\[
\bar{K}(j|k) = \Phi'(j|k) [\bar{K}(j+1|k) - \bar{K}(j+1|k) \bar{B}(j|k) (\bar{r}(j|k) + \bar{V}(j|k)) - 1 \bar{B}'(j|k) \bar{K}(j+1|k) \bar{B}(j|k)] \Phi(j|k) + \bar{V}(j|k)
\]  

(2.3.2)

satisfying the boundary condition.

\[
\bar{K}(N|k) = \begin{bmatrix} Q(N) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]  

(2.3.3)

where the parameters \( \bar{r}(j|k), \bar{B}(j|k), \Phi(j|k), \bar{d}(j+1|k), \) and \( \bar{V}(j|k) \) will be defined below.

The open-loop feedback optimal control actually applied at time \( k \) is given by

\[
u^*(k) = u^O(k|k)
\]  

(2.3.4)

To find the open-loop feedback optimal control sequence, we have to solve the open-loop control problem for \( k=0,1,\ldots,N-1 \). We shall show in Section 3.4 that we can write the open-loop feedback optimal control as

\[
u^*(k) = \Phi'(k)\bar{K}(k|k) + u_c(k)
\]  

(2.3.5)
where the 1 x n (row) vector

$$\Phi'(k) \triangleq \left[ \tilde{x}(k|k) + \tilde{b}'(k|k) \tilde{q}(k+1|k) \tilde{b}(k|k) \right]^{-1} \tilde{b}'(k|k) \tilde{q}(k+1|k) \Phi(k|k)$$

is defined as the optimal open-loop feedback adaptive gain. We shall call the scalar

$$u_c(k) \triangleq \left[ \tilde{x}(k|k) + \tilde{b}'(k|k) \tilde{q}(k+1|k) \tilde{b}(k|k) \right]^{-1} \tilde{b}'(k|k) \tilde{q}(k+1|k) \Phi(k|k)$$

the adaptive control correction term.

The structure of the overall open-loop feedback optimal control system is given in Fig. 2.2. The digital computer implementation of the open-loop feedback optimal control algorithm is straightforward. In Fig. 2.3, we give a flow chart description for on-line computation of the open-loop feedback optimal control. We summarize below all the equations needed for the computation of the optimal open-loop control sequence.

**a. Identification Equations**

Since the simultaneous state and parameter estimation is a non-linear filtering problem, we shall only give the approximate expressions for the conditional expectations of the state and parameter estimates and their error covariance matrices as generated by the extended Kalman filter algorithm [17], [33], [18]. The structure of the extended Kalman
Fig. 2.2 Structure of the Open-Loop Feedback Optimal Control System
Fig. 2.3 Flowchart of the On-Line Open-Loop Feedback Optimal Control Computation Algorithm
filter is given in Fig. 2.4. We assume that \( U^*(0,k-1) \) has been chosen and that \( Z_k \) is available. Then the estimates of the augmented state vector of \( S_2 \) Eq. (2.2.10) are generated via the equations

\[
\begin{bmatrix}
\hat{x}(i+1| i+1) \\
\hat{a}(i+1| i+1) \\
\hat{b}(i+1| i+1)
\end{bmatrix} =
\begin{bmatrix}
I_{3n} - G(i+1) \tilde{C} \hat{A}(i| i) \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
\hat{x}(i| i) \\
\hat{a}(i| i) \\
\hat{b}(i| i)
\end{bmatrix} + G(i+1) Z(i+1)
\tag{2.3.8}
\]

where

\[
\hat{A}(i| i) \Delta \begin{bmatrix}
\hat{A}(i| i) & 0 & u^*(i) I_n \\
0 & I_n & 0 \\
0 & 0 & I_n
\end{bmatrix}
\tag{2.3.9}
\]

with the initial estimation vector given by

\[
\begin{bmatrix}
\hat{x}(0| 0) \\
\hat{a}(0| 0) \\
\hat{b}(0| 0)
\end{bmatrix} =
\begin{bmatrix}
x_0 + \Sigma_{x_0} C' (C_0 x_0 C' + \Theta(0)^{-1}(z(0) - C x_0) \\
0 \\
0
\end{bmatrix}
\tag{2.3.10}
\]

The filter gain matrix \( G(i+1) \) in (2.3.8) satisfies the relation

\[
G(i+1) = \Sigma(i+1| i) \tilde{C} (\Sigma(i+1| i) \tilde{C} + \Theta(i+1))^{-1} i=0,1,\ldots,k-1 \tag{2.3.11}
\]

\[
= \Sigma(i+1| i+1) \tilde{C} \Theta^{-1}(i+1) \tag{2.3.12}
\]

provided the indicated inverse exists. The extended Kalman filter gain matrix cannot be precomputed since from Eq. (2.2.12)

\[
\Sigma(i+1| i) = \Sigma(i; \hat{x}(i| i), \hat{a}(i| i), u^*(i)) \Sigma(i| i) \Sigma(i; \hat{x}(i| i), \hat{a}(i| i), u^*(i)) + \tilde{C}(i)
\tag{2.3.13}
\]

i=0,1,\ldots,k-1

\[
= \Sigma(i+1| i+1) \tilde{C} \Theta^{-1}(i+1) \tag{2.3.12}
\]

provided the indicated inverse exists. The extended Kalman filter gain matrix cannot be precomputed since from Eq. (2.2.12)

\[
\Sigma(i+1| i) = \Sigma(i; \hat{x}(i| i), \hat{a}(i| i), u^*(i)) \Sigma(i| i) \Sigma(i; \hat{x}(i| i), \hat{a}(i| i), u^*(i)) + \tilde{C}(i)
\tag{2.3.13}
\]

i=0,1,\ldots,k-1

\[
= \Sigma(i+1| i+1) \tilde{C} \Theta^{-1}(i+1) \tag{2.3.12}
\]
Fig. 2.4 Structure of the Extended Kalman Filter
is a function of the current estimates of the state $x(i)$ and parameter $a(i)$, and hence, dependent on the observations. The matrix $\tilde{C}(i)$ is given by

$$ \tilde{C}(i) \Delta \begin{bmatrix} \tilde{C}(i) & 0 & 0 \\ 0 & \tilde{A}(i) & 0 \\ 0 & 0 & \Gamma(i) \end{bmatrix} \quad i = 0, 1, \ldots, k-1 \quad (2.3.14) $$

$\Sigma(i+1|i+1)$ is a $3n \times 3n$ symmetric positive semi-definite matrix given by

$$ \Sigma(i+1|i+1) = \Sigma(i+1|i) - G(i+1) \tilde{C} \Sigma(i+1|i) \quad i = 0, 1, \ldots, k-1 $$

$$ = [I - G(i+1) \tilde{C}] \Sigma(i+1|i) [I - G(i+1) \tilde{C}]' + G(i+1) O(i+1) G'(i+1) $$

with the initial condition

$$ \Sigma(0|0) \Delta \begin{bmatrix} \Sigma_{x0} & -\Sigma_{x0} C'(C \Sigma_{x0} C' + O(0))^{-1} C \Sigma_{x0} & 0 & 0 \\ 0 & \Sigma_{ao} & 0 \\ 0 & 0 & \Sigma_{bo} \end{bmatrix} \quad (2.3.15) $$

**b. Predictor Equations**

It will be shown in Appendix A that for the open-loop control policy we have the following deterministic dynamic equations for $j \geq k$

$$ \hat{x}(j+1|k) = \hat{A}(j|k) \hat{x}(j|k) + \hat{B}(j|k) u(j) \quad (2.3.17) $$

$$ \hat{A}(j+1|k) = \hat{A}(j|k) \quad (2.3.18) $$

$$ \hat{B}(j+1|k) = \hat{B}(j|k) \quad (2.3.19) $$
given the initial conditions $\hat{x}(k|k)$, $\hat{a}(k|k)$, $\hat{b}(k|k)$ from Eq. (2.3.8).

The submatrices of $\Sigma(j+1|k)$ are given by

\[
\Sigma_{xx}(j+1|k) = \hat{A}(j|k)\Sigma_{xx}(j|k)\hat{A}'(j|k) + \hat{X}(j|k)\Sigma_{xa}(j|k)\hat{X}'(j|k) + \hat{X}(j|k)\Sigma_{xax}(j|k)\hat{X}'(j|k)
\]

\[
+ u(j)\hat{A}(j|k)\Sigma_{xb}(j|k) + u(j)\Sigma_{xa}(j|k)\hat{A}'(j|k) + u(j)\hat{X}(j|k)\Sigma_{ab}(j|k)
\]

\[
+ u(j)\Sigma_{ab}(j|k)\hat{X}'(j|k) + \hat{X}(j|k)\Sigma_{aa}(j|k)\hat{X}'(j|k) + u^2(j)\Sigma_{bb}(j|k) + \Sigma(j)
\]

where

\[
\hat{A}(j|k) = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{n-1} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \vdots & \mathbf{0} \end{bmatrix}, \quad \hat{X}(j|k) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}
\]

\[
\Sigma_{xx}(j+1|k) = \hat{A}(j|k)\Sigma_{xx}(j|k)\hat{A}'(j|k) + \hat{X}(j|k)\Sigma_{xa}(j|k)\hat{X}'(j|k) + \hat{X}(j|k)\Sigma_{xax}(j|k)\hat{X}'(j|k)
\]

\[
+ u(j)\hat{A}(j|k)\Sigma_{xb}(j|k) + u(j)\Sigma_{xa}(j|k)\hat{A}'(j|k) + u(j)\hat{X}(j|k)\Sigma_{ab}(j|k)
\]

\[
+ u(j)\Sigma_{ab}(j|k)\hat{X}'(j|k) + \hat{X}(j|k)\Sigma_{aa}(j|k)\hat{X}'(j|k) + u^2(j)\Sigma_{bb}(j|k) + \Sigma(j)
\]

\[
\Sigma_{aa}(j+1|k) = \Sigma_{aa}(j|k) + \Delta(j)
\]

\[
\Sigma_{ab}(j+1|k) = \Sigma_{ab}(j|k)
\]

\[
\Sigma_{bb}(j+1|k) = \Sigma_{bb}(j|k) + \Gamma(j)
\]

given the initial conditions $\Sigma_{aa}(k|k)$, $\Sigma_{ab}(k|k)$, $\Sigma_{bb}(k|k)$, $\Sigma_{xx}(k|k)$, $\Sigma_{xa}(k|k)$ and $\Sigma_{xb}(k|k)$ from Eq. (2.3.15).

\[c. \quad \text{Parameter Equations}\]

To compute the open-loop feedback optimal control we need to determine the parameters $\Phi(j|k)$, $\psi(j|k)$, $\theta(j+1|k)$, $\tilde{a}(j|k)$, $\tilde{b}(j|k)$, $\tilde{r}(j|k)$ for $j=k, k+1, \ldots, N-1$ using the equations below, \(2.3.26 - 2.3.34\).

\[\text{\(e_1, e_2, \ldots, e_n\) denote the natural basis vectors in } \mathbb{R}^n\]
\[
\mathbf{\Sigma}_a(j|k) \triangleq \begin{bmatrix}
\Sigma_{xb}(j|k)e_1 \\
\vdots \\
\Sigma_{xb}(j|k)e_n
\end{bmatrix} \in \mathbb{R}^{n \times n}
\]

\[
\mathbf{\Sigma}_a(j|k) = \begin{bmatrix}
\Sigma_{xa}(j|k)e_1 \\
\vdots \\
\Sigma_{xa}(j|k)e_n
\end{bmatrix} \in \mathbb{R}^{n \times n}
\]

\[
\mathbf{\Sigma}_{ab}(j|k) = \begin{bmatrix}
\Sigma'_{ab}(j|k)e_1 \\
\vdots \\
\Sigma'_{ab}(j|k)e_n
\end{bmatrix} \in \mathbb{R}^{n \times (2n+1)}
\]

\[
\mathbf{A}^\prime(j|k) = \begin{bmatrix}
\Sigma_{ab}(j|k)P_{xx}(j+1|k)e_n \\
\vdots \\
\Sigma_{ab}(j|k)P_{xx}(j+1|k)e_n
\end{bmatrix} \in \mathbb{R}^{n \times (2n+1)}
\]

(2.3.26)
\[ A^{\dagger}(j|k) = \begin{bmatrix} \hat{A}(j|k) & 0 & \cdots & 0 \\ e_{n-1-1}^{e'}(j|k) & \hat{A}(j|k) & \cdots & 0 \\ e_{n-1-1}^{e'}(j|k) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n-1-1}^{e'}(j|k) & 0 & \cdots & \hat{A}(j|k) \end{bmatrix} \quad (n(2n+1) \times n(2n+1)) \] (2.3.29)

\[ \hat{\Phi}(j|k) = A^{\dagger}(j|k) A(j|k) \tilde{d}^{-1}(j|k) \] (2.3.30)

\[ W(j|k) = \begin{bmatrix} \Omega(j) + P_{nn}(j+1|k)\Sigma_{aa}(j|k) & 0 & \cdots & 0 & e_{n-1}^{e'P_{xx}(j+1|k)} \hat{A}'(j|k) & \cdots & e_{n-1}^{e'P_{xx}(j+1|k)} \hat{A}'(j|k) \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hat{A}(j|k)P_{xx}(j+1|k)e_{n-1}^{e'} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \hat{A}(j|k)P_{xx}(j+1|k)e_{n-1}^{e'} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (n(2n+1) \times n(2n+1)) \] (2.3.31)
where

\[ p_{nn}(j+1 \mid k) = e_n^t P_{nn}(j+1 \mid k) e_n \]

\[ \mathcal{V}(j \mid k) = q(j \mid k) - d(j+1 \mid k) \tilde{r}^{-1}(j \mid k) d'(j+1 \mid k) \]  \hspace{1cm} (2.3.32)

Since we are given \( r(j) > 0, q(j) > 0 \), and \( P_{bb}(j \mid k), P_{xx}(j \mid k) > 0 \) from Eq. (2.3.24) and

\[ P_{xx}(j \mid k) = \hat{A}'(j \mid k) P_{xx}(j+1 \mid k) \hat{A}(j \mid k) + Q(j), P_{xx}(N \mid k) = Q(N) \]

\[ j = k, \ldots, N-1 \]  \hspace{1cm} (2.3.33)

we can define the "modified control weighting" to be

\[ \tilde{r}(j \mid k) = r(j) + tr_{bb}(j \mid k) P_{xx}(j+1 \mid k) > 0 \]  \hspace{1cm} (2.3.34)

and it is positive definite. We remark that the matrix \( P_{xx}(j \mid k) \) is dependent upon observation, and, thus, cannot be precomputed.
CHAPTER 3

METHOD OF SOLUTION

3.1 Cost Transformation

Suppose at \( j = k, k = 0, 1, \ldots, N-1 \) the system is at \( x(k) \), then we can rewrite the main cost functional Eq. (2.1.21) as

\[
\overline{J}(u) = \frac{1}{2} E \left\{ \sum_{j=k}^{N-1} x^*(j)Q(j)x^*(j) + r(j)(u^*(j))^2 \right\}
\]

\[
+ \frac{1}{2} E \left\{ x'(N)Q(N)x(N) \right\} + \sum_{j=k}^{N-1} x'(j)Q(j)x(j) + r(j)u^2(j) \}
\]

\( k = 0, 1, \ldots, N-1 \) (3.1.1)

Using Bellman's principle of optimality, we have the equivalent minimization problem at time \( k \).

\[
\overline{J}_k \triangleq J(U^*(0,k-1), U(k,N-1))
\]

\[
= \frac{1}{2} E \left\{ x'(N)Q(N)x(N) \right\} + \sum_{j=k}^{N-1} x'(j)Q(j)x(j) + r(j)u^2(j) \}
\]

\( k = 0, 1, \ldots, N-1 \) (3.1.2)

The optimal closed-loop control policy uses the a priori known statistics of the future measurements. The a priori probability density functions are those at time \( j = k \).

The suboptimal closed-loop control policy we are solving consists of replacing the closed-loop controls with open-loop controls for \( j \geq k \). It does not take into account the knowledge that future measurements will be made. Therefore, the objective of the problem becomes
to find a future control sequence such that the average value of the
cost-to-go given by

\[
\overline{J}_k = \frac{1}{2} E \left\{ x'(N) Q(N) x(N) \right\} + \sum_{j=k}^{N-1} x'(j) Q(j) x(j) | Z_k \}
\]

\[
+ \frac{1}{2} \sum_{j=k}^{N-1} x(j)^2 \quad k = 0, 1, \ldots, N-1
\]

conditioned on the total available data at the present time \( k \), is mini-
mized (open-loop) subject to the constraint equations \( (2.1.1)-(2.1.2) \).

We can take \( u(j) \) out of the expectation in \( (3.1.3) \) since the
future control sequence \( \{ u(j) \} \) is assumed to be deterministic, and the
optimal open-loop control sequence \( \{ u^O(j|k) \} \) is only based upon the
measurement statistic \( Z_k \). It is possible now to formulate exactly the
stochastic control problem Eqs. \( (2.1.1), (2.1.2), \) and \( (2.1.21) \) as a com-
pletely deterministic optimization problem.

Using Eqs. \( (2.2.2)-(2.2.9) \) we then obtain the conditional cost
\( (3.1.3) \) as

\[
\overline{J}_k = \frac{1}{2} \hat{\Sigma}(N|k) Q(N) \hat{\Sigma}(N|k) + \frac{1}{2} \text{tr} [Q(N) \hat{\Sigma}_{xx}(N|k)]
\]

\[
+ \frac{1}{2} \sum_{j=k}^{N-1} \{ x'(j|k) Q(j) x(j|k) + \text{tr} [Q(j) \hat{\Sigma}_{xx}(j|k)] + r(j) u^2(j) \}
\]

\[ k = 0, 1, \ldots, N-1 \] (3.1.4)

since for any matrix \( M \)

\[
E \left\{ x'(j) M x(j) | Z_k \right\} = \hat{\Sigma}(j|k) M \hat{\Sigma}(j|k) + E \left\{ e'_x(j|k) M e_x(j|k) \right\}
\]

(3.1.5)

We have obtained, therefore, in Eq. \( (3.1.4) \) the deterministic form of the
expected value of the cost-to-go.
3.2 Open-Loop Control Problem Definition

To complete the formulation of the deterministic open-loop control problem, we will need the deterministic dynamical equations satisfied by \( \tilde{x}(j|k) \) and \( \Sigma_{xx}(j|k) \) (\( j \geq k \)), the respective estimates of the state vector and its error covariance matrix conditioned upon the output measurements \( Z_k \) and the past control history \( U^*(0,k-1) \). We shall, however, develop only the approximate expressions for \( \tilde{x}(j|k) \) and \( \Sigma_{xx}(j|k) \) in Appendix A. These expressions and the identification equations (2.3.8)-(2.3.16) are then used to define a completely deterministic optimal control problem for the \( k \)th step, whose solution would yield the optimal open-loop controls for \( k \leq j \leq N-1 \).

Given:  
\[
\begin{align*}
\hat{x}(j+1|k) &= \hat{A}(j|k)\hat{x}(j|k) + \hat{B}(j|k)u(j) \\
\end{align*}
\]  
(3.2.1)

where

\[
\begin{align*}
\hat{A}(j+1|k) &= \hat{A}(j|k) \\
\hat{B}(j+1|k) &= \hat{B}(j|k) \\
\hat{\Sigma}(j+1|k) &= \hat{F}(j,\hat{A}(j|k),\hat{x}(j|k),u(j))\hat{\Sigma}(j|k)\hat{F}^t(j,\hat{A}(j|k),\hat{x}(j|k),u(j)) + \hat{\Sigma}(j) \\
\end{align*}
\]  
(3.2.4)

where

\[
\begin{align*}
\hat{F}(j,\hat{A}(j|k),\hat{x}(j|k),u(j)) \triangleq \\
\begin{pmatrix}
\hat{A}(j|k) & \hat{x}(j|k) & u(j) \cdot I_n \\
0 & I_n & 0 \\
0 & 0 & I_n \\
\end{pmatrix}
\end{align*}
\]  
(3.2.5)
The initial conditions are \( \hat{x}(k|k) \), \( \hat{p}(k|k) \), and \( \Sigma(k|k) \) specified by the extended Kalman filter, (2.3.8)-(2.3.16).

The aim is to find a deterministic control sequence \( \{u(k), \ldots, u(N-1)\} \) such that it minimizes in \( (N-k) \) steps the average value cost-to-go

\[
\mathcal{J}_k = \frac{1}{2} \left( \hat{x}'(N|k)Q(N)\hat{x}(N|k) + \text{tr}[\Pi(N)\Sigma(N|k)] + \sum_{j=k}^{N-1} \hat{x}'(j|k)Q(j)\hat{x}(j|k) \right.
+ \left. \text{tr}(\hat{p}(j)\Sigma(j|k)) + r(j)u^2(j) \right) \tag{3.2.7}
\]

where

\[
\hat{\Pi}(N) = \begin{bmatrix} Q(N) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \Sigma(N|k) \end{bmatrix} \quad \text{and} \quad \hat{\Pi}(j) = \begin{bmatrix} Q(j) & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \Sigma(j|k) \end{bmatrix} \tag{3.2.8}
\]

and \( \hat{\Pi}(N), \hat{\Pi}(j) \geq 0 \) and \( r(j) > 0 \) subject to the constraint equations (3.2.1)-(3.2.4). We shall also assume that \( Q(j) \) and \( Q(N) \) are not both the zero matrix. The terminal states \( \hat{x}(N|k) \) and \( \Sigma(N|k) \) are not specified.

We see from (3.2.7) that the conditional cost-to-go depends on the estimate \( \hat{x}(j|k) \) and the error covariance matrix \( \Sigma_{xx}(j|k) \) generated by the predictor equations (3.2.1)-(3.2.4), and \( u(j) \), an arbitrary deterministic input for \( k \leq j \leq N-1 \). The difference equation (3.2.1) for \( \hat{x}(j|k) \) involves \( \hat{a}(j|k) \), \( \hat{b}(j|k) \) and \( u(j) \). The difference equation (3.2.4) for \( \Sigma_{xx}(j|k) \) involves the error covariances \( \Sigma_{bb}(j|k), \Sigma_{xb}(j|k), \Sigma_{xa}(j|k), \Sigma_{aa}(j|k), \Sigma_{ab}(j|k) \) and the estimates \( \hat{a}(j|k), \hat{b}(j|k), \hat{a}(j|k) \), and the control \( u(j) \).
We have then a well-posed deterministic optimal control problem. The
deterministic formulation allows us to use the discrete Minimum Prin-
ciple [34] to derive the necessary conditions for optimality (in the
open-loop feedback sense). We shall, however, present an alternative
method of solution via dynamic programming in the next section [35].

3.3 Open-Loop Control Problem Solution

In this section we shall derive the result stated in Section 2.3,
using the dynamic programming method for the discrete optimal control
problem defined by Eqs.(3.2.1)-(3.2.4) and the cost functional (3.2.7) to
be minimized is

$$
\bar{J}_k = \frac{1}{2} \{ \hat{R}'(N|k)\hat{Q}(N)\hat{R}(N|k) + \text{tr} \ \hat{Q}(N)\hat{\Sigma}(N|k) \}
+ \sum_{j=k}^{N-1} L(\hat{R}(j|k),\hat{\Sigma}(j|k),u(j),j)
$$

(3.3.1)

where

$$
L(\hat{R}(j|k),\hat{\Sigma}(j|k),u(j),j) = \frac{1}{2} \{ \hat{R}'(j|k)\hat{Q}(j)\hat{R}(j|k) + \text{tr} \ \hat{Q}(j)\hat{\Sigma}(j|k) + r(j)u^2(j) \}
$$

(3.3.2)

To use the standard dynamic programming algorithm, we shall
define the "conditional open-loop optimal cost-to-go" at \( j = i \), for
\( i \in [k,N-1] \). The superscript circle denotes open-loop optimal. We shall
define \( J^o_i(k) \) as the minimum cost remaining along the optimal
trajectory starting at time \( i \) and the initial "states" \( \hat{R}(i|k) \) and \( \hat{\Sigma}(i|k) \)
given by (3.2.1)-(3.2.4). Hence we have the functional equation
\[
\bar{J}_{i|k}^O(\hat{X}(i|k), \Sigma(i|k)) = \min \left\{ \frac{1}{2} \{ \hat{X}'(N|k)\hat{Q}(N)\hat{X}(N|k) + \text{tr} \hat{Q}(N)\Sigma(N|k) \} \right. \\
\left. \sum_{j=i}^{N-1} \{L(\hat{X}(j|k), \Sigma(j|k), u(j), j) \} \right. \\
\left. + \sum_{j=i}^{N-1} \{ \Sigma(i|k) \Sigma(i+1|k) \} \right. \\
\left. i = k, k+1, \ldots, N-1 \right\} \tag{3.3.3}
\]

\[
\bar{J}_{i+1|k}^O(\hat{X}(i+1|k), \Sigma(i+1|k)) = \min \left\{ \{ L(\hat{X}(i+1|k), \Sigma(i+1|k), u(i+1), i+1) \} \right. \\
\left. + \bar{J}_{i+1|k}^O(\hat{X}(i+1|k), \Sigma(i+1|k)) \right. \\
\left. i = k, k+1, \ldots, N-1 \right\} \tag{3.3.4}
\]

using Eq. (3.3.2). The method of dynamic programming yields necessary conditions based on the optimality principle or condition Eq. (3.3.4).

We remark that \(\mathcal{J}_{i|k}(\cdot, \cdot)\) is defined as a function on \(M(3n\times 3n) \times \mathbb{R}^n\).

From the error covariance equation (3.2.4) \(\Sigma(j|k) \geq 0, \ j = i, i+1, \ldots, N-1\) if and only if \(\Sigma(i|k) \geq 0, \ i = k, k+1, \ldots, N-1\). Thus, from Eq. (3.3.3) we obtain that

\[
\bar{J}_{i|k}^O(\hat{X}, \Sigma) \geq 0, \text{ if } \Sigma \geq 0 \tag{3.3.5}
\]

Hence, if \(Q(i) \geq 0\) and \(r(i) > 0\), then \(\bar{J}_{i|k}^O \geq 0\) and \(\Sigma \geq 0\) in Eqs. (3.3.3) and (3.2.4).

The terminal time is \(N\). Let us define

\[
\bar{J}_{N-1|k}^O(\hat{X}(N-1|k), \Sigma(N-1|k)) \triangleq \text{optimal value of } \bar{J} \text{ for one-stage control process starting at } i=N-1 \text{ and using an optimal } u(N-1).
\]

Using (3.3.5) the conditional optimal cost-to-go is given by

\[
\bar{J}_{N-1|k}^O(\hat{X}(N-1|k), \Sigma(N-1|k)) = \min \left\{ R(\hat{X}(N-1|k), \Sigma(N-1|k), u(N-1)) \right\} \tag{3.3.6}
\]
where
\[ R(\hat{x}(N-1|k), \Sigma(N-1|k), u(N-1)) \triangleq \mathcal{L}(\hat{x}(N-1|k), \Sigma(N-1|k), u(N-1), N-1) \]
\[ + \mathcal{O}(N|k) \left( \hat{x}^0(N|k), \Sigma^0(N|k) \right) \]
(3.3.7)

But,
\[ \mathcal{O}(N|k) \left( \hat{x}^0(N|k), \Sigma^0(N|k) \right) = \frac{1}{2} \hat{x}^0(N|k) \mathcal{Q}(N) \hat{x}^0(N|k) + \frac{1}{2} \text{tr} \mathcal{Q}(N) \Sigma^0_{xx}(N|k) \]
(3.3.8)

Hence we obtain
\[ R(\hat{x}(N-1|k), \Sigma(N-1|k), u(N-1)) = \hat{x}'(N-1|k) \mathcal{Q}(N-1) \hat{x}(N-1|k) + \text{tr} \mathcal{Q}(N-1) \Sigma^0_{xx}(N-1|k) \]
\[ + r(N-1) u^2(N-1) + \hat{x}'(N|k) \mathcal{Q}(N) \hat{x}(N|k) + \text{tr} \mathcal{Q}(N) \Sigma^0_{xx}(N|k) \]
(3.3.9)

Using the system equations (3.2.1)-(3.2.6) we get
\[ R(\hat{x}(N-1|k), \Sigma(N-1|k), u(N-1)) = \hat{x}'(N-1|k) \mathcal{Q}(N-1) \hat{x}(N-1) + \text{tr} \mathcal{Q}(N-1) \Sigma^0_{xx}(N-1|k) \]
\[ + r(N-1) u^2(N-1) + \hat{x}'(N|k) \mathcal{Q}(N) \hat{x}(N|k) u(N-1) \]
\[ + u(N-1) \hat{b}'(N-1|k) \mathcal{Q}(N) \hat{b}(N-1|k) + u^2(N-1) \hat{b}'(N-1|k) \mathcal{Q}(N) \hat{b}(N-1|k) \]
\[ + \text{tr} \{ \mathcal{Q}(N) [\hat{a}(N-1|k) \Sigma^0_{xa}(N-1|k) \hat{a}'(N-1|k) \]
\[ + 2\hat{a}(N-1|k) \Sigma^0_{aa}(N-1|k) \hat{x}'(N-1|k) + 2\hat{a}(N-1|k) \Sigma^0_{xb}(N-1|k) u(N-1) \]
\[ + \hat{x}(N-1|k) \Sigma^0_{ab}(N-1|k) \hat{x}'(N-1|k) + 2\hat{x}(N-1|k) \Sigma^0_{bb}(N-1|k) u(N-1) \]
\[ + u^2(N-1) \Sigma^0_{bb}(N-1|k) + \mathcal{E}(N-1) \} \]\n(3.3.10)

* We shall drop the factor of 1/2 in \( R(\cdot, \cdot, \cdot) \) for notational convenience.
If we consider \( \hat{x}^{(N-1|k)} \) as fixed, then we can minimize Eq. (3.3.6) with respect to \( u^{(N-1)} \) only to yield \( u^O^{(N-1|k)} \). Note that the optimal open-loop control will depend on the state \( \hat{x}^{(N-1|k)} \). Performing the minimization of \( R \) gives

\[
\frac{\partial R}{\partial u^{(N-1)}} = 2r^{(N-1)}u^{(N-1)} + 2\hat{b}'^{(N-1|k)}Q^{(N)}\hat{A}^{(N-1|k)}\hat{x}^{(N-1|k)} + 2u^{(N-1)}\hat{b}'^{(N-1|k)}Q^{(N)}\hat{b}^{(N-1|k)} + 2tr[Q^{(N)}\hat{A}^{(N-1|k)}\Sigma_{xb}^{(N-1|k)}] + Q^{(N)}\hat{x}^{(N-1|k)}\Sigma_{ab}^{(N-1|k)} + u^{(N-1)}Q^{(N)}\Sigma_{bb}^{(N-1|k)}
\]

\[
\frac{\partial R}{\partial u^{(N-1)}} = 0 = [r^{(N-1)} + trQ^{(N)}\Sigma_{bb}^{(N-1|k)}]Q^{(N)}\hat{A}^{(N-1|k)}\hat{x}^{(N-1|k)} + \hat{b}'^{(N-1|k)}Q^{(N)}\hat{b}^{(N-1|k)}Q^{(N-1|k)}
\]

\[
\frac{\partial R}{\partial u^{(N-1)}} = 0 = [r^{(N-1)} + trQ^{(N)}\Sigma_{bb}^{(N-1|k)}]Q^{(N)}\hat{A}^{(N-1|k)}\hat{x}^{(N-1|k)} + \hat{b}'^{(N-1|k)}Q^{(N)}\hat{b}^{(N-1|k)}Q^{(N-1|k)}
\]

Therefore the optimal open-loop control is given by

\[
u^O^{(N-1|k)} = -Z^{-1}_{uu}^{(N-1|k)}\{\hat{b}'^{(N-1|k)}Q^{(N)}\hat{A}^{(N-1|k)}\hat{x}^{(N-1|k)} + tr[Q^{(N)}\hat{A}^{(N-1|k)}\Sigma_{xb}^{(N-1|k)}] + Q^{(N)}\hat{x}^{(N-1|k)}\Sigma_{ab}^{(N-1|k)}\}\}
\]

\[
(3.3.11)
\]

provided the indicated inverse exists, and

\[
Z_{uu}^{(N-1|k)} = r^{(N-1)} + \hat{b}'^{(N-1|k)}Q^{(N)}\hat{b}^{(N-1|k)} + trQ^{(N)}\Sigma_{bb}^{(N-1|k)}
\]

\[
(3.3.12)
\]

It can be shown that we can write (3.3.12) in the form of Eq. (2.3.1) using Eqs. (2.3.26)-(2.3.34)

\[
u^O^{(N-1|k)} = -(\tilde{r}^{(N-1|k)} + \hat{b}'^{(N-1|k)}\tilde{k}^{(N|k)}\hat{b}^{(N-1|k)})^{-1}\hat{b}'^{(N-1|k)}
\]

\[
\tilde{k}^{(N|k)}\hat{b}^{(N-1|k)} = \begin{pmatrix} \hat{x}^{(N-1|k)} \\ \vdots \\ \hat{x}^{(N-1|k)} \end{pmatrix} \begin{pmatrix} \hat{A}^{(N-1|k)} \\ \vdots \\ \hat{A}^{(N-1|k)} \end{pmatrix}^{-1} \begin{pmatrix} \hat{b}^{(N-1|k)} \\ \vdots \\ \hat{b}^{(N-1|k)} \end{pmatrix} = \begin{pmatrix} \hat{x}^{(N-1|k)} \\ \vdots \\ \hat{x}^{(N-1|k)} \end{pmatrix} \begin{pmatrix} \hat{A}^{(N-1|k)} \\ \vdots \\ \hat{A}^{(N-1|k)} \end{pmatrix}^{-1} \begin{pmatrix} \hat{b}^{(N-1|k)} \\ \vdots \\ \hat{b}^{(N-1|k)} \end{pmatrix}
\]

\[
(3.3.14)
\]
where \( \hat{K}(N|k) \) is given by Eq. (2.3.3). Substituting this value of \( u^0(N-1|k) \) 
Eq. (3.3.12) in Eqs. (3.3.6)-(3.3.10), and do some manipulating we obtain 
the conditional optimal cost-to-go

\[
2\cdot J^0_{N-1|k}(\hat{X}(N-1|k), \Sigma_{N-1|k}) = \hat{X}'(N-1|k)Q(N-1)\hat{X}(N-1|k) + trQ(N-1)\Sigma_{xx}(N-1|k)
\]

\[
- \hat{u}_{uu}^{-1}(N-1|k)\{\hat{X}'(N-1|k)Q(N)\hat{A}(N-1|k)\hat{X}(N-1|k)
\]

\[
+ tr[Q(N)\hat{A}(N-1|k)\Sigma_{xb}(N-1|k)] + \hat{Q}(N)\hat{X}(N-1|k)\Sigma_{ab}(N-1|k)\}
\]

\[
+ \hat{X}'(N-1|k)Q(N)\hat{X}(N-1|k) + \hat{Q}(N)\Sigma_{xx}(N-1|k)\hat{X}(N-1|k)
\]

\[
+ tr[Q(N)\hat{A}(N-1|k)\Sigma_{xa}(N-1|k)] + \hat{Q}(N)\Sigma_{aa}(N-1|k)\hat{X}'(N-1|k)
\]

\[
+ \hat{X}'(N-1|k)Q(N)\hat{A}(N-1|k)\Sigma_{xa}(N-1|k)\hat{X}'(N-1|k) + \hat{Q}(N)\Sigma_{aa}(N-1|k)\}
\]

(3.3.15)

\[
= \hat{X}'(N-1|k)Q(N-1)\hat{X}(N-1|k) + tr[\Sigma_{xx}(N-1|k)\hat{A}'(N-1|k)]
\]

\[
Q(N)\hat{A}(N-1|k) + \hat{Q}(N-1)\} + tr[Q(N)\hat{X}'(N-1|k)]
\]

\[
\Sigma_{aa}(N-1|k)\hat{X}'(N-1|k) + 2Q(N)\hat{A}(N-1|k)\Sigma_{xa}(N-1|k)
\]

\[
\hat{X}'(N-1|k) + \hat{Q}(N)\Sigma_{aa}(N-1|k)\}
\]

\[
Q(N)\hat{A}(N-1|k)\hat{X}(N-1|k) - \hat{u}_{uu}^{-1}(N-1|k)\hat{D}(N-1|k)
\]

\[
Q(N)\hat{A}(N-1|k)\hat{X}(N-1|k) + tr[Q(N)\hat{A}(N-1|k)]
\]

\[
\Sigma_{xb}(N-1|k) + \hat{Q}(N)\Sigma_{ab}(N-1|k)\}
\]

(3.3.16)

which has the closed form
\[ J_{N-1|k}(\hat{x}(N-1|k), \Sigma(N-1|k)) = \frac{1}{2} < \begin{pmatrix} \hat{x}(N-1|k) \\ \cdots \\ \hat{x}(N-1|k) \\ \Sigma(N-1|k) \\ \cdots \\ \Sigma(N-1|k) \\ \hat{x}(N-1|k) \\ \cdots \\ \Sigma(N-1|k) \end{pmatrix}, \begin{pmatrix} \Sigma(N-1|k) \\ \cdots \\ \Sigma(N-1|k) \\ \Sigma(N-1|k) \\ \cdots \\ \Sigma(N-1|k) \end{pmatrix} > \\
\quad + \frac{1}{2} \text{tr} \left[ \sum_{\Sigma_{xx}} (N-1|k) (\hat{A}'(N-1|k) P_{xx}(N|k) \hat{A}(N-1|k) + Q(N-1)) \right] \\
\quad + \frac{1}{2} \text{tr} \sum_{\Sigma(N-1)} Z(N-1); \quad P_{xx}(N|k) = Q(N) \tag{3.3.17} \]

if we choose \( \hat{x}(N-1|k) \) such that

\[
\begin{align*}
\hat{x}(N-1|k) &= \hat{\Phi}'(N-1|k) \hat{x}(N|k) + \hat{\Phi}(N-1|k) + \hat{\Sigma}(N-1|k), \\
\hat{\Sigma}(N|k) &= \hat{\Sigma}(N-1|k) \hat{\Sigma}(N-1|k) \hat{\Sigma}(N-1|k) \\
\hat{\Phi}'(N-1|k) &= \hat{\Phi}'(N-1|k) \hat{\Phi}(N-1|k) \hat{\Sigma}(N-1|k) \hat{\Sigma}(N-1|k) \hat{\Phi}(N-1|k) \\
\end{align*} \tag{3.3.18} \]

where \( \hat{x}(N|k) \) is given by (2.3.3) and the parameters are computed from Eqs. (2.3.26)-(2.3.34). We have thus far assumed that \( Z_{uu}(N-1|k) \) is nonzero. In fact, in order that the control law (3.3.14) minimizes the cost-to-go, \( Z_{uu}(N-1|k) \) must be positive definite.

In Equation (3.3.17) we have determined the expression for the conditional optimal cost-to-go at step \( N-1 \). Now using Equation (3.3.5) we have

\[
J_{N-2|k}(\hat{x}(N-2|k), \Sigma(N-2|k)) = \min_{u(N-2)} \left\{ L(\hat{x}(N-2|k), \Sigma(N-2|k), u(n-2), N-2) \right\} \\
+ J_{N-1|k}(\hat{x}(N-1|k), \Sigma(N-1|k)) \tag{3.3.19} \]

Using Equation (3.3.17), this reduces to a form exactly identical to Eq. (3.3.6) except for the indices. Thus we have, comparing Equation (3.3.9)
\[
R(\hat{x}(N-2|k), F_{2}(N-2|k)) = \hat{x}'(N-2|k)Q(N-2)\hat{x}(N-2|k)
+ \text{tr } Q(N-2)F_{xx}(N-2|k) + r(N-2)u^{2}(N-2)
\]
\[
+ <\begin{bmatrix}
\hat{x}^{O}(N-1|k) \\
...... \\
\hat{o}^{O}(N-1|k)
\end{bmatrix}, \begin{bmatrix}
\hat{x}^{O}(N-1|k) \\
...... \\
\hat{o}^{O}(N-1|k)
\end{bmatrix}> + s(N-1)
\]
\[\text{(3.3.20)}\]

where the scalar
\[
s(N-1) = \text{tr}[\Sigma^{O}_{xx}(N-1|k)P^{O}_{xx}(N-1|k) + P^{O}_{xx}(N|k)\Sigma(N-1)] \tag{3.3.21}\]

and
\[
P^{O}_{xx}(N-1|k) = \tilde{X}^{O'}(N-1|k)Q(N)\tilde{X}^{O}(N-1|k) + Q(N-1) \tag{3.3.22}\]

and \(\tilde{X}(N-1|k)\) is given by (3.3.18). The cycle now repeats. Thus, by

induction on \(i\) we have shown that the optimal open-loop control sequence

is given by
\[
u^{O}(i|k) = -[(\tilde{x}(i|k) + b'(i|k)\tilde{v}(i+1|k)\tilde{\nu}(i|k)]^{-1}b'(i|k)\tilde{v}(i+1|k)\Phi(i|k)
\]
\[+ \tilde{\nu}^{-1}(i|k)A'(i+1|k)\begin{bmatrix}
\hat{x}^{O}(i|k) \\
...... \\
\hat{o}^{O}(i|k)
\end{bmatrix} \quad i = k, k+1, \ldots, N-1 \tag{3.3.23}\]

and the conditional optimal (minimum) open-loop cost-to-go is given by
\[
\bar{J}_{i|k}(\hat{x}(i|k), \Sigma(i|k)) = \frac{1}{2} <\begin{bmatrix}
\hat{x}(i|k) \\
...... \\
\hat{o}(i|k)
\end{bmatrix}, \begin{bmatrix}
\hat{x}(i|k) \\
...... \\
\hat{o}(i|k)
\end{bmatrix}> 
+ \frac{1}{2} s(i) \quad i = k, k+1, \ldots, N-1 \tag{3.3.24}\]
where

\[ s(i) = \text{tr}\left[ \sum_{i\leq x} (i|k) P_{xx}(i|k) + \sum_{j=1}^{N-1} P_{xx}(j+1|k) E(j) \right] \quad (3.3.25) \]

and \( \tilde{K}(i|k) \) satisfies the matrix difference Equation (2.3.2) and the parameters are computed from Equations (2.3.26)-(2.3.34).

The matrix \( \tilde{K}(i|k) \) is a symmetric matrix. Taking the transpose of both sides of Equation (2.3.2) we find that

\[
\tilde{K}'(i|k) = \Phi'(i|k) (\tilde{K}'(i+1|k) - \tilde{K}'(i+1|k) \tilde{E}(i|k) \tilde{E}'(i|k) - \tilde{K}'(i|k) \tilde{E}'(i+1|k) \tilde{E}(i|k)) + \tilde{V}(i|k) \\
\]

\[ i = k, k+1, \ldots, N-1 \quad (3.3.26) \]

since \( \tilde{V}(i|k) \) is symmetric. Comparing this with Equation (2.3.2), we observe that both \( \tilde{K}(i|k) \) and \( \tilde{K}(i|k) \) are solutions of the same difference equation. At \( i = N \), we have the boundary condition \( \tilde{K}'(N|k) = \tilde{Q}(N) \).

Since \( \tilde{Q}(N) \) is symmetric, \( \tilde{Q}(N) = \tilde{Q}'(N) \), we conclude that

\[ \tilde{K}(N|k) = \tilde{K}'(N|k) = \tilde{Q}(N) \quad (3.3.27) \]

Since \( \tilde{K}(i|k) \) and \( \tilde{K}'(i|k) \) are solutions of the same difference equation with the same boundary conditions, we conclude \( \tilde{K}(i|k) = \tilde{K}'(i|k) \) from the uniqueness of solutions of difference equations.

We note that in the conditional optimal cost-to-go equation (3.3.24), that it depends on the initial estimation of the state \( \tilde{X}(i|k) \) and \( \tilde{X}(i|k) \), and the random disturbance \( \tilde{E}(j) \) which is forcing the system. The presence of the plant noise \( (\tilde{E}(j) \neq 0) \), therefore, increases the cost-to-go on the average, since \( \text{tr}[P_{xx}(j+1|k) E(j)] \) is nonnegative if \( P_{xx}(j+1|k) \) and \( E(j) \) are positive semidefinite.
3.4 Feedback Interpretation

In this section we will show that the open-loop feedback optimal sequence applied is

\[ u^*(k) = \phi'(k) \hat{x}(k|k) + u_c(k) \; ; \; k = 0, 1, ..., N-1 \]  

(3.4.1)

Let us rewrite Eq. (3.3.23) at \( i = k \).

\[
\begin{align*}
u^o(k|k) &= -\{ (\hat{r}(k|k) + \hat{b}'(k|k) \hat{r}(k+1|k) \hat{b}(k|k) )^{-1} \hat{b}'(k|k) \hat{r}(k+1|k) \hat{c}(k|k) \\ &\quad + \hat{r}^{-1}(k|k) \hat{d}'(k+1|k) \}\begin{bmatrix} I_n & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{r}(k|k) \\ \vdots \\ \hat{c}(k|k) \end{bmatrix} \\ &\quad - \{ (\hat{r}(k|k) + \hat{b}'(k|k) \hat{r}(k+1|k) \hat{b}(k|k) )^{-1} \hat{b}'(k|k) \hat{r}(k+1|k) \hat{c}(k|k) \\ &\quad + \hat{r}^{-1}(k|k) \hat{d}'(k+1|k) \}\begin{bmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} \hat{r}(k|k) \\ \vdots \\ \hat{c}(k|k) \end{bmatrix} \\ &\quad + \hat{r}^{-1}(k|k) \hat{d}'(k+1|k) \}
\end{align*}
\]

(3.4.2)

\[ k = 0, 1, ..., N-1 \]

By definitions (2.3.6) and (2.3.7) Eq. (3.4.1) is immediately verified.

The adaptive control gain \( \phi'(k) \) is, by definition, independent of the current estimate of the state vector \( \hat{x}(k|k) \). It depends on \( \hat{b}(k|k) \) and \( \hat{b}(j|k) \), \( \hat{a}(k|k) \) and \( \hat{a}(j|k) \), \( \Sigma_{bb}(k|k) \) and \( \Sigma_{bb}(j|k) \), \( \Sigma_{ab}(k|k) \) and \( \Sigma_{ab}(j|k) \), and \( \Sigma_{aa}(k|k) \) and \( \Sigma_{aa}(j|k) \) evaluated along the open-loop feedback optimal trajectory for \( k < j \leq N-1 \).

The control correction term \( u_c(k) \) is independent of \( \hat{x}(k|k) \). It depends on \( \hat{b}(k|k) \) and \( \hat{b}(j|k) \), \( \hat{a}(k|k) \) and \( \hat{a}(j|k) \), \( \Sigma_{bb}(k|k) \) and \( \Sigma_{bb}(j|k) \), \( \Sigma_{ab}(k|k) \) and \( \Sigma_{ab}(j|k) \), \( \Sigma_{xa}(k|k) \) and \( \Sigma_{xa}(j|k) \), \( \Sigma_{bb}(k|k) \) and \( \Sigma_{bb}(j|k) \), and \( \Sigma_{ab}(k|k) \) and \( \Sigma_{ab}(j|k) \) all evaluated along the open-loop feedback optimal control trajectory for
If the cross error covariances $\Sigma_{xb}(k|k)$, $\Sigma_{xa}(k|k)$, and $\Sigma_{ab}(k|k)$ are zero, then the adaptive control gain $u_c(k) = 0$ in Eq. (3.4.1).

In Eqs. (3.4.1) and (3.4.2) we have the explicit variation of the adaptive gain as a function of the future expected uncertainty of the parameters. The OLFO control correction term is affected by the estimation accuracy of the $a$ and $b$ vector through $\Sigma_{xa}(\cdot|k)$, $\Sigma_{xb}(\cdot|k)$, $\Sigma_{ab}(\cdot|k)$, $\Sigma_{aa}(\cdot|k)$, and $\Sigma_{bb}(\cdot|k)$. We note that the uncertainty in the state vector $\hat{x}$ given by $\Sigma_{xx}(\cdot|k)$ does not affect the OLFO control calculation, and, hence, consistent with the results of the standard Separation Theorem.

If $\Sigma_{bb}(k|k) = 0$ and $\Sigma_{aa}(k|k) = 0$, then we have "identified" $b$ and $a$, that is, $\hat{b}(k|k) = b(k)$ and $\hat{a}(k|k) = a(k)$. We also have then $\Sigma_{xb}(k|k) = 0$, $\Sigma_{xa}(k|k) = 0$, and $\Sigma_{ab}(k|k) = 0$. It can be shown by induction from Eq. (2.3.2) and Eqs. (2.3.26)-(2.3.34) that if we define in this case

$$\hat{K}(k|k) = \begin{bmatrix}
K_{11}(k) & K_{12}(k) & K_{13}(k) \\
K_{21}(k) & K_{22}(k) & K_{23}(k) \\
K_{31}(k) & K_{32}(k) & K_{33}(k)
\end{bmatrix}$$

(3.4.3)

then $K_{11}(k)$ satisfies the matrix Riccati equation

$$K_{11}(k) = A'(k)K_{11}(k+1) - K_{11}(k+1)b'(k)(r(k) + b'(k)K_{11}(k+1)b(k))^{-1}b'(k)K_{11}(k+1)A(k) + Q(k),$$

$$K_{11}(N) = Q(N)$$

(3.4.4)

The optimal open-loop feedback adaptive gain is then effectively

$$\phi'(k) = -[r(k) + b'(k)K_{11}(k+1)b(k)]^{-1}b'(k)K_{11}(k+1)A(k) \phi^*(k)$$

(3.4.5)
Fig. 3.1 Combined Estimation and Control of Linear Stochastic System with Known Dynamics
which is the truly optimal gain for the linear-quadratic-Gaussian problem given by the Separation Theorem [1]. Under these assumptions, the adaptive control correction term \( u_c(k) = 0 \), and the stochastic optimal control is, therefore,

\[
u^*(k) = -[r(k) + b'(k)K_{11}(k+1)b(k)]^{-1}b'(k)K_{11}(k+1)\hat{\alpha}(k)\hat{\beta}(k|k) \quad (3.4.6)\]

The structure of the truly optimal stochastic control is given in Fig. 3.1. Therefore, if at time \( k \), the identification of the parameters \( a(k) \) and \( b(k) \) has a very high level of confidence, i.e., \( \Sigma_{bb}(k|k) \approx 0 \), and \( \Sigma_{aa}(k|k) \approx 0 \), then the optimal open-loop feedback control will act nearly optimal, and use the generated estimates \( \hat{a}(k|k) \) and \( \hat{b}(k|k) \) as if they were the correct parameter values.
4.1 Existence and Uniqueness of the O.L.F.O. Solution

In obtaining the dynamic programming solution to the deterministic open-loop control problem Eq. (3.2.1)-(3.2.6), the convexity condition (3.3.13)

\[ Z_{uu}(i|k) > 0 \]  \hspace{1cm} (4.1.1)

where

\[ Z_{uu}(i|k) = r(i|k) + tr\mathbb{P}_{bb}(i|k)\mathbb{P}_{xx}(i+1|k) + \mathbb{P}'(i|k)\mathbb{K}(i+1|k)\mathbb{E}(i|k) \]  \hspace{1cm} (4.1.2)

is required. If \( Z_{uu}(i|k) < 0 \), then a bounded optimal solution does not exist. Also, if \( Z_{uu}(i|k) \) is singular, the optimal solution will not be unique. If we assume that neither of the foregoing cases hold, then we have the following theorem.

**Theorem 4.1** (Uniqueness and Sufficiency):

If \( Z_{uu}(i|k) > 0; \ i = k, k+1, \ldots, N-1 \)

then the optimal open-loop control of the deterministic problem exists, is unique, and is given by Eq. (3.3.23).

**Proof:** The proof follows directly from the derivation of the above dynamic programming equations.

We shall use Eqs. (3.3.24) and (3.3.5) to show the existence and uniqueness of \( \mathbb{K}(j|k) \), \( j = k, k+1, \ldots, N-1; \ k = 0, 1, \ldots, N-1 \) in Eq. (2.3.2). We will show by induction the following Lemma.
Lemma 4.1.1:

\[ \tilde{r}(j|k) + \hat{E}'(j^1k) \tilde{K}(j+1,k) \tilde{E}(j^1k) > 0; \quad j = k, k+1, \ldots, N-1 \]  

(4.1.3)

Proof: We have \( \Omega(N) \geq 0, \ r(j) > 0 \) for all \( j \). In particular, \( r(N-1) > 0 \).

Using Eq. (2.3.34) we obtain

\[
\tilde{r}(N-1|k) + \hat{E}'(N-1|k) \tilde{K}(N|k) \tilde{E}(N-1|k)
\]

\[ = r(N-1) + \text{tr}_{\hat{E}(N-1|k) \Omega(N)} + \hat{E}'(N-1|k) \hat{K}(N|k) \tilde{E}(N-1|k) > 0 \]

(4.1.4)

Assume

\[
\tilde{r}(\ell|k) + \hat{E}'(\ell|k) \tilde{K}(\ell+1|k) \tilde{E}(\ell|k) > 0; \quad \ell = i, i+1, \ldots, N-1
\]

(4.1.5)

Let \( \tilde{r}(j) = 0, \ j = k, \ldots, N-1 \). Then, by the induction hypothesis, Eq. (3.3.24) and Eq. (3.3.5) imply that

\[
\hat{J}_{i/k} \hat{G}_{\ell}(i-1|k) \hat{J}_{\ell}(i-1|k) = \frac{1}{2} \text{tr}_{\hat{E}(i-1|k) \Omega(i|k)} + \frac{1}{2} \hat{E}'(i-1|k) \hat{K}(i|k) \hat{E}(i-1|k) > 0
\]

(4.1.6)

if we choose

\[
\hat{J}(i-1|k) = \begin{bmatrix}
\hat{E}_{bb}(i-1|k) & \hat{E}_{ba}(i-1|k) & \hat{E}_{bb}(i-1|k) \\
\hat{E}_{ab}(i-1|k) & \hat{E}_{aa}(i-1|k) & \hat{E}_{ab}(i-1|k) \\
\hat{E}_{bb}(i-1|k) & \hat{E}_{ba}(i-1|k) & \hat{E}_{bb}(i-1|k)
\end{bmatrix}
\]

(4.1.7)

and since \( r(i-1) > 0 \), we have then from Eq. (3.3.24)

\[
\tilde{r}(i-1|k) + \hat{E}'(i-1|k) \tilde{K}(i|k) \tilde{E}(i-1|k)
\]

\[ = r(i-1) + 2 \hat{J}_{i/k} \hat{G}_{\ell}(i-1|k) \hat{J}_{\ell}(i-1|k) > 0 \]

(4.1.8)

Thus assertion in Eq. (4.1.3) is proved by induction.
By Lemma 4.1.1, the optimal open-loop control \( \{u^O(j|k)\}_{j=k}^{N-1} \) exists and is unique for all \( k = 0,1,\ldots,N-1 \), and, therefore, the open-loop feedback optimal control \( \{u^*(k)\}_{k=0}^{N-1} \) exists and is unique.

4.2 Asymptotic Behavior

In this section we study the asymptotic properties of the overall system by considering the behavior of \( \Sigma_{bb}(k|k) \) and \( \Sigma_{aa}(k|k) \) as \( k \to \infty \). We assume that the corresponding deterministic system of \( S_1 \) is completely controllable and completely observable. If the nonlinear filter worked properly then the estimates given by the extended Kalman filter would approach the actual values of the parameters, and the observation noise would lose its effect on the estimates as time increases. The filter gain in Eq. (2.3.12) would, therefore, decrease, since the predicted error covariance is expected to go to zero if there are no plant disturbances. However, since the filter is only an approximation to the optimal Bayesian filter, a bias error may occur. The extended Kalman filter behaves as if this error did not exist. The propagated error becomes smaller than the real errors. To tell the filter that we have nonlinearities, we can introduce fictitious driving noises into the augmented state equation, and hence, make use of all the measurements.

Lemma 4.2.1: Let \( \delta(k) = 0 \), that is, there is no stochastic variation in parameter vector \( \underline{a}(k) \) and the \( \underline{a}(k) \) vector is constant. Then, given any
control sequence, the error covariance $\Sigma_{aa}$ is monotonically decreasing. *

$$\Sigma_{aa}(k+1|k+1) \leq \Sigma_{aa}(k|k) \quad (4.2.1)$$

**Proof:** From Eq. (2.3.15)

$$\Sigma_{aa}(k+1|k+1) = \Sigma_{aa}(k|k) - \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] G(k+1) \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right]^{T} + \Phi(k+1)G'(k+1)$$

$$\quad + \Phi(k+1)G'(k+1) \quad (4.2.2)$$

where $G(k+1)$ is given by Eq. (2.3.12). The Lemma then follows immediately.

Intuitively, since

$$\hat{a}(k+1) = \hat{a}(k)$$

the uncertainty in $\hat{a}(k)$ cannot grow.

As a result of the Lemma 4.2.1, there exists then a $\Sigma_{aa}$ such that

$$\lim_{k \to \infty} \Sigma_{aa}(k|k) = \Sigma_{aa} \quad (4.2.3)$$

**Lemma 4.2.2:** For $b(k+1) = b(k)$, given any control sequence we have from Eq. (2.3.15)

$$\Sigma_{bb}(k+1|k+1) \leq \Sigma_{bb}(k|k) \quad (4.2.4)$$

**Proof:** The proof is similar to that for Lemma 4.2.1. There exists then a $\Sigma_{bb}$ such that

$$\lim_{k \to \infty} \Sigma_{bb}(k|k) = \Sigma_{bb} \quad (4.2.5)$$

*The matrix $P$ is said to be small than $M$ if for all nonzero vectors, the scalar quantity $x'Px < x'Mx$. 
In analogy with the deterministic case, we can thus say that the parameters \( a \) and \( b \) are observable since the variance of the estimation error of \( a \) and \( b \) can be decreased by operation on \( z \).

It can be shown that if \( z(k) = \mathbf{0} \), \( y(k) = \mathbf{0} \), and the system completely observable, then for any bounded but nonzero control \( u(k) \), \( k = 0, 1, \ldots \) \cite{16}, \cite{29}

\[
\lim_{k \to \infty} \Sigma_{aa}(k|k) = 0 \quad (4.2.6)
\]

\[
\lim_{k \to \infty} \Sigma_{bb}(k|k) = 0 \quad (4.2.7)
\]

Since \( \Sigma(k|k) \geq 0 \), this result implies that \( \lim_{k \to \infty} \Sigma_{xb}(k|k) = 0 \)

\[
\lim_{k \to \infty} \Sigma_{xa}(k|k) = 0, \quad \text{and} \quad \lim_{k \to \infty} \Sigma_{ab}(k|k) = 0.
\]

Hence, we can design a reasonable controller for a completely observable system with unknown parameters.

\[
a(k+1) = a(k)
\]

\[
b(k+1) = b(k)
\]

using an ad-hoc control law \( \zeta_k(x, a, b, \Sigma_{aa}, \Sigma_{bb}) \) for \( k \geq 0 \) given by (Fig. 4.1)

\[
(1) \quad \zeta_k(x, a, b, \Sigma_{aa}, \Sigma_{bb}) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times M_{nn} \times M_{nn} \to \mathbb{R}
\]

\[
\quad x \in \mathbb{R}^n, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \quad \Sigma_{aa} \in M_{nn}, \quad \Sigma_{bb} \in M_{nn}
\]

\[
(2) \quad \zeta_k(x, a, b, \Sigma_{aa}, \Sigma_{bb}) \neq 0, \quad \Sigma_{bb} \neq 0, \quad \Sigma_{aa} \neq 0, \quad x \neq 0
\]

\[
(3) \quad \zeta_k(x, a, b, 0, 0) = -(x(k) + b'K(k+1)b)^{-1}b'K(k+1)a \times
\]

Condition 2 satisfies Eqs. (4.2.6) and (4.2.7) and Condition 3 implies that the ad-hoc control will converge to the optimal control when
Fig. 4.1 Block Diagram of the ad-hoc Control Scheme

\[ u(k) = -(r(k) + \hat{b}(k+1) \hat{b}_k(k+1) A \hat{\theta}) \]
a and b become known. Hence, the ad-hoc control scheme for system with unknown parameters can provide reasonable simulation results.

Let us now assume an observable system $S_l$, where

$$a(k+1) = a(k), \ b(k+1) = b(k)$$

that is, the parameters are constant and not growing. We want to control the system over a finite interval $N$. In the beginning when the uncertainty in $b(k)$ is large ($\sum_{bb}$ big) $\hat{r}(j|k)$ is big the adaptive control gain $\hat{q}(k)$ is small for both the stable and unstable systems. The trajectory of the overall control system then approximates that of the input-free trajectory of system $S_l$. The initial guess on $b(k)$ is not changed since little control input is applied. For the not exponentially stable system $\hat{q}(k) \sim 0$ at the beginning. The low control magnitude nevertheless starts the identification of $a(k)$. It is expected that the resulting in stability of the system will cause large inputs to be applied which results in the identification of $b(k)$. The high control magnitude will be used mainly for identification of the parameters. The control will be nonzero as long as the identification process is not completed. We, therefore, expect that for the unstable system, the estimate of $b(k)$ will be identified before the control magnitude goes to zero.

In the exponentially stable systems, we have nonzero control in the beginning. Small control (even zero) may be utilized to bring the state toward zero. Hence, little energy is needed to keep the trajectory near zero, and identification of $b(k)$ is expected to be slow. Identification of $a(k)$ is expected to be reasonable, since the entire output will be due to mainly the product $A(k)x(k)$, although as $k \to \infty$ the observation approaches white noise. Hence, one may end up with good control but bad
estimates, since little control effort is spent for identification and control purposes. We remark that the above interpretation of the derived equations assumes that the nonlinear estimator is working satisfactorily, that the deviations of the estimates from the actual values are not large.

Finally, we shall consider the problem of controlling the time-invariant system $S_l$ with unknown parameters over an infinite time period $(N \to \infty)$. Assume that the system is controllable and observable, then the window-shifting approach suggested in [16] can be employed. By Lemmas 4.2.1 and 4.2.2, $(y(k) = 0$ and $\hat{\xi}(k) = 0)$ the estimates in $a$ and $b$ will converge asymptotically and hence, the time-varying adaptive control system tends to a time-invariant control system.

4.3 Enforced Separation Scheme

In this section we shall discuss the ad-hoc control design in which the Separation Theorem is arbitrarily enforced. This controller is a direct approximation of the stochastic optimal controller resulting from Separation Theorem, which in the absence of uncertainty about the plant parameters of $S_l$ minimizes the cost functional

$$J = \frac{1}{2} E \{x'(N)Q(N)x(N) + \sum_{j=0}^{N-1} x'(j)Q(j)x(j) + r(j)u^2(j)\}$$

where the expectation is taken over the underlying random variables $x(0)$, $\xi(\cdot)$, and $\theta(\cdot)$. The approximate minimization is based on the present estimates of the system parameters $\hat{a}(k|k)$ and $\hat{b}(k|k)$. We assume the parameter values to be known $(\Sigma_{aa}(\cdot|k) = \Sigma_{bb}(\cdot|k) = 0)$. Thus at each time step $k$, $0 \leq k \leq N$ we have the conditional cost given by Eq. (3.1.4) to be minimized subject to the dynamics Eqs. (3.2.1) - (3.2.4).
\( \Sigma_{xx}(\cdot|k) \) is now independent of \( u(\cdot) \), the equivalent functional minimization is given by

\[
\bar{J} = \frac{1}{2} \hat{X}'(N|k)Q(N)\hat{X}(N|k) + \frac{1}{2} \sum_{j=k}^{N-1} \hat{X}'(j|k)Q(j)\hat{X}(j|k) + r(j)u^2(j) \quad (4.3.2)
\]

The structure of the resulting control law has the form

\[
u(k|k) = -q(k|k)\hat{X}(k|k) \quad (4.3.3)
\]

where \( q(k|k) \) is the optimal deterministic gain which arbitrarily uses the current parameter estimates for the unknown system parameter values,

\[
q(k|k) = (r + \hat{b}'(k|k)K(k+1|k)\hat{b}(k|k))^{-1}\hat{b}'(k|k)K(k+1|k)\hat{a}(k|k) \quad (4.3.4)
\]

\[
K(k|k) = \hat{a}'(k|k)[\hat{K}(k+1|k) - K(k+1|k)\hat{b}(k|k)K(k+1|k)] + Q(k) \quad (4.3.5)
\]

We shall denote this suboptimal open-loop feedback design the enforced separation scheme. The control law is identical with the linear-quadratic-Gaussian case with known parameters, except that the actual parameter values of \( a(k) \) and \( b(k) \) are arbitrarily replaced by their updated estimates. We shall use the extended Kalman filter to generate the estimates. Optimality of the cascaded form feedback controller, thus, cannot be claimed any more.

As in the open-loop feedback optimal design, the enforced separation scheme controller is time-varying, because the feedback control gains must be recomputed to make use of the updated estimates. At each time step \( k \) a Riccati-type equation needs to be solved backwards in time, since \( \hat{a}(k|k) \neq \hat{a}(k+1|k+1) \) and \( \hat{b}(k|k) \neq \hat{b}(k+1|k+1) \). The enforced separation
scheme differs from the open-loop feedback optimal design, essentially, in that the control does not depend explicitly on the goodness of the parameter estimation process. Since arbitrarily enforcing Separation Theorem does not require the propagation of the error covariance matrices and predictor equations of \( \hat{\sigma}(j|k) \) and \( \hat{\sigma}(j|k) \), the computational requirements are that much simpler compared with the open-loop feedback approach. We emphasize that both designs are suboptimal closed-loop control systems. In the limit as the parameter estimates converges to the time parameter vectors \( a \) and \( b \), from Eq. (4.2.6), so will each suboptimal control policy asymptotically approach the truly optimal solution when the actual parameters are known.
CHAPTER 5
INTERPRETATION OF THE RESULTS

In this chapter we shall discuss further the O.L.F.O. approach and the results on adaptive systems. The adaptive control problem we stated in Section 2.1 and solved is one of optimal nonlinear stochastic control. In the formulation of computationally feasible solution to the optimal control problem, we have, in essence, forced separation in the control action. The separation in the open-loop feedback approach yields a suboptimum controller. Restricting the form to a reasonable closed-loop stochastic control system of Fig. 2.1, the open-loop feedback optimal control system can be viewed as a separation of the overall control system into an estimator and a zero-memory controller. The first subsystem is the learning device where the states and the parameters are identified in real time and can be designed independently of the controller objectives. The controller subsystem computes the on-line optimal control for a deterministic system and is parameter adaptive.

The results we derived in Section 3.3 do not correspond to the strict separation of optimum estimation and optimum deterministic control as is in the case of uncertain linear systems with known parameters and quadratic cost criteria. [6],[5] We expect that the Separation Theorem does not hold in the adaptive control problem. The value of the optimal open-loop feedback control depends on both the estimate (by examining the adaptive control gain vector \( \hat{\phi}'(k) \)) of the parameter and their error covariance matrices. The effect of identification error is, therefore, taken into account in the deterministic control problem we have formulated.
and solved. The design bears the adaptive properties and violates the pure Separation Theorem.

The system $S_1$ contains an input signal $u(k), k = 0, 1, \ldots, N-1$. It is expected that the choice of this input will affect how well the system identification can be performed. Let $\xi(k), \delta(k), \gamma(k) = 0$, then we obtain for the time-invariant system

$$x(k+1) = A^{k+1}x(0) + \sum_{i=0}^{k} A^{k-i}B u(i)$$

(5.1)

$$z(k+1) = CA^{k+1}x(0) + CB \sum_{i=0}^{k} A^{k-i}u(i) + \theta(k+1)$$

(5.2)

For example, if $x(0) = 0$, then a nonzero $u(k)$ is required for the system to identify the parameters. From Eq. (2.1.1), it is obvious that the larger the control input $u(k)$, the larger the contribution to the state trajectory at $x(k+1)$. This implies that the observation will contain a large amount of information about the gain parameter $b(k)$. Large values of control input will, therefore, help in the identification of $b(k)$. The control signal can also be used to regulate the signal-to-noise ratio at the sensor. But, large input is discouraged in our quadratic control-penalty term in the cost functional. Therefore, the dual nature of the control input is clearly emphasized in our formulation. A reasonable adaptive control sequence must then be a compromise between the desire to get accurate estimates and the desire to minimize the cost.

Let us consider further the open-loop feedback optimal solution. The original state weighting matrix was $Q(j)$, and the original control weighting was $r(j)$. The effect of the parameter uncertainties in the
open-loop feedback optimal control problem is to transform it, heuristically speaking, to a linear quadratic tracking problem with modified weightings, where \( u^0(j|k) \) is the optimal control for the following control system [37]

\[
\tilde{x}(j+1|k) = A^+(j|k)\tilde{x}(j|k) + \tilde{b}(j|k)u(j|k)
\]

with the cost functional

\[
J = \tilde{x}(N|k)\tilde{Q}(N)\tilde{x}(N|k) + \sum_{j=k}^{N-1} \left\{ \tilde{x}'(j|k)\tilde{W}(j|k)\tilde{x}(j|k) + \tilde{r}(j|k)u^2(j|k) \right\} + 2\tilde{x}'(j|k)d(j+1|k)u(j|k)
\]

Comparing this problem with the original linear quadratic state-regulator problem, we can call \( \tilde{r}(j|k) \) the modified relative weighting on the control.

We remark that the scalar weighting \( \tilde{r}(j|k) \) in Eq. (2.3.34) is related directly to the conditional error covariance \( \Sigma_{bb}(j|k) \). The modified relative weighting on the control \( \tilde{r}(j|k) \) indicates that a low level of confidence in the estimate of the gain parameter \( \hat{b}(j|k) \) or that \( \Sigma_{bb}(j|k) \) is big will weight the control heavily in Eq. (5.5) so that little energy will be expended. Hence, the more the gain parameter uncertainty (now and in the future) the larger \( \tilde{r}(j|k) \), which implies the smaller the value of the adaptive gain \( \tilde{b}(k) \) in Eq. (2.3.6). The control gain is, therefore, adjusted by the level of uncertainty of the estimation of \( b(k) \). We note that the parameter uncertainty \( \Sigma_{bb}(j|k) \) is modified by the matrix \( P_{xx}(j+1|k) \) in the computation of \( \tilde{r}(j|k) \) in Eq. (2.3.34).
In Eq. (2.3.33) if \( \| \hat{A}(j|k) \| < 1 \), then \( \| P_{xx}(j|k) \| \approx \| Q(j) \| \)
for \( j > k \). If, however, \( \| \hat{A}(j|k) \| > 1 \), then \( \| P_{xx}(j|k) \| >> \| Q(j) \| \), and, thus there is much more contribution from the trace term to the value of \( \tilde{r}(j|k) \) in Eq. (2.3.34). We remark that at the same level of parameter uncertainty, the more unstable the system, the larger \( P_{xx}(j|k) \), and hence the larger the value of the \( \tilde{r}(j|k) \). The result is the smaller the adaptive control gain \( \hat{\theta}(k) \) at the initial stages the more the system model is unstable.

In the discussion thus far we have not examined the effect of uncertainty in \( a(k) \) (\( \Sigma_{aa}(k|k) \neq 0 \)). Recall that from Eqs. (2.3.1)-(2.3.3), (2.3.31)-(2.3.32) the computation of the O.L.F.O. adaptive control is affected not only by \( \Sigma_{bb}(j|k) \), but also by \( \Sigma_{aa}(j|k) \). The original state weighting matrix \( Q(j) \) is modified in Eq. (2.3.31). If the system is asymptotically stable, then \( \| Q(j) \| + \| P_{nn}(j+1|k)\Sigma_{aa}(j|k) \| \approx \| Q(j) \| \). The state weighting matrix is not affected by the uncertainty in \( a(k) \). Hence, the O.L.F.O. will be cautious in applying the control to the system in the beginning due to uncertainty in \( b \). Intuitively, this is reasonable, since the system is stable, and will decay to zero asymptotically. If the system is not asymptotically stable, then the state weighting matrix is modified by \( \| Q(j) \| + \| P_{nn}(j+1|k)\Sigma_{aa}(j|k) \| >> \| Q(j) \| \). The net effect is that the O.L.F.O. will act with large control magnitudes to regulate the system. This is not easily seen heuristically from the equations. The interaction between the uncertainty in \( b(k) \) and \( a(k) \) is not completely predictable from the equations (2.3.31)-(2.3.32), (2.3.2) for the adaptive gain, since \( P_{xx}(j+1|k) \) is observation dependent.
Finally, we want to discuss the optimization of the (deterministic) conditional cost functional (3.1.4). The open-loop covariance matrix should be a function of the conditional probability density functions. The reformulation would have been exact. Since $A(k)$ is unknown, the estimation problem is an infinite dimensional one. We are forced by computational limitation to approximate the conditional expectations $\hat{\mu}(k|k)$ and $\hat{\mu}(j|k)$ and $\Sigma^{x}(k|k)$ and $\Sigma^{x}(j|k)$ using the extended Kalman filter. The optimal open-loop control sequence we have derived in Chapter 3 is at best an approximation to the optimal conditional open-loop control law. If $A(k)$ is known, then the exact conditional means and error covariances can be generated by the optimal linear filter [16]. Hence, in the absence of rigorous analytical results on the goodness of the approximation we have developed, we shall turn to simulation experiments to provide further clues to the adaptive control problem under consideration.
We have derived the open-loop feedback optimal control sequence in Chapter 3 via dynamic programming for the problem stated in Section 2.1. Based on the equations (2.3.8)-(2.3.16) for the identifier and Eqs. (2.3.5)-(2.3.7) for the feedback gain plus correction term controller we discussed qualitatively the asymptotic behavior of the identifier in Chapter 4 and of the overall control system in Chapter 5. In this chapter we shall report the results of simulated experimentation made on some dynamical systems. The main purpose of the simulation studies is to verify the qualitative properties that we predicted and to provide the quantitative measures on the convergence rate of both the O.L.F.O. control and the enforced separation schemes to the truly optimal stochastic control when the parameters are known. The simulation studies will compare the performance measure of (1) the truly optimal stochastic control system when the full dynamics are known, (2) the O.L.F.O. adaptive control system, and (3) the enforced separation scheme.

We shall consider specifically the first-order linear dynamical system described by the stochastic difference equation

\[ x(k+1) = ax(k) + bu(k) + \xi(k) \]  \hspace{1cm} (6.1)

with noisy measurements given by

\[ z(k) = cx(k) + \theta(k) \]  \hspace{1cm} (6.2)

We assume that the unknown parameters are constant.

The initial values \( x(0), a(0), \) and \( b(0) \) are assumed to be Gaussian random variables with known statistics.
\[ E\{x(0)\} = x_0, \quad \text{cov}\{x(0),x(0)\} = \Sigma_{x0} \quad (6.3) \]
\[ E\{a(0)\} = a_0, \quad \text{cov}\{a(0),a(0)\} = \Sigma_{a0} \quad (6.4) \]
\[ E\{b(0)\} = b_0, \quad \text{cov}\{b(0),b(0)\} = \Sigma_{b0} \quad (6.5) \]

where \( \Sigma_{x0} \), \( \Sigma_{a0} \), and \( \Sigma_{b0} \) are positive semidefinite. The scalar zero-mean white Gaussian driving noise sequence has the known statistics
\[
E\{\xi(k)\} = 0, \quad \text{cov}\{\xi(j),\xi(k)\} = \Xi(j)\delta_{jk} \quad (6.6)
\]

and the scalar zero-mean white Gaussian observation noise has the known statistics
\[
E\{\theta(k)\} = 0, \quad \text{cov}\{\theta(j),\theta(k)\} = \Theta(k)\delta_{jk} \quad (6.7)
\]

where \( \Xi(\cdot) \geq 0 \) and \( \Theta(\cdot) > 0 \). The random variables \( \{x(0),a(0),b(0),\xi(\cdot),\theta(\cdot)\} \) are mutually independent.

The objective of the problem is to find an optimal control sequence such that the expected cost functional
\[
J = E\left\{ \frac{1}{2} q(N) x^2(N) + \frac{1}{2} \sum_{k=0}^{N-1} q(k) x^2(k) + r(k)u^2(k) \right\} \quad (6.8)
\]
is minimized based on some information set.

We can now apply the theoretical results of the open-loop feedback optimal control obtained in Chapters 2 and 3 to Eqs. (6.1)-(6.8). A digital computer program was written, in which the identification equations (2.3.8)-(2.3.16), and the O.L.F.O. parameters equations (2.3.17)-(2.3.19), (2.3.23)-(2.3.25), (2.3.27)-(2.3.34), and adaptive control equations (2.3.1)-(2.3.2) are programmed as individual subroutines. The computer program listing is contained in Appendix B. In the computer program, if we let \( a_o = a, b_o = b, \) and \( \Sigma_{ao} = \Sigma_{bo} = 0, \) then the simulation
results would correspond to 1) the optimal closed-loop stochastic control for the system defined by Eqs. (6.1)-(6.8) with known parameters. If we now at each time step k assume that $\Sigma_{aa}(k|k) = \Sigma_{bb}(k|k) = 0$, and use the instantaneous estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ of parameters $a(k)$ and $b(k)$ to compute the suboptimal feedback control, Eqs. (3.4.4)-(3.4.6) the simulation results in this case would correspond to the system 3) in which the Separation Theorem is arbitrarily enforced. Hence, the algorithm we have is a general stochastic control simulation program that readily accommodates changes in the simulation model and initial conditions, and can be trivially modified for n-dimensional systems.

Using a plotting subroutine, we then plotted for a series of sample runs the resulting trajectory for 1) the stochastic control system with known parameters (where Separation Theorem holds), 2) the open-loop feedback optimal control and 3) the enforced separation scheme; the changes in the estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ in the open-loop feedback optimal and in the enforced separation schemes; the changes in the optimal feedback gain $\Phi^*(k)$, the adaptive gain $\phi(k)$, and the ad-hoc gain, and lastly, the applied control sequence in the three different stochastic control systems. In each sample run, to evaluate the cost functional for the three separate stochastic control systems, we computed the quadratic terms in $x(k)$ and $u(k)$, and these are also plotted versus $k$.

We note that we do not present here a comparison between the suboptimal control schemes and the true closed-loop stochastic control for systems with unknown parameters. The latter can be obtained only if we are able to compute the discrete probability distributions. The comparison of the system responses reported here will only indicate the
Table 6.1
Summary of the Monte Carlo Simulation (Sampling)
Sample size = 20, \( c = 1, b_0 = 1, x_o = 0, \)
\( E_{x_0} = 3, \bar{E} = 0.04, r = 1 \)

<table>
<thead>
<tr>
<th>Simulation</th>
<th>( a_o )</th>
<th>( \Sigma a_o )</th>
<th>( \Sigma b_o )</th>
<th>( \theta )</th>
<th>( q )</th>
<th>Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>U1</td>
<td>1.2</td>
<td>0.0049</td>
<td>0.25</td>
<td>1.0</td>
<td>10</td>
<td>C.1</td>
</tr>
<tr>
<td>U2</td>
<td>1.2</td>
<td>0.0009</td>
<td>0.25</td>
<td>1.0</td>
<td>10</td>
<td>C.2</td>
</tr>
<tr>
<td>U3</td>
<td>1.2</td>
<td>0.0009</td>
<td>0.25</td>
<td>1.0</td>
<td>2</td>
<td>C.3</td>
</tr>
<tr>
<td>U4</td>
<td>1.2</td>
<td>0.0049</td>
<td>0.25</td>
<td>1.0</td>
<td>2</td>
<td>C.4</td>
</tr>
<tr>
<td>U5</td>
<td>1.2</td>
<td>0.0009</td>
<td>0.25</td>
<td>4.0</td>
<td>10</td>
<td>C.5</td>
</tr>
<tr>
<td>U6</td>
<td>1.2</td>
<td>0.0009</td>
<td>0.25</td>
<td>9.0</td>
<td>10</td>
<td>C.6</td>
</tr>
<tr>
<td>U7</td>
<td>1.2</td>
<td>0.0049</td>
<td>0.0</td>
<td>1.0</td>
<td>10</td>
<td>C.7</td>
</tr>
<tr>
<td>U8</td>
<td>1.2</td>
<td>0.0049</td>
<td>0.25</td>
<td>1.0</td>
<td>10</td>
<td>C.8</td>
</tr>
<tr>
<td>S1</td>
<td>0.8</td>
<td>0.0049</td>
<td>0.25</td>
<td>1.0</td>
<td>10</td>
<td>C.9</td>
</tr>
<tr>
<td>S2</td>
<td>0.8</td>
<td>0.0049</td>
<td>0.25</td>
<td>4.0</td>
<td>10</td>
<td>C.10</td>
</tr>
</tbody>
</table>
loss in performance from the control viewpoint when simultaneous parameter identification is necessary. By examining these simulated time responses, we can obtain clues to the interaction between identification and control in an actual system. By evaluating the cost functional for the resulting trajectory and control sequence, we can numerically compare the performance of the open-loop feedback optimal design with the enforced separation design (in the closed-loop sense).

Since only first-order systems will be considered, the most number of unknowns we can have is three (x, a, and b). The uncertainty in b corresponds to uncertainty in the plant d. c. gain, while the uncertainty in a corresponds to uncertainty in the plant time constant. Not knowing a exactly is equivalent to not knowing the pole location of the system function. The only information about the system comes from the observation of the state trajectory obtained in the presence of a random disturbance. For each control system design, we used the same sample random sequence in the simulation via the Monte Carlo method, which required the establishment of statistical population for the uncertain quantities, repetitive calculation of performance using random samples from these populations, and averaging over the ensemble of results. We repeat (consecutively) the entire simulation run 20 times with the same initial conditions. A summary of the simulation experiments is given in Table 6.1. The comparison for the expected costs using controls 1), 2), and 3) is given in Appendix C.

The simulation results of Ul using the crude Monte Carlo method are shown in Figs. 6.1-6.5. We assume that the a priori distribution of a(0) is given by
a(0) \sim N[1.2, 0.0049]

From the plot of state trajectories in Fig. 6.1, we see that the open-loop feedback optimal trajectory has an overshoot at \(k = 1\) due to the large control \(u(0)\). The large control magnitude is used for identification and control purposes. The average of the parameter estimates \(\hat{a}(k|k)\) and \(\hat{b}(k|k)\) are shown in Figs. 6.2-6.3. They approach nearly the initial mean values. The identification of \(b\) was better using the O.L.F.O. control than the enforced separation scheme. In the plot of the feedback gains, Fig. 6.4 we have the experimental result that the open-loop feedback initial adaptive gains are nonzero and, surprisingly, large compared to the truly optimal feedback gain. The open-loop feedback optimal control sequence proved to be more "aggressive" on the average than the enforced separation design as seen from Fig. 6.5. Both the open-loop feedback optimal and the enforced separation control sequence were able to stabilize the system, although the identification was not exact. Not shown here is the O.L.F.O. correction term which does go to zero as \(k \to N\) in all the simulation runs.

In Figs. 6.6-6.7 we plot the quadratic terms in \(x(k)\) and \(u(k)\) vs. \(k\) averaged over 20 sample runs. They indicate the large contribution to the O.L.F.O. average cost occurs at \(k = 1\). A detailed examination of the computer data also showed that the sample variance of the cost is huge, thus implying that possibly a small number of statistically bad runs contributed to making the sample mean large. The detailed computations of the sample run costs and cumulative average costs are given in Table C.1.
Fig. 6.1 Comparison of the average response of the unstable systems UI when the parameters $a(k)$ and $b(k)$ are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme). The sample noise sequence was the same. Sample size = 20.
Fig. 6.2  Average Behavior of the Estimate of $a(k)$ for the Unstable Systems Ul

Fig. 6.3  Average Behavior of the Estimate of $b(k)$ for the Unstable Systems Ul
Fig. 6.4  Comparison between the Average Behavior of the Optimal Feedback Gain (when the parameters are known) and the two Suboptimal Feedback Gains (when the parameters are unknown) for the Unstable Systems UI
Fig. 6.5  Comparison between the Average Behavior of the Optimal Stochastic Control (when the parameters are known) and the two Suboptimal Stochastic Controls (when the parameters are unknown) for the Unstable Systems U1
Fig. 6.6 Comparison of the Average Behavior of $x^2(k)$ for the Unstable Systems UI

Fig. 6.7 Comparison of the Average Behavior of $u^2(k)$ for the Unstable Systems UI
In simulation run U2 the initial uncertainty in $a_0$ was reduced ($\sigma_{a_0} = 0.07$ in U1 and $\sigma_{a_0} = 0.03$ in U2). When $\Sigma_{a_0} = 0$, the estimation problem becomes linear, so decreasing $\Sigma_{a_0}$ would tend to reduce the non-linearity in the system. Simulation results showed that the identifier and the controller exhibited similar average behavior in all three cases as in U1. The open-loop feedback control gains and controls are a little larger than they were in U1. This tells us to look at the effect of $\Sigma_{a_0}(j|k)$ in Eq. (2.3.31). The initial overshoot in $x(k)$ is also little larger as a result.

Since the open-loop feedback optimal controller seemed too anxious to reduce the covariance in $x$, we ran simulations U3 and U4 with state weightings decreased to 2. The only noticeable effect is that the control magnitudes are smaller on the average than in U2 and U1. The average costs for the open-loop feedback optimal control remained greater than the enforced separation scheme.

By using Eq. (6.8) to evaluate the average costs, we are then comparing the open-loop feedback optimal and the enforced separation scheme in terms of performance index in the optimal closed-loop sense. Hence, in experiments U5 and U6 the measurement noise covariance was increased to 4 and then 9. We know that the open-loop feedback optimal design optimalizes the open-loop cost functional in the open-loop feedback sense. By increasing the measurement uncertainty, the estimation process would tend to ignore the data in the beginning and rely on the predicted observation. The system will then act more according to its average behavior (ignoring the zero-mean random processes) and, hence, more in the open-loop sense. We expect that the controls are not going
to be good when the noise covariances become large. This is seen from the larger costs incurred in U5 and U6 as compared to U2.

In simulation U5, the controller is more cautious on the average. The initial control is smaller than it was in U2, but takes on bigger values at \( k = 2 \) and \( k = 3 \) as the controller notices the divergence phenomenon in the state \( x(k) \). The rate of convergence of the estimates was slower in this system. As \( \theta \) was increased to 9, the initial measurements are used with even less confidence. The system does not respond to the divergence phenomenon until later, and thus, the initial large overshoots due to large control magnitudes are removed. The trajectory in \( x(k) \) showed instability in the beginning and larger variations than previous simulation runs. The biggest control occurred at \( k = 3 \), which produced the largest deviation in \( x \) at \( k = 4 \).

The simulation plots of S1 for stable systems using the crude Monte Carlo method are given in Figs. 6.8 - 6.12. We assume that the a priori distribution for \( a(0) \) is

\[
a(0) \sim N[0.8, 0.0049]
\]

From Fig. 6.8, we see that the state trajectory \( x(k) \) is essentially input-free. The initial open-loop feedback optimal control is nonzero, but small as expected since the modified control weighting Eq. (2.3.34) is large due to the initial uncertainty in \( b_0 \). The systems are asymptotically stable, hence, the control magnitudes are kept small. The estimates of parameters \( a \) and \( b \) are further off from the true values than in the unstable systems. But, the suboptimal controls do converge very fast to the truly optimal control. Again, the exact identification
Fig. 6.8  Comparison of the average response of the stable systems $S_1$ when the parameters $a(k)$ and $b(k)$ are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme). The sample noise sequence was the same. Sample size = 20.
**Fig. 6.9** Average Behavior of the Estimate of $a(k)$ for the Stable Systems S1

**Fig. 6.10** Average Behavior of the Estimate of $b(k)$ for the Stable Systems S1
Fig. 6.11 Comparison between the Average Behavior of the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for the Stable Systems $S1$.
Fig. 6.12  Comparison between the Average Behavior of the Optimal Stochastic Control (When the parameters are known) and the two Suboptimal Stochastic Controls (When the parameters are unknown) for the Stable Systems SI
Fig. 6.13 Comparison of the Average Behavior of $x^2(k)$ for the Stable Systems S1

Fig. 6.14 Comparison of the Average Behavior of $u^2(k)$ for the Stable Systems S1
of the unknown parameters is not necessary for good control. The averaged estimates in Figs. 6.9 - 6.10 are not good, since small magnitude input sequence is used, and, thus, little identification is accomplished. In Figs. 6.13 - 6.14, the quadratic terms in x(k) and u(k) are plotted. The greater part of the open-loop costs comes from the initial state deviations and large controls at the beginning.

By increasing the measurement noise covariance in S1 from 1.0 to 4.0, the aggregate and average costs increased for all three control strategies. The system was made to act more open-loop, as little confidence is placed on the initial data. The average behavior of the identifier and the controller was not changed greatly since the systems are asymptotically stable. From simulations S1 and S2, we have a comparisons of the average costs. The results show that O.L.F.O. control incurred a little smaller cost than the enforced separation scheme.

Finally, in Figs. 6.15 - 6.19 we present a set of single run plots for an unstable system and in Figs. 6.20 - 6.24 plots for a typical stable system.
Fig. 6.15 Comparison of the typical response of the unstable system when the parameters $a(k)$ and $b(k)$ are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme)
Fig. 6.16 Estimate of $a(k)$ for an Unstable System

Fig. 6.17 Estimate of $b(k)$ for an Unstable System
Fig. 6.18 Comparison between the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for an Unstable System
Fig. 6.19 Comparison between the Optimal Stochastic Control (When the parameters are known) and the Suboptimal Stochastic Controls (When the parameters are unknown) for an Unstable System
Fig. 6.20 Comparison of the typical response of the stable system when the parameters $a(k)$ and $b(k)$ are known (optimal stochastic control) and when the parameters are unknown (O.L.F.O. method and enforced separation scheme)
Fig. 6.21 Estimate of $a(k)$ for a Stable System

Fig. 6.22 Estimate of $b(k)$ for a Stable System
Fig. 6.23 Comparison between the Optimal Feedback Gain (When the parameters are known) and the two Suboptimal Feedback Gains (When the parameters are unknown) for a Stable System
Fig. 6.24 Comparison between the Optimal Stochastic Control (When the parameters are known) and the Suboptimal Stochastic Controls (When the parameters are unknown) for a Stable System
In Chapter 6 we described the digital computer simulation results in detail. In this Chapter we will interpret the results in light of the qualitative properties discussed in Chapters 4 and 5. From the simulations we found that nonzero control were applied at the beginning in both the stable and unstable systems using the open-loop feedback approach. The control input is used for identification and control purposes. This high-gain nature of the adaptive control is the response of the open-loop feedback approach to systems with more unknown parameters.

The rate of convergence seems to be dependent upon the stability of the system. The more unstable the system, the faster the estimates tend to the true parameter values. Further, the rate of convergence also depends on the initial guess $\hat{\theta}(0|0)$ and $\hat{\alpha}(0|0)$, more true in the case of stable systems than the unstable systems. The initial guess on $b(k)$ and $a(k)$ hence affect the OLFO trajectory of the stable systems more than the unstable systems.

Large controls help in the identification of the unknown parameters. From Eqs. (2.3.15) and (2.3.31) - (2.3.33), we note that the larger the control $u^*(k)$, the faster $\Sigma_{aa}(k|k)$ and $\Sigma_{bb}(k|k)$ decrease. The estimates $\hat{a}(k|k)$ and $\hat{b}(k|k)$ of the parameter vectors $a(k)$ and $b(k)$ themselves will depend on the particular control law from Eqs. (2.3.8) - (2.3.14), since the recursive filter contains $u^*(k)$ as a parameter. This is verified by the simulation results. Conversely, the goodness of the estimates $\hat{\theta}(k|k)$, $\hat{\alpha}(k|k)$, and $\hat{b}(k|k)$ will affect the control law actually used.
For both the unstable and stable systems we remark that exact identification of \( a(k) \) and \( b(k) \) is not necessary from the control viewpoint. This was shown experimentally to be more so in the case of stable systems. Simulation results showed that the open-loop feedback optimal control systems can work well even if the parameter estimates are bad. The use of feedback also reduces the effect of parameter variations or the system's sensitivity to parameter inaccuracy. We recall that our objective functional rewards the system for good control performance, but not for good estimation of parameters.

Simulation results via Monte Carlo method compare the average cost incurred from combined identification and control, using, first, the open-loop feedback technique and, second, the enforced separation scheme. Since the sample size is arbitrarily chosen, we cannot make any precise conclusions. From statistical theory, we can say with probability 0.99 for sample size 20 independent of distributions that 7/10 of the values will fall in the range of the sample. The experimental results seem to indicate that in the stable systems, the open-loop feedback optimal method on the average incurred a smaller performance index than the ad-hoc scheme. For unstable systems, the results seem to indicate that the enforced separation scheme will incur smaller average cost in doing the job than the open-loop feedback optimal control design. This imbalance seems to originate in the large control magnitude that the open-loop feedback optimal technique uses to probe the parameters and stabilize the system in the beginning.

Finally, we shall discuss the computation feasibility of the OLFO control algorithm in real time. In the estimator Eqs. (2.3.8) - (2.3.16) at
each time step $k$, $k=0,1,\ldots,N-1$, we solve forward in time a 1-step $3n$ vector difference equation to generate the current state and parameter estimates, and a 1-step $3nx3n$ matrix difference equation to propagate the covariance matrices, which must be computed on-line since they depend on the measurements. Next we compute the parameters $\Phi(j|k)$, $\Lambda_t(j|k)$, $\bar{\Delta}(j|k)$, $\bar{\Phi}(j|k)$, and $\bar{\Psi}(j|k)$ which involve some 1-step computations, Eqs. (2.3.27) - (2.3.32) and solve forward in time two $(N-k)$-steps $n$-vector difference equations (2.3.18) - (2.3.19), three $(N-k)$-steps $nxn$ matrix difference equations (2.3.23) - (2.3.25), and lastly both a $(2n+1)nx(2n+1)n$ matrix difference equation (2.3.2) and a matrix difference equation (2.3.33) which will have to be computed on-line backward in $(N-k)$ steps. The O.L.F.O. control is then obtained from Eqs. (2.3.1) and (2.3.4) using Eqs. (2.3.26) - (2.3.27). For the scalar system we have simulated and reported in Chapter 6, the actual computation of the O.L.F.O. control sequence for $N=30$ was about 0.06 second. The digital computer used was IBM 370/155. The total storage requirement corresponds to storing on-line $\hat{x}(k|k)$, $\hat{a}(k|k)$, $\hat{b}(k|k)$ and $\hat{\Sigma}(k|k)$ which require a total of $(3n+(3n+\frac{1}{2})3n)$ memory locations, since $\hat{\Sigma}(k|k)$ is symmetric. We conclude that the O.L.F.O. control algorithm is recursive and computationally easy to implement.

The computational requirement of the enforced separation scheme will be less. Since we assume that at each time step, the parameter values are exact, there is no need to propagate the error covariance matrices associated with the unknown parameters. This translates into a saving both in memory and execution time for on-line adaptive control implementation.
CHAPTER 8
CONCLUSIONS

We have considered in detail both analytically and experimentally, the problem of controlling a discrete linear system $S_1$, subject to stochastic disturbances, on the basis of noisy measurements. In addition, the system has a number of unknown parameters which may also vary in a stochastic manner. We assumed that the structure of the dynamical system is known. All of the underlying uncertain quantities are modeled as random variables with known statistics. It is assumed that the controller has perfect recall. The aim is to control the system such that the expected value of a quadratic cost functional of the state and control is minimized. Since the truly optimal closed-loop adaptive control solution given by Bellman's equation cannot be easily implemented because of the "curse of dimensionality", we sought to use suboptimal but simple and computationally feasible adaptive control algorithms.

The results of previous work on the control of a linear discrete-time stochastic system with unknown and stochastically varying control gains by Tse and Athans [16],[29] are extended to a larger class of problems where the poles are also unknown. We have investigated this class of problems in great detail, both from a theoretical and a simulation point of view. We adopted the open-loop feedback control philosophy, techniques, and theory in considering this broader class of problems. The O.L.F.O. approach led to a feedback controller design that has the desired adaptive properties. The analytical results showed that the gains of the O.L.F.O.
The adaptive control system were "modulated" by the current and future uncertainty (error covariance matrices) of the parameter estimation. Therefore, the standard Separation Theorem did not hold in this class of problems.

The O.L.F.O. control system consists of an identifier and a feedback gain plus correction term controller. Since this adaptive control problem involves nonlinear estimation, the exact solution cannot be realized by a finite dimensional system. Thus, we used a nonlinear extended Kalman filter to generate the approximate conditional means and the associated error covariances of the state and unknown parameters based on noisy data. The identification technique has to be on-line. It was shown that the identifier can be designed apart from control purposes. We have, therefore, in the O.L.F.O. approach forced some kind of a one-way separation [7],[42], although the identification and control aspects of the problem are not necessarily independent.

Any interaction between identification and control is explicitly exhibited in the form of constraints imposed by the identification technique on the optimal control problem.

From the equations we derived, qualitative properties of the overall O.L.F.O. control system were discussed. The derived theoretical results are in a form that can be easily programmed on a digital computer for on-line applications. Simulation studies on some constant first-order dynamic systems were made to obtain some quantitative measures on the convergence and the aggregate performance characteristics of the overall O.L.F.O. control system. It was found experimentally that the
average behavior of the adaptive control system was drastically different for stable and unstable systems. For the unstable system, the O.L.F.O. control converged greater to the true stochastic optimal closed-loop system than the stable system. However, excellent control can result even if the identification algorithm did not identify the parameters accurately. In optimizing control systems, the goodness of identification is not rewarded.

In Chapter 4 we also developed a second approximation to the optimal closed-loop solution in the form of a cascade of the extended Kalman filter with a deterministic actuator. This arbitrary use of the Separation Theorem led to a feedback controller that performed on the average little inferior than the O.L.F.O. control for the stable systems, but was much superior than the O.L.F.O. control for the unstable systems. The performance comparison was based on the evaluation of the original cost functional provided by the Monte Carlo technique for lack of analytical tools. The O.L.F.O. adaptive control system was found to be somewhat high gain. The evaluation of the cost functional also provided some quantitative measures on the loss in performance due to identification and control when compared with the truly optimal control system. It must be stressed that simulation is inherently an imprecise technique and can only provide numerical data about the performance of the system, and is useful for lack of more satisfactory techniques of analysis.

In the following we discuss some possible extensions of the research related to the class of problems we considered.
(A) **Different Cost Criteria**

We have considered only the quadratic cost criteria. In our reformulation of the stochastic control problem into a deterministic control problem, the quadratic criteria is preserved, and we can thus derive explicit results. Theoretically, we can extend the O.L.F.O. approach to the more general case where the cost criteria is not necessarily quadratic, e.g., exponential. The identification equations will remain unchanged, but the open-loop control problem to be formulated will be different from Eqs. (3.2.1) - (3.2.6).

We note that in general the performance indexes used specified the cost of the system operation in terms of error and energy, but do not give us information about the transient-response characteristics of the system. Therefore, to assure satisfactory transient characteristics, we need secondary criteria relating to response characteristics in order to influence the choice of cost weighting elements.

(B) **Vector Control**

The O.L.F.O. approach can be directly extended to $u(k) \in \mathbb{R}^r$ and $B(k)$ is a $n \times r$ input gain matrix. First, the identifier equations need to be derived which will generate the current estimate of the state and unknown matrices $A(k)$ and $B(k)$ and their associate error covariance matrices. An open-loop control problem can be formulated and solved as in Chapters 2 and 3. The results should be similar to those we derived, but the equations will be more complicated.
(C) **Convergence Rate**

We have not analyzed in detail the convergence rate of the sub-optimal O.L.F.O. control system to the optimal system. We have turned to simulation results to provide the basis for quantitative estimates about the convergence rate for stable and unstable systems. An error bound for the identifier and the controller would prove necessary to analyze and compare the expected value of the cost functional incurred by using the open-loop feedback optimal versus the enforced separation scheme.

(D) **Control with Unknown Parameters and Imperfectly Known Distrubances**

We have the matrices \( \bar{A}(k) \) and \( \bar{b}(k) \), \( k = 0,1,\ldots \) partially known but satisfying some difference equations. The noise vectors \( \bar{\xi}(k), \bar{\theta}(k), \bar{\delta}(k), \) and \( \bar{\gamma}(k) \); \( k = 0,1,\ldots \) are independent Gaussian vectors with unknown means and/or covariances. It is necessary then to perform adaptive filtering to recover the true means and covariances of the noise processes.

(E) **Second-Order Filter**

Since the identifier can be designed independently of the feedback controller, we can use the second-order filter to generate the approximate conditional estimates and covariance matrices. The second-order filter will in general remove the bias error contained in the extended Kalman filter due to multiplicative effects of nonlinearities in the plant equation and in the level of the driving noise.
APPENDIX A

FORMULATION OF THE OPEN-LOOP CONTROL PROBLEM

In this appendix we shall derive the dynamic equations satisfied by \( \hat{x}(j|k) \) and \( \Sigma_{xx}(j|k) \). We will then have completed the formulation of a deterministic optimal control problem from the stochastic control problem with unknown parameters. We index the present time by \( k \). Since all the noise sequences are assumed to be white and uncorrelated, then for \( j > k \).

\[
E\{ x(j) | Z_k \} = 0, E\{ \hat{\delta}(j) | Z_k \} = 0, E\{ y(j) | Z_k \} = 0
\]  
(A.1)

The future control sequence is restricted to be deterministic and we assumed that there are no more observations, Eqs. (A.1) and (2.1.) imply that for \( j > k \) prediction beyond time \( k \):

\[
\hat{x}(j+1|k) = \hat{\alpha}(\hat{\alpha}(j|k)) \hat{x}(j|k) + \hat{b}(j|k) u(j)
\]  
(A.2)

where the parameters are constant

\[
\hat{\alpha}(j+1|k) = \hat{\alpha}(j|k)
\]  
(A.3)

\[
\hat{b}(j+1|k) = \hat{b}(j|k)
\]  
(A.4)

with the initial values defined at \( j = k \), \( \hat{x}(k|k) \), \( \hat{\alpha}(k|k) \), \( \hat{b}(k|k) \) which is to be obtained from the extended Kalman filter for the nonlinear augmented system \( S_2 \). Since it is assumed that the control sequence \( U^*(0,k-1) \) has been applied to the system, the parameters in Eq. (2.2.10) are known.
The error vectors for \( j \geq k \) are given by

\[
e_{x}(j+1|k) = \hat{A}(a(j|k))\hat{x}(j|k) + \hat{b}(j|k)u(j) - \hat{A}(a(j|k)b(j|k)u(j) - \xi(j) \tag{A.5}
\]

To the first order approximation we can write (A.5) using Eq. (2.2.1) - (2.2.7)

\[
e_{x}(j+1|k) = \hat{A}(a(j|k))e_{a}(j|k) + \hat{x}(j|k)e_{x}(j|k) + u(j)e_{b}(j|k) - \xi(j) \tag{A.6}
\]

where

\[
e_{a}(j+1|k) = e_{a}(j|k) - \delta(j) \tag{A.7}
\]

\[
e_{b}(j+1|k) = e_{b}(j|k) - \gamma(j) \tag{A.8}
\]

The initial error at time \( j = k \) depends, however, only on
\[
\{\xi(i), \delta(i), \gamma(i), i \leq k-1\} \text{ and } \{\theta(i), i \leq k\}, \text{ but not on } \{\xi(i), \delta(i), \\
\gamma(i), i \geq k\}. \text{ Assuming all the noise sequences to be zero-mean,}
\text{ Gaussian, white, and uncorrelated with } Z_{k-1} \text{ for } j \geq k, \text{ the noise co-}
\text{variances are}
\]

\[
E\{\xi(j)\xi'(j)|Z_{k}\} = \Xi(j) \tag{A.9}
\]

\[
E\{\delta(j)\delta'(j)|Z_{k}\} = \Delta(j) \tag{A.10}
\]

\[
E\{\gamma(j)\gamma'(j)|Z_{k}\} = \Gamma(j) \tag{A.11}
\]

Since the initial error and the future noise sequence are independent, from Eq. (A.6)-A.11) and Eq. (2.2.8) we have for \( j \geq k \)

\[
\Sigma(j+1|k) = \hat{P}(j|k)\Sigma(j|k)\hat{P}'(j|k) + \tilde{\Sigma}(j) \tag{A.12}
\]

where \( \hat{P}(j|k) \) and \( \tilde{\Sigma}(j) \) are defined in Eqs. (3.2.5) - (3.2.6). The initial conditions are those given at \( j = k \).
We should note that \( \hat{x}(k|k) \), \( \hat{a}(k|k) \) and \( \hat{b}(k|k) \) are the approximate conditional means of \( x(k), a(k), \) and \( b(k) \) respectively, while \( \Sigma(k|k) \) is the approximate conditional error covariance of the augmented state vector \( \begin{bmatrix} \hat{x}(k|k) \\ \hat{a}(k|k) \\ \hat{b}(k|k) \end{bmatrix} \). These approximate conditional estimates and error covariances are generated by the extended Kalman filter [17], [20] given the past control \( U^*(0,k-1) \) has been chosen. The identification equations represent an optimum first-order estimator for the augmented states system \( S_2 \), and are summarized in Eqs. (2.3.8)-(2.3.16).

Assuming that the state and parameter estimates and their associated error covariances are known along with the past control sequence, we can then formulate a entirely deterministic (open-loop) control problem at time \( k, k=0,1,\ldots, N-1 \). We have then the deterministic dynamic system given by Eqs. (A.2)-(A.4) and (A.12).
APPENDIX B

FORTRAN SOURCE PROGRAM LISTING

The entire computer simulation program is included for the sake of completeness. It consists of the FORTRAN language routines (modules) — MAIN, PLOT1, GNOISE, RAND, PJ, OLFO, PARAM, ESTIM, GRAPH, and STAT. Each routine is documented with a descriptive preamble.

The compiler used is FORTRAN IV G LEVEL 20 of the M.I.T. Information Processing Center.
C**********************************************************************
C ADAPTIVE CONTROL OF A SCALAR DISCRETE TIME-INvariant
C LINEAR SYSTEM (A,B,C) WITH UNKNOWN DYNAMICS Driven BY
C ADDITIVE WHITE NOISE. THE MEASUREMENTS ARE NOISY.
C PLANT NOISE IS XI AND HAS STATISTICS N(O,Q)
C MEASUREMENT NOISE IS THETA AND HAS STATISTICS N(O,R)
C
ICODE= CODE FOR THE THREE CASES UNDER CONSIDERATION
C 1. SYSTEM WITH KNOWN DYNAMICS - SET SAA(0)=0, SBB(C)=0
C BHA T=B, AHA T=A
C 2. SYSTEM WITH UNKNOWN DYNAMICS- O.L.E.O. APPROACH
C 3. SYSTEM WITH UNKNOWN DYNAMICS- 'SEPARATION PRINCIPLE'
C FOR K *GT* 0
C COMPUTE THE ESTIMATES-XHAT, AHA T,BHA T
C COMPUTE THE ERROR COVARIANCE MATRICES
C SXX,SXA,SXB,SAA,SAB,SBB
C COMPUTE THE FORWARD DIFFERENCE EQUATIONS
C COMPUTE THE PARAMETERS--- PHI,BT,RT,V
C COMPUTE THE BACKWARD DIFFERENCE EQUATIONS
C
C SUBROUTINES CALLED---
C ESTIM, PARAM, OLF)
C
C**********************************************************************
C MAIN PROGRAM
C ALLOWS MONTE CARLO SIMULATION
I MPLICIT REAL*8 (A-H,O-Z)
REAL*8 KK
INTEGER T,T1
COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
COMMON /WHK/ SXX,SXA,SXB,SAA,SAB,SBB
COMM CN /HMK/ SXXP
COMMON /JPE/ KT,ICODE, MC
COMMON /SWK/ QJ,RJ
C FOR DIFFERENT FINAL TIME THE DIMENSION STMENT HAVE TO CHANGE
DIMENSION CK(20,31)
DIMENSION XK(20,31)
DIMENSION COST(3)
DIMENSION XIK(20,30), THEK(20,31)
C T IS THE FINAL TIME
READ (5,101) T
T1=T+1
C SET THE INDICATOR FOR NOISE DATA - 1 YES, 2 NO
NSYS=2
95 CONTINUE
READ (5,101) MC
C MC IS THE NUMBER OF SAMPLE RUNS
C F,QJ ARE THE STATE WEIGHTINGS,RJ IS THE CONTROL WEIGHTING
READ (5,100) F,QJ,RJ
READ (5,100) C
READ (5,100) Q,R
91 CONTINUE
CALL PLOT1(T,T1)
96 CONTINUE
C PARAMETERS FOR SEPARATE CASE RUNS
READ (5,101) ICODE
IF (ICODE .EQ. T) GO TO 95
C A PRIORI PROBABILITY DISTRIBUTIONS
READ (5,100) XO,SXXO
READ (5,100) AO,SAAO
READ (5,100) B0,SBB0
100 FORMAT (6F10.5)
101 FORMAT (110)
104 FORMAT (* A=", F15.6)
102 FORMAT (* B=", F15.6)
103 FORMAT (* C=", F15.6)
105 FORMAT (* Q=",F15.6 *)
106 FORMAT (* R=", F15.6)
107 FORMAT (* X0=",F15.6)
108 FORMAT (* A0=",F15.6)
109 FORMAT (* B0=",F15.6)
110 FORMAT (* SXXO=",F15.6, * SAAO=",F15.6, * SBB0=",F15.6)
111 FORMAT (' X(0)=', F15.6)
112 FORMAT (' QN(1)=', F15.6, ' Q(J)=', F15.6, ' R(J)=', F15.6)
113 FORMAT ('1', I10)
114 FORMAT (' COST 1=', F15.6, ' COST 2=', F15.6, ' COST 3=', F15.6)
115 FORMAT (8F10.5)
116 FORMAT (' EXPECTED COST =', F15.6)
117 FORMAT (' COST', I5, ' =', F15.6)
118 FORMAT ('1')
119 FORMAT (' INCREMENTAL COSTS')
120 FORMAT (10 F12.3)
121 FORMAT ('')
5 CONTINUE
WRITE (6,113) ICODE
WRITE (6,112) F,QJ,RJ
WRITE (6,103) C
WRITE (6,105) Q
WRITE (6,106) R
WRITE (6,107) X0
WRITE (6,108) A0
WRITE (6,109) B0
WRITE (6,110) SXX0, SAAO, SBB0
C COST(ICODE)=0.
C COSTL=0.
SIGMAA=DSQRT(SAAO)
SIGMAB=DSQRT(SBB0)
SIGMAX=DSQRT(SXX0)
SIGMAV=DSQRT(Q)
SIGMAW=DSQRT(P)
C SET THE RUN INDICATOR
KRUN=0
CONTINUE
WITH DIFFERENT NOISE SEQUENCE
IXV=IXV1
IXW=IXW1

SYSTEM PARAMETERS
CALL GNOISE(IA,SIGMAA,AO,A)
CALL GNOISE(IB,SIGMAB,BO,B)
CALL GNCIS~(IX,SIG~AX,XO,XO)
XO IS THE STARTING TRAJEKTORY
WRITE (6,104) A
WRITE (6,102) B
WRITE (6,111) X)

SET THE TIME STEP INDICATOR
KT=0

COMPUTE Y(0)
X=XC
CALL GNOISE(IXW,SIGMAW,0.0,THETA)
THEK(KRUN+1,KT+1)=THETA
Y=C*X+THE

COMPUTE THE ESTIMATES AT (0/0)
GAINX=SXXO*C/(C*SXXO*C+R)
GAINA=0
GAINB=0
XHAT=XO+GAINX*(Y-C*X0)
AHAT=AO
BHAT=BO

SET THE CROSS COVARIANCES TO ZERO
SXX=SXXO-GAINX*C*SXXO
SXA=0.
SXB=0.
SAA=SAAO
SAB=0.
SBB=SBB0

CONTINUE
XX(KRUN+1,KT+1)=X
CK(KRUN+1,KT+1)=X*X*QJ/2.

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C FORM THE OUTPUT PL0T MATRIX
CALL PLOT2(X,Y)
C
ACTUAL COST COMPUTATION
COST (ICODE)=COST (ICODE)+X*X*QJ/2.
C
IF (KT .GE. T) G0 TO 99
C COMPUTE THE 'OPTIMUM' CONTROL GAIN
CALL PJ(KT,T,AHAT,QJ,PXX,F)
NAMELIST/BUG2/PXX
CALL OLFO(QJ,RJ,F,GAIN,UC,KT,T,ICODE,&34)
C COMPUTE THE ADAPTIVE CONTROL
U=GAIN*XHAT+UC
C PLOT THE CONTROL
CALL PLOT3(U,UC,GAIN)
C
CK(KRUN+1,KT+1)=CK(KRUN+1,KT+1)+U*U*RJ/2.
COST (ICODE)=COST (ICODE)+U*U*RJ/2.
C
NAMELIST /BUG1/ KT, THETA,X, XI,Y,XHAT,AHAT,BHAT,SXX,SXA,SXB,SA,
* SAB,SR,B ,A,B,X)
C
RECORD X(K+1)
CALL GNOISE(IXV,SIGMAV,0.0,XI)
XK(KRUN+1,KT+1)=XI
X=A*X+B*U+XI
KT=KT+1
C DATA Y(K+1)
CALL GNOISE(IXW,SIGMAW,0.0,THETA)
THEK(KRUN+1,KT+1)=THETA
Y=C*X+THETA
C COMPUTE THE ESTIMATES OF STATE AND PARAMETERS AT (K/K)
C COMPUTE THE ERROR COVARIANCE MATRIX
CALL ESTIM(U,Y,C,Q,R)
GO TO 98
99 CONTINUE
KRUN=KRUN+1
C
COST1=COST(ICODE)-COSTL
WRITE (6,117) KRUN,COST1
COSTL=COST(ICODE)
IF (KRUN .GE. MC) GO TO 82
C
IXV1=IXV
IXW1=IXW
C
GO TO 93
82 CONTINUE
C
C AVERAGING OVER THE SAMPLE RUNS
COST(ICODE)=COST(ICODE)/MC
WRITE(6,113)
WRITE (6,119)
DO 60 I=1,MC
WRITE (6,121)
WRITE (6,120) (CK(I,J),J=1,T1)
60 CONTINUE
CALL PLOT5(XK,CK)
IF (ICODE .NE. 3) GO TO 96
WRITE (6,114) COST(1),COST(2),COST(3)
C OUTPUTS ARE THE SET OF PLOTS
CALL PLOT4(SXXO,SAAO,SBB0)
IF (NSYS .GT. 1) GO TO 91
CALL STAT(XIK,31,MC)
CALL STAT(THEK,31,MC)
NSYS=2
C SIMULATION WITH DIFFERENT INITIAL CONDITIONS OR ANOTHER SYSTEM
GO TO 91
94 WRITE (6,BUG1)
WRITE (6,101) KRUN
C
END
SUBROUTINE PLOT1(T, T1)

C******************************************************************************
C PLOTTING GRAPHS SUBPROGRAM
C SUBROUTINE CALLED-
C GRAPH(MODIFIED VERSION OF PRTPLT OF I.P.C.)
C******************************************************************************
IMPLICIT REAL*8 (A-H, O-Z)
COMMON /RICK/ XHAT, YHAT, BHAT, GAINX, GAINA, GAINB, PXX(100)
COMMON /WHK/ SX, SXA, SXB, SAA, SAB, SBB
COMMON /JPE/ KT, ICODE, MC
COMMON /SWK/ QJ, RJ
INTEGER T, T1
DIMENSION XS(31, 4), US(30, 4)
DIMENSION XK(20, 31), XV(31, 4)
DIMENSION XPLOT(31, 4), APLT(31, 4), BPLT(31, 4)
DIMENSION UPLOT(30, 4)
DIMENSION SPLT(30, 4), UCPLT(30, 4)
DIMENSION CV(31, 4), CK(20, 31)
DIMENSION CPLT(31, 4)
DIMENSION XHPLOT(31, 4), YPLOT(31, 4)
DIMENSION GXPLOT(101, 4), GAPLOT(101, 4), GRPLOT(101, 4)
DIMENSION SXPLT(31, 4), SAPLOT(31, 4), SBPLOT(31, 4)
DIMENSION EXPLOT(31, 4)
C INITIALIZING THE MATRICES TO ZERO
DO 1 I=1, T1
DO 1 J=1, 4
XPLOT(I, J)=0
APLOT(I, J)=0
BPLT(I, J)=0
XS(I, J)=0.
SXPLT(I, J)=0.
SAPLOT(I, J)=0.
SBPLOT(I, J)=0.
XHPLT(I, J)=0.
CPLT(I, J)=0.
EXPLT(I, J)=0.
1 CONTINUE
1 CONTINUE
DO 2 I=1, T
DO 2 J=1, 4
UPlOT(I, J)=0.
VCPlOT(I, J)=0.
GPlOT(I, J)=0.
US(I, J)=0.
2 CONTINUE
RETURN
ENTRY PlOT2(X, Y)
C TO PLOT THE TRAJECTORY FOR THE THREE CASES
XPlOT(KT+1, 1)=KT
XPlOT(KT+1, ICODE+1)=X +XPlOT(KT+1, ICODE+1)
XS(KT+1, 1)=KT
XS(KT+1, ICODE+1)=X*X +XS(KT+1, ICODE+1)
C TO PLOT THE A ESTIMATE FOR THE THREE CASES
APLOT(KT+1, 1)=KT
APLOT(KT+1, ICODE+1)=AHAT +APLOT(KT+1, ICODE+1)
C TO PLOT THE B ESTIMATES
BPLOT(KT+1, 1)=KT
BPLOT(KT+1, ICODE+1)=BHAT +BPLOT(KT+1, ICODE+1)
YPLOT(KT+1, 1)=KT
YPLOT(KT+1, ICODE+1)=Y
XHPlOT(KT+1, 1)=KT
XHPlOT(KT+1, ICODE+1)=XHAT +XHPlOT(KT+1, ICODE+1)
SXPlOT(KT+1, 1)=KT
SXPlOT(KT+1, ICODE+1)=SX +SXPlOT(KT+1, ICODE+1)
SAPLOT(KT+1, 1)=KT
SAPLOT(KT+1, ICODE+1)=SA +SAPLOT(KT+1, ICODE+1)
SBPlOT(KT+1, 1)=KT
SBPlOT(KT+1, ICODE+1)=SB +SBPlOT(KT+1, ICODE+1)
GXPlCT(KT+1, 1)=KT
GAPLOT(KT+1, 1)=KT
GBPlOT(KT+1, 1)=KT
GXPlOT(KT+1, ICODE+1)=GAINX
GAPLOT(KT+1, ICODE+1)=GAINA
GHPLOT(KT+1,ICODE+1) = GAIN
C PLOT(KT+1,1) = KT
C PLOT(KT+1,ICODE+1) = X * X * QJ/2 + C PLOT(KT+1,ICODE+1)
EX PLOT(KT+1,1) = KT
EX PLOT(KT+1,ICODE+1) = (XHAT - X) ** 2 + EXPLOT(KT+1,ICODE+1)
RETURN
ENTRY PLOT3(U, UC, GAIN)
C TO PLOT THE OPTIMAL FEEDBACK OR ADAPTIVE GAIN
G PLOT(KT+1,1) = KT
G PLOT(KT+1,ICODE+1) = - GAIN * G PLOT(KT+1,ICODE+1)
UC PLOT(KT+1,1) = KT
UC PLOT(KT+1,ICODE+1) = UC + UC PLOT(KT+1,ICODE+1)
UPLOT(KT+1,1) = KT
UPLOT(KT+1,ICODE+1) = U + U PLOT(KT+1,ICODE+1)
US(KT+1,1) = KT
US(KT+1,ICODE+1) = U * U + US(KT+1,ICODE+1)
C PLOT(KT+1,ICODE+1) = U * U * RJ/2 + C PLOT(KT+1,ICODE+1)
RETURN
ENTRY PLOT5(XK, CK)
J = I CODE+1
DO 92 I = 1, T1
X PLOT(I, J) = X PLOT(I, J) / MC
A PLOT(I, J) = A PLOT(I, J) / MC
B PLOT(I, J) = B PLOT(I, J) / MC
X S(I, J) = X S(I, J) / MC
X H PLOT(I, J) = X H PLOT(I, J) / MC
S X PLOT(I, J) = S X PLOT(I, J) / MC
S A PLOT(I, J) = S A PLOT(I, J) / MC
S B PLOT(I, J) = S B PLOT(I, J) / MC
C PLOT(I, J) = C PLOT(I, J) / MC
EX PLOT(I, J) = EX PLOT(I, J) / MC
92 CONTINUE
DO 80 K = 1, T1
C V(K, 1) = K - 1
X V(K, 1) = K - 1
C V(K, J) = 0.
XV(K,J)=0.0
DO 81 I=1,MC
CV(K,J)=CV(K,J)+(CK(I,K)-CPLT(K,J)**2
XV(K,J)=XV(K,J)+(XK(I,K)-XPLT(K,J)**2
81 CONTINUE
CV(K,J)=CV(K,J)/MC
XV(K,J)=XV(K,J)/MC
80 CONTINUE
DO 90 I=1,T
GPLOT(I,J)=GPLOT(I,J)/MC
UCPLOT(I,J)=UCPLOT(I,J)/MC
UPLOT(I,J)=UPLOT(I,J)/MC
US(I,J)=US(I,J)/MC
90 CONTINUE
RETURN
ENTRY PLOT4(SXXO,SAAO,SBBO)
NL1=2*T+1
NL2=2*T-1
YMAX=1.
YMIN=-1.
CALL GRAPH(1,XPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
XMAX=YMAX
XMIN=YMIN
YMAX=1.25
YMIN=0.75
CALL GRAPH(2,APLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
YMAX=1.25
YMIN=0.75
CALL GRAPH(3,BPLOT,T1,4,NL1,0,T1,4,YMAX,YMIN)
YMAX=1.5
YMIN=0.5
CALL GRAPH(4,GPLOT,T,4,NL2,0,T,4,YMAX,YMIN)
YMAX=0.45
YMIN=-0.5
CALL GRAPH(5,UPLOT,T,4,NL2,0,T,4,YMAX,YMIN)
YMAX=0.04
YMIN=-.06
CALL GRAPH (6, UCFL0T, T, 4, NL2, 0, T, 4, YMAX, YMIN)
YMAX=SXXO
YMIN=0.0
CALL GRAPH (7, Xv, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=SXXO+.0000J1
YMIN=0.0
CALL GRAPH (8, XL0T, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=SXXO+.0000J1
YMIN=0.0
CALL GRAPH (9, CXLOT, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=SXXO+.0000J1
YMIN=0.0
CALL GRAPH (10, SPLOT, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=XMAX
YMIN=XMIN
CALL GRAPH (11, XHLOT, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=SXXO
YMIN=0.0
CALL GRAPH (12, EXPLOT, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=10.0
YMIN=0.0
CALL GRAPH (13, CV, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=10.0
YMIN=0.0
CALL GRAPH (14, CLOT, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=5.0
YMIN=0.0
CALL GRAPH (20, XS, T1, 4, NL1, 0, T1, 4, YMAX, YMIN)
YMAX=5.0
YMIN=0.0
CALL GRAPH (21, US, T, 4, NL2, 0, T, 4, YMAX, YMIN)
RETURN
FND
SUBROUTINE GNOISE(IX,S,AM,V)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*4 Y
A=0.0
DO 50 I=1,12
CALL RAND(IX,IY,Y)
IX=IY
50 A=A+Y
V=(A-6.)*S+AM
RETURN
END
SUBROUTINE RAND(IX, IY, YFL)

IY=IX*65539

IF (IY<56746)

IY=JY+2147483647+1

YFL=IY

YFL=YFL*.4656613E-9

RETURN

END
SUBROUTINE PJ(KT,N,AHAT,QJ,PXX,F)
C***************************************************************
C SOLVING A BACKWARD DIFFERENCE EQUATION NAMED PXX
C COMPUTES PXX(N/K) BACKWARDS TO PXX(J+1/K)
C F IS THE TERMINAL STATE WEIGHTING
C QJ IS THE STATE WEIGHTING MATRIX
C RJ IS THE CONTROL WEIGHTING
C***************************************************************
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION PXX(N)
PXX(N)=F/2.
J=N-1
IF (J .EQ. KT) GO TO 2
1 CONTINUE
PXX(J)=AHAT*PXX(J+1)*AHAT+QJ/2.
IF (J .EQ. (KT+1)) GO TO 2
J=J-1
GO TO 1
2 CONTINUE
RETURN
END
SURROUTINE J1FO(QJ,RJ,F1G~JN,UC,KT,~,ICNDE,*).
C SOLVE THE MATRIX DIFFERENCE EQUATION FOR K BACKWARDS IN TIME
C GAIN IS THE ADAPTIVE CONTROL GAIN
C IC IS THE CONTROL CORRECTION TERM
C KT IS THE PRESENT TIME INDEX
C N IS THE FINAL TIME INDEX
C
IMPLICIT REAL*8 (A-H,O-Z)

COMMON /RTKXHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
COMMON /WHAT,XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
DIMENSION D(3),8K(3)
DIMENSION K(3,3),K1(3,3),K2(3,3)
DIMENSION BKP(3),PHI(3,3),V(3,1)

K IS THE RICCATI MATRIX

IF (KT .EQ. (!-1)) GO TO 14
C SOLVING A TIME-BACKWARD DIFFERENCE EQUATION
DO 1 M=1,3
CONTINUE
K(l,l)=F
IF (ICNDE::.FE.31) GO TO 15
CALL PARA~(QJ,RJ,
JT,BT,RT,PHI,V,n,SAA,SAR,SHB)
14 CONTINUE
CALL PARA~(QJ,RJ,
JT,BT,RT,PHI,V,D,ZERO,lERO,ZERO)
15 CONTINUE
RK(M) = 0.
DO 2 I = 1, 3
RK(M) = BT(I) * K(I, M) + 3K(M)
2 CONTINUE
BKB = RT
DO 3 M = 1, 3
BKB = RK(M) * BT(M) + BKB
C CHECK THE INVERSE
IF (BKB) 16, 16, 13
13 CONTINUE
DO 4 L = 1, 3
DO 4 M = 1, 3
KBBK(L, M) = (BK(L) * AK(M)) / BKB
4 CONTINUE
DO 5 L = 1, 3
DO 5 M = 1, 3
K1(L, M) = K(L, M) - KBBK(L, M)
5 CONTINUE
DO 6 L = 1, 3
DO 6 M = 1, 3
K2(L, M) = 0
DO 6 I = 1, 3
K2(L, M) = PHI(I, L) * K1(I, M) + K2(L, M)
6 CONTINUE
DO 7 L = 1, 3
DO 7 M = 1, 3
K(L, M) = V(L, M)
DO 7 I = 1, 3
K(L, M) = K2(L, I) * PHI(I, M) + K(L, M)
7 CONTINUE
NAMELIST /bug2/ PHI, JT, BT, PT, V, K, D, SXA, SXB, BK, BKB, KBBK, K1, K2,
1 AHAT, BAHAT, SAA, SAB, SBR
IF (JT .GT. (KT + 1)) GO TO 9
8 CONTINUE
C OBTAINED K(K+1/K)
JT = JT - 1
IF (ICODE .EQ. 3) GO TO 17
CALL PARAM(QJ,RJ, JT,RT,PHI,V,D,SAA,SAB,SBB)
GO TO 18
17 CONTINUE
CALL PARAM(QJ,RJ, JT,RT,PHI,V,D,ZERO,ZERO,ZERO)
C DATA ARE *T(K/K),BT(K/K),PHI(K/K),SXR(K/K),SXK(K/K), D(K+1)
18 CONTINUE
DO 10 M=1,3
BK(M)=0
DO 10 I=1,3
BK(M)=BT(I)*K(I,M)+BK(M)
10 CONTINUE
BKB=RT
DO 11 M=1,3
BKP(M)=0
DO 11 I=1,3
BKP(M)=BKP(M)+BK(I)*PHI(I,M)
11 CONTINUE
12 CONTINUE
C CONTROL GAIN
SAIN =-BKP(1)/BKB-D(1)/RT
IF (ICODE .NE. 2) GO TO 19
C ADAPTIVE CONTROL CORRECTION TERM
UC =-(BKP(2)/BKB+D(2)/RT)*SXK-(BKP(3)/BKB)*SXK
GO TO 20
19 CONTINUE
UC=0.
20 CONTINUE
NAMELIST /BUG3/EKP
RETURN
16 WRITE (6,BUG2)
RETURN 1
END
SUBROUTINE PARAM(QJ,RJ, JT,BT,RT,PHI,V,D,SAA,SAB,SBB)

C**************************************************************************************
C COMPUTATION OF PARAMETERS AT (J/K).
C FOR THE TIME-ININVARIANT CASE THERE IS NO DYNAMICS IN THE
C PREDICTOR EQUATIONS
C DEFINITION OF RT,RT,PHI,V,D
C**************************************************************************************

IMPLICIT REAL*8 (A-H,O-Z)
COMMON /RTK/ XHAT,AHAT,BHAT,GAINX,GAINA,GAINB,PXX(100)
DIMENSION PHI(3,3),W(3,3),AT(3,3),BT(3),D(3)

C FORWARD DIFFERENCE EQUATIONS
SBBJ=SBB
SAAJ=SAA
SABJ=SAB
AJ=AHAT
BJ=BHAT

C DEFINITION OF VARIABLES
BT(1)=BJ
BT(2)=SBBJ
BT(3)=SABJ
D(1)=2.*SABJ*PXX(JT+1)
D(2)=2.*AJ*PXX(JT+1)
D(3)=0.
RT=RJ+SRBJ*PXX(JT+1)*2.
AT(1,1)=AJ
AT(1,2)=0
AT(1,3)=0
AT(2,1)=SA3J
AT(2,2)=AJ
AT(2,3)=0
AT(3,1)=SAAJ
AT(3,2)=0
AT(3,3)=AJ
DO 1 I=1,3
DO 1 J=1,3
PHI(I,J)=AT(I,J)-((BT(I)*D(J))/RT)

PARM0001
PARM0002
PARM0003
PARM0004
PARM0005
PARM0006
PARM0007
PARM0008
PARM0009
PARM0010
PARM0011
PARM0012
PARM0013
PARM0014
PARM0015
PARM0016
PARM0017
PARM0018
PARM0019
PARM0020
PARM0021
PARM0022
PARM0023
PARM0024
PARM0025
PARM0026
PARM0027
PARM0028
PARM0029
PARM0030
PARM0031
PARM0032
PARM0033
PARM0034
PARM0035
PARM0036
1 CONTINUE
   W(1,1)=QJ+2.*SAAJ*PXX(JT+1)
   W(1,2)=0
   W(1,3)= 2.*AJ*PXX(JT+1)
   W(2,1)=0
   W(2,2)=0
   W(2,3)=0
   W(3,1)=2.*AJ*PXX(JT+1)
   W(3,2)=0
   W(3,3)=0
   DO 2 I=1,3
   DO 2 J=1,3
   V(I,J)=W(I,J)-((D(I)*D(J))/PT)
2 CONTINUE
NAMELIST /BUG/ XHAT,AHAT,BHAT,SXX,SXA,SXB,W,AT
RETURN
END
SUBROUTINE ESTIM(U,Y,C,Q,R)
C*********************************************************************************C***********************************************************************************
C IDENTIFICATION EQUATIONS FOR A LINEAR DISCRETE TIME SCALEP
C SYSTEM WITH UNKNOWN PARAMETERS DRIVEN BY ADDITIVE WHITE NOISES
C EXTENDED KALMAN FILTER ALGORITHM
C PREDICTED ERROR COVARIANCES--- SXXP, SXAP, SXBP, SAAP, SARP, SBBP
C UPDATED ESTIMATES--- XHAT, AHAT, BHAT
C UPDATED ERROR COVARIANCES--- SXX, SXA, SXB, SAA, SAB, SBB
C U IS THE KNOWN INPUT
C Y IS THE NOISY MEASUREMENT
C Q IS THE PLANT NOISE COVARIANCE MATRIX
C R IS THE OBSERVATION NOISE COVARIANCE MATRIX
C*********************************************************************************C***********************************************************************************
C INPLICIT REAL*8 (A-H,O-L)
C COMMON /RTK/ XHAT, AHAT, BHAT, GAINX, GAINA, GAINB, PXX(100)
C COMMON /WHK/ SXX, SXA, SXB, SAA, SAB, SBB
C COMMON /HMK/ SXXP
C SXXP=AHAT*SXX*AHAT+2.*(AHAT*SXA*XHAT+AHAT*SXB*U+XHAT*SAB*U)+
* XHAT*SAA*XHAT+U*U*SBB+Q
C SXAP=AHAT*SXA*XHAT+SAA+U*SAB
C SXBP=AHAT*SXB*XHAT+SAB+U*SBB
C SAAP=SAA
C SABP=SAB
C SBBP=SBB
C COMPUTE THE GAIN VECTOR
C GAINX=SXXP*C/(C*SXXP*C+R)
C GAINA=SXAP*C/(C*SXXP*C+R)
C GAINB=SXBP*C/(C*SXXP*C+R)
C COMPUTE THE RESIDUAL
C RES=Y-C*(AHAT*XHAT+BHAT*U)
C COMPUTE THE CURRENT ESTIMATES
C XHAT=AHAT*XHAT+BHAT*U+GAINX*RES
C AHAT=AHAT+GAINA*RES
C BHAT=BHAT+GAINB*RES
C COMPUTE THE ERROR COVARIANCE MATRIX
C SXX=SXXP-GAINX*C*SXXP

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$SXA = SXAP \times \text{GAINX} \times C \times SXAP$

$SXR = SXBP \times \text{GAINX} \times C \times SXBP$

$SAA = SAAP \times \text{GAINA} \times C \times SXAP$

$SAH = SABP \times \text{GAINA} \times C \times SXAP$

$SBB = SBBP \times \text{GAINB} \times C \times SXBP$

NAMELIST /BUG/ GAINX, GAINA, GAINB, SXAP, SXBP, SAAP, SABP, SBBP, RES

RETURN

END
SUBROUTINE GRAPH (NO, B, N, M, NL, NS, KX, JX, YMAX, YMIN)

PURPOSE

PLT SEVERAL CROSS-VARIABLES VERSUS A BASE VARIABLE

USAGE

CALL PRTPLT(NO, B, N, M, NL, NS, KX, JX)

DESCRIPTION OF PARAMETERS

NO - CHART NUMBER (3 DIGITS MAXIMUM)

B - MATRIX OF DATA TO BE PLOTTED. FIRST COLUMN REPRESENTS
    BASE VARIABLE AND SUCCESSIVE COLUMNS ARE THE CROSS-
    VARIABLES (MAXIMUM IS 9).

N - NUMBER OF ROWS IN MATRIX B

M - NUMBER OF COLUMNS IN MATRIX B (EQUAL TO THE TOTAL
    NUMBER OF VARIABLES). MAXIMUM IS 10.

NL - NUMBER OF LINES IN THE PLOT. IF 0 IS SPECIFIED, 50
    LINES ARE USED. THE NUMBER OF LINES MUST BE EQUAL TO
    OR GREATER THAN N.

NS - CODE FOR SORTING THE BASE VARIABLE DATA IN ASCENDING
    ORDER

    0 SORTING IS NOT NECESSARY (ALREADY IN ASCENDING
    ORDER).

    1 SORTING IS NECESSARY.

KX - DIMENSION OF B MATRIX FROM DIMENSION STATEMENT.
    IT MUST BE OF THE FORM B(KX, JX)

JX - DIMENSION OF B MATRIX FROM DIMENSION STATEMENT.
    IT MUST BE OF THE FORM B(KX, JX)
C
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION JUT(101),YPR(11),IANG(9),A(1000),B(KX,JX)
INTEGER IJUM,'L','/','IANG '','','','+','-','X','Y','#','O','&','/
INTEGER IUT
DATA BLANK /' '/

C
998 FORMAT (1H1,//)
999 FORMAT (8E15.6)
WRITE (6,998)
WRITE (6,999) ((3(I,J),J=1,M),I=1,N)
C
STORE COLUMNWISE
I=1
 DO 39 J=1,N
  DO 39 K=1,N
  A(I)=B(K,J)
  I=I+1
39 CONTINUE
C
1 FORMAT(1H1,60X,7H CHART, I3, //)
2 FORMAT(1H ,F11.4,5X,101A1)
3 FORMAT(1H )
4 FORMAT (1H ,E15.9,1X,101A1)
7 FORMAT(1H ,16X,101H ! I I I I)
1 I I I I
8 FORMAT (1H ,9X,5(G12.4,8X),G12.4)
9 FORMAT(1H ,19X,4(G12.4,8X),G12.4)
C
C
C
NLL=NL
C
C
C
SORT BASE VARIABLE DATA IN ASCENDING ORDER
C
TEST NLL

PRINT TITLE

20 WRITE(6,1)NO

DEVELOP BLANK AND DIGITS FOR PRINTING

FIND SCALE FOR BASE VARIABLE

\[ XSCAL = \frac{A(N) - A(1)}{(FLOAT(NLL - 1))} \]

FIND SCALE FOR CROSS-VARIABLES

SCALING DONE ON THE MAX OR MIN OF THE TWO

\[ M1 = N + 1 \]
\[ YYMIN = A(M1) \]
\[ YYMAX = YYMIN \]
\[ M2 = M \times N \]

DO 40 J = M1, M2
IF (A(J) .GT. YYMAX) YYMAX = A(J)
IF (A(J) .LT. YYMIN) YYMIN = A(J)
CONTINUE

\[ YMAX = DMAX1(YYMAX, YMAX) \]
\[ YMIN = DMIN1(YYMIN, YMIN) \]
\[ YSCAL = (YMAX - YMIN) / 100.0 \]

FIND BASE VARIABLE PRINT POSITION

\[ XB = A(1) \]
\[ L = 1 \]
\[ MY = M - 1 \]

DO 80 I = 1, NLL
F=I-1
XPR=XB+F*XSCAL
NSIZE = 2
IF(XPR .GE. (-99999.)*.AND.XPR.LE.99999.) NSIZE = 1
IF(A(L)-XPR*XSCAL*0.5) 50,50,70
C
C FIND CROSS-VARIABLES
C
50  DO 55 IX=1,101
55  OUT(IX)=BLANK
57  CONTINUE
   DO 60 J=1,MY
      JJ = MY + 1 - J
   LL = L + JJ*N
      JP=((A(LL)-YMIN)/YSCAL+1.0
      OUT(JP) = IANG(JJ)
60  CONTINUE
C
C PRINT LINE AND CLEAR, OR SKIP
C
   IF(L.EQ.N) GO TO 61
   L=L+1
   IF(A(L)-XPR*XSCAL*0.5) 57,57,61
61  GO TO (62, 63), NSIZE
62  WRITE(6,2) XPR, (OUT(IZ), IZ = 1, 101)
   GO TO 80
63  WRITE(6,4) XPR, (OUT(IZ), IZ = 1, 101)
   GO TO 80
70  WRITE(6,3)
80  CONTINUE
C
C PRINT CROSS-VARIABLES NUMBERS
C
WRITE(6,7) YPR(1)=YMIN
   DO 90 KN=1,9
90 YPR(KN+1)=YPR(KN)+YSCAL*10.0
YPR(11)=YMAX
WRITE(6,8) (YPR(IP), IP=1,11,2)
WRITE(6,9) (YPR(IP), IP=2,10,2)
RETURN
END
SUBROUTINE STAT(X, NS, MC)
C********************************************************************
C THIS SUBROUTINE COMPUTES THE STATISTICS OF THE PSEUDO-RANDOM
C VARIABLE USED IN THE MONTE CARLO SIMULATION.
C MATRIX X CONTAINS MC SAMPLE RUNS OF THE RANDOM SEQUENCE OF
C LENGTH NS.
C COR IS THE CORRELATION MATRIX
C COV IS THE COVARIANCE MATRIX
C X MEAN IS THE SAMPLE MEAN VECTOR
C********************************************************************
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION XMEAN(31), COR(31,31), COV(31,31)
DIMENSION X(MC, NS)
100 FORMAT (8F15.6)
101 FORMAT (' AVERAGE VALUES OF THE NOISE AT TIME K')
102 FORMAT (' CORRELATION MATRIX')
103 FORMAT (' COVARIANCE MATRIX')
104 FORMAT ('1')
106 FORMAT ('0')
    DO 7 I=1, MC
        WRITE (6,106) (X(I,J), J=1, NS)
    7 CONTINUE
    DO 1 J=1, NS
        XMEAN(J)=0.
        DO 1 I=1, MC
            XMEAN(J)=X(I,J)/MC+XMEAN(J)
        1 CONTINUE
    DO 2 I=1, NS
        DO 2 J=1, NS
            COR(I,J)=0.
            DO 2 N=1, MC
                COR(I,J)=COR(I,J)+X(N,I)*X(N,J)/MC
            2 CONTINUE
WRITE (6,101) XMEAN, COR, COV
END
COV(I,J) = COR(I,J) - XMEAN(I) * XMEAN(J)
3 CONTINUE
   WRITE (6,104)
   WRITE (6,101)
   WRITE (6,100) (XMEAN(J),J=1,NS)
   WRITE (6,102)
   DO 5 I=1,NS
   WRITE (6,100) (COR(I,J), J=I,NS)
5 CONTINUE
   WRITE (6,104)
   WRITE (6,103)
   DO 6 I=1,NS
   WRITE (6,100) (COV(I,J), J=I,NS)
6 CONTINUE
   RETURN
END
APPENDIX C

MONTE CARLO SIMULATION TABLES

\[ J_{\text{OLFO}} = \text{performance index using the open-loop feedback optimal approach} \]

\[ J_{\text{SEP}} = \text{performance index using the enforced separation scheme} \]

\[ \overline{J}_{\text{OLFO}} = \text{cumulative average of } J_{\text{OLFO}} \]

\[ \overline{J}_{\text{SEP}} = \text{cumulative average of } J_{\text{SEP}} \]

\[ J_{\text{OPT}} = \text{performance index using the truly optimal stochastic control} \]
<table>
<thead>
<tr>
<th>Run No</th>
<th>a(0)</th>
<th>b(0)</th>
<th>x(0)</th>
<th>J_{OLFO}</th>
<th>J_{SEP}</th>
<th>\bar{J}_{OLFO}</th>
<th>\bar{J}_{SEP}</th>
<th>J_{OPT}</th>
</tr>
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<td>93.4</td>
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<td>73.966</td>
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<td>288.91</td>
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<td>102.97</td>
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<td>-2.583</td>
<td>349.022</td>
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<td>306.09</td>
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<td>1.610</td>
<td>-0.982</td>
<td>707.907</td>
<td>179.586</td>
<td>309.34</td>
<td>148.74</td>
<td>137.411</td>
</tr>
<tr>
<td>20</td>
<td>1.229</td>
<td>0.651</td>
<td>2.095</td>
<td>161.136</td>
<td>139.322</td>
<td>301.93</td>
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<td>Average Cost</td>
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<td></td>
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<td>Run No.</td>
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Average Cost

84.81  45.97  25.22
Table C.4

Monte Carlo Simulation U4

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Average Cost: 85.26, 45.95, 25.22
Table C.5

Monte Carlo Simulation US

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Monte Carlo Simulation U6

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## Table C.7

Monte Carlo Simulation U7

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Average Cost | 155.09 | 144.18 | 109.93 |
### Table C.9

**Monte Carlo Simulation S1**

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**Average Cost**

- $a(0)$
- $b(0)$
- $x(0)$
- $J_{OLFO}$
- $J_{SEP}$
- $\bar{J}_{OLFO}$
- $\bar{J}_{SEP}$
- $J_{OPT}$

Average Cost: 34.99 35.65 30.55
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