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# RELIABILITY ANALYSIS APPLIED TO STRUCTURAL TESTS

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## SUMMARY

Although full-scale fatigue testing is now widely adopted in modern aircraft design practice, the current fatigue-life assessment procedures do not utilise all of the test data that is obtained, and they only partly take account of the probability of failure of the structure during the period in which it is being progressively weakened by the fatigue crack.

The present paper is concerned with the application of reliability theory to predict, from structural fatigue test data, the risk of failure of a structure under service conditions because its load-carrying capability is progressively reduced by the extension of a fatigue crack.

The procedure is applicable to both safe-life and fail-safe structures and, for a prescribed safety level, it will enable an inspection procedure to be planned or, if inspection is not feasible, it will evaluate the life to replacement.

The theory has been further developed to cope with the case of structures with initial cracks, such as can occur in modern high-strength materials which are susceptible to the formation of small flaws during the production process.

The method has been applied to a structure of high-strength steel and the results are compared with those obtained by the current life estimation procedures. This has shown that the conventional methods can be unconservative in certain cases, depending on the characteristics of the structure and the design operating conditions.

The suitability of the probabilistic approach to the interpretation of the results from full-scale fatigue testing of aircraft structures is discussed and the assumptions involved are examined.



## INTRODUCTION

In recent years the development of high-performance aircraft using new high-strength materials and more refined methods of stress analysis to satisfy the ultimate strength requirement has led to the fatigue performance of aircraft structures becoming a progressively more important factor.

Basic studies of the fatigue behaviour of complete structures, such as those described in references 1 and 2, have shown that a full-scale fatigue test of the structure under representative loading conditions is essential to identify the fatigue critical areas and accurately represent the complex stress conditions under fatigue loading.

Although full-scale fatigue testing is now widely adopted in aircraft design practice, this usually consists of applying to a single test specimen a loading sequence representing the service load history.

Complete failure under the test load sequence or the appearance of a crack of a particular length is defined as failure and the results are applied to determine a life under the service loading conditions.

However, such an arbitrary criterion of failure does not consider the increasing risk of static failure to which the structure is subjected as it is progressively weakened by the growing fatigue crack. The actual risk of failure could therefore differ considerably from that obtained by the currently used methods of life estimation.

Furthermore the difficulty of detecting very small cracks with current techniques, together with the susceptibility of the modern high-strength materials to the formation of flaws in production, may result in some probability of cracks existing in airframes prior to entering service.

This paper is concerned with applying reliability analysis to calculate the probability of survival as a function of life from the results of the full-scale fatigue test, including the case of structures which may be initially cracked.

## NOMENCLATURE

Footnotes for the nomenclature are found at the end of the list.

$a$ — crack length (this may refer to crack length at surface, crack depth, or some other specified dimension of crack front)

$a_F$ — crack length for complete collapse under mean load (or crack length at which slope of crack propagation curve becomes infinite)

$a_0$ — length of the largest crack that will not be detected during production process

$a_D$ — length of largest crack that will not be detected during in-service inspections

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| $a_c$ | length of initial crack in any structure which is cracked at beginning of its service life |
| † $F_t(t_1)$ | probability of variate $t$ exceeding some particular value $t_1$ |
| $h_l$ | period of operation (or service life) to extend a crack to length $l$ in structure which contained initial crack of length $l_c$, $h_l = n_l - n_c$ |
| $l$ | relative crack length $a/a_F$ ($l$ is dimensionless and has same value whether "a" refers to crack length at surface or to crack depth) |
| $l_o, l_D, l_c$ | relative crack lengths corresponding to $a_o$, $a_D$, $a_c$, respectively |
| $\tilde{l}_N, \tilde{l}_n$ | median values of distributions of $l$ at life $N$ and relative life $n$ |
| $L(n)$ | probability of survival to life $n$ (also called the survivorship function) |
| $L_F(n), L_s(n)$, $L_I(n), L_I^*(n)$, $L_{sL}(n), L_{s,\mu}(n)$ | survivorship functions at relative life $n$, corresponding to risk functions, $r_F(n)$, $r_s(n)$, $r_I(n)$, $r_I^*(n)$, $r_{sL}(n)$, and $r_{s,\mu}(n)$, respectively |
| $L_s(h)$ | survivorship function at relative service life $h$ corresponding to risk function $r_s(h)$ for structures with initial crack |
| $N$ | life of structure expressed as number of load applications or hours of operation |
| $N_i$ | life to first formation of fatigue crack (also called life to inital failure) |
| $H$ | service life of structure which was initially cracked expressed as number of load applications or hours of operation |
| $h$ | relative service life of structure which was initially cracked, $H/N_i$ |
| $N_l$ | life to produce crack length $l$ in any structure |
| $\tilde{N}_i$ | median of the distribution of $N_i$ |
| $n$ | relative life, $N/\tilde{N}_i$ |

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| $n_l$ | relative life to crack length $l$ for any structure |
| $n_{l,z}$ | life of structure which has life $z$ times median life at same crack length $l$ |
| $n_F$ | relative life to complete collapse of structure under mean load |
| $n_o, n_D, n_c$ | relative lives to produce crack lengths of $l_o$, $l_D$, and $l_c$, respectively |
| $\tilde{n}_l, \tilde{n}_F, \tilde{n}_o, \tilde{n}_D, \tilde{n}_c$ | medians of distributions of $n_l$, $n_F$, $n_o$, $n_D$, and $n_c$, respectively |
| $n_s$ | relative life corresponding to particular life $N_s$ |
| $\overline{N}_L$ | estimated mean fatigue life obtained from structural fatigue test |
| $n_{I(1)}, n_{I(2)}, n_{I(m)}$ | relative lives to 1st, 2d, and mth inspections carried out to detect fatigue cracks |
| $p_R, \mu_R$ | probability density function of residual strength $R$ with mean value $\mu_R$ |
| †$p_x(x_1)$ | probability density function of variate $x$ at particular value $x_1$ |
| †$P_x(x_1)$ | probability distribution of variate $x$ at particular value $x_1$, $P_x(x_1) = \Pr\{x \leq x_1\}$ |
| $P(N)$ | probability of failure up to life $N$ |
| $R(l)$ | static strength of structure containing fatigue crack of relative length $l$ |
| $r(N)$ | probability of failure in remaining fleet at Nth load application or risk of failure at life $N$ |
| $r(n)$ | risk of failure at relative life $n$ for unit change in $z$ |
| $r(h)$ | risk of failure after period of operation $h$ in population of structures which contain initial cracks for unit change in $z$ |

$r(h_s \mid l_o)$ — risk of failure after a period of operation $h_s$ in population of structures all of which contain initial crack of length $l_o$

$r(h_s \mid p(l_c))$ — risk of failure after period of operation $h_s$ in population of structures all of which contain initial cracks with probability distribution of initial crack lengths given by $p(l_c)$

$r_s(n_s)$ — risk of static fracture due to fatigue at particular life $n_s$, defined as failure at life $n_s$ from fatigue crack in structure which is still able to sustain applied service load exceeding mean load

$r_{s,\mu}(n_s)$ — risk of static fracture due to fatigue at life $n_s$, assuming no variability in residual static strength of structures all containing cracks of given length

$r_F(n_s)$ — risk of fatigue fracture at life $n_s$, defined as failure at life $n_s$ due to fatigue crack reaching such extent that structure is unable to sustain mean load

$r_{FT}(n_s)$ — the total risk of fatigue failure at life $n_s$, $r_{FT}(n_s) = r_s(n_s) + r_F(n_s)$

$r_{sL}(n)$ — risk of failure at life $n$ as calculated by conventional safe-life procedure

‡$r_I(n_s;l_D,n_I)$ — risk of fatigue failure at life $n_s$ in population of structures which have all been previously inspected at life $n_I$ with inspection procedure which detects crack lengths greater than $l_D$

‡$r_I^*(n_s;l_D,n_I)$ — risk of fatigue failure at life $n_s$ when cracks of length exceeding $l_D$ are detected by inspection at $n_I$ and are then repaired and structures returned to service

‡$r_I^*(n_s;l_D,n_s)$ — risk of fatigue failure at life $n_s$ with continuous inspection procedure by which cracks with length exceeding $l_D$ are detected and are then repaired and structures returned to service

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| † $r_I^*(h_s \mid p(l_c); l_D, h_s)$ | risk of fatigue failure after period of operation $h_s$ in population of structures all initially cracked with distribution of initial crack lengths given by $p(l_c)$ and continuously inspected to detect crack lengths exceeding $l_D$; after cracks are detected they are repaired and structures returned to service |
| † $r_I^*(n_s; l_D, r_{max})$ | risk of fatigue failure at life $n_s$ with inspection procedure detecting crack lengths greater than $l_D$ at inspection intervals designed to limit risk below some specified value $r_{max}$; after cracks are detected they are repaired and structures returned to service |
| † $r_I^*(h_s \mid p(l_c); l_D, r_{max})$ | risk of fatigue failure after period of operation $h_s$ in population of structures all initially cracked with distribution of initial crack lengths given by $p(l_c)$ and inspected to detect crack lengths exceeding $l_D$ at inspection intervals designed to limit risk below some specified value $r_{max}$; after cracks are detected they are repaired and structures returned to service |
| † $r_D^*(n_{I(m)}; l_D, n_{I(m-1)})$ | probability of detecting cracks by inspection at life $n_{I(m)}$ in population of structures previously inspected at $n_{I(m-1)}$ with an inspection procedure detecting crack lengths exceeding $l_D$; after cracks are detected they are repaired and structures returned to service |
| † $r_D^*(h_{I(m)} \mid p(l_c); l_D, h_{I(m-1)})$ | probability of detecting cracks by inspection after period of operation $h_{I(m)}$ in population of structures all initially cracked with distribution of initial crack lengths given by $p(l_c)$ and previously inspected at $h_{I(m-1)}$ to detect crack lengths exceeding $l_D$; after cracks are detected they are repaired and structures returned to service |
| $S$ | applied service load |
| $S_{Ult}$ | ultimate design load |

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| $S_m$ | mean load on structure |
| $U$ | gust velocity |
| $Y$ | relative service load, $S/S_{Ult}$ |
| † $\mu, \sigma^2$ | general symbols for mean and variance of population; used with suffix to denote variate |
| $\mu_o$ | mean strength (failing load) of uncracked structures |
| $\mu_R(l)$ | mean strength of structures containing cracks of length $l$ |
| $\tilde{l}_n$ | median crack propagation curve for population of structures, $\tilde{l}_n = g(\tilde{n}_l)$ |
| $\frac{\mu_R(l)}{\mu_o}$ | mean residual strength expressed nondimensionally as function of crack length $l$, $\frac{\mu_R(l)}{\mu_o} = \phi(l)$ |
| $x(l)$ | relative strength of any structure containing crack length $l$, $x(l) = \frac{R(l)}{\mu_R(l)}$ |
| $z$ | comparative life or life factor of structure with life to crack length $l$ of $z$ times median life to same crack length, $z = \frac{N_{l,z}}{\tilde{N}_l}$ or $\frac{n_{l,z}}{\tilde{n}_l}$ |

†Where no confusion can arise subscript for variate may be omitted.

‡Actual dimension of detectable crack $a_D$ may be specified instead of relative crack length $l_D$.

## INTERPRETATION OF FATIGUE TEST RESULTS

With the present practice of fatigue certification by full-scale testing, the data provided by the test specimen representing the median structure of the population includes

(1) Location of the fatigue critical areas
(2) The median crack propagation curve
(3) The life to final failure under the test load sequence
(4) Residual strength data from static failure of the cracked specimen under the test load sequence, which include the failing load and the extent of fatigue cracking

## CURRENT APPROACHES TO SAFETY IN FATIGUE

The current practice is to obtain from these results a mean fatigue life $\overline{N}_L$ corresponding to failure at some arbitrarily selected point on the crack propagation curve.

For a safe-life structure, $\overline{N}_L$ may be the life at which the specimen broke in the fatigue test or the life at which it would be estimated to fail under some specified load such as limit load. For a fail-safe structure, $\overline{N}_L$ is often taken to be the test life at which the fatigue failure became readily detectable by the inspection procedures that would be used in service.

In order to allow for variability in fatigue performance for either structure, the estimated mean life $\overline{N}_L$ is divided by a scatter factor to obtain a safe operating period for replacement or inspection of the structure. The scatter factor is obtained by using an assumed probability distribution of fatigue life with an acceptable probability of failure.

## DIFFICULTY WITH CURRENT METHODS

The difficulty with the previously discussed procedure is that although the safe life to replacement or inspection is based on failure at a given point on the crack growth curve, there is, in service, an increasing risk of failure as the fatigue crack extends and the structure may fail at any stage of the crack propagation.

This difficulty is well illustrated by the measurement of the collapse load of Mustang wings that were fatigue tested to destruction under a random load sequence (ref. 1). In figure 29 of reference 1, the relative frequency distribution is presented for the load at failure as determined by experiment. For the twelve structures tested the results indicate a wide range in the failing load from 30 percent to 60 percent of the ultimate load of the virgin structure. This means that for a given life the safety level in service may be significantly different from that indicated by the fatigue test result.

Clearly the effect will depend on the shape of the crack growth curve and on the service load spectrum; however to investigate the question further an example of an ultrahigh-strength steel welded structure has been taken. The crack propagation and residual strength curves of this structure are shown in figure 1 and indicate a reasonably typical safe-life construction in that once a fatigue crack has developed there is a very marked reduction in strength which leads rapidly to failure.

The probability of survival has been calculated for this structure by the conventional method, taking two rather extreme cases for the definition of failure as follows:

(1) Failure occurs at the limit load. This is a relatively high value of the load, being near the upper limit of loads at which failure would be expected in service. $\tilde{N}_L = \tilde{N}_{sL}$.

(2) Failure occurs at the mean load. This is the lowest load at which service failure can occur and it will give a lower limit to the definition of failing load. $\tilde{N}_L = \tilde{N}_F$.

The probabilities of survival corresponding to definitions (1) and (2), $L_{sL}$ and $L_F$, have been evaluated for the two load spectra shown in figure 2 by a log normal distribution of fatigue life.

If $N_l$ is the fatigue life to any crack length $l$ and $\tilde{N}_l$ is the median value, then

$$z = \frac{N_l}{\tilde{N}_l}$$

has a logarithmic normal distribution and

$$L_F(N) = \int_{N/\tilde{N}_F}^{\infty} p_z(z)\, dz \tag{1}$$

$$L_{sL}(N) = \int_{N/\tilde{N}_{sL}}^{\infty} p_z(z)\, dz \tag{2}$$

The results are plotted for the manoeuvre load spectrum and the gust load spectrum in figures 3 and 4, respectively. For both spectra, $L_F$ is considerably more than $L_{sL}$; this indicates that the point on the crack growth curve at which failure is defined will have a significant effect on the safety level.

## RELIABILITY ANALYSIS OF FATIGUE FAILURE

Consider a more representative model of the fatigue process in which a structure progressively weakened by the fatigue crack may be broken by a service load at any stage of the crack propagation. The structure may survive this risk and continue in service until the fatigue crack has reached the stage where the crack propagation curve is rising practically vertical. The residual strength of the structure then drops rapidly until it reaches the mean load when failure must ensue. This is essentially a case where failure occurs by the fatigue process alone and in this paper the failure is termed "fatigue fracture."

The risk of failure in this mode has been considered in the section "Interpretation of Fatigue Test Results" where the probability of survival $L_F(N)$ at the life $N$ has been derived in equation (1) as

$$L_F(N) = \int_{N/\tilde{N}_F}^{\infty} p_z(z)\, dz$$

and the corresponding risk of failure is readily obtained as

$$r_F(N) = \frac{p_z\left(N/\tilde{N}_F\right)}{\int_{N/\tilde{N}_F}^{\infty} p_z(z)\, dz} \tag{3}$$

In addition to the risk due to fatigue fracture, there is the risk of failure due to chance occurrence of a service load on a structure weakened by fatigue cracking although the structure is still able to maintain the steady load. Current methods fail to take full account of this risk which is called herein the "risk of static fracture due to fatigue" and denoted as $r_S(N)$.

The total probability of fatigue failure at $N$ is therefore given by

$$r_{FT}(N) = r_S(N) + r_F(N) \tag{4}$$

If it is desired to indicate a specified value of the service life, $N_S$ may be used rather than $N$; therefore, an alternative form of equation (4) is

$$r_{FT}(N_S) = r_S(N_S) + r_F(N_S)$$

## RELIABILITY ANALYSIS WITH VARIABILITY IN FATIGUE STRENGTH

First consider the risk of static fracture due to fatigue in the simplified case where there is no variability in static strength but a characteristic distribution of fatigue life at any given crack length. Next consider the probability of failure in the fleet at the Nth load cycle (i.e., the risk of failure at life N) of structures all containing cracks of the same crack length a which may be expressed nondimensionally in terms of the crack length $a_F$ at which the structure would fail under the mean load; that is, $l = a/a_F$.

Let $S_N$ denote the Nth service load and $R(l)$ the residual strength of structures with crack length $l$. $R(l)$ is a decreasing function of $l$ and may be expressed nondimensionally in terms of the ultimate strength $\mu_o$ of an uncracked structure as

$$\frac{R}{\mu_o} = \phi(l) \tag{5}$$

Hence

$$\begin{aligned} \Pr\left\{\text{Failure at life } N \mid \text{crack length } l\right\} &= p_F(N/l) \\ &= \Pr\left\{S_N \geqq R(l)\right\} \\ &= \Pr\left\{S_N \geqq \mu_o\phi(l)\right\} \\ &= F_s\left(\mu_o\phi[l]\right) \end{aligned} \tag{6}$$

where $F_s(s)$ is the probability of exceeding any service load s. The total probability of failure in the fleet at life N (i.e., the risk of failure at N) is then obtained by summing over all crack lengths from $l = 0$ to $l = 1$

$$\begin{aligned} r_{s,\mu}(N) &= \int_0^1 p_F(N/l)\, p(l)\, dl \\ &= \int_0^1 F_s\left(\mu_o\phi[l]\right) p(l)\, dl \end{aligned} \tag{7}$$

where $r_{s,\mu}(N)$ denotes the risk of static fracture at the life N assuming that there is no variability in the static strength at a given crack length.

The probability density function $p(l)$ of the crack length $l$ at any given life N is not known but this difficulty is overcome by transposing the variate from crack length at a given life to life at a given crack length. This is done by using the model of the fatigue process shown in figure 5 in which it is assumed that for any structure the life $N_l$ bears a constant ratio z to the median life $\tilde{N}_l$ at the same crack length $l$,

$$N_l = z\tilde{N}_l$$

or by expressing life nondimensionally in terms of the median life to initial failure $\tilde{N}_i$

$$\frac{N_l}{\tilde{N}_i} = n_l = z\tilde{n}_l \tag{8}$$

where z is constant for any structure and is called the life factor. By considering the shaded element in figure 5 it can be seen that structures with crack lengths between $l$ and $l + dl$ at N have initial lives between $n_i$ and $n_i + dn_i$. Hence

$$p(l)\ dl = p(n_i)\ dn_i$$

$$= p(z)\ dz$$

since $z = \frac{n_i}{\tilde{n}_i}$. This expression neglects the effect on the probability density function of $n_i$ of the very few structures that have failed between $n_i$ and $n_s$.

If the equation of the median crack propagation curve

$$l = g(\tilde{n}_l) = g(n_l/z) \tag{9}$$

is used, equation (7) can now be transformed by changing the variable of crack length $l$ to one of fatigue life represented by the life factor z. Taking $z = n$ at $l = 0$ and $z = \frac{n}{\tilde{n}_F}$ at $l = 1$, equation (7) can now be written as

$$r_{s,\mu}(n) = \int_{n/\tilde{n}_F}^{n} F_s\left\{\mu_o\phi\left(g\left[\frac{n}{z}\right]\right)\right\} p(z)\ dz \tag{10}$$

## RELIABILITY ANALYSIS WITH VARIABILITY IN FATIGUE STRENGTH AND STATIC STRENGTH

In the preceding section it was assumed that there was no variability in the residual strength property, whereas, in general, at any crack length $l$, the residual strength $R(l)$ will have a probability distribution about a mean value $\mu_R(l)$. If the dimensionless variate $x(l) = \frac{R(l)}{\mu_R(l)}$ is assumed to have a characteristic distribution which applies for all values of crack length, then

$$R(l) = \mu_R(l)\, x(l)$$

and $\mu_R(l)$ can be expressed as a decreasing function of $l$ from equation (5) as

$$R(l) = \mu_0 \phi(l)\, x(l)$$

This is analogous to equation (6), and integrating over all crack lengths gives as before

$$r_S\big(n \mid x(l)\big) = \int_{n/\tilde{n}_F}^{n} F_S\left\{x\mu_0\phi\left(g\left[\frac{n}{z}\right]\right)\right\} p(z)\ dz \tag{11}$$

To obtain the total risk of static fracture at $n$, integrate over all values of $x(l)$ from 0 to $\infty$ to get

$$r_S(n) = \int_0^{\infty} \int_{n/\tilde{n}_F}^{n} F_S\left\{x\mu_0\phi\left(g\left[\frac{n}{z}\right]\right)\right\} p(z)\, p(x)\ dz\ dx \tag{12}$$

This equation is the general expression for the risk of static fracture by fatigue at life $n$. As stated earlier an alternative expression using $n_S$ instead of $n$ may be adopted where the risk at a specified value $n_S$ of the service life is desired. This expression is

$$r_S(n_S) = \int_0^{\infty} \int_{n_S/\tilde{n}_F}^{n_S} F_S\left\{x\mu_0\phi\left(g\left[\frac{n_S}{z}\right]\right)\right\} p(z)\, p(x)\ dz\ dx$$

## PROBABILITY DISTRIBUTION OF THE LOAD AT FAILURE

It is of interest to consider the probability distribution of the load at failure since this indicates how the risk of failure is being affected by the changing residual strength of aircraft in the fleet.

The condition for investigation is the probability that at a given life $n_S$ structures will fail with a residual strength less than some specified value $R_0$.

Requiring

$$R \leqq R_0$$

or

$$x = \frac{R}{\mu_R} \leqq \frac{R_0}{\mu_R}$$

then substituting

$$\mu_R = \mu_o \phi(l)$$

$$x \leqq \frac{R_o}{\mu_o \phi(l)}$$

or

$$x \leqq \frac{x_o}{\phi(l)}$$

where

$$x_o = \frac{R_o}{\mu_o}$$

and transposing the variate from crack length $l$ to the life factor $z$ give

$$x \leqq \frac{x_o}{\phi\left\{g\left[\frac{n_s}{z}\right]\right\}}$$

From equation (11)

$$\Pr\left\{\text{Static fracture at } n_s \text{ with the collapse load} \leqq R_o\right\}$$

$$= r_s\left\{n_s \mid R \leqq \mu_o x_o\right\}$$

$$= \int_{n_s/\tilde{n}_F}^{n_s} \int_{x=0}^{x=x_o/\phi\left\{g\left[\frac{n_s}{z}\right]\right\}} F_s\left\{x\mu_o\phi\left(g\left[\frac{n_s}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz \qquad (13)$$

where

$$x_o = \frac{R_o}{\mu_o}$$

Since the total probability of static fracture due to fatigue at $n_s$ is given by $r_s(n_s)$, the required probability distribution for the load at failure at a specified life $n_s$ is as follows:

$$\Pr\left\{\text{Failing load} \leqq \mu_0 x_0 \quad \text{at life} \quad n_s\right\}$$

$$= \int_{n_s/\tilde{n}_F}^{n_s} \int_0^{x_0/\phi\left\{g\left[\frac{n_s}{z}\right]\right\}} \frac{F_s\left\{x\mu_0\phi\left(g\left[\frac{n_s}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz}{r_s(n_s)} \tag{14}$$

## APPLICATION OF THE METHOD

To illustrate the method of reliability analysis and to compare the results according to the various risk functions in equations (2), (1), (10), and (12), the risk of failure has been calculated for a nonredundant high-strength steel structure. Sample test data for the structure are shown in figure 1.

The crack propagation curve has been determined from the results of a representative full-scale fatigue test in which fractographic examination of the fracture surface of the critical failures has been used to determine the crack dimensions at various stages of the test life. Although the curve in figure 1 is based on the crack length at the surface of the material, use of the nondimensional relative crack length $l = \frac{a}{a_F}$ enables it to represent also the crack depth or any other leading dimension of the crack front.

The residual strength curve $\frac{\mu_R}{\mu_o} = \phi(l)$ has been estimated from the relationship $l = \frac{A}{\left(\frac{\mu_R}{\mu_o}\right)^2}$ based on fracture mechanics theory, where A is a constant depending primarily on the fracture toughness of the material and the shape of the crack front.

The variability in residual strength about the mean value $\mu_R$ was assumed to follow the three parameter Weibull distribution, and with representative data on small steel specimens (ref. 3), the following expression was obtained for the probability distribution of the relative residual strength $x = \frac{R}{\mu_R}$:

$$P_x(x) = \Pr\left\{\frac{R}{\mu_R} \leqq x\right\} = 1 - \exp - \left\{\frac{x - 0.824}{1.017 - 0.824}\right\}^{2.55}$$

The crack length at failure under limit load, according to the relevant fatigue test data used, is approximately 0.08 in., giving a crack depth of 0.04 in. for a semicircular crack.

The distribution of fatigue life about the median value was assumed to be log normal with variance $\sigma^2_{\log N}$ of 0.02.

Two service load spectra were assumed as shown in figure 2. Spectrum I is a spectrum of manoeuvre load derived from data on U.S. jet fighter operations in reference 4. A median life to initial failure of 2000 hours was assumed to correspond to the fatigue test result, and an ultimate load factor of 10 was assumed, which gives a mean load of 10 percent of the design ultimate.

Spectrum II was based on thunderstorm gust load data from reference 5 giving the probability of exceeding a gust load $U$ as $F_u(U) = e^{-0.197U}$. Expressing load nondimensionally as

$$Y = \frac{S}{S_{Ult}}$$

where $S$ is the load due to a gust velocity $U$ and $S_{Ult}$ is the load corresponding to the ultimate design gust velocity of 99 fps with the mean load of the aircraft assumed to be 20 percent of the design ultimate, gives the following equation for the gust load spectrum:

$$F_s(Y) = e^{-24.4(Y-0.2)}$$

A life to initial failure of 20 000 hours was assumed as typical of this type of spectrum.

The four different risk functions of equations (1), (2), (10), and (12) have been evaluated by using numerical analysis techniques (ref. 6) for both spectra I and II. The corresponding probabilities of survival to life $n$ have been calculated from the relationship $L(n) = e^{-\int_0^n r(t)\,dt}$ and are plotted for spectrum I and spectrum II in figures 3 and 4, respectively.

These results show that conventional safe-life estimates as represented by $L_{sL}$ ($L_{sL}$ corresponds to static fracture of a fatigue cracked structure under limit load and is in accordance with current life estimation procedures) can be inaccurate since they fail to take proper account of the risk of static fracture of the structure weakened by the growing fatigue crack.

Comparison of $L_s$ and $L_{s,\mu}$ indicates that the variability in residual strength has a significant effect on the probability of survival (or failure). The probability of survival $L_F$ refers to failure due to the fatigue fracture extending to the stage where the structure is not able to sustain the steady mean load. The risk from this type of failure is often small but as mentioned previously it must be included in the total risk.

## RISK OF FAILURE IN STRUCTURES INITIALLY CRACKED

With the high-strength materials of low ductility now being introduced into aircraft construction there is a difficulty of detecting very small cracks with current nondestructive inspection (NDI) techniques. This factor together with the susceptibility of these high-strength materials to the formation of flaws in the production process may result in a probability of cracks existing in a number of aircraft structures before they go into service.

### STRUCTURES WITH INITIAL CRACKS OF CONSTANT LENGTH

In the most adverse case, all structures are assumed to be cracked in the fatigue critical areas to a relative crack length $l_o$ which corresponds to the maximum length of crack that will escape detection. According to this assumption all structures start their service life with a crack of length $l_o$ present.

In the model of the fatigue process illustrated in figure 5, all the crack propagation curves can be regarded as radiating from a single point or pole P. If all structures are initially cracked to the same length $l_o$, this corresponds to shifting the pole to the point P' with coordinates $(\tilde{n}_o, l_o)$ as shown in figure 6. Each structure now starts its service life h at the life $n_o$ which would have produced a fatigue crack of length $l_o$ in this particular structure. This infers that the initial crack or defect induces the same stress field as a fatigue crack of the same dimensions in the area being considered. It may be regarded as a fair assumption since under repeated loading the defect will rapidly initiate a fatigue crack which can be expected to give rise to a similar stress field as that which would result if the crack had been produced by fatigue alone.

Referring to figure 6 shows that for any structure which has a life factor $z = n_l/\tilde{n}_l$, the service life $h_l$ to any crack length $l$ is given by

$$h_l = n_l - n_o = z\tilde{n}_l - z\tilde{n}_o = z(\tilde{n}_l - \tilde{n}_o)$$

For the median values,

$$\tilde{h}_l = \tilde{n}_l - \tilde{n}_o$$

Hence

$$h_l = z\tilde{h}_l \qquad (15)$$

Therefore, the same model of the crack propagation process applies as for structures without initial cracks except that the origin is shifted to $(\tilde{n}_o, l_o)$, the service life is given by $h_s = (n_s - n_o) = z(\tilde{n}_s - \tilde{n}_o) = z\tilde{h}_s$, and the equation of the median crack propagation curve is transformed to

$$l = g(\tilde{h}_l + \tilde{n}_o) = g\left(\frac{h_l}{z} + \tilde{n}_o\right) \tag{16}$$

The risk of failure is therefore obtained in the same way as for structures initially uncracked, and by integrating over crack lengths from $l = l_o$ to $l = 1$, the following equation is obtained from equation (7):

$$r_{s,\mu}(n \mid l_o) = \int_{l_o}^{1} F_s\left\{\mu_o \phi(l)\right\} p(l)\, dl \tag{17}$$

Hence if the variable is changed from one of crack length to one of fatigue life at a given crack length as represented by the life factor $z$, the following equation is obtained from equations (17) and (16):

$$r_{s,\mu}(h_s \mid l_o) = \int_{h_s/(\tilde{n}_F - \tilde{n}_o)}^{\infty} F_s\left\{\mu_o \phi\left[g\left(\frac{h_s}{z} + \tilde{n}_o\right)\right]\right\} p(z)\, dz \tag{18}$$

where $r_{s,\mu}(h_s \mid l_o)$ denotes the risk of failure at a particular operating life $h_s$ of structures having initial cracks of length $l_o$ and having no variability in residual strength.

The corresponding expression when there is a probability distribution of residual strength $x$ given by $p(x)$ can be derived from equation (18) as

$$r_s(h_s \mid l_o) = \int_0^{\infty} \int_{h_s/(\tilde{n}_F - \tilde{n}_o)}^{\infty} F_s\left\{x \mu_o \phi\left[g\left(\frac{h_s}{z} + \tilde{n}_o\right)\right]\right\} p(z)\, p(x)\, dz\, dx \tag{19}$$

where $r_s(h_s \mid l_o)$ denotes the risk of failure at service life $h_s$ for structures which are all cracked to a length $l_o$ at the start of their service life.

The risk of failure by fatigue fracture for this case follows from the expression given in equation (3) and is

$$r_F(h_s \mid l_o) = \frac{p_z\left(\frac{h_s}{\tilde{n}_F - \tilde{n}_o}\right)}{\int_{h_s/(\tilde{n}_F-\tilde{n}_o)}^{\infty} p_z(z)\, dz} \tag{20}$$

The corresponding probabilities of survival can then be calculated as before.

## STRUCTURES WITH INITIAL CRACKS OF VARIOUS LENGTHS

In the general case the population of structures will contain cracks ranging from zero length up to the detectable length $l_o$ and it can be assumed that there is a probability of a structure containing a crack of length $l_c$ between 0 and $l_o$ as given by the probability density function $p(l_c)$.

Consider the fraction of the population $p(l_c)\, dl_c$ which has initial crack lengths between $l_c$ and $l_c + dl_c$. The probability of failure at $h_s$ for these structures is given by $r(h_s \mid l_c)$ according to equation (19). Their contribution to the total risk of failure in the population at service life $h_s$ is therefore,

$$\Delta r = r(h_s \mid l_c)\, p(l_c)\, dl_c \tag{21}$$

Since $h_s$ is the same for all structures whatever their initial crack length $l_c$, the total risk of failure for all structures at service life $h_s$ may be calculated by integrating equation (21) over all values of initial crack length from $l_c = 0$ to $l_c = l_o$. Then

$$r_s\left(h_s \mid p(l_c)\right) = \int_{l_c=0}^{l_c=l_o} r(h_s \mid l_c)\, p(l_c)\, dl_c \tag{22}$$

As was done in the derivation of $r_s(h_s \mid l_o)$ in equation (19), the variable of initial crack length $l_c$ is expressed as the corresponding life $\tilde{n}_c$ on the median crack propagation curve, with $l_c = g(\tilde{n}_c)$ and

$$p(l_c)\, dl_c = p(\tilde{n}_c)\, d\tilde{n}_c$$

Then, since $\tilde{n}_c = \tilde{n}_i$ when $l_c = 0$ and $\tilde{n}_c = \tilde{n}_o$ when $l_c = l_o$, the following equation is obtained from equation (22) by substituting $r(h_s \mid l_c)$ from equation (19):

$$r_s\left(h_s \mid p(l_c)\right) = \int_{\tilde{n}_c=\tilde{n}_i}^{\tilde{n}_c=\tilde{n}_o} \int_{x=0}^{x=\infty} \int_{z=h_s/(\tilde{n}_F-\tilde{n}_c)}^{z=\infty} F_s\left\{x\mu_o\phi\left[g\left(\frac{h_s}{z} + \tilde{n}_c\right)\right]\right\} p(z)\, p(x)\, p(\tilde{n}_c)\, dx\, dz\, d\tilde{n}_c \tag{23}$$

Similarly the risk of fatigue fracture can be derived from equation (20) as

$$r_F\left(h_s \mid p(l_c)\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \frac{\left[p_z\left\{h_s/\left(\tilde{n}_F - \tilde{n}_c\right)\right\} p\left(\tilde{n}_c\right) d\tilde{n}_c\right]}{\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{h_s/\left(\tilde{n}_F-\tilde{n}_c\right)}^{\infty} p_z(z)\, p\left(\tilde{n}_c\right) dz\, d\tilde{n}_c} \tag{24}$$

## PROBABILITY DISTRIBUTION OF THE FAILING LOAD

The probability distribution of the failing load can be determined for the case of structures with initial cracks by a simple extension of the method developed in the section "Probability Distribution of the Load at Failure."

If one is interested in structures with residual strength $R$ less than some specified value $R_o$, then as in the aforementioned section this corresponds to structures with

$$x \leq \frac{x_o}{\left(\mu_R/\mu_o\right)} \qquad \left(x_o = \frac{R_o}{\mu_o}\right) \tag{25}$$

Consider structures with initial cracks of length $l_c$ corresponding to a life of $\tilde{n}_c$ on the median crack propagation curve. Now from equation (16)

$$\frac{\mu_R}{\mu_o} = \phi(l) = \phi\left[g\left(\frac{h_l}{z} + \tilde{n}_c\right)\right]$$

Hence substituting this equality into equation (25) gives the following equation:

$$x \leq \frac{x_o}{\phi\left[g\left(\frac{h_l}{z} + \tilde{n}_c\right)\right]} \tag{26}$$

Thus, for structures with initial cracks of length $l_c$ it follows from equation (19) that the probability of failure with residual strength $\frac{R}{\mu_o}$ less than some given fraction $x_o$ of the virgin strength is given by

$$r_s\left\{h_s \,\middle|\, \begin{matrix} l_c \\ R \leq \mu_o x_o \end{matrix}\right\} = \int_{h_s/\left(\tilde{n}_F-\tilde{n}_c\right)}^{\infty} \int_0^{x_o/\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]} F_s\left\{x\mu_o\phi\left[g\left(\frac{h_s}{z} + \tilde{n}_c\right)\right]\right\} p(x)\, p(z)\, dx\, dz \tag{27}$$

The total risk of static fracture due to fatigue at $h_s$ is given by $r_s(h_s \mid l_c)$ and therefore it follows that



C.6

$$\Pr\left\{\text{Failing load} \leqq \mu_0 x_0 \quad \text{at life} \quad h_s\right\}$$

$$= \int_{h_s/(\tilde{n}_F-\tilde{n}_c)}^{\infty} \int_0^{x_0/\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]} \frac{F_s\left\{x\mu_0\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]\right\} p(x)\, p(z)\, dx\, dz}{r_s(h_s \mid l_c)} \qquad (28)$$

Where the population of structures have initial cracks with a probability distribution of crack length represented by $p(l_c)$ it follows from equation (23) that the probability of failure with relative strength $R/\mu_R$ less than $x_0$ is given by an analogous expression to equation (27) as follows

$$r_s\left\{h_s \,\middle|\, \begin{matrix} p(l_c) \\ R \leqq x_0\mu_0 \end{matrix}\right\} = \int_{\tilde{n}_i}^{\tilde{n}_0} \int_{h_s/(\tilde{n}_F-\tilde{n}_c)}^{\infty} \int_0^{x_0/\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]} F_s\left\{x\mu_0\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]\right\} p(x)\, p(z)\, p(\tilde{n}_c)\, dx\, dz\, d\tilde{n}_c \qquad (29)$$

If equation (29) is divided by $r_s(h_s \mid p(l_c))$, the total risk of static fracture due to fatigue at $h_s$, the probability that $R \leqq x_0\mu_0$ at $h_s$ is obtained as follows:

$$P_{x_0}\left\{h_s \,\middle|\, \begin{matrix} p(l_c) \\ x_0 \end{matrix}\right\} = \frac{\int_{\tilde{n}_i}^{\tilde{n}_0} \int_{h_s/(\tilde{n}_F-\tilde{n}_c)}^{\infty} \int_0^{x_0/\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]} F_s\left\{x\mu_0\phi\left[g\left(\frac{h_s}{z}+\tilde{n}_c\right)\right]\right\} p(x)\, p(z)\, p(\tilde{n}_c)\, dx\, dz\, d\tilde{n}_c}{r_s(h_s \mid p(l_c))} \qquad (30)$$

## APPLICATION

The foregoing theory has been applied to calculate the risk of failure for the ultrahigh-strength steel structures considered previously for which the crack propagation and residual strength curves are shown in figure 1. The load spectrum used in the calculations was the manoeuvre load spectrum shown in figure 2 as spectrum I.

For the case of structures all initially cracked to the same extent, the relative crack length $l_0$ has been taken as 0.075 from a consideration of the crack detection capability of the NDI techniques used in production.

For the case where it is assumed that there is a continuous probability distribution of initial crack size, an exponential distribution of initial crack length $l_c$ has been adopted with the probability density function

$$p(l_c) = 26.2e^{-20.6 l_c} \qquad (0 \leqq l_c \leqq 0.075) \qquad (31)$$

The exponential distribution has been adopted since it follows from the physically realistic assumption that the occurrence of a defect in a small element of the material follows a uniform probability law over the whole volume.

The detectable crack length $l_D$ for in-service inspections has been taken as 0.15.

As stated in the section "Structures With Initial Cracks of Constant Length," the theory assumes that the initial defect produces the same stress field as a fatigue crack the same size as the defect. In applying fracture mechanics theory to deduce crack propagation and residual strength characteristics, the depth of the crack is the important parameter; whereas for crack detection, the length of the crack exposed at the surface is the controlling factor. However, with the nondimensional relative crack length

$$l = \frac{a}{a_F} \tag{32}$$

it is immaterial whether crack length or crack depth is taken since both yield the same value of $l$, provided the shape of the crack front does not change markedly as the crack propagates.

In establishing the detectable relative crack lengths $l_o$ and $l_D$, it has been assumed that the crack length exposed at the surface which will be detected by the best available methods is 0.02 inch for production-line conditions and 0.04 inch for in-service inspections. Assuming a semicircular crack front, which is often characteristic of cracks originating at a surface, gives corresponding crack depths of 0.01 and 0.02 inch.

A value of $a_F$ of 0.132 inch was obtained from typical crack propagation data by determining the crack depth at which the crack propagation curve becomes vertical since this is virtually equivalent to failure at mean load. The relative crack lengths $l_o$ and $l_D$ given previously were thus obtained from equation (32).

With these input data, the risk functions $r_s^*(h \mid 0.01'')$ and $r_s^*\left(h \mid p(l_c)\right)$ for the two cases of constant initial crack depth of 0.075 and an exponential distribution of initial crack depths have been evaluated from equations (19) and (23) and are plotted in figures 7 and 9, respectively. The corresponding survivorship functions are plotted in figures 8 and 10. The probability distribution of the failing load at various service lives $h_s$ has been calculated from equation (28) and the results are presented in figure 11.

It is apparent that the presence of initial cracks greatly increases the risk of failure at a given life. Also the risk of failure at the beginning of the service life is finite in this case as distinct from the case where all structures are without cracks initially. This arises because with all structures cracked initially every member of the fleet is exposed to the risk of static fracture from the outset.

## SAFETY BY INSPECTION

As inspection techniques become more highly developed, increasing applications are likely to be found in monitoring structural safety. However, inspections of a complex aircraft structure are both time consuming and costly, and the efficient planning of inspection intervals is becoming an essential requirement. The reliability approach by calculating the risk of failure as a function of life enables the effect of any inspection procedure to be investigated and suitable inspection intervals to be planned.

### CONTINUOUS INSPECTION

The optimum effect of inspection is, of course, obtained when every structure is inspected continuously. As soon as cracks reach the detectable length $l_D$, remedial action is taken and therefore the risk of fatigue fracture is eliminated.

The risk of failure is then equal to the risk of static fracture by fatigue which is determined by calculating the probability of failure for structures with crack lengths between $l = 0$ and $l = l_D$.

If structures are repaired and replaced when cracks are detected, there is no reduction in size of the fleet and the risk of failure at any life $n_s$ is obtained by integrating in equation (12) between the limits $z = \frac{n_s}{\tilde{n}_D}$ to $z = n_s$ since this corresponds to integrating over crack lengths between 0 and $l_D$. (See fig. 5.)

Hence the risk of failure for "continuous inspection with replacement" is given by

$$r_I^*(n_s; l_D, n_s) = \int_{n_s/\tilde{n}_D}^{n_s} \int_0^{\infty} F_s\left\{x\mu_o\phi\left(g\left[\frac{n_s}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz \tag{33}$$

The corresponding result for structures which are initially cracked is found in a similar manner from equation (20); that is,

$$r_I^*\left(h_s \mid p(l_c); l_D, h_s\right) = \int_{\tilde{n}_c=\tilde{n}_i}^{\tilde{n}_c=\tilde{n}_o} \int_0^{\infty} \int_{h_s/(\tilde{n}_D-\tilde{n}_c)}^{\infty} F_s\left\{x\mu_o\phi\left[g\left(\frac{h_s}{z} + \tilde{n}_c\right)\right]\right\} p(z)\, p(x)\, p(\tilde{n}_c)\, dz\, dx\, d\tilde{n}_c \tag{34}$$

When cracked structures are not repaired but are taken out of service after detection, there is a continual depletion of the population since at life $n_s$ all structures which have a life less than $n_s$ at crack length $l_D$ are eliminated by inspection; that is, the distribution of fatigue life $p(z)$ is truncated at $z = \frac{n_s}{\tilde{n}_D}$ and hence the proportion of the population remaining at life $n_s$ is given by $\int_{n_s/\tilde{n}_D}^{\infty} p(z)\, dz$.

Therefore, for "inspection without replacement" the risk of failure at $n_S$ (which is the probability of failure in the fleet remaining at $n_S$) is derived from equation (33) as

$$r_I(n_S;l_D,n_S) = \frac{r_I^*(n_S;l_D,n_S)}{\int_{n_S/\tilde{n}_D}^{\infty} p(z)\ dz} \tag{35}$$

In a similar way the risk of failure for inspection without replacement in a population of structures which are initially cracked follows from equation (34) as

$$r_I\left(h_S \mid p(l_C);l_D,h_S\right) = \frac{r_I^*\left(h_S \mid p(l_C);l_D,h_S\right)}{\int_{\tilde{n}_C=\tilde{n}_i}^{\tilde{n}_C=\tilde{n}_O} \int_{h_S/\left(\tilde{n}_D-\tilde{n}_C\right)}^{\infty} p(z)\, p\left(\tilde{n}_C\right)\ dz\ d\tilde{n}_C} \tag{36}$$

## INSPECTION FOR LIMITED RISK

In practice, it is usually not economic or even feasible to inspect structures continuously but inspection is carried out at predetermined intervals. A method is proposed for the efficient planning of inspection intervals in which, when the risk of static fracture by fatigue reaches a prescribed upper limit, an inspection is carried out. The risk of failure is reduced at this stage to the same value as the risk of failure with continuous inspection, but it rises as the life continues until it again reaches the prescribed risk limit when a second inspection is carried out.

Repeated application of this process ensures that each inspection is equally effective in maintaining the risk of failure below a prescribed upper limit. The application of the procedure is shown in a subsequent section, and the expression for the risk function is presented in the appendix.

## CRACK DETECTION RATE

It is important to determine the probability of cracks being detected at each inspection since this gives the fraction of the fleet that can be expected to require repair and modification before continuing in service.

Reference to the model of the fatigue process in figure 5 shows that in the first inspection at life $n_{I(1)}$ all structures with crack lengths between $l = l_D$ and $l = 1$ are eliminated. These correspond to structures which have values of $z$ between $z = \frac{n_{I(1)}}{\tilde{n}_D}$

and $z = \frac{n_{I(1)}}{\tilde{n}_F}$. Hence the fraction of the population in which cracks are expected to be revealed at the first inspection is given by

$$r_D^*\left(n_{I(1)};l_D\right) = \int_{n_{I(1)}/\tilde{n}_F}^{n_{I(1)}/\tilde{n}_D} p(z)\ dz \tag{37}$$

Or in general for the mth inspection, the probability of cracks being detected in a structure is given by

$$r_D^*\left(n_{I(m)};l_D,n_{I(m-1)}\right) = \int_{n_{I(m-1)}/\tilde{n}_D}^{n_{I(m)}/\tilde{n}_D} p(z)\ dz \tag{38}$$

where $r_D^*\left(n_{I(m)};l_D,n_{I(m-1)}\right)$ denotes the probability of finding cracks at the mth inspection at life $m_{I(m)}$ following the previous inspection at life $n_{I(m-1)}$. It is assumed that cracks with a length greater than $l_D$ will be detected and that structures in which cracks have been detected will be repaired and returned to service.

For structures with initial crack lengths $l = l_o$ it can be seen by reference to figure 6 that the probability of detecting cracks is

$$r_D^*\left(h_{I(m)}\mid l_o;l_D,h_{I(m-1)}\right) = \int_{h_{I(m-1)}/\left(\tilde{n}_D-\tilde{n}_o\right)}^{h_{I(m)}/\left(\tilde{n}_D-\tilde{n}_o\right)} p(z)\ dz \tag{39}$$

where, with a similar notation as for equation (38), $r_D^*\left(h_{I(m)}\mid l_o;l_D,h_{I(m-1)}\right)$ denotes the probability of detection at the mth inspection after a period of operation in service of $h_{I(m)}$, following a previous inspection at $h_{I(m-1)}$. It is again assumed that all cracks with a length exceeding $l_D$ will be detected and $\tilde{n}_D$ and $\tilde{n}_o$ denote the lives on the median crack propagation curve corresponding to crack lengths of $l_D$ and $l_o$.

If the population of structures has a continuous distribution $p(l_c)$ of initial crack lengths between $l_c = 0$ and $l_c = l_o$ the probability of detection can be derived from equation (39) by integrating over the initial crack lengths from $l_c = 0$ to $l_c = l_o$,

$$r_D^*\left(h_{I(m)}\mid p(l_c);l_D,h_{I(m-1)}\right) = \int_0^{l_o}\int_{h_{I(m-1)}/\left(\tilde{n}_D-\tilde{n}_c\right)}^{h_{I(m)}/\left(\tilde{n}_D-\tilde{n}_c\right)} p(z)\,p(l_c)\ dz\ dl_c$$

or

$$r_D{}^*\left(h_{I(m)} \mid p(l_c); l_D, h_{I(m-1)}\right) = \int_{\tilde{n}_i}^{\tilde{n}_o} \int_{h_{I(m-1)}/\left(\tilde{n}_D - \tilde{n}_c\right)}^{h_{I(m)}/\left(\tilde{n}_D - \tilde{n}_c\right)} p(z)\, p\left(\tilde{n}_c\right) dz\, d\tilde{n}_c \qquad (40)$$

expressing $l_c$ in terms of the corresponding life $\tilde{n}_c$ according to the median crack propagation curve, and integrating with $\tilde{n}_c = n_i$ at $l_c = 0$ and $\tilde{n}_c = \tilde{n}_o$ at $l_c = l_o$.

## APPLICATION

The foregoing theory has been applied to demonstrate the effect of planned inspection procedures for the case of a high-strength steel structure under a manoeuvre load spectrum (spectrum I in fig. 2) which has been considered previously.

The risk function for fatigue failure with continuous inspection has been calculated by using numerical analysis procedures (ref. 6) for the three cases of structures without initial cracks, structures with initial cracks of constant length $l_o$, and structures with a distribution of initial crack sizes given by the probability density function $p(l_c)$. The risk functions for periodic inspection with limited risk have been calculated for the same three cases. The results have been plotted in figures 12, 7, and 9, respectively, and the corresponding survivorship functions are shown in figures 13, 8, and 10. The inspection intervals for inspection with limited risk for each of the three cases are shown in table I together with the expected detection rate at each inspection which has been calculated according to the procedure developed in the preceding section.

With periodic inspection, the risk function returns to the continuous inspection curve at each inspection. The continuous inspection curve therefore has a basic significance since it indicates the maximum extent to which the risk of failure can be reduced by inspection.

## DISCUSSION OF RESULTS

Consider the results of applying the foregoing theory to the case of the high-strength steel structure described previously with particular reference to the suitability of the fail-safe and safe-life procedures.

### RISK OF FATIGUE FAILURE

Reference to the risk functions $r_{sL}$ and $r_F$ in figure 14 illustrates the difficulty with the conventional approach. As the life extends, the difference in these two risks becomes considerable, although as was stated in the section "Interpretation of Fatigue

Test Results" they merely represent two rather extreme conditions in the application of the conventional safe-life approach.

In fact, the risks $r_{sL}$ and $r_F$ differ only in the point on the crack growth curve at which failure is taken to occur. This difference introduces a problem in the interpretation of the fatigue test result since the structure under a representative test load sequence may well fail at a rather different stage of the crack propagation curve as compared with the structures that happen to fail at a relatively short fatigue life in service.

This can be seen by reference to the curves of the probability distribution of the failing load in figure 15. These show that at lives typical of service operation (n = 1.0 to 1.25), the expected value of the failing load, for the few structures that fail, is relatively high, being above the limit load, whereas at longer lives the expected value of the failing load is considerably reduced. Therefore the fatigue test specimen, representing the average structure, is likely to fail at loads considerably below those at which service failures will occur.

The basic difficulty is that neither $r_{sL}$ nor $r_F$ represents the true situation in that they do not take account of the fact that there is some probability of failure at all points along the crack propagation curve as the fatigue crack extends. This effect (the risk of static fracture) is taken account of by $L_{s,\mu}(n)$ which, as can be seen in figures 3 and 4, gives an increased probability of failure for the example taken.

Another effect of considerable importance in considering static fracture due to fatigue is the variability in residual static strength of cracked structures since this may have a significant effect on the probability of failure (or survival) depending on the severity of the loading spectrum. This is shown by the comparison between $L_{s,\mu}$ and $L_s$ for the two load spectra as shown in figures 3 and 4. The probability of survival $L_s$ calculates the increasing risk of failure as the fatigue crack extends in the same way as $L_{s,\mu}$ but it also includes the effect of the variability in residual static strength.

The probability of survival $L_s$ can be applied with equal validity to calculate the probability of survival for structures with initial cracks as outlined in the section "Risk of Failure in Structures Initially Cracked." This has been done for example of the high-strength steel structure taken previously and the results for two cases of initial cracking are shown in figures 8 and 10 where it will be noted that, for an equivalent probability of survival, the fatigue life is greatly reduced by the presence of initial cracks. The shortcomings of the conventional methods of life calculation are more marked in this case, since for all structures the whole of the service life involves the propagation of a fatigue crack with continual exposure to the progressively increasing risk of static fracture due to fatigue.

## PROBABILITY DISTRIBUTION OF THE FAILING LOAD

Curves showing the probability distribution of the collapse load for static fracture by fatigue for the high-tensile steel structure are shown at a series of lives in figure 15. In the early stages of the life when only small cracks are present the majority of the structures that fail do so from occurrence of a high load in excess of the design limit load. At longer lives, however, when a large percentage of the fleet has developed more extensive fatigue cracks, failure tends to take place by the occurrence of the much more frequent lower loads. The curves for the probability distribution of the failing load have a well defined "knee" which marks the transition from failures of structures with low static strength properties (according to the Weibull distribution of relative strength which has a lower limit at $x = 0.82$) to structures with low fatigue strength and hence larger crack lengths at any given life.

With the corresponding curves in figure 11, for all structures with initial cracks of a 0.010-inch depth, this knee does not occur. In this case, at any particular life, all structures have substantial cracks and the extent of these is largely independent of the fatigue strength so that the probability distribution of static strength is the controlling factor for all values of failing load.

## THE EFFECT OF INSPECTION

The effect of inspection on the risk of failure and probability of survival for initially uncracked structures is shown in figures 12 and 13. Although it is not usually a feasible procedure in practice, continuous inspection has an important basic significance which warrants some consideration here.

The risk function for continuous inspection slowly approaches an upper limiting value when there is no repair and replacement of structures in which cracks are detected ("inspection without replacement"). This situation arises because as the initial cracks are propagated by fatigue to the detectable length these structures are eliminated by inspection and a stage is therefore reached where the increase in risk due to the extension of fatigue cracks is offset by the continual removal from service of structures with detectable cracks and high risk of failure.

In the more practical case where structures are repaired and returned to service after detection of cracks ("inspection with replacement") the risk function goes through a maximum value and then eventually approaches zero. The explanation of this behaviour appears to be that, as fatigue cracks extend, the number of cracked structures replaced by sound structures increases until a stage is reached where this counteracts and then outweighs the increasing risk of static fracture by fatigue in the dwindling members of the original fleet.

With this model therefore the original fleet is eventually replaced by new structures which are taken to be free of any fatigue weakness and the risk of fatigue failure decreases to zero. If the service life were to be prolonged to this stage, however, other areas of the structure would become fatigue critical and their risk of failure would have to be considered.

In practice, cracked structures or components are often replaced by new members from the same population as the structures or components in the original fleet. This model of the fatigue process ("inspection with renewal") would show a behaviour intermediate between the two procedures considered above.

The risk functions for continuous inspection of structures with initial cracks are presented in figures 7 and 9 and these show a similar behaviour to that found with initially uncracked structures although for the case of a continuous distribution of initial crack size in figure 9 the peak of the "inspection with replacement" curve is much flatter because of the wider range of crack sizes that results.

Turning now to the practical case of periodic inspections designed to limit the risk of failure below a specified value $r_{max}$, it can be seen from figures 12, 7, and 9 that in all cases the risk of failure fluctuates between the risk for continuous inspection and the specified maximum value $r_{max}$.

For inspection with replacement it can be seen that because of the peak in the curve for the risk function with continuous inspection, the inspection intervals for limited risk at first decrease with each inspection and then increase.

This effect is clearly shown for the three cases considered by the inspection intervals given in table I which also lists the expected fraction of the fleet in which cracks will be detected at each inspection.

The curves showing the corresponding survivorship functions for inspection with limited risk are shown in figures 13, 8, and 10, and it is apparent that inspection for limited risk can give a comparable performance to the ideal case of continuous inspection. At the cost of decreasing the inspection intervals, the probability of survival can be increased by reducing the maximum allowable risk $r_{max}$, although this must always exceed the maximum risk for continuous inspection for the inspection procedure with limited risk to be possible.

## APPLICATION

The reliability approach to structural design has received increasing attention in recent years and it is proposed here that the safety against fatigue of aircraft structures is one of the most important and promising fields of application.

## DEVELOPMENT OF THE RELIABILITY APPROACH TO FATIGUE

Early work on the probabilistic approach to fatigue of aircraft structures was mainly concerned with efforts to establish the fail-safe philosophy on a more quantitative basis by considering the probability of failure of the structure during the crack propagation stage.

One of the first papers on this subject was concerned with the fail-safe operation of transport aircraft (ref. 7), and a similar approach was used subsequently (refs. 8 and 9) in efforts to develop a proposal for ensuring the airworthiness of fail-safe structures.

In references 10 and 11 reliability analysis was applied to derive the probability of failure for a fail-safe structure by using a sophisticated model to represent the effect of multiple redundancies in the structure.

Probably influenced by the successful application of reliability techniques to electronic systems, the reliability approach to structural safety in general received increasing attention and several papers dealing with the basic development of the philosophy (refs. 12 to 15) also dealt at some length with its application to the fatigue of structures.

The reliability approach to structural design has received increasing attention more recently and papers (some relating to the aspect of fatigue) have been represented at a number of International Conferences (refs. 16 to 24).

However, a major difficulty in applying reliability theory to the fatigue of structures is the extensive amount of data required since this is not normally available. The present paper seeks to overcome this difficulty by presenting an approach which allows representative data to be used in conjunction with the full utilisation of the information which can be obtained from the full-scale tests now widely adopted in aircraft design practice.

## RELIABILITY ANALYSIS WITH FULL-SCALE TESTING

The method proposed in this paper calculates the probability of failure of a structure at each stage of the life with data obtained from full-scale tests on the actual structure in conjunction with other representative data. It therefore estimates the risk of failure in the fleet, and hence the probability of failure (or survival) up to any required life, taking account of the flight loads to be encountered, the progressive reduction in strength due to the growing fatigue crack, and the variability in static and fatigue strength.

The inspection or replacement of structures in service can then be planned to achieve a prescribed safety level using basic data from the fatigue test without requiring any arbitrary decision as to the crack length that constitutes failure or as to whether a structure is "fail safe" or not.

## Application of the Method

With the risk function having been calculated, the life $n_I$ to reach the allowable risk $r_{max}(n_I)$ is determined as the life for inspection or replacement.

From the physical nature of the failure as revealed by the fatigue test and the risk function for continuous inspection with the detectable crack length, a judgement can be made whether to rely on inspection or on replacement.

If replacement is decided on all structures are replaced at $n_I$ and the process can be repeated with the constant inspection interval $n_I$ until the probability of survival has been reduced to the minimum allowable value.

If inspection is adopted the inspection intervals are calculated as described in the section "Inspection for Limited Risk" and the process is continued up to the life $n_S$ at which the probability of survival has been reduced to the minimum allowable value. The fraction of defective structures that can be expected to be revealed at each inspection can be calculated from equation (39). Also the probability distribution of the failing load can be calculated and used to estimate the average value of the failing load at the life for any inspection, from which an indication of the average crack length can be obtained.

It is clear from figures 13 and 10 that the safe operating life can be greatly extended by this type of inspection procedure and therefore as the service life continues other fatigue-prone areas of the structure revealed in the fatigue test may need to be included in the analysis in the same way.

## Basic Assumptions

The following basic assumptions are involved:

(a) The service load S is independent of the failing load of the structure R. This assumption infers that any increase in flexibility of the structure as a fatigue crack extends does not affect its response to the applied loads.

(b) There is no correlation between the residual strength of a cracked structure and its fatigue strength. This is supported by the fact that in a complex structure static ultimate load failure usually occurs in a different area and by a different mechanism to fatigue failure.

(c) The relative residual strength $x = \frac{R(l)}{\mu_R(l)}$ of structures cracked to some crack length $l$ has a characteristic probability distribution which applies for any value of $l$. For the monolithic structure considered in the section on page 289, the fracture mechanics relationship $R(l) = K\sqrt{\frac{2}{\pi l}}$ is assumed to apply. It can be shown from this that $R(l)$ has the same probability distribution as the fracture toughness $K$ and it is therefore the same for all crack lengths.

(d) The distribution of fatigue life $N_{l,z}$ at a given crack length $l$ has a log normal distribution. The log normal distribution is often used in making safe-life estimates and it has been supported as a good approximation by comprehensive surveys of fatigue test data (refs. 25 and 26).

(e) At all points on the crack propagation curve of any structure, the fatigue life $N_{l,z}$ bears a constant ratio to the median life $\tilde{N}_l$ at the same crack length $\frac{N_{l,z}}{\tilde{N}_l} = z$. It can be shown that this follows from the properties of the log normal distribution of fatigue life assumed in assumption (d).

(f) As structures fail by fatigue and are thus eliminated from the population there is no change in shape of the probability density functions of fatigue life $z$, relative strength $x$, or initial crack length $l_c$. In practice some distortion of these functions will occur but for the small probabilities of failure considered it is regarded as a reasonable assumption.

## Input Data

The following data are required:

(a) The service load spectrum $F_s(s)$ which can usually be estimated from the considerable body of flight load data available.

(b) The mean value of the ultimate failing load $\mu_o$ which can usually be obtained from the results of static strength tests on the structure.

(c) The probability distribution of relative strength $x = \frac{R(l)}{\mu_R(l)}$ which must be estimated from representative data (as was done for the case of the high-strength steel structure by using data from high-tensile steel specimens) and the results from component testing during the design stage.

(d) The median crack propagation curve for the structure $\tilde{l}_n = g(\tilde{n}_l)$; it is proposed to rely on the crack propagation curve obtained in the full-scale fatigue test of the structure.

## CONCLUDING REMARKS

From a reliability analysis of the fatigue failure in aircraft structures under service loading conditions it is concluded that the current procedures for obtaining safety are not entirely adequate. These methods do not take full account of the probability of failure of the structure during the period in which it is being progressively weakened by the growing fatigue crack and they are therefore subject to inaccuracies which may be significant depending on the structural design parameters and the service conditions.

It is also concluded that a reliability approach to the safety in fatigue of aircraft structures must be considered, using the results available from the structural tests and design analysis in conjunction with other representative data.

Such an approach is quite feasible although an extensive body of data and a number of assumptions are involved which warrant some development and testing of the procedure in practice.

However, the reliability approach has major potential advantages by enabling the safety of both safe-life and fail-safe structures to be determined on a quantitative basis, including the planning of efficient inspection procedures and allowance for the possibility of initial flaws in the material where appropriate.

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## APPENDIX

## TABULATION OF RISKS OF FAILURE AND PROBABILITY OF CRACK DETECTION

For simplicity the risk functions in the body of the paper have been expressed in terms of the dimensionless variate z and they have been compared on a common basis in the various figures using the dimensionless variate $N_S/\tilde{N}_i$. However, in this appendix they are expressed in a form more suitable for practical application, the risk of failure per hour using the relation:

$$r(N_S)\ dN_S = r(z)\ dz$$

$$r(N_S) = r(z) \frac{dz}{dN_S}$$

where $N_S$ is the service life in hours.

If the risk of failure were to be required in units other than hours – such as load applications, for example – the dimensional variable $N_S$ (or for cracked structures $H_S$) would have to be expressed in those units.

The footnotes for this appendix are included at the end of the appendix.

### STRUCTURES WITH NO INITIAL CRACKS

#### No Inspection

Risk with safe-life analysis.- Risk of failure per hour at $N_S$ hours, based on an estimated mean life $\tilde{N}_L$ determined from a fatigue test as the life to some crack length L at which failure occurred, is given by

$$r_L(N_S) = \frac{\frac{1}{\tilde{N}_L} p_z\left(\frac{n_S}{\tilde{n}_L}\right)}{\int_{n_S/\tilde{n}_L}^{\infty} p_z(z)\ dz} \tag{A1}$$

where $\tilde{N}_L$ is the estimated mean life to the crack length L expressed in hours.

Risk of fatigue fracture[a].- Risk of failure per hour by fatigue fracture at a life of $N_S$ hours can be given by

$$r_F(N_S) = \frac{\frac{1}{\tilde{N}_F} p_z\left(\frac{n_S}{\tilde{n}_F}\right)}{\int_{n_S/\tilde{n}_F}^{\infty} p_z(z)\, dz} \tag{A2}$$

where $\tilde{N}_F$ is the median of the life in hours to complete collapse under the mean load.

Risk of static fracture due to fatigue.- Risk of failure per hour by static fracture due to fatigue at a life of $N_S$ hours is given by

$$r_S(N_S) = \int_{x=0}^{x=\infty} \int_{z=n_S/\tilde{n}_F}^{z=n_S} F_S\left\{x\mu_O\phi\left(g\left[\frac{n_S}{z}\right]\right)\right\} p(z)\, p(x)\, dz\, dx \tag{A3}$$

where $F_S(s)$ denotes here the probability of exceeding a service load $s$ per hour of operation[b].

Probability distribution of the failing load.-

$$\Pr\left\{\begin{matrix}\text{At life } N_S \text{ hours that the loads causing} \\ \text{static fracture due to fatigue} \leqq \mu_O x_O\end{matrix}\right\}$$

$$= \int_{x=n_S/\tilde{n}_F}^{z=n_S} \int_{x=0}^{x=x_O\Big/\phi\left\{g\left[\frac{n_S}{z}\right]\right\}} \frac{F_S\left\{x\mu_O\phi\left(g\left[\frac{n_S}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz}{r_S(N_S)} \tag{A4}$$

where $r_S(N_S)$ is given by equation (A3), and $F_S(s)$ is taken as the probability of exceeding a service load $s$ per hour of operation[b].

Periodic Inspection at $N_{I(1)}, N_{I(2)}, \ldots, N_{I(m)}$ Hours

Risk of fatigue fracture with replacement[c d].- Risk of failure per hour by fatigue fracture at a life of $N_S$ hours with structures repaired and returned to service after cracks have been detected is given by

$$r_F^*\left(N_S; l_D, N_{I(m)}\right) = \frac{1}{\tilde{N}_F} p_z\left(\frac{n_S}{\tilde{n}_F}\right) \qquad \left(N_S > N_{I(m)} \frac{\tilde{n}_F}{\tilde{n}_D}\right)$$

$$= 0 \qquad \text{(Otherwise)}$$

where $\tilde{N}_F$ is the median of the life in hours to complete collapse of structures under the mean load.

Note: For continuous inspection the risk of fatigue fracture is zero in this case.

Risk of static fracture due to fatigue with replacement [c].- Risk of failure per hour by static fracture due to fatigue at a life of $N_s$ hours with structures repaired and returned to service after cracks have been detected is given by

$$r_I^*\left(N_s;l_D,N_{I(m)}\right) = \int_{z=n_{I(m)}/\tilde{n}_D}^{z=n_s} \int_{x=0}^{x=\infty} F_s\left\{x\mu_o\phi\left(g\left[\frac{n_s}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz \qquad (A5)$$

where $F_s(s)$ denotes here the probability of exceeding a service load $s$ per hour of operation [b].

Note: For continuous inspection substitute $N_s$ for $N_{I(m)}$ and $n_s$ for $n_{I(m)}$.

Probability of detecting cracked structures with replacement [c].- Probability of detection at the mth inspection with structures repaired and returned to service after cracks have been detected is given by

$$r_D^*\left(N_{I(m)};l_D,N_{I(m-1)}\right) = \int_{n_{I(m-1)}/\tilde{n}_D}^{n_{I(m)}/\tilde{n}_D} p(z)\, dz - \Pr\left\{\text{Fatigue fracture between } N_{I(m-1)} \text{ and } N_{I(m)}\right\}$$

$$= \int_{n_{I(m-1)}/\tilde{n}_D}^{n_{I(m)}/\tilde{n}_D} p(z)\, dz$$

Since it follows that where an inspection procedure is feasible, the probability of fatigue fracture is relatively insignificant compared to the probability of crack detection.

Note: For continuous inspection the probability of detection per hour at any life $N_s$ hours is given by

$$r_D^*(N_s;l_D,N_s) = \frac{1}{\tilde{N}_D}\, p_z\left(\frac{n_s}{\tilde{n}_D}\right)$$

where $\tilde{N}_D$ is the median of the life in hours to the detectable crack length $l_D$.

Probability distribution of the failing load with replacement.-

$$\Pr\left\{\begin{matrix}\text{At life of } N_s \text{ hours following mth inspection, the loads} \\ \text{causing static fracture due to fatigue} \leqq \mu_o x_o\end{matrix}\right\}$$

$$= \int_{z=n_{I(m)}/\tilde{n}_D}^{z=n_s} \int_{x=0}^{x=x_o/\phi\left\{g\left[\frac{n_s}{z}\right]\right\}} \frac{F_s\left\{x\mu_o\phi\left(g\left[\frac{n_s}{z}\right]\right)\right\} p(x)\, p(z)\, dx\, dz}{r_I^*\left(N_s;l_D,N_{I(m)}\right)} \tag{A6}$$

where $r_I^*\left(N_s;l_D,N_{I(m)}\right)$ is given by equation (A5), and $F_s(s)$ is taken as the probability of exceeding a service load $s$ per hour of operation[b].

Note: For continuous inspection substitute $N_s$ for $N_{I(m)}$ and $n_s$ for $n_{I(m)}$.

## STRUCTURES WITH INITIAL CRACKS (PROBABILITY DENSITY OF CRACK LENGTHS $p(l_c)$)

### No Inspection

Risk of fatigue fracture[a].- Risk of failure per hour by fatigue fracture at a service life of $H_s$ hours is given by

$$r_F\left(H_s \mid p(l_c)\right) = \frac{\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \frac{1}{\tilde{N}_F - \tilde{N}_c}\, p_z\left\{\frac{h_s}{\tilde{n}_F - \tilde{n}_c}\right\} p\left(\tilde{n}_c\right)\, d\tilde{n}_c}{\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_s/\tilde{n}_F-\tilde{n}_c}^{z=\infty} p(z)\, p\left(\tilde{n}_c\right)\, dz\, d\tilde{n}_c} \tag{A7}$$

where $\tilde{N}_F$ is the median of the life in hours to complete collapse of initially uncracked structures under the mean load, and $\tilde{N}_c$ is the median of the life in hours to produce a crack of length $l_c$ for initially uncracked structures.

Risk of static fracture due to fatigue.- Risk of failure per hour by static fracture due to fatigue after a service life of $H_s$ hours is given by

$$r_s\left(H_s \mid p(l_c)\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{x=0}^{x=\infty} \int_{z=h_s/\left(\tilde{n}_F-\tilde{n}_c\right)}^{z=\infty} F_s\left\{x\mu_o\phi\left(g\left[\frac{h}{z} + \tilde{n}_c\right]\right)\right\} p(z)\, p(x)\, p\left(\tilde{n}_c\right)\, dz\, dx\, d\tilde{n}_c \tag{A8}$$

where $F_s(s)$ denotes here the probability of exceeding a service load $s$ per hour of operation[b].

Probability distribution of the failing load.-

$$\Pr\left\{\begin{array}{l}\text{At service life } H_s \text{ hours that the loads causing}\\ \text{static fracture due to fatigue} \leq \mu_o x_o\end{array}\right\}$$

$$= \frac{\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_s/(\tilde{n}_F-\tilde{n}_c)}^{z=\infty} \int_{x=0}^{x=x_o/\phi\left(g\left[\frac{h_s}{z}+\tilde{n}_c\right]\right)} F_s\left\{x\mu_o\phi\left(g\left[\frac{h_s}{z} + \tilde{n}_c\right]\right)\right\} p(x)\, p(z)\, p(\tilde{n}_c)\, dx\, dz\, d\tilde{n}_c}{r_s\left(H_s \mid p(l_c)\right)} \qquad \text{(A9)}$$

where $r_s\left(H_s \mid p(l_c)\right)$ is given by equation (A8), and $F_s(s)$ is taken as the probability of exceeding a service load $s$ per hour of operation[b].

Periodic Inspection at $H_{I(1)}, H_{I(2)}, \ldots, H_{I(m)}$ Hours

Risk of fatigue fracture with replacement[e d].- Risk of failure per hour by fatigue fracture after a service life of $H_s$ hours with structures repaired and returned to service after cracks have been detected is given by

$$r_F^*\left(H_s \mid p(l_c); l_D, H_{I(m)}\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\left(h_s\tilde{n}_D - h_{I(m)}\tilde{n}_F\right)/\left(h_s - h_{I(m)}\right)} \frac{1}{\tilde{N}_F - \tilde{N}_c} p_z\left(\frac{h_s}{\tilde{n}_F - \tilde{n}_c}\right) p(\tilde{n}_c)\, d\tilde{n}_c \qquad \left(H_s > H_{I(m)} \frac{\tilde{n}_F - 1}{\tilde{n}_D - 1}\right)$$

$$= 0 \qquad \text{(Otherwise)}$$

where $\tilde{N}_F$ is the median of the life in hours to complete collapse of uncracked structures under the mean load, and $\tilde{N}_c$ is the median of the life in hours to produce a crack of length $l_c$ for initially uncracked structures.

Risk of static fracture due to fatigue with replacement[e].- Risk of failure per hour by static fracture due to fatigue after a service life of $H_s$ hours with structures repaired and returned to service after cracks have been detected is given by

$$r_I^*\left(H_s \mid p(l_c); l_D, H_{I(m)}\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{x=0}^{x=\infty} \int_{z=h_{I(m)}/(\tilde{n}_D-\tilde{n}_c)}^{z=\infty} F_s\left\{x\mu_o\phi\left(g\left[\frac{h_s}{z} + \tilde{n}_c\right]\right)\right\} p(z)\, p(x)\, p(\tilde{n}_c)\, dz\, dx\, d\tilde{n}_c \qquad \text{(A10)}$$

where $F_s(s)$ denotes here the probability of exceeding a service load $s$ per hour of operation[b].

Note: For continuous inspection substitute $H_s$ for $H_{I(m)}$ and $h_s$ for $h_{I(m)}$.

Probability of detecting cracked structures with replacement[e].- Probability of detecting cracked structures at the mth inspection with structures repaired and returned to service after cracks have been detected is given by

$$r_D^*\left(H_{I(m)} \mid p(l_c);l_D,H_{I(m-1)}\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_{I(m-1)}/(\tilde{n}_D-\tilde{n}_c)}^{z=h_{I(m)}/(\tilde{n}_D-\tilde{n}_c)} p(z)\, p(\tilde{n}_c)\, dz\, d\tilde{n}_c - \Pr\left\{\begin{matrix}\text{Fatigue fracture between} \\ H_{I(m-1)} \text{ and } H_{I(m)}\end{matrix}\right\}$$

$$= \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_{I(m-1)}/(\tilde{n}_D-\tilde{n}_c)}^{z=h_{I(m)}/(\tilde{n}_D-\tilde{n}_c)} p(z)\, p(\tilde{n}_c)\, dz\, d\tilde{n}_c$$

Since it follows that when an inspection procedure is feasible the probability of fatigue fracture is relatively insignificant compared to the probability of crack detection.

Note: For continuous inspection the probability of detection per hour at any service life $H_s$ is given by

$$r_D^*\left(H_s \mid p(l_c);l_D,H_s\right) = \int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \frac{1}{\tilde{N}_D - \tilde{N}_c}\, p_z\left(\frac{h_s}{\tilde{n}_D - \tilde{n}_c}\right) p(\tilde{n}_c)\, d\tilde{n}_c$$

where $\tilde{N}_D$ and $\tilde{N}_c$ are the median values of the lives in hours to produce crack lengths of $l_D$ and $l_c$, respectively, in initially uncracked structures.

Probability distribution of the failing load with replacement.-

$$\Pr\left\{\begin{matrix}\text{At a service life } H_s \text{ hours following the mth inspection} \\ \text{that the loads causing static fracture due to fatigue} \leq \mu_o x_o\end{matrix}\right\}$$

$$= \frac{\displaystyle\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_{I(m)}/(\tilde{n}_D-\tilde{n}_c)}^{z=\infty} \int_{x=0}^{x=x_o/\phi\left[g\left(\frac{n_s}{z}+\tilde{n}_c\right)\right]} F_s\left\{x\mu_o\phi\left(g\left[\frac{n_s}{z} + \tilde{n}_c\right]\right)\right\} p(x)\, p(z)\, p(\tilde{n}_c)\, dx\, dz\, d\tilde{n}_c}{r_I^*\left(H_s \mid p(l_c);l_D,H_{I(m)}\right)}$$

where $r_I^*\left(H_s \mid p(l_c);l_D,H_{I(m)}\right)$ is given by equation (A10), and $F_s(s)$ is taken as the probability of exceeding a service load $s$ per hour of operation[b].

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[a]The term in the denominator of this expression is a normalising factor resulting from the truncation of the $z$ distribution by the removal from the population of the structures that fail by fatigue fracture. However, it is very close to unity for the probabilities of survival that are acceptable in practice.

[b]In the body of the paper where $r_s(n_s)$ has been compared with other risk functions using the dimensionless variate $N_s/\tilde{N}_i$, $F_s(s)$ has been taken as the probability of exceeding a service load $s$ in a time interval $\tilde{N}_i$.

[c]When there is no replacement of those structures in the fleet in which cracks have been detected, the corresponding probabilities and risk functions are obtained by dividing by the normalising factor $\int_{n_{I(m)}/\tilde{n}_D}^{\infty} p(z)\, dz$. For continuous inspection, $n_{I(m)}$ is replaced by $n_s$.

[d]When an inspection procedure is applied, the effect on the risk function resulting from truncation of the $z$ distribution, by elimination of structures that fail by fatigue fracture, is so small that it has been neglected here.

[e]When there is no replacement of those structures in the fleet in which cracks have been detected the corresponding probabilities and risk functions are obtained by dividing by the factor

$$\int_{\tilde{n}_c=1}^{\tilde{n}_c=\tilde{n}_o} \int_{z=h_{I(m)}/(\tilde{n}_D-\tilde{n}_c)}^{z=\infty} p(z)\, p(\tilde{n}_c)\, dz\, d\tilde{n}_c$$

For continuous inspection $h_{I(m)}$ is replaced by $h_s$.

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## TABLE I

## INSPECTION INTERVALS AND DETECTION RATES WITH INSPECTION FOR LIMITED RISK

| Inspection number, m | Uncracked structures, $a_c = 0.0$, $r_{max} = 0.025$ | | | Structures with initial cracks, $a_c = a_o = 0.010$ in., $r_{max} = 0.05$ | | | Structures with distribution of initial cracks, $p(l_c) = 2.22e^{-20.61c}$, $r_{max} = 0.05$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | $n_{I(m)}$ | $N_{I(m)}$, hr | $r_D(n_{I(m)})$ | $n_{I(m)}$ | $N_{I(m)}$, hr | $r_D(n_{I(m)})$ | $n_{I(m)}$ | $N_{I(m)}$, hr | $r_D(n_{I(m)})$ |
| 1 | 1.44 | 2880 | 0.0005 | 0.17 | 340 | 0.030 | 0.30 | 600 | 0.018 |
| 2 | 1.86 | 3520 | .0055 | .19 | 380 | .032 | .42 | 840 | .055 |
| 3 | | | | .22 | 440 | .076 | .54 | 1080 | .052 |
| 4 | | | | .23 | 460 | .032 | .67 | 1340 | .057 |
| 5 | | | | .26 | 520 | .111 | .80 | 1600 | .046 |
| 6 | | | | .29 | 580 | .122 | | | |
| 7 | | | | .32 | 640 | .120 | | | |
| 8 | | | | .39 | 780 | .224 | | | |
| 9 | | | | .51 | 1020 | .185 | | | |
| 10 | | | | .73 | 1460 | .063 | | | |

$\tilde{N}_i$ — median of life to initial failure, 2000 hr
$N_{I(m)}$ — life in hours to mth inspection
$n_{I(m)}$ — relative life to mth inspection, $N_{I(m)}/\tilde{N}_i$
$r_D(n_{I(m)})$ — probability of detectable cracks at mth inspection
$a_o$ — depth of smallest crack detectable during production
$a_c$ — depth of initial crack in any structure
$l$ — relative crack length (or depth), $a/a_F$

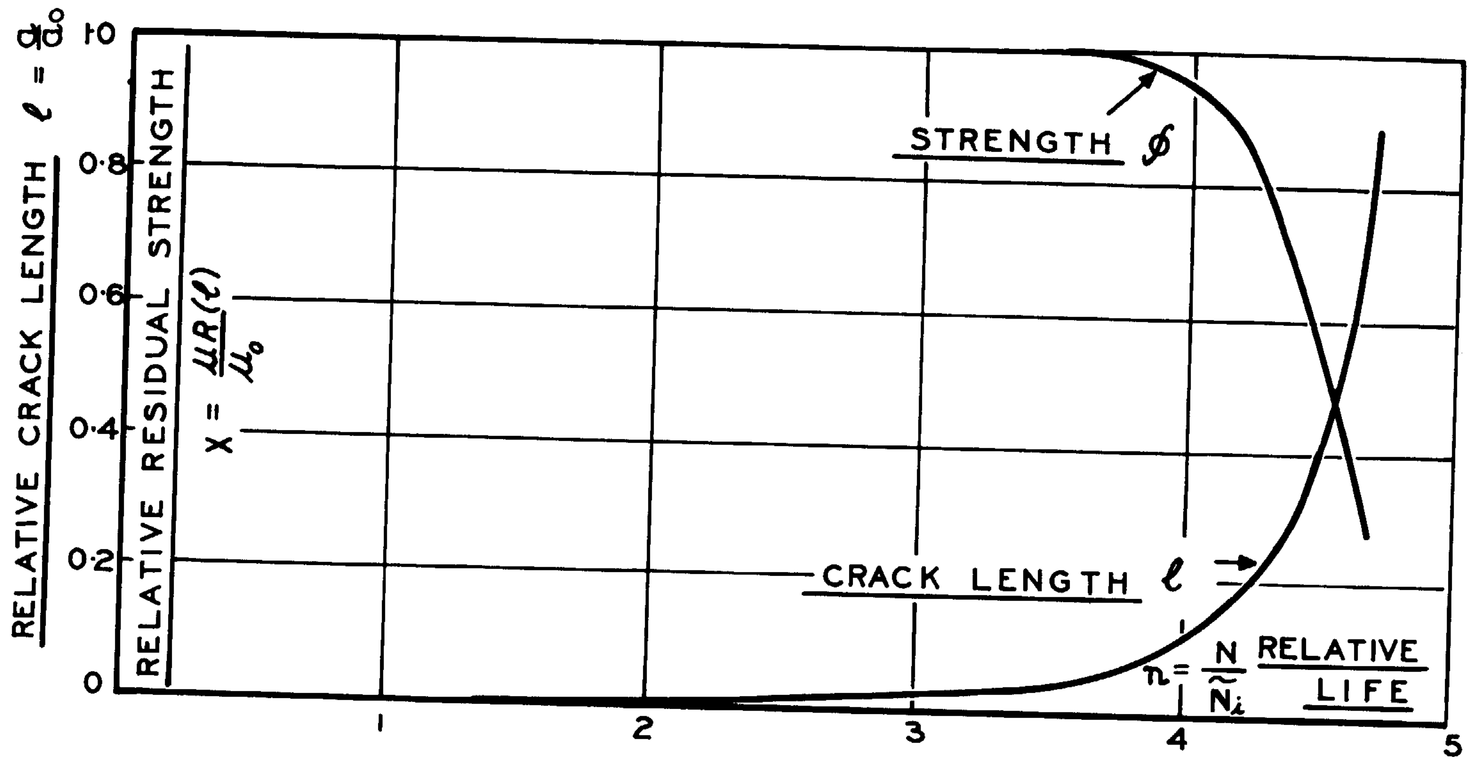


Figure 1.- Fatigue characteristics of high-tensile steel structure.

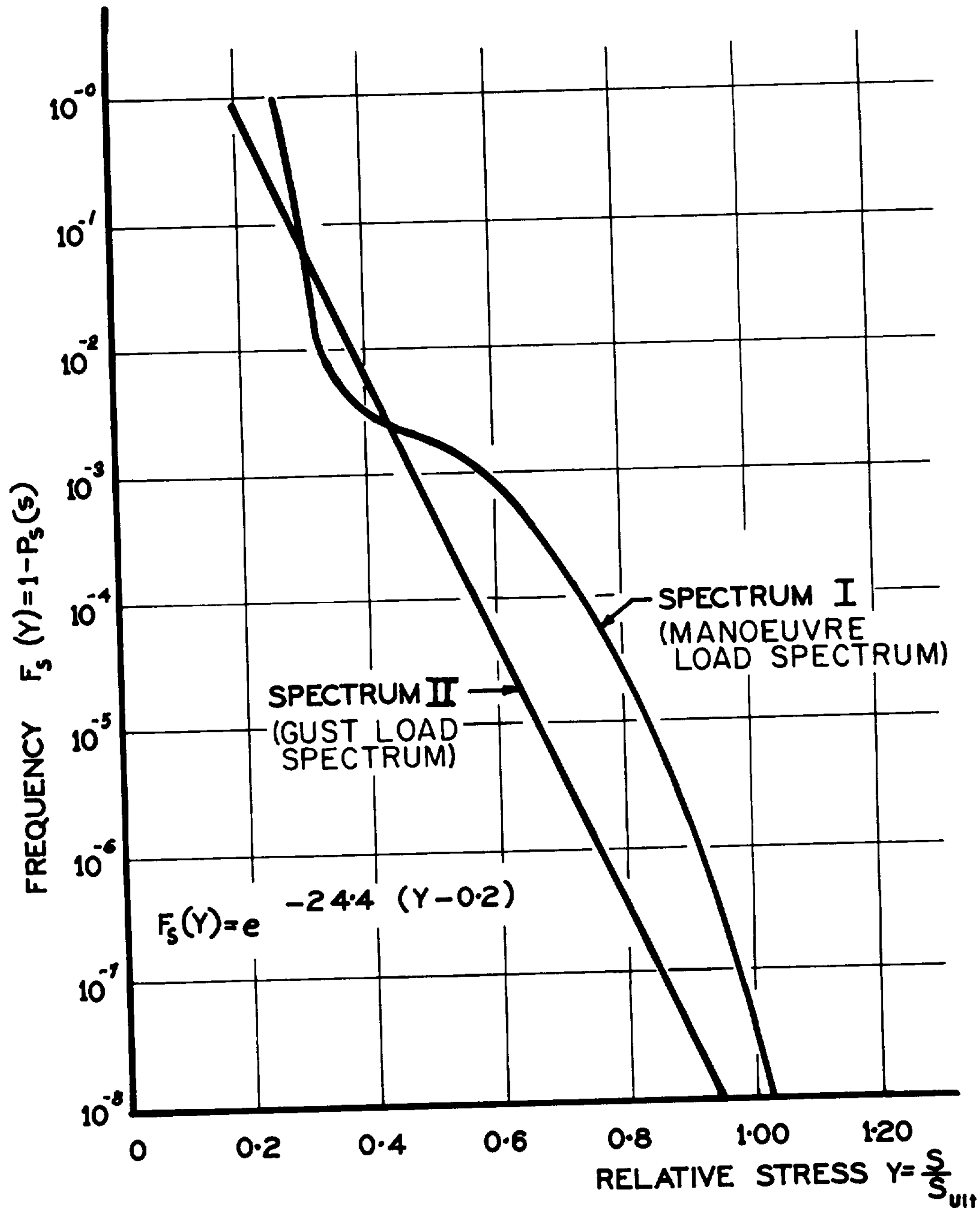


Figure 2.- Load spectra.

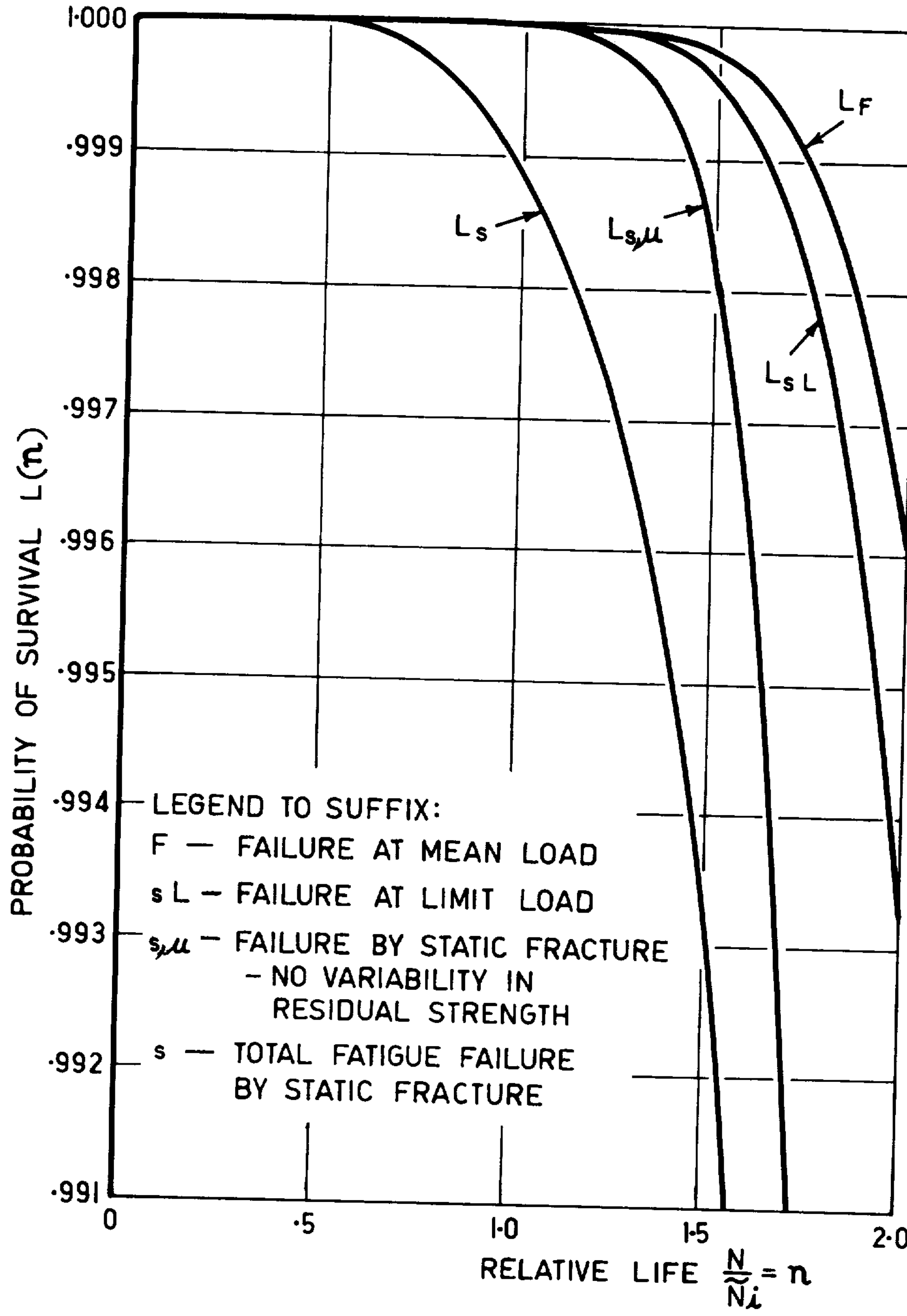


Figure 3.- Probability of survival for spectrum I.

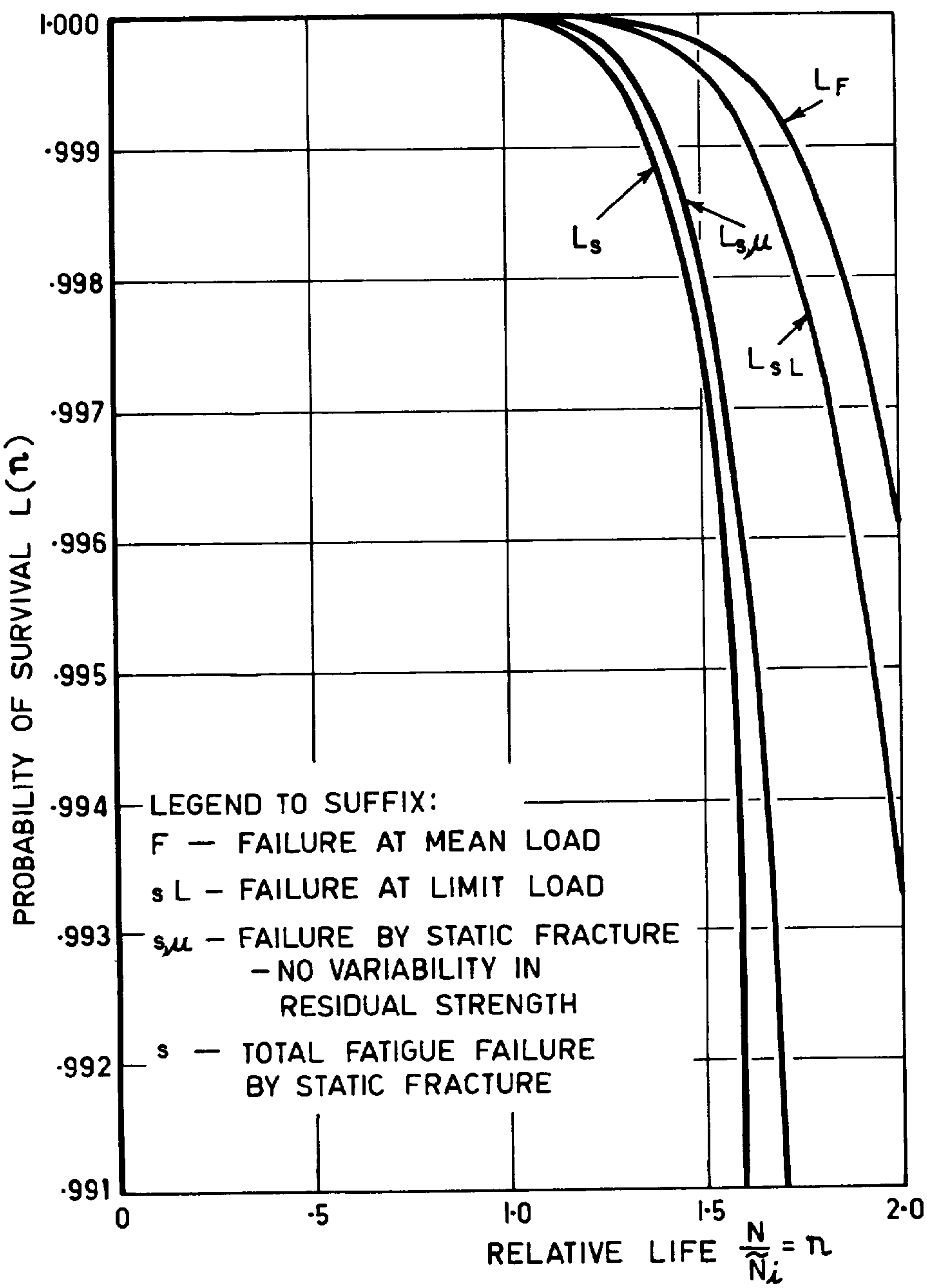


Figure 4.- Probability of survival for spectrum II.

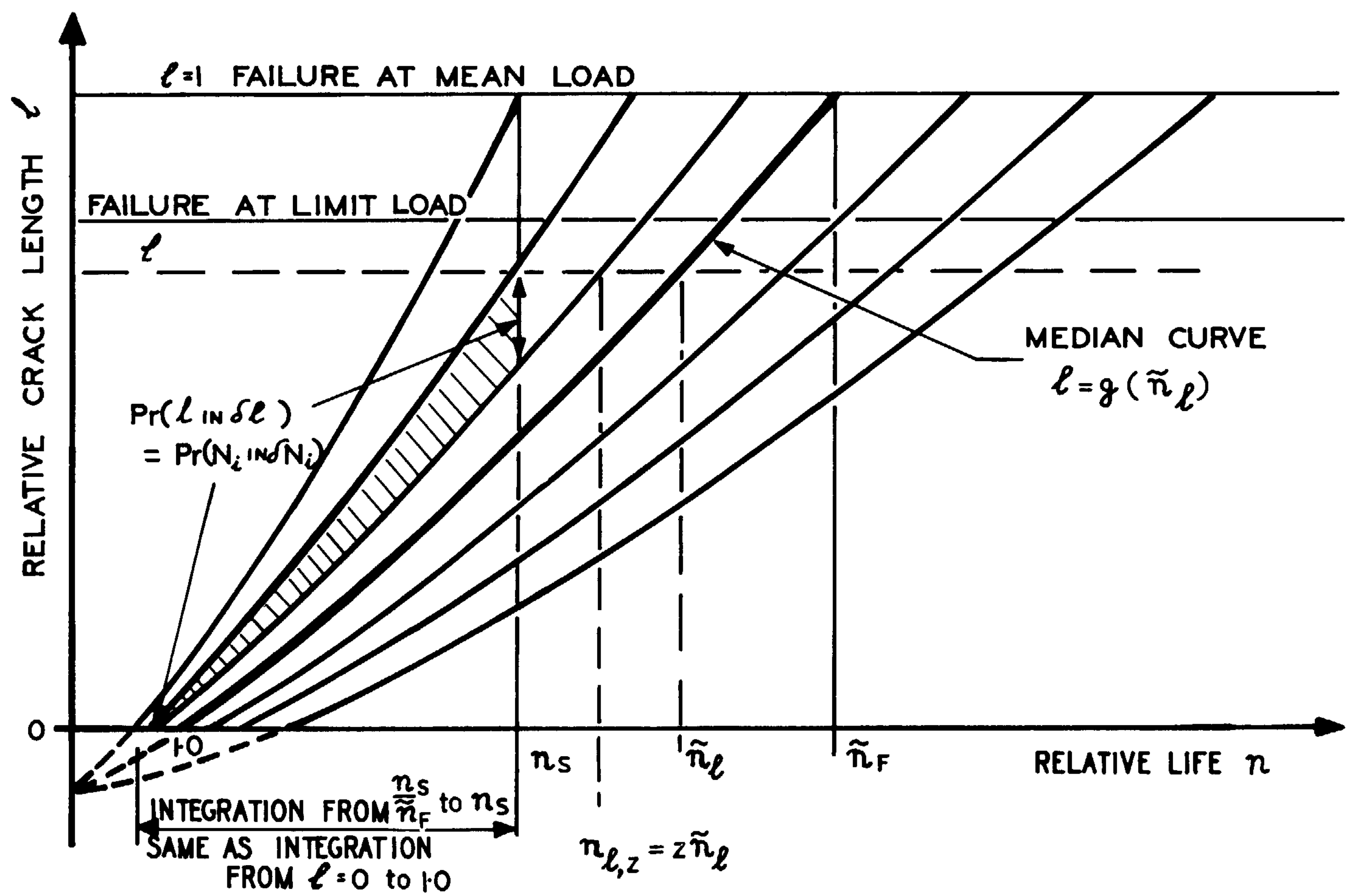


Figure 5.- Calculation of risk of failure due to fatigue.

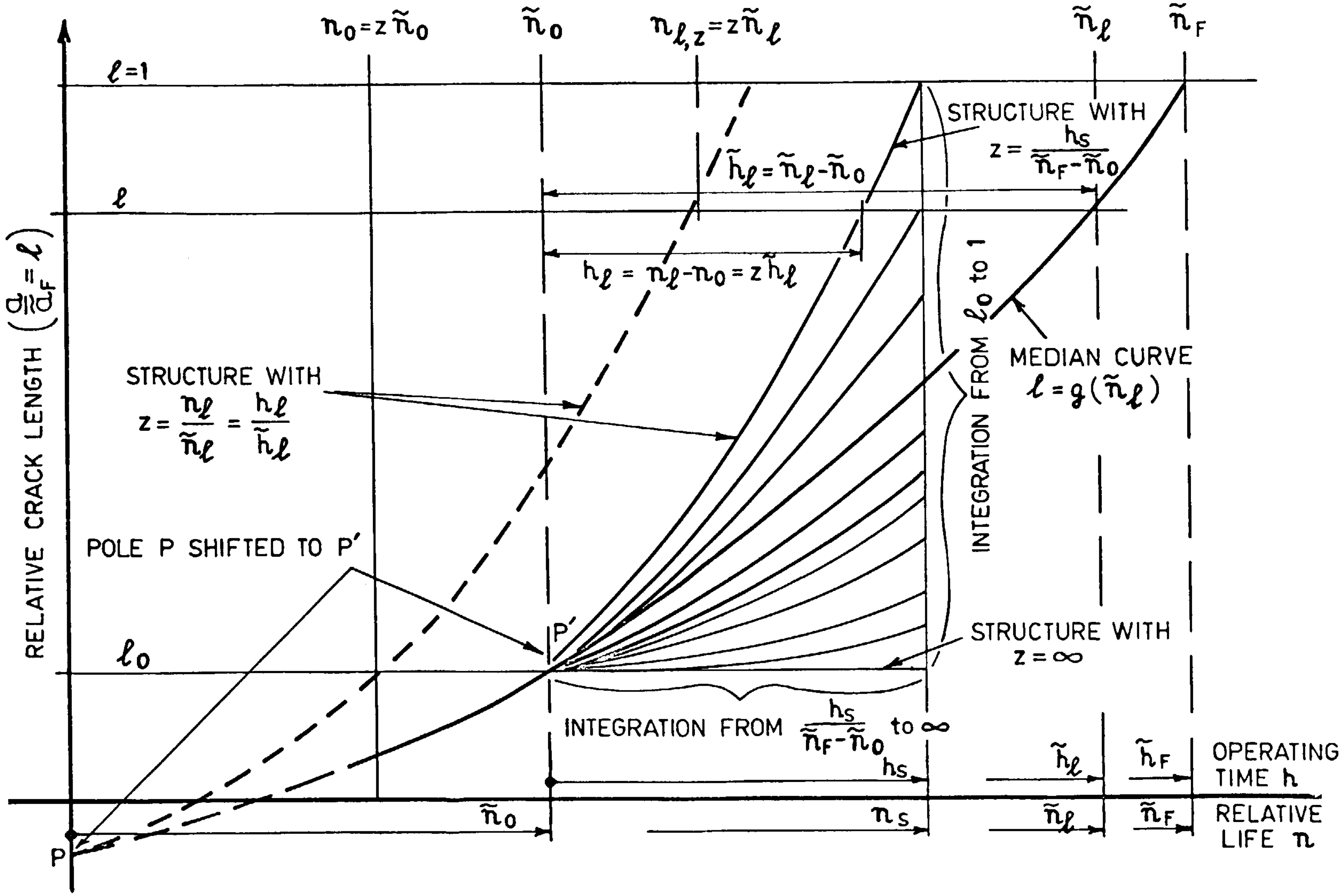


Figure 6.- Risk of fatigue failure in initially cracked structures.

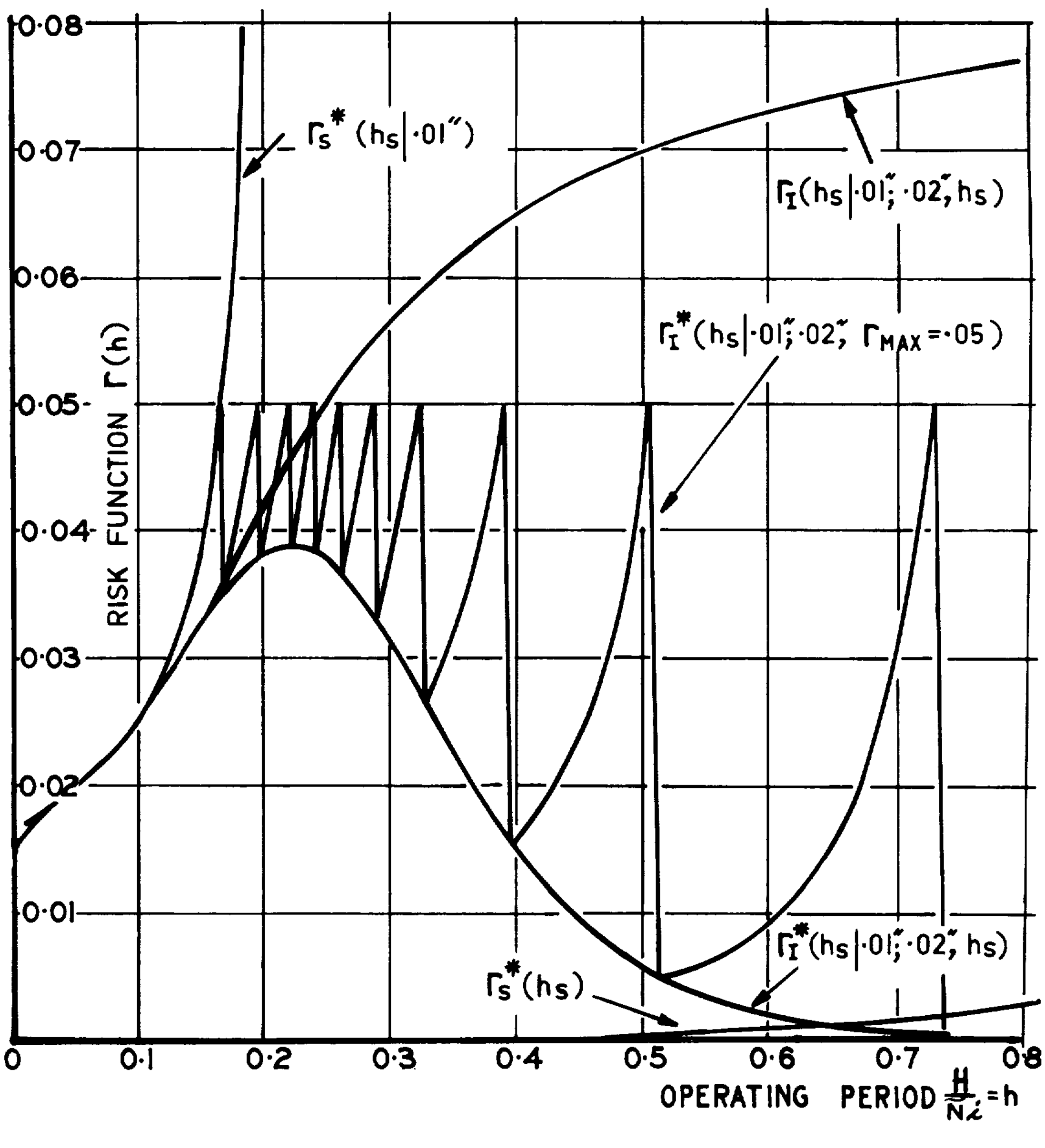


Figure 7.- Risk function for structures with initial crack depth $a_0$ = 0.01 in. for various inspection procedures.

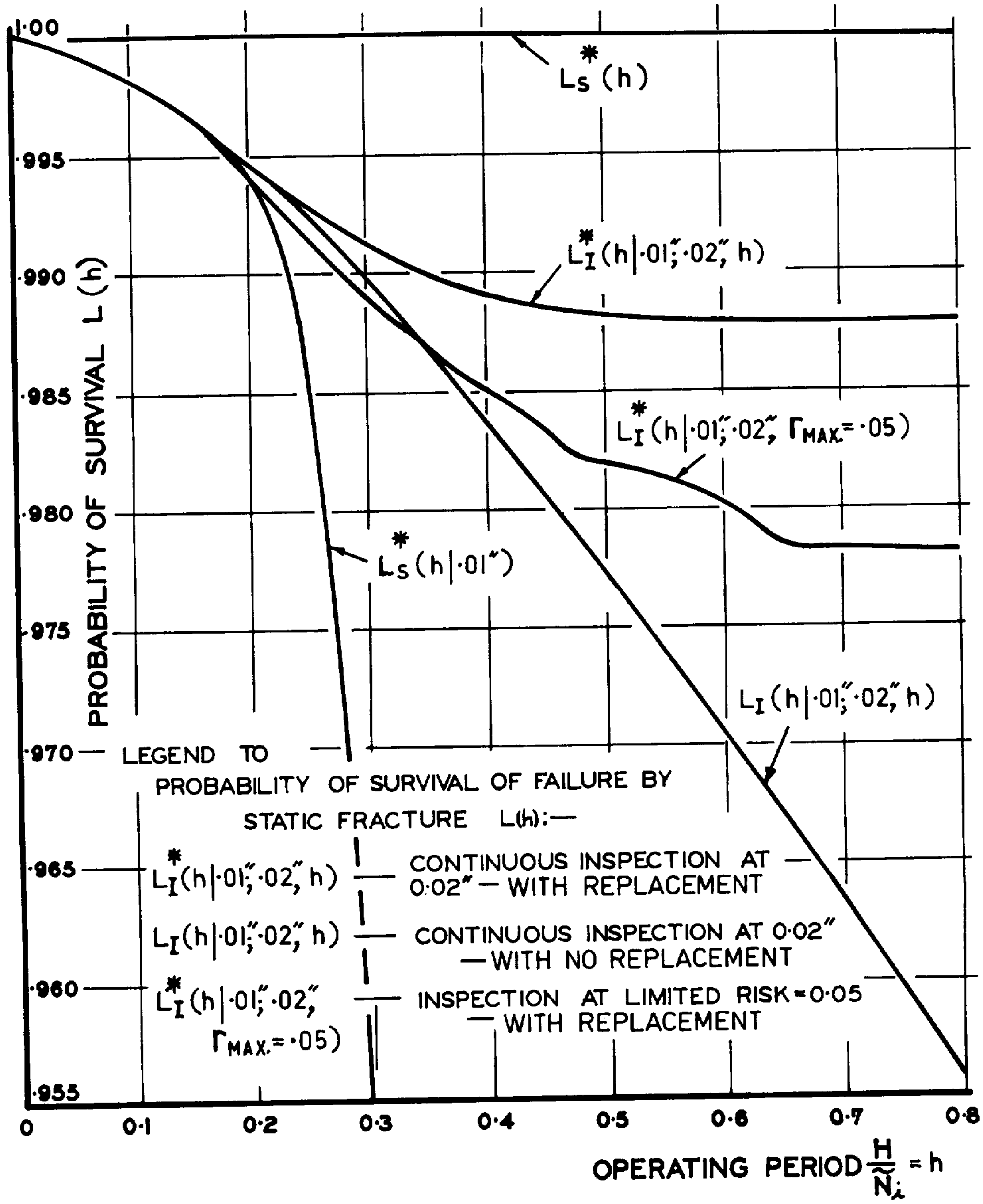


Figure 8.- Probability of survival of structures with initial crack depth $a_0$ = 0.01 in. for various inspection procedures.

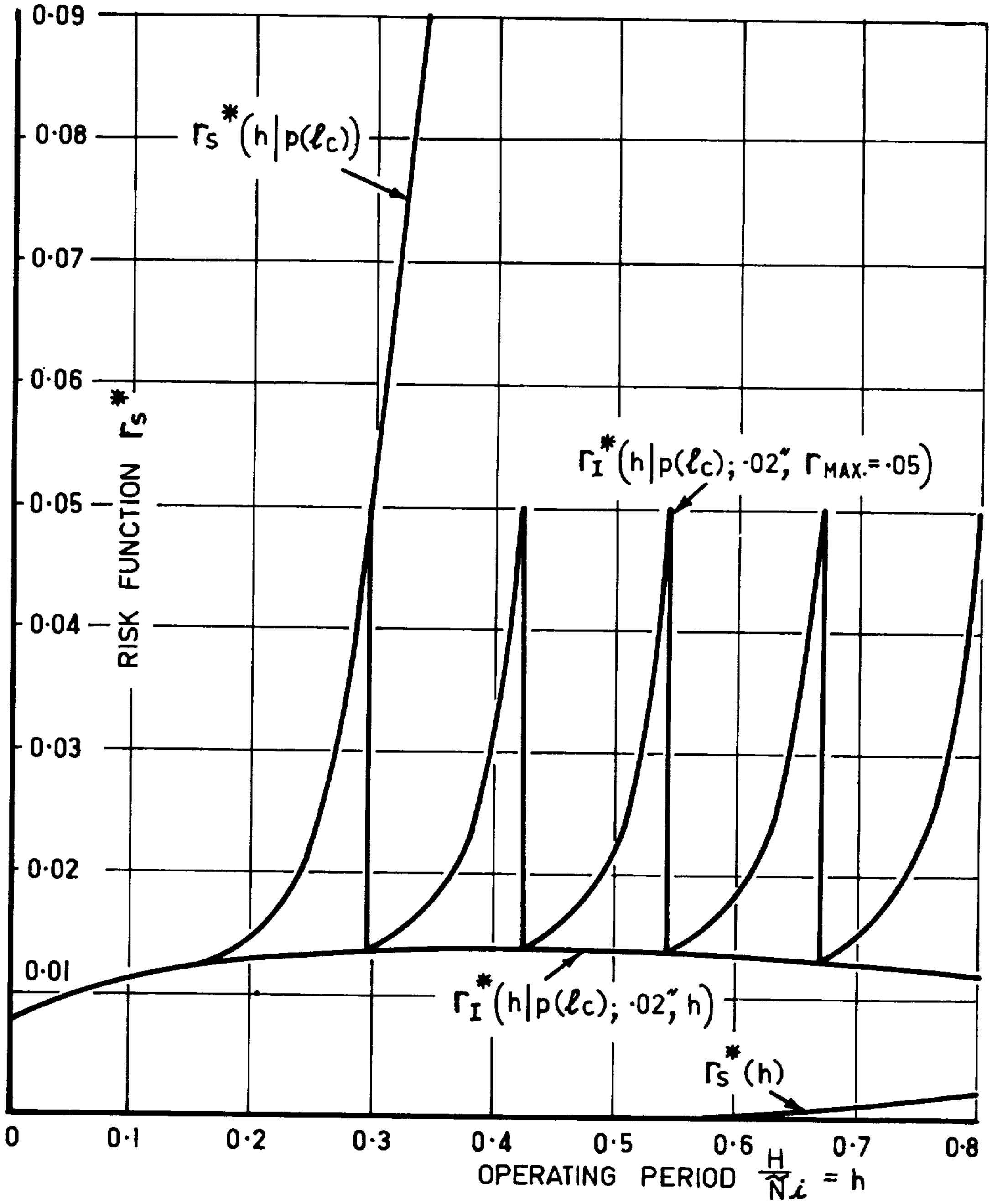


Figure 9.- Risk function for structures with variable initial crack depth $l_c$ for various inspection procedures.

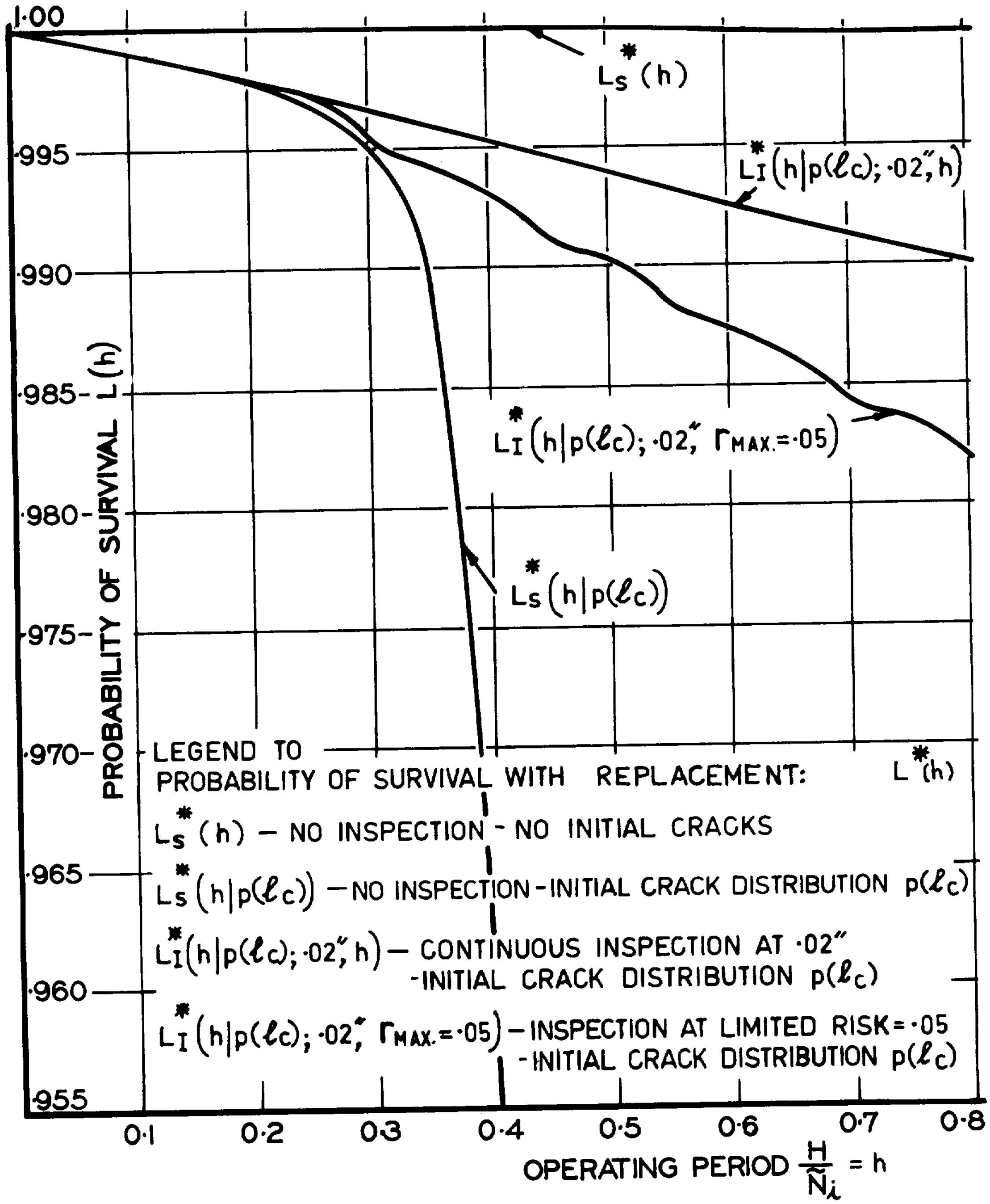


Figure 10.- Probability of survival of structures with variable initial crack depth $l_c$ for various inspection procedures.

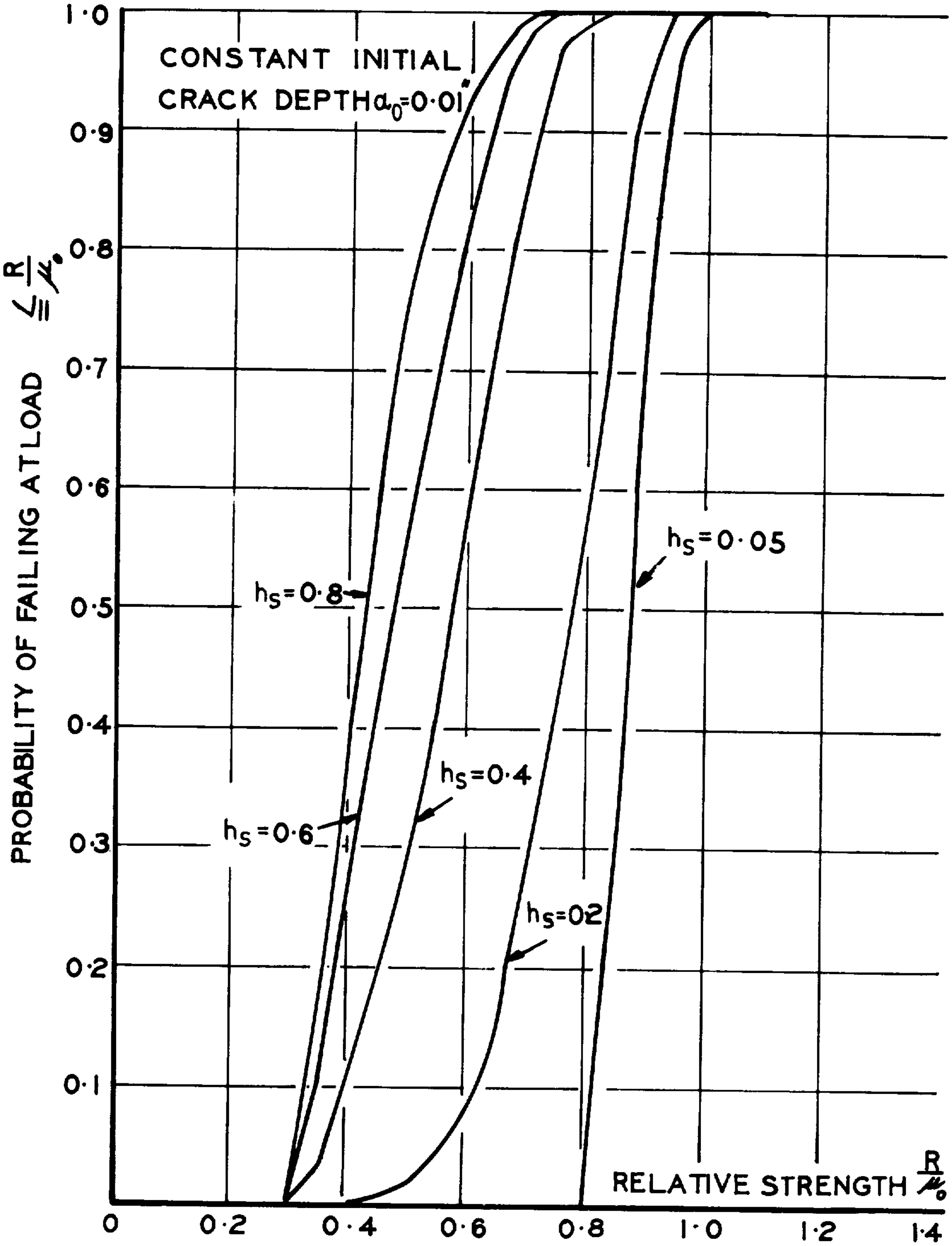


Figure 11.- Probability distribution of the failing load with spectrum I. Cracked structures.

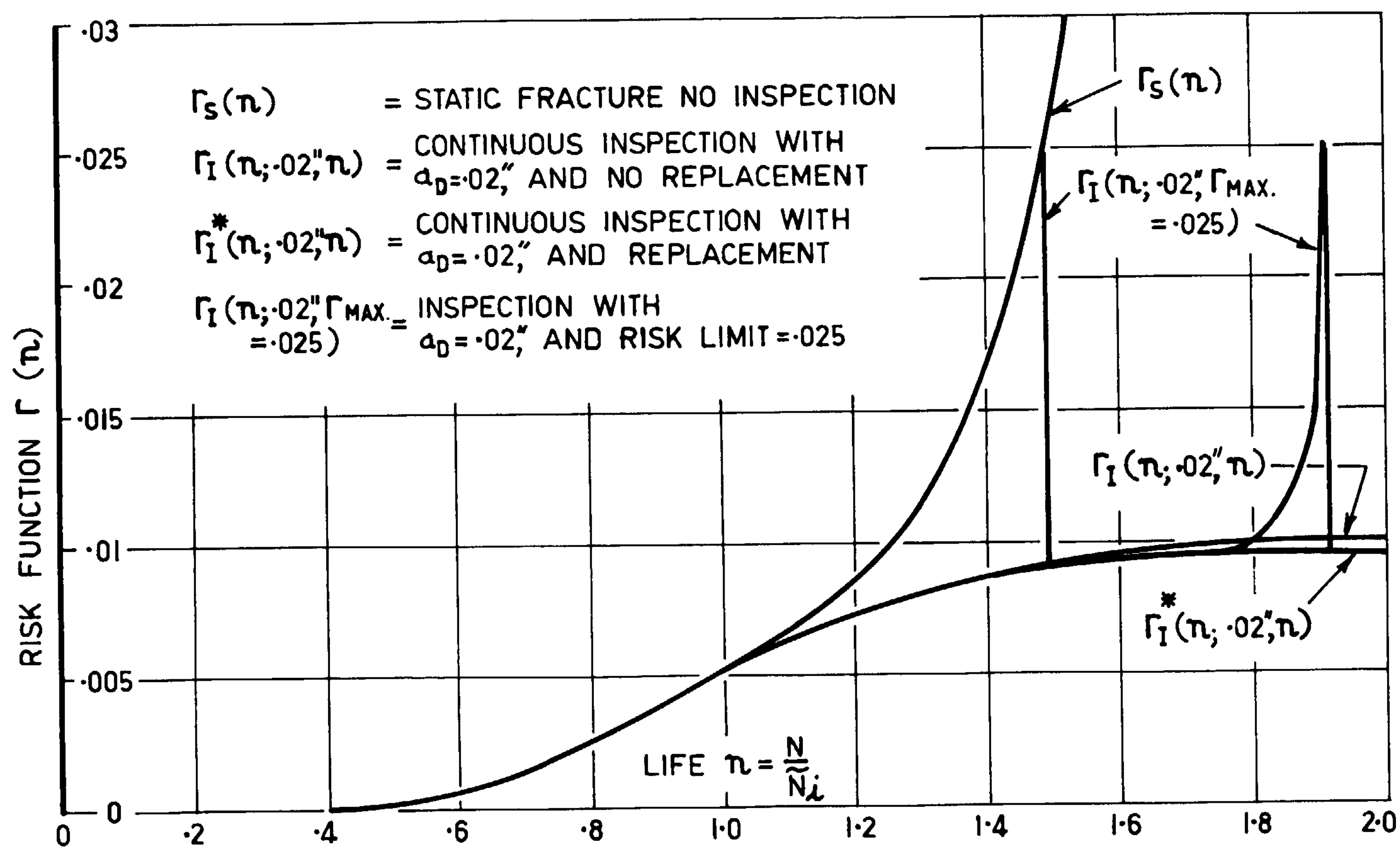


Figure 12.- Risk functions for structures without initial cracks for various inspection procedures.

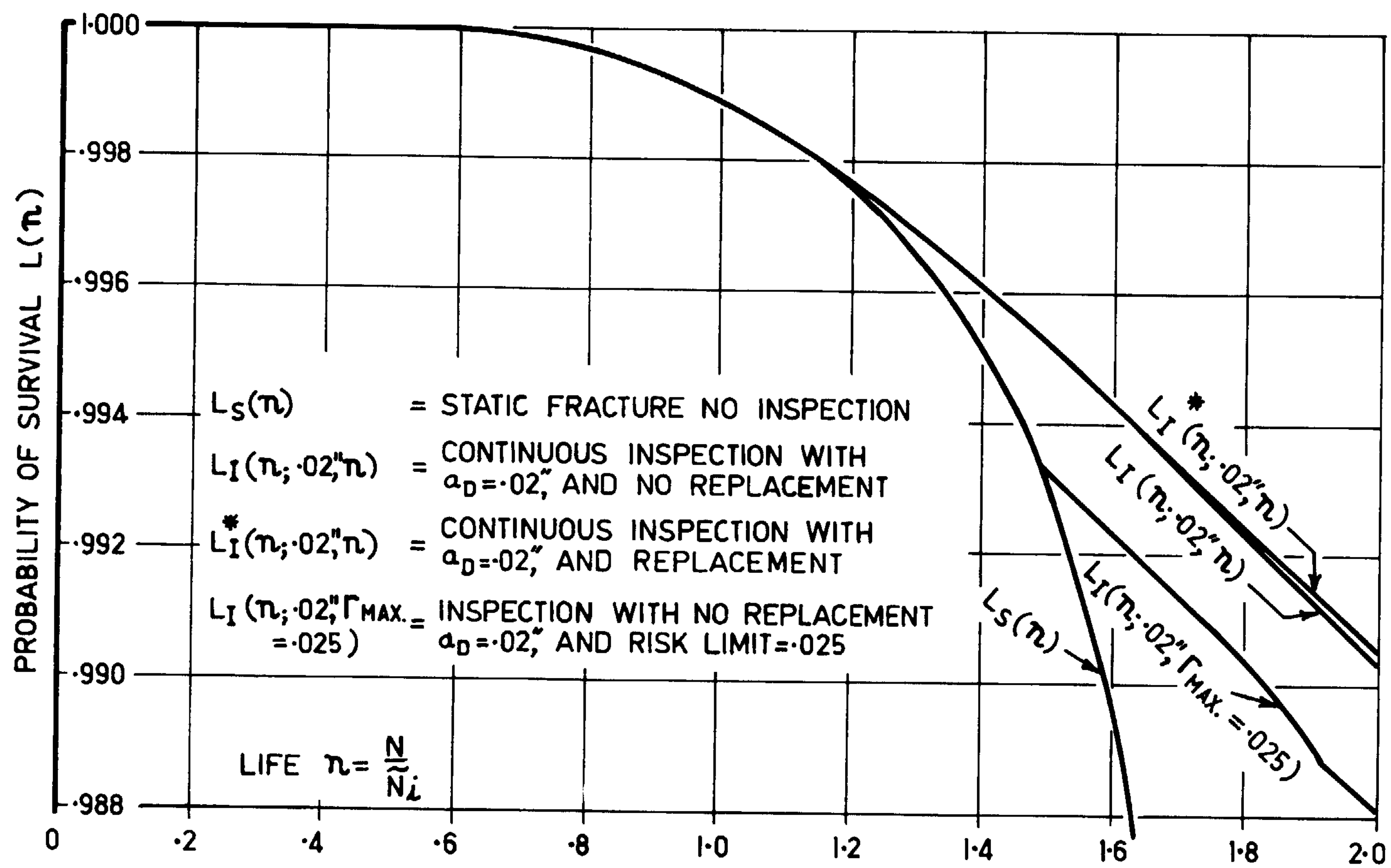


Figure 13.- Survivorship functions for structures without initial cracks for various inspection procedures.

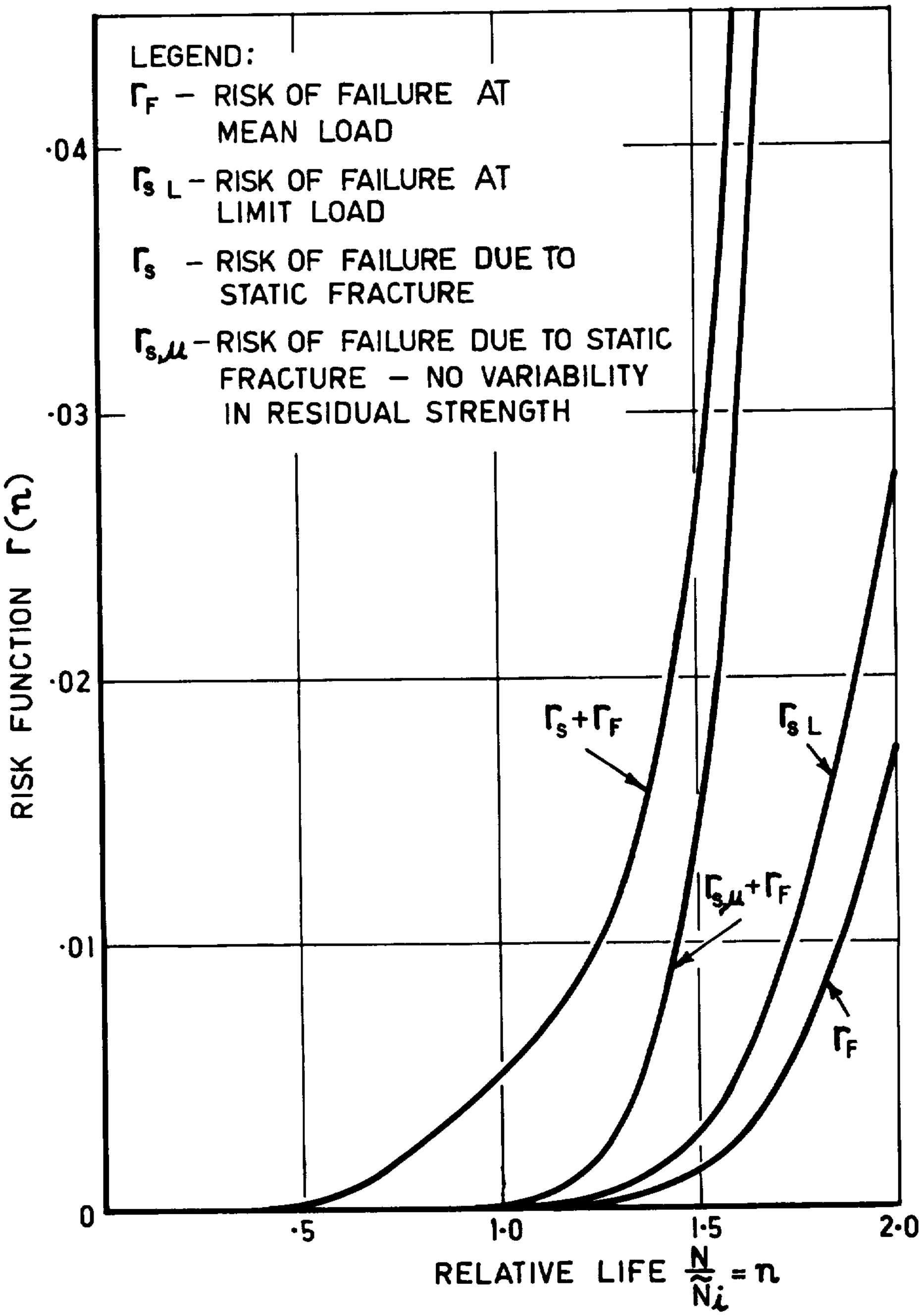


Figure 14.- Risk of failure for spectrum I.

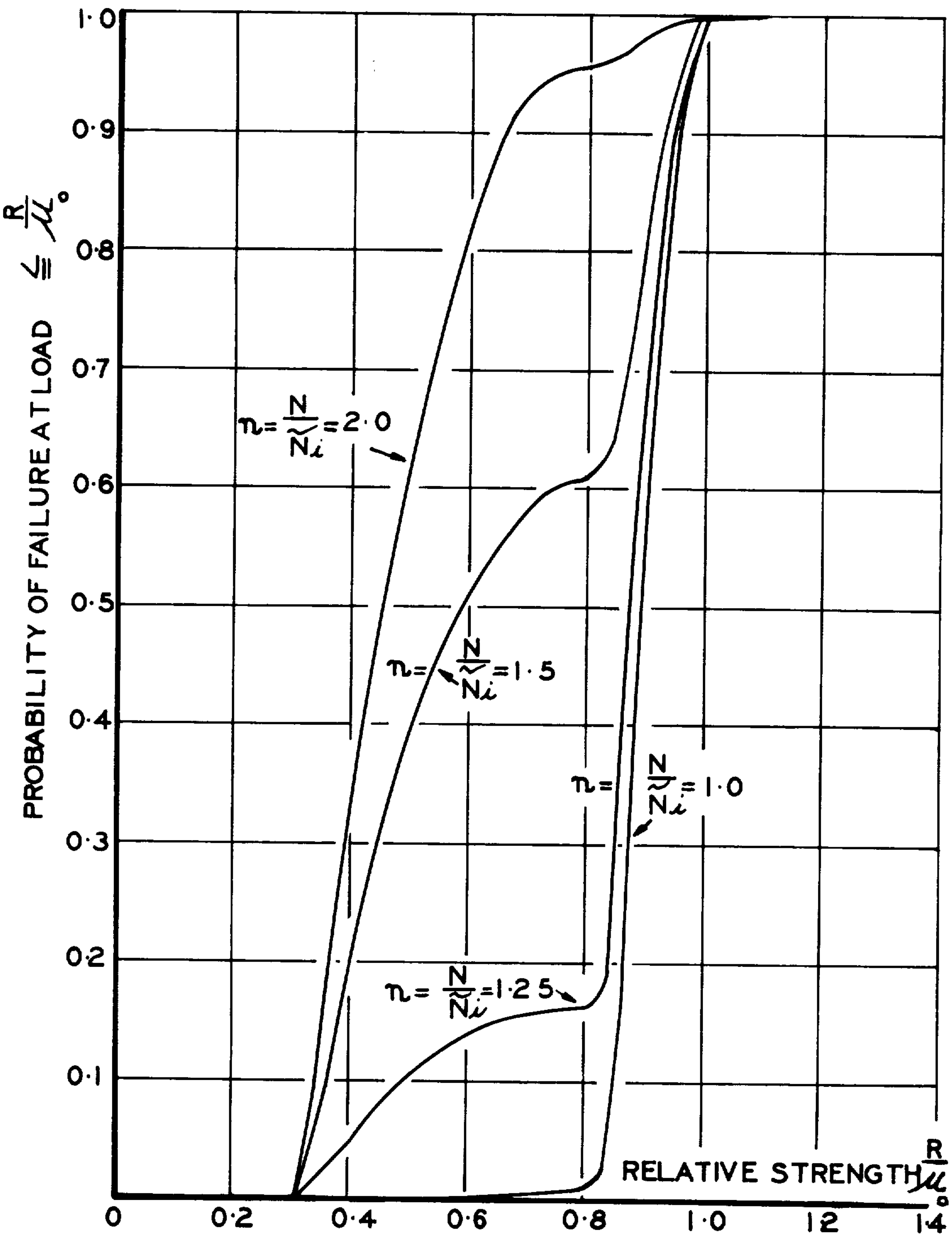


Figure 15.- Probability distribution of failing load with spectrum I. Uncracked structures.