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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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TOPICAL REPORT

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ABSTRACT

This report consists of a series of investigations on problems of the type which arise in the control of switched electrical networks. The main results concern the controllability and observability of these systems. Subsequent work will address itself to the stabilization of these systems, building on the basic theory given here.

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System Theory on Group Manifolds and Coset Spaces*

R. W. Brockett**

Abstract

The purpose of this paper is to study questions regarding controllability, observability, and realization theory for a particular class of systems for which the state space is a differentiable manifold which is simultaneously a group or, more generally, a coset space. We show that it is possible to give · her explicit expressions for the reachable set and the set of indistinguishable states in the case of autonomous systems. We also establish a type of state space isomorphism theorem. These results parallel, and in part specialize to, results available for the familiar case described by $\dot{x}(t) = Ax(t)+Bu(t)$; $y(t)=Cx(t)$. Our objective is to reduce all questions about the system to questions about Lie algebras generated from the coefficient matrices entering in the description of the system and in that way arrive at conditions which are easily visualized and tested.

Contents

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1. Introduction

A standard assumption in modern control theory is that the state space is a vector space. This assumption is both valid and natural in many situations, but there is a significant class of problems for which it cannot be made. Typical of these are certain problems which arise in the control of the attitude of a rigid body. The state space in this case is not a vector space. Linearization often destroys the essence of the problem -- even if one can work locally -- and in any case new and different methods are needed for treating global questions.

In this paper we substitute the following hypothesis for the usual vector space assumptions. We let $\mathcal{F}$ and $\mathcal{C}$ be matrix groups and study

$$\dot{X}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)X(t) \quad ; \quad y(t) = \mathcal{C}X(t) \quad ; \quad X \in \mathcal{F}$$

where A and B_i belong to the Lie algebra associated with $\mathcal{F}$, the u_i are the controls, and the notation $\mathcal{C}X(t)$ is to be interpreted as being a coset in $\mathcal{F}$. We also study vector systems of a similar type whereby we can view their evolution as occuring in a coset space. The results concern the explicit construction of the reachable set and a characterization of observability which is easily tested. Our main point is that this class of systems is in many ways not more difficult than linear systems of the usual type in $\mathbb{R}^n$.

There is a moderately large literature on the use of Chow's results [1] and related ideas to study controllability, including the work of Hermann, Kucera, Hermes, Haynes, and Lobry (see [2-6]). This work is relevant here but we are directly interested in controllability only in-so-far as it contributes to the identification of a framework in which we can study a full range of system-theoretic questions, including observability and realization theory. Notice that it is impossible to pass directly from controllability results to observability results in the present set up because there is no clear notion of duality. The main motivation for this work came from some work on Lie algebraic methods in differential equations (see [7-10]) and, above all, from being confronted with certain physical problems where linear theory was simply inadequate.

Some unpublished work [18,19] by Jurdjevic and Sussmann is related to this paper. In particular they give in [19] an alternative proof of our theorem 5 and make a serious study of the unsymmetric case (treated only superficially in theorem 7 here). We also mention a recent paper by Elliott [20].

2. Examples

We postpone the development of the subject long enough to present a few simple examples which will help justify why the assumptions are set up the way they are.

Example 1: (Control Systems Design) Consider the problem of determining the gain, k in the system

$$\dot{x}(t) = Ax(t) - k(t)Bcx(t)$$

so as to achieve good performance relative to an index of the form

$$\eta = \int_0^\infty x'(t)Mx(t)dt \quad ; \quad M = M' \geq 0$$

If a particular initial state is chosen and k( ) is selected so as to minimize η, then the performance might be bad relative to some other initial state. In cases where the initial state is not known it is much more realistic to pick a collection of initial state vectors and to pick k in such a way as to minimize a weighted average of the individual performances. In fact, just to insure stability it is necessary to average over at least n linearly independent initial states. If exactly n are chosen then k should be regarded as controlling the evolution of the matrix equation

$$\underline{\dot{\Phi}}(t) = (\underline{A}-k(t)\underline{bc})\underline{\Phi}(t) \quad ; \quad \underline{\Phi}(0) = [\underline{x}_1,\underline{x}_2,\ldots,\underline{x}_n]$$

The state space is then the space of nonsingular n by n matrices, $\mathcal{G}\ell(n)$

Example 2: (Rigid body control) The orientation of a rigid body relative to some fixed set of axes is described by a 3 by 3 orthogonal matrix $\underline{A}$ which satisfies the differential equation

$$\begin{bmatrix} \dot{a}_{11}(t) & \dot{a}_{12}(t) & \dot{a}_{13}(t) \\ \dot{a}_{21}(t) & \dot{a}_{22}(t) & \dot{a}_{23}(t) \\ \dot{a}_{31}(t) & \dot{a}_{32}(t) & \dot{a}_{33}(t) \end{bmatrix} = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) \\ -\omega_3(t) & 0 & \omega_1(t) \\ \omega_2(t) & -\omega_1(t) & 0 \end{bmatrix} \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \\ a_{31}(t) & a_{32}(t) & a_{33}(t) \end{bmatrix}$$

The ω's themselves are usually controlled via the equations

$$\dot{\omega}_1(t) = [(I_2-I_3)/I_1]\omega_2(t)\omega_3(t) + n_1(t)/I_1$$

$$\dot{\omega}_2(t) = [(I_3-I_1)/I_2]\omega_1(t)\omega_3(t) + n_2(t)/I_2$$

$$\dot{\omega}_3(t) = [(I_1-I_2)/I_3]\omega_1(t)\omega_2(t) + n_3(t)/I_3$$

The state space for the first set of equations is $\mathscr{SO}(3)$ -- the set of 3 by 3 orthogonal matrices -- the state space for the second set of equations is R^3 -- cartesian 3-space. For our present purpose suppose that the center of mass of the body is fixed and suppose that the observed output of this system is a pencil beam of light generated by a light source which is mounted in the body along a line passing through the center of mass. In this case the output is $\mathscr{C}X(t)$ where $\mathscr{C}$ is a subgroup which corresponds to a rotation about the pencil beam (an undetectable motion).

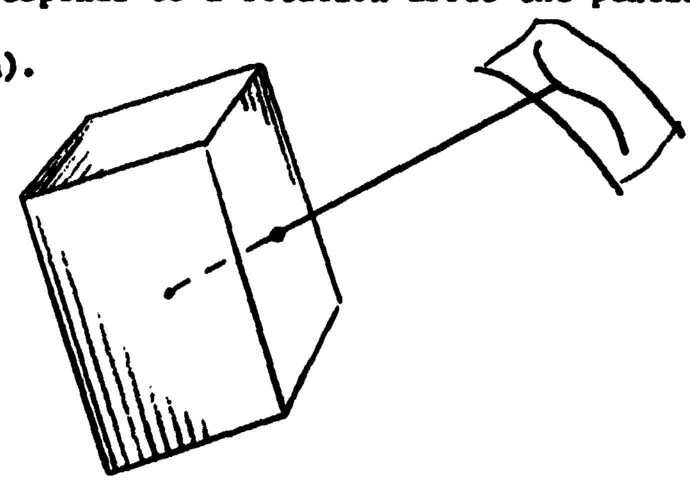

Figure 1 : Illustration of the observability of a rigid body.

Example 3 : (A model for DC to DC conversion) The electrical network shown in figure 1 contains switches which are to be manipulated in such a way as to transfer the energy stored on the capacitor 1 to capacitor 2. In order to have a sensible physical model we demand that there be exactly one path through the inductor at all times.

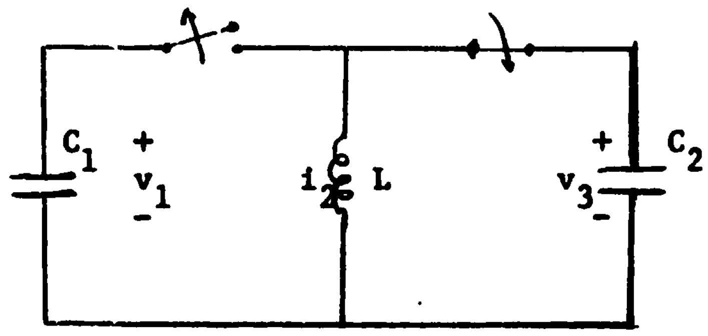

Figure 2 : An electrical network for which energy is conserved.

The equations of motion are

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & s_1(t) & 0 \\ -s_1(t) & 0 & s_2(t) \\ 0 & -s_2(t) & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

where $x_1 = v_1\sqrt{C_1}$, $x_3 = \sqrt{C_2}\, v_3$ and $x_2 = i_2\sqrt{L}$ and s_1 and s_2 are depend on the switch positions and take on the values 1 or zero. We have $s_1 = 1$ and $s_2 = 0$ and if the switch on the left is closed and we have $x_1 = 0$, $s_2 = 1$ if the switch on the input is closed.

3. Lie Algebras and Lie Groups

Let $\mathbb{R}^{n\times n}$ denote the set of real n by n matrices; $\mathbb{R}^{n\times n}$ is a vector space of dimension n^2. By a Lie algebra $\mathscr{L}$ in $\mathbb{R}^{n\times n}$ we understand a subset of $\mathbb{R}^{n\times n}$ which is a vector space and which has the property that if A and B belong to $\mathscr{L}$ thus so does $[A,B] = AB-BA$. If $\mathscr{L}_1$ and $\mathscr{L}_2$ are Lie algebras in $\mathbb{R}^{n\times n}$ and their intersection $\mathscr{L}_1 \cap \mathscr{L}_2$ is also a Lie algebra since if A and B belongs to $\mathscr{L}_1$ and $\mathscr{L}_2$ and both are algebras then $[A,B]$ belongs to both $\mathscr{L}_1$ and $\mathscr{L}_2$. The union $\mathscr{L}_1 \cup \mathscr{L}_2$ of two Lie algebras, the sum $\mathscr{L}_1 + \mathscr{L}_2$ of two Lie algebras and the commutator $[\mathscr{L}_1, \mathscr{L}_2]$ of two Lie algebras are not necessarily Lie algebras.

Given an arbitrary subset of $\mathbb{R}^{n\times n}$ we can add additional elements to it so as to imbed it in a Lie algebra. To obtain the smallest Lie algebra which contains a given set $\mathscr{N}$ we first add to $\mathscr{N}$ all linear combinations of elements in $\mathscr{N}$ so as to get a real vector space $\mathscr{N}_1$. Then commute elements in $\mathscr{N}_1$ to get $\mathscr{N}_2 = \mathscr{N}_1 + [\mathscr{N}_1, \mathscr{N}_1]$ if this is not contained in $\mathscr{N}_1$ then we form $\mathscr{N}_3 = \mathscr{N}_2 + [\mathscr{N}_1, \mathscr{N}_2]$ etc. Clearly this process stops in a finite number of steps since at each stage we increase the dimension of the vector space by at least one and the dimension is upper bounded by n^2. We call this Lie algebra the Lie algebra generated by $\mathscr{N}$ and denote it by $\{\mathscr{N}\}_A$.

If $\mathscr{M}$ is a set of nonsingular matrices in $\mathbb{R}^{n\times n}$ we let $\{\mathscr{M}\}_G$ denote the multiplicative matrix group generated by $\mathscr{M}$, i.e. the smallest group in $\mathbb{R}^{n\times n}$ which contains $\mathscr{M}$ and is closed under multiplication and inversion. If $\mathscr{N}$ is a linear subspace of $\mathbb{R}^{nxn}$ then the set

$$\mathscr{M} = \{M : M = e^{N_1}e^{N_2}\ldots e^{N_p} \quad ; \quad N_i \in \mathscr{N} \quad ; p = 0,1,2,\ldots\}$$

contains no singular matrices since $\det(\exp N_i) = \exp(\operatorname{tr} N_i) > 0$. Clearly $\mathscr{M}$ is closed under multiplication and inversion and in our notation

$$\mathscr{M} = \{\exp \mathscr{N}\}_G$$

Let $\mathscr{L}$ be a Lie algebra. At each point M in $\{\exp \mathscr{L}\}_G$ there is a one to one map ϕ_M from a neighborhood of 0 in $\mathscr{L}$ onto a neighborhood of M in $\{\exp \mathscr{L}\}_G$ which is defined by

$$\phi_M : \mathscr{L} \rightarrow \{\exp \mathscr{L}\}_G \quad ; \quad \phi_M(L) = e^L M$$

This map has a smooth inverse which shows that $\{\exp \mathscr{L}\}_G$ is a locally Euclidean space of dimension equal to the dimension of $\mathscr{L}$. We may check that the maps ϕ_M^{-1} satisfy the conditions for a C^∞ manifold in the sense of reference [11] (page 97). Thus we may give $\{\exp \mathscr{L}\}_G$ the structure of a differentiable manifold. This justifies our referring to $\{\exp \mathscr{L}\}_G$ as a <u>group manifold</u>.

If $\mathscr{A}$ is a linear subspace of $\mathbb{R}^{nxn}$ which is not necessarily a Lie algebra we might inquire as to the relation between $\{\exp \mathscr{A}\}_G$ and $\{\exp \{\mathscr{A}\}_A\}_G$. Clearly the latter contains the former. The following theorem claims that they are identical.

<u>Theorem 1</u> : Let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_p$ be a collection of linear subspaces of $\mathbb{R}^{n\times n}$. Then

$$\{\exp\mathscr{A}_1, \exp\mathscr{A}_2, \ldots, \exp\mathscr{A}_p\}_G = \{\exp\{\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_p\}_A\}_G$$

Before proving Theorem 1 is is appropriate to make a few remarks about its relationship to the controllability literature. (Perhaps a glance at theorem 5 would help at this point.) In considering equation

$$\dot{X}(t) = (\sum_{i=1}^{m} u_i(t)A_i)X(t)$$

as a differential equation in $\mathscr{G}\ell(n)$ it is clear from the theorem of Frobenius [12] that the solution passing through $X \in \mathscr{G}\ell(n)$ lies in $\{\exp\{A_i\}_A\}_G X$ because at each point X in $\mathscr{G}\ell(n)$, $\{A_iX\}_A$ is an involative vector field which contains X and spans the tangent space of $\{\exp\{A_i\}_A\}_G X$ at X. Wei and Norman [9] confirm this fact by giving (locally valid) formulas for the solution of this differential equation in terms of the functions $u_i(\cdot)$ and the structural constants of the Lie algebra generated by the A_i (without pointing out the differential geometric interpretation of the result). On the other hand, if we regard $\dot{X}(t) = (\sum_{i=1}^{m} u_i(t)A_i)X(t)$ as a control problem then the natural question is not what manifold contains the solution, but rather what set can be attained from a given point, given freedom over the choice of $(u_1, u_2, \ldots, u_m)$. The results of Chow [1] (see also Hermann [2]) are applicable here. Chow showed under a suitable regularity condition that the set of points reachable for the vector system $\dot{x}(t) = \sum_{i=1}^{m} u_i(t)f_i[x(t)]$ using piecewise constant controls as the same as those points reachable for

$$\dot{x}(t) = \sum_{i=1}^{n} v_i(t) g_i[x(t)]$$

where $\{g_i(x)\}$ is a basis for the involative distribution generated by $\{f_i(x)\}$. That is $\{g_i(x)\}$ spans a vector space which includes $\{f_i(x)\}$ and is closed under the Lie bracket operation

$$[f,g] = \frac{\partial f}{\partial x} g - \frac{\partial g}{\partial x} f$$

In our case the Lie bracket of A_iX and A_jX is $[A_i,A_j]X$. Thus we see that for the differential equation in question the reachable set includes $\{\exp\{A_i\}_A\}_G$ and the theorem of Frobenius insures that it includes nothing more.

The proof of theorem 1 given below could be shortened considerably by the use of these ideas. The reason for preferring the longer proof given here is that it is constructive, it is self-contained (nothing harder than the implicit function theorem is used) and it has the merit of proving a theorem about n by n matrices using the notation and tools natural to that subject.

Proof : We give a proof which relies on an implicit-function theorem which, under suitable hypothesis, insures the existence of a solution of α equations in $\beta < \alpha$ unknowns. (See reference [13] pages 29-30.) We also need the Baker-Hausdorff formula which asserts that

$$e^{At}Le^{-At} = L + [At,L] + \frac{1}{2}[At,[At,L]] + \frac{1}{3!}[At,[At,[At,L]]] + \dots$$

Note that the norm of the (n+1)th term in this series is less than $||L||2^n||A||^n/n!$ so that the series is majorized by the stories $||L||[1+2||A|| + (2||A||)^2/2!+\dots] = ||L||\cdot e^{2||At||}$ and hence is absolutely and uniformly convergent on $-T \leq t \leq T$ for all T.

Let $\{A_1, A_2, \ldots, A_n\} \subset \mathscr{A}_1 \cup \mathscr{A}_2 \cup \ldots \mathscr{A}_p$ be a basis for $\mathscr{A}_1 + \mathscr{A}_2 + \ldots \mathscr{A}_p$ and let $\mathscr{L}$ be the Lie algebra generated by $\{A_1, A_2, \ldots, A_r\}$. Assume this algebra is of dimension q. There exists a basis for $\mathscr{L}$ which consists of terms of the form

$$L_1 = A_1$$

$$L_2 = A_2$$

$$\cdots\cdots$$

$$L_r = A_r$$

$$L_{r+1} = [A_{k(r+1)}, A_{\ell(r+1)}]$$

$$L_{r+2} = [A_{k(r+2)}, A_{\ell(r+2)}]$$

$$\cdots\cdots$$

$$L_{r+s} = [A_{k(r+s)}, A_{\ell(r+s)})$$

$$L_{r+s+1} = [A_{k(r+s+1)}, [A_{\ell(r+s+1)}, A_{m(r+s+1)}]]$$

$$\cdots\cdots$$

$$L_q = [\ldots[A_{k(q)}, \ldots, A_{m(q)}]\ldots]$$

We are quite explicit here because at certain points in our proof it is necessary to regard these expressions as formal expressions as opposed to matrices.

We introduce the following special notation. The operator EXP maps formal expressions into matrices. It is defined on A_i and its commutators (i.e. formal expressions such as $A_i, [A_i, A_j]$, $[A_i, [A_j, A_k]]$ etc.)

as follows

$$\mathrm{EXP}\ A_i t = e^{A_i t}$$

$$\mathrm{EXP}[A_i,A_j]t = \begin{cases} e^{A_i\sqrt{t}}e^{A_j\sqrt{t}}e^{-A_i\sqrt{t}}e^{-A_j\sqrt{t}} & t \geq 0 \\ e^{A_j\sqrt{|t|}}e^{A_i\sqrt{|t|}}e^{-A_j\sqrt{|t|}}e^{-A_i\sqrt{|t|}} & t < 0 \end{cases}$$

The definition is completed by recursion. If B is a commutator expression then

$$\mathrm{EXP}[A_i,B]t = \begin{cases} e^{A_i\sqrt{t}}(\mathrm{EXP}\ B\sqrt{t})e^{-A_i\sqrt{t}}(\mathrm{EXP}\ B\sqrt{t})^{-1} & t \geq 0 \\ \mathrm{EXP}\ B\sqrt{|t|}e^{A_i\sqrt{|t|}}(\mathrm{EXP}\ B\sqrt{|t|})^{-1}e^{-A_i\sqrt{|t|}} & t < 0 \end{cases}$$

and

$$\mathrm{EXP}[B,A_i]t = \begin{cases} (\mathrm{EXP}\ B\sqrt{t})e^{A_i\sqrt{t}}(\mathrm{EXP}\ B\sqrt{t})^{-1}e^{-A_i\sqrt{t}} & t \geq 0 \\ e^{A_i\sqrt{|t|}}(\mathrm{EXP}\ B\sqrt{|t|})e^{-A_i\sqrt{|t|}}(\mathrm{EXP}\ B\sqrt{|t|})^{-1} & t < 0 \end{cases}$$

It is an easy calculation to verify that $\mathrm{EXP}[A_i,A_j]t = I+[A_i,A_j]t+o(t)$ where $o(t)/t$ goes to zero as t goes to zero. We now show that in general if B is a commutator expression then

$$\mathrm{EXP}\ Bt = I + Bt + o(t)$$

To carry out this proof we use induction. Assume that the result is true for commutator expressions B with n-1 brackets we will show it true for those containing n brackets. We write B as $B = [A,C]$ with C having n-1 brackets. (Similar calculations cover the case $B = [C,A]$.) By the induction hypothesis

$$\mathrm{EXP}\ Ct = I + Ct + o(t)$$

Of course EXP Ct can be expanded in a convergent power series involving fractional powers of t. If G(t) denotes the terms involving powers of t between t^1 and t^2 and if F is the coefficient of t^2 then

$$\text{EXP } Ct = (I+Ct+G(t)+Ft^2+o(t^2))$$

The power series expansion for the inverse of EXP Ct is then

$$(\text{EXP } Ct)^{-1} = I-Ct-G(t)+(C^2-F)t^2+o(t^2))$$

as is verified by multiplication with the expression for EXP Ct itself. Now for t nonnegative

$$\begin{aligned}\text{EXP } Bt &= \text{EXP}[A,C]t \\ &= (I+A\sqrt{t}+A^2t/2+o(t))(I+C\sqrt{t}+G(\sqrt{t})+Ft+o(t)) \\ &\quad (I-A\sqrt{t}+A^2t/2+o(t))(I-C\sqrt{t}-G(\sqrt{t})+C^2-F)t+o(t)) \\ &= I+[A,C]t+Ft+A^2t/2+A^2t/2-A^2t+(C^2-F)t-C^2t+o(t) \\ &= I+[A,C]t+o(t)\end{aligned}$$

and the case $t < 0$ leads to the same result.

Well known properties of the matrix exponential function let one conclude that for $|t| > 0$ EXP Bt is continuously differentiable. The above argument shows that EXP Bt is differentiable with respect to t in a neighborhood of $t = 0$ and

$$\frac{d}{dt} \text{EXP } Bt\Big|_{t=0} = B$$

Hence we have for the basis elements L_ν,

$$\frac{d}{dt} \text{EXP } L_\nu t \Big|_{t=0} = L_\nu$$

Now consider a function of $u = (u_1, u_2, \ldots, u_q)$ and $v = (V_1, v_2, \ldots, v_q)$ which maps $\mathbb{R}^q \times \mathbb{R}^q$ into $\mathbb{R}^{nxn}$ and which is defined by

$$F(u,v) = -I+(\text{EXP } L_1u_1)(\text{EXP } L_2u_2)\ldots(\text{EXP } L_qu_q)e^{-L_1v_1-L_2v_2\ldots-L_qv_q}$$

Clearly $F(0,0) = 0$. Now the linear approximation of F at $(u,v) = (0,0)$ is given by

$$F_{(u,v)}(u,v)\Big|_{(0,0)}(\delta u,\delta v) = L_1\delta u_1+L_2\delta u_2+\ldots+L_q\delta u_q-L_1\delta v_1-L_2\delta v_2-\ldots-L_q\delta v_q$$

so that the range space of $F_{(u,v)}(u,v)\big|_{(0,0)}(u,0)$ is the q-dimensional subspace of R^{nxn} spanned by $\{L_i\}$. Now $F(u,v) + I$ is a finite product of exponentials which we write as

$$F(u,v)+I = e^{A_{i_1}u_{i_1}^{p_1}}e^{A_{i_2}u_{i_2}^{p_2}}\ldots e^{A_{i_\nu}u_{i_\nu}^{p_\nu}}e^{-L_1v_1-L_2v_2-\ldots L_qv_q}$$

Since the Baker-Hausdorff formula lets one write

$$e^{A_it}A_ke^{-A_it} = A_k + [A_i,A_k]t + \ldots$$

we see that $e^{A_it}A_ke^{-A_it}$ belongs to the Lie algebra generated by the A's. Moreover, it is continuous with respect to t and at $t = 0$ takes on the value A_k. Using this result repeatedly we see that for each $\{u_{i_k}\}$ we can find R_{i_k} in $\mathscr{L}$ such that

$$e^{A_{i_1}u_{i_1}^{p_1}}e^{A_{i_2}u_{i_2}^{p_2}}\cdots e^{A_{i_k}u_{i_k}^{p_k}}A_{i_k}e^{A_{i_{k+1}}u_{i_{k+1}}^{p_{k+1}}}\cdots e^{A_{i_\nu}u_{i_\nu}^{p_\nu}}e^{L_1v_1+L_2v_2\dots+L_qv_q}$$

$$= (F(u,v)+I)R_{i_k}(u_{i_1},u_{i_2}\dots u_{i_\nu})$$

simply by pushing A_k past the exponentials one at a time. Clearly $R_{i_k}(0,0,\dots,) = A_{i_k}$. Thus we see that for a and b small

$$F_{(u,v)}(u,v)\Big|_{(a,b)}(\delta u,\delta v) = (F(a,b)+I)\Big(\sum_{i=1}^{q} S_i(a,b)\delta u_i+H_i(a,b)\delta v_i\Big)$$

for some $S_i(a,b)$ and $H_i(a,b)$ in $\mathscr{L}$. Since $S_i(0,0) = H_i(0,0) = L_i$ and since S_i and H_i depend continuously on their arguments this establishes that the Jacobian of the map $F : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^{n\times n}$ must have rank q in a neighborhood of (0,0) and hence by the implicit function theorem cited earlier there exists an $\varepsilon > 0$ and a map $\phi : R^q \to R^q$ such that if $||v|| < \varepsilon$ then

$$F(\phi(v),v) = 0$$

Since $F(u,v) = 0$ implies that

$$\mathrm{EXP}\ L_1u_1\mathrm{EXP}\ L_2u_2\dots\mathrm{EXP}\ L_qu_q = e^{L_1v_1+L_2v_2+\dots+L_qv_q}$$

We conclude that there exists $\varepsilon_1 > 0$ such that if L ε and $||L|| < \varepsilon_1$ we can write

$$e^L = e^{A_{i_1}u_{i_1}}e^{A_{i_2}u_{i_2}}\dots e^{A_{i_\nu}u_{i_\nu}}$$

Now for any $L \in \mathscr{L}$ it follows that $||\frac{1}{m} L|| < \varepsilon_1$ for some integer m and thus we can express L as expL/m·expL/m...expL/m. Likewise we can express $e^{L_1}e^{L_2}\dots e^{L_q}$ in this form.

Let $\mathscr{K}$ and $\mathscr{L}$ be Lie algebras in $\mathbb{R}^{nxn}$. It can happen that $\{\exp\mathscr{K}\}_G$ is a bounded subset of $\mathbb{R}^{nxn}$ which is not closed, and it can happen that the closure of $\{\exp\mathscr{K}\}_G$ equals $\{\exp\mathscr{L}\}_G$ with $\mathscr{K} \neq \mathscr{L}$. The skew-line on the torus [11] is an easy example. Also, $\{\exp\mathscr{K}\}_G$ is not necessarily simply connected. Nonetheless, we have the following result which we deduce from theorem 1 rather than sending the reader to the literature.

Corollary 1 : If $\mathscr{K}$ and $\mathscr{L}$ are Lie algebras in $\mathbb{R}^{nxn}$ then $\{\exp\mathscr{K}\}_G \subset \{\exp\mathscr{L}\}_G$ if and only if $\mathscr{K} \subset \mathscr{L}$ and $\{\exp\mathscr{K}\}_G = \{\exp\mathscr{L}\}_G$ if and only if $\mathscr{K} = \mathscr{L}$.

Proof : For both statements the sufficiency is obvious. To establish necessity in the first case notice that if $\{\exp\mathscr{K}\}_G \subset \{\exp\mathscr{L}\}_G$ then by theorem 1

$$\{\exp\mathscr{K}\}_G = \{\exp\mathscr{K},\ \exp\mathscr{L}\}_G = \{\exp\{\mathscr{K},\mathscr{L}\}_A\}_G$$

Suppose $\mathscr{L}$ is of dimension n. To obtain a proof by contradiction, suppose that $\mathscr{K}$ is not contained in $\mathscr{L}$. Then $\{\mathscr{K},\mathscr{L}\}_A$ is of dimension n+1 or greater. Then $\exp\mathscr{L}$ is an n dimensional manifold and $\{\exp\{\mathscr{K},\mathscr{L}\}\}$ is not which contradicts $\{\exp\mathscr{L}\} = \{\exp\{\mathscr{L},\mathscr{K}\}\}$. To establish necessity in the second case repeat this argument verbatum but with "contained in" replaced by "equals" both verbally and symbolically.

The notation $ad_A^0 B = B$, $ad_A^1 B = [A,B]$, $ad_A^2 B = [A,[A,B]]$ etc. is standard. If $\mathscr{K}$ and $\mathscr{L}$ are Lie algebras we use the notation $\{ad_{\mathscr{L}}\mathscr{K}\}_A$ to denote the Lie algebra generated by $\mathscr{K}$ under commutation with elements of $\mathscr{L}$. That is

$$\{ad_{\mathscr{L}}\mathscr{K}\}_A = \{\mathscr{K},[\mathscr{L},\mathscr{K}],\ldots,[\mathscr{L},[\mathscr{L}\ldots[\mathscr{L},\mathscr{K}]\ldots]]\ldots\}_A$$

This algebra may also be described as the intersection of all Lie algebras which contain $\mathscr{K}$ and are closed under commutation with $\mathscr{L}$. If $\mathscr{F}$ and $\mathscr{G}$ are groups we introduce an analogous notation. The smallest group which contains $\mathscr{F}$ and all products of the type GFG^{-1} for G in $\mathscr{G}$ and F in $\mathscr{F}$ will be denoted by $\{AD_{\mathscr{G}}\mathscr{F}\}_G$. This group may be described as the intersection of all groups which contain $\mathscr{F}$ and are closed under conjugation with elements of $\mathscr{G}$. If $\mathscr{G}$ is $\{\exp\mathscr{L}\}_G$ and $\mathscr{F}$ is $\{\exp\mathscr{K}\}_G$ then clearly $\{AD_{\mathscr{G}}\mathscr{F}\}_G$ consists of products of terms of the form

$$M = e^{L_1}e^{K_1}e^{-L_1}e^{L_2}e^{K_2}e^{-L_2}\cdots e^{L_m}e^{K_m}e^{-L_m}$$

<u>Theorem 2</u>: Let $\mathscr{K}$ and $\mathscr{L}$ be Lie algebras in $\mathbb{R}^{n\times n}$. Then

$$\{\bigcup_{M\in\{\exp\mathscr{L}\}_G} M\mathscr{K}M^{-1}\}_A = \{ad_{\mathscr{L}}\mathscr{K}\}_A$$

and

$$\{AD_{\{\exp\mathscr{L}\}_G}\{\exp\mathscr{K}\}_G\}_G = \{\exp\{ad_{\mathscr{L}}\mathscr{K}\}_A\}_G$$

<u>Proof</u>: From the Baker-Hausdorff formula we see at once that if L belongs to $\mathscr{L}$ and K belongs to $\mathscr{K}$ then $e^{L}Ke^{-L}$ belongs to $\{ad_{\mathscr{L}}\mathscr{K}\}_A$. Thus the right side of the first equality in question contains the left. On the other hand, expressions of the following type belong to the left side.

$$e^{\alpha L}\frac{1}{\alpha}Ke^{-\alpha L}-\frac{1}{\alpha}K = [L,K] + o(\alpha)$$

$$e^{\alpha L}(\frac{1}{\alpha}[K,L]+o(\alpha))e^{-\alpha L}-\frac{1}{\alpha}[K,L]+o(\alpha)=[L[L,K]]+o(\alpha)$$

etc.

Since $\mathbb{R}^{n\times n}$ is a finite dimensional space and since

$$\{\bigcup_{M \in \{\exp \mathscr{L}\}_G} M\mathscr{K}M^{-1}\}_A$$

is a linear subspace, it is closed. Thus [L,K] [L[L,K]]... etc. belongs to this set and the first equality is seen to hold.

The second statement is obtained by exponentiating the first. This gives

$$\{\exp\{\bigcup_{M \in \{\exp \mathscr{L}\}_G} M\mathscr{L}M^{-1}\}_A\}_G = \{\exp\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A\}_G$$

but since $e^L e^K e^{-L} = \exp(e^L K e^{-L})$ we see that

$$\{\exp\{\bigcup_{M \in \{\exp \mathscr{L}\}_G} M\mathscr{L}M^{-1}\}_A\}_G = \{AD_{\{\exp \mathscr{L}\}_G}\{\exp \mathscr{L}\}_G\}_G$$

so the result follows.

The next theorem states a purely group theoretic result which although easily proven, is stated formally because we need it in our study of observability.

<u>Theorem 3</u> : Let $\mathscr{H}$ and $\mathscr{R}$ be subgroups of a group $\mathscr{G}$. Let $\mathscr{P}$ be the subset of $\mathscr{G}$ defined as

$$\mathscr{P} = \{P : RPR^{-1} \in \mathscr{H} \text{ ; all } R \in \mathscr{R}\}$$

Then $\mathscr{P}$ is a subgroup of $\mathscr{H}$, $\mathscr{R}\mathscr{P}$ is a subgroup of $\mathscr{G}$ and $\mathscr{P}$ is a normal subgroup of $\mathscr{R}\mathscr{P}$. Thus $\mathscr{R} \cap \mathscr{P}$ is a normal subgroup of $\mathscr{R}$ and $\mathscr{R}\mathscr{P}/\mathscr{P}$ is isomorphic with $\mathscr{R}/\mathscr{R} \cap \mathscr{P}$.

<u>Proof</u> : Suppose P_1 and P_2 belong to $\mathscr{P}$ then for each R in $\mathscr{R}$ there exists $H_1(R)$ and $H_2(R)$ in H such that $RP_1R^{-1}(RP_2^{-1}R^{-1}) = H_1(R)\cdot[H_2(R_2)]^{-1}$. Since

$\mathscr{H}$ is a group this means $RP_1P_2^{-1}R^{-1}$ belongs to $\mathscr{H}$ and thus that $\mathscr{P}$ is a subgroup of $\mathscr{G}$. Clearly it is a subgroup of $\mathscr{H}$ since the choice $R = I$ is possible. To see that $\mathscr{R}\mathscr{P}$ is a group, note that if R_1 and R_2 belong to $\mathscr{R}$ and P_1 and P_2 belong to $\mathscr{P}$ then

$$R_1P_1(R_2P_2)^{-1} = R_1P_1R_2^{-1}R_2P_2^{-1}R_2^{-1} = R_1R_2^{-1}(R_2P_1R_2^{-1})(R_2P_2^{-1}R_2^{-1})$$

Since $\mathscr{R}$ is a group and since $\mathscr{P}$ is a group which has the property that if P belongs to $\mathscr{P}$ then so does RPR^{-1} for each R in $\mathscr{R}$, we see that this product belongs to $\mathscr{R}\mathscr{P}$. Clearly $\mathscr{P}$ is a normal subgroup of $\mathscr{R}\mathscr{P}$ since $RP\mathscr{P}P^{-1}R^{-1} = \mathscr{P}$ for each RP in $\mathscr{R}\mathscr{P}$. By the second isomorphism theorem (Rotman [14], page 26) $\mathscr{R} \cap \mathscr{P}$ is normal in $\mathscr{R}$ and $\mathscr{R}\mathscr{P}/\mathscr{P} \simeq \mathscr{R}/\mathscr{R} \cap \mathscr{P}$.

We now state and prove a Lie algebraic analog of this theorem. Algebraic tests for observability will be derived from this result.

Theorem 4 : Let $\mathscr{H}$ and $\mathscr{L}$ be Lie algebras in $\mathbb{R}^{n\times n}$. Let $\mathscr{P}$ be defined as

$$\mathscr{P} = \{P : RPR^{-1} \in \{\exp\mathscr{H}\}_G, \text{ all } R \in \{\exp\mathscr{L}\}_G$$

If $\mathscr{K}$ is a Lie algebra in $\mathbb{R}^{n\times n}$ then $\{\exp\mathscr{K}\}_G \subset \mathscr{P}$ if and only if $\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A \subset \mathscr{H}$. There exists a unique Lie algebra $\mathscr{K}_1$ such that $\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}_1\}_A \subset \mathscr{H}$ and $\mathscr{K}_1$ contains all other Lie algebras having this property.

Proof : Suppose $\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A \subset \mathscr{H}$. Then for L_i in $\mathscr{L}$ and K_i in $\mathscr{K}$ we see from theorem 2 that $\{\exp\mathscr{H}\}_G$ contains

$$RPR^{-1} = e^{L_1}e^{L_2}\dots e^{L_p}e^{K_1}e^{K_2}\dots e^{K_q}e^{-L_p}\dots e^{-L_2}e^{-L_1}$$

By the hypothesis and corollary 1 $\{\exp\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A\}_G \subset \{\exp\mathscr{H}\}_G$.

On the other hand, if for all L_i in $\mathscr{L}$ and all K_i in $\mathscr{K}$ we have

$$RPR^{-1} = e^{L_1}e^{L_2}\ldots e^{L_p}e^{K_1}e^{K_2}\ldots e^{K_q}e^{-L_p}\ldots e^{-L_2}e^{-L_1} \in \{\exp\mathscr{K}\}_G$$

then since $\{\exp\mathscr{K}\}_G$ is a group we see that $\{AD_{\{\exp\mathscr{L}\}_G}\{\exp\mathscr{K}\}_G\}_G \subset \{\exp\mathscr{K}\}_G$ and again from theorem 2 and corollary 1 we see that $\{ad_{\mathscr{L}}\mathscr{K}\}_A \subset \mathscr{K}$.

Finally, notice that if $\{ad_{\mathscr{L}}\mathscr{K}_1\}_A \subset \mathscr{K}$ and $\{ad_{\mathscr{L}}\mathscr{K}_2\}_A \subset \mathscr{K}$ then $\{ad_{\mathscr{L}}(\mathscr{K}_1+\mathscr{K}_2)\}_A \subset \mathscr{K}$ thus there is a largest Lie algebra with this property.

4. Controllability on Group Manifolds

The first question of a system theoretic character which we investigate is that of controllability. Since we want to emphasize global results we work with the most elementary type of evolution equation appropriate to our present steeing, namely

$$\dot{X}(t) = (A + \Sigma u_i(t)B_i)X(t)$$

The choice of control affects the direction in which X moves. However A is a constant over which there is no control. This evolution equation has the property that the change of variables $X \to XP$ for P nonsingular, leaves the equation unchanged. This invariance gives the vector field which a given choice of $\{u_i(t)\}$ establishes on $G\ell(n)$ a particular simple form.

Theorem 5 : Consider the linear dynamical system

$$\dot{X}(t) = (\sum_{i=1}^{m} u_i(t)B_i)X(t) \quad ; \quad \underline{X} = n \text{ by } n \text{ matrix}$$

Given a time $t_a > 0$ and given two nonsingular matrices X_1 and X_2, there exists piecewise continuous controls which steer the state from X_1 at $t = 0$ to X_2 at $t = t_a$ if and only if $X_2X_1^{-1}$ belongs to $\{\exp\{B_i\}_A\}_G$.

Proof : (Sufficiency) Theorem 1 asserts that any matrix M in $\{\exp\{B_i\}_A\}_G$ can be written as a finite product, say

$$M = e^{B_{i_1}\alpha_1} e^{B_{i_2}\alpha_2} \cdots e^{B_{i_m}\alpha_m}$$

Suppose $X_2X_1^{-1} = M$. Divide the interval $0 \leq t \leq t_a$ up into m equal intervals $[t_i,t_{i+1})$ whereby $t_i = i\cdot t_a/m$. Let $t_a/m = \beta^{-1}$. On the interval $[0,t_1)$ all controls are zero except the i_mth control which takes on the value $\alpha_m\beta$. On the interval $[t_1,t_2)$ all controls are zero except the i_{m-1}th which takes on the value $\alpha_{m-1}\beta$, etc. down to the last interval on which all controls are zero except the i_1st which takes on the value $\alpha_1\beta$. Since the differential equation is linear and constant on each of the subintervals the solution is a product of exponentials and the result follows.

(Necessity) To show that X_2 cannot be reached from X_1 unless $X_2X_1^{-1}$ is of the form $e^{L_a}e^{L_{m-1}}\dots e^{L_1}$ we assume the contrary and obtain a contradiction. Suppose that $u_1(\cdot),\dots,u(\cdot)$ is a control which steers the system from X_1 at $t = 0$ to X_2 at $t = t_a$ by theorem 1 of [9] we know that there exists a sequence of times $t_o,t_1,t_2,\dots,t_m$ such that on each of the subintervals $[t_i,t_{i+1}]$ the transition matrix of

$$\dot{X}(t) = (\sum_{i=1}^{q} u_i(t)B_i)x(t)$$

can be written as $e^{H_i(t)}$ for some $H_i(\ )$ in $\mathscr{L}$. Thus we can write

$$X(t_a) = e^{L_m}e^{L_{m-1}}\dots e^{L_1}X_o$$

which establishes the contradiction.

As an application which emphasizes the ease w' which we can study global questions using this theorem we observe the following results relating to the classical groups. Here J is given by

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

and a matrix is called symplectic if $\theta'J\theta = J$.

<u>Theorem 6</u> : Consider the system of Theorem 5. Given a time $t_a > 0$ and given two nonsingular n by n matrices X_1 and X_2 with $\det X_1X_2 > 0$, there exists a piecewise continuous control which steers the state from X_1 at $t = 0$ to X_2 at $t = t_a$ if $\{B_i\}_A$

i) spans $\mathbb{R}^{nxn}$

ii) spans the n^2-1 dimensional subspace of $\mathbb{R}^{nxn}$ consisting of the zero trace matrices and $\det X_1 = \det X_2$.

iii) spans the $n(n+1)/2$ dimensional subset of $\mathbb{R}^{nxn}$ consisting of the set of matrices which satisfy $JA+A'J = 0$ and $X_2X_1^{-1}$ is symplectic

iv) spans the $n(n-1)/2$ dimensional subset of $\mathbb{R}^{nxn}$ consisting of all skew-symmetric matrices and $X_2X_1^{-1}$ is orthogonal.

<u>Proof</u> : As is well known any nonsingular matrix can be written as θR with $\theta'\theta = I$ and $R = R' > 0$. Also real orthogonal matrices with positive determinants and real symmetric positive definite matrices have real logarithms. Moreover in case iii) the factors in the polar representation inherit the property of the group itself, which is to say that the and R in the polar representation of a symplectic matrix are symplectic. To complete the proof we need only invoke theorem 5 since the previous remarks justify our writing $X_2X^{-1} = e^{\Omega}e^{S}$ with $\Omega = -\Omega'$ and $S = S'$ both in the appropriate Lie algebras.

The results of theorems 5 and 6 are somewhat unsatisfactory in that the A term is absent. The following theorem describes one way in which this can be relaxed.

Theorem 7 : Consider the linear dynamical system

$$\dot{X}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)X(t) \quad ; \quad X = n \text{ by } n \text{ matrix}$$

Suppose that $[ad_A^k B_i, B_j] = 0$ for $i, j=1, 2, \ldots \nu$ and $k=0, 1, \ldots n^2-1$. Let $\mathscr{H}$ be the linear subspace of $R^{n\times n}$ spanned by $ad_A^k B_i$ for $i=1,2,\ldots \nu$ and $k=0, 1, \ldots n^2-1$. Then given a time $t_a > 0$ and two n by n matrices X_1 and X_2 there exist continuous controls which steer the system from the state X_1 at $t=0$ to the state X_2 at $t = t_a$ if and only if there exists H in $\mathscr{H}$ such that

$$X_2 = e^{At_a}e^{H}X_1$$

Proof : First of all, notice that

$$\begin{aligned}
\frac{d^k}{dt^k}[e^{At}B_i e^{-At}, B_j]\Big|_{t=0} &= \frac{d^{k-1}}{dt^{k-1}}[e^{At}[A,B_i]e^{-At}, {}_j]\Big|_{t=0} \\
&= \frac{d^{k-2}}{dt^{k-2}}[e^{At}(ad_A^2 B_i)e^{-At}, B_j]\Big|_{t=0} \\
&\quad \cdots\cdots\cdots \\
&= [e^{At}(ad_A^k B_i)e^{-At}, B_j]\Big|_{t=0} \\
&= [ad_A^k B_i, B_j]
\end{aligned}$$

Thus $[e^{At}B_i e^{-At}, B_j]$ is identically zero if $[ad_A^k B_i, B_j] = 0$ for $k=0,1,2,\ldots$. However, ad_A is a linear operator from an n^2 dimensional space into itself so that by the Cayley-Hamilton theorem all powers above n^2-1 are linearly dependent on the first n^2-1. Thus under the hypothesis of the theorem statement $[e^{At}B_ie^{-At},B_j]$ vanishes identically. Also

$$0 = e^{At}B_ie^{-At}B_j - B_je^{At}B_ie^{-At}$$

$$= e^{A\sigma}(e^{At}B_ie^{-At})B_je^{-A\sigma} - e^{A\sigma}B_je^{At}B_ie^{-At}e^{-A\sigma}$$

Now let $t+\sigma = \beta$ and $\gamma = \sigma$. Thus for all β and γ

$$0 = [e^{A\beta}B_ie^{-A\beta}, e^{A\gamma}B_je^{-A\gamma}]$$

For the purpose of solving the differential equation we introduce $Z(t) = e^{-At}X(t)$ and observe that

$$\dot{Z}(t) = (\sum_{i=1}^{\nu} u_i(t)e^{-At}B_ie^{At})Z(t)$$

But recall (see e.g. Martin [15]) that the solution of $\dot{Z}(t)=B(t)Z(t)$ is $\exp\int_0^t B(\sigma)d\sigma$ if $[B(t),B(\sigma)]$ vanishes for all t and σ. Thus we can write

$$Z(t) = \exp(\int_0^t \sum_{i=1}^{n} u_i(t)e^{-At}B_ie^{At}dt)Z(0)$$

It is a well known and frequently used fact (e.g. [16] page 79) that the image space of the map taking continuous functions into $\mathbb{R}^p$ according to the rule $x = L(u) = \int_0^{t_1} e^{A\sigma}bu(\sigma)d\sigma$, is spanned by the first p derivatives of $e^{At}b$ evaluated at zero. Using this fact here we see that for each H in $\mathscr{H}$ and each $t_a > 0$ we have a continuous u defined on $[0,t_a]$ such that

$$Z(t_a) = e^{H}Z(0)$$

Therefore in terms of X we see that we can reach at t_a using $\dot{X}$ which can be expressed as $e^{At_a}e^{H}x(0)$ with H in $\mathscr{H}$.

As an application of this result we derive a familiar relationship.

<u>Example 4</u> : Consider the system in $\mathbb{R}^n$

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{i=1}^{m} b_i u_i(t) \quad ; \quad x(0) = \text{given}$$

Related to this is the matrix system in $R^{(n+1)x(n+1)}$.

$$\dot{X}(t) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} X(t) + \sum_{i=1}^{m} u_i(t) \begin{bmatrix} 0 & b_i \\ 0 & 0 \end{bmatrix} X(t)$$

Let A and B_i be the matrices appearing in this expression. In this case $[Ad_A^k(B_i),B_j]$ vanishes as required and so the reachable set from x(0) = I is

$$\mathscr{R}(t) = \exp\begin{bmatrix} At & 0 \\ 0 & 0 \end{bmatrix}\{\exp H \; ; \; H \in \mathscr{H}\}$$

where $\mathscr{H}$ is the subspace spanned by $Ad_A^k B_i$. A computation gives

$$Ad_A^k B_i = \begin{bmatrix} 0 & A^k b_i \\ 0 & 0 \end{bmatrix}$$

so that the reachable set at t is

$$\mathscr{R} = \{X : X = \begin{bmatrix} e^{At} & H \\ 0 & 1 \end{bmatrix} ; \; H \in \text{Range}, B, AB, \ldots A^{n-1}B\} \quad .$$

where we have used the fact that $e^{At}H = H$ for all t and all H in Range $(B, AB, \ldots, A^{n-1}B)$.

5. Observability

In order to get a theory having a scope comparable to linear theory, it is necessary to treat observability. The choice of an appropriate form of the observational equation is critical for the success of the overall theory. As it turns out, the natural choice is indicated by the second example in section 2.

Let $\mathscr{F}$ be a matrix group and let $\mathscr{C}$ be a subgroup. Consider the system evolving in $\mathscr{F}$

$$\dot{X}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)X(t) \quad ; \quad y(t) = \mathscr{C}X(t)$$

by which we mean that instead of observing $X(t)$ directly, we observe what equivalence class $X(t)$ belongs to with respect to the equivalence relation in $\mathscr{F}$ defined by $\mathscr{C}$. Thus $y(t)$ takes on values in the coset space $\mathscr{F}/\mathscr{C}$ which is generally not a group manifold. (see section 7)

We call two states X_1 and X_2 <u>distinguishable</u> if there exists some control which gives rise to different outputs for the two starting states. In general the zero control is not adequate to distinguish between all states which are distinguishable as contrasted with the situation one finds for linear systems.

<u>Theorem 8</u> : Let $\mathscr{C}$ be a matrix group and suppose that the set of points $\mathscr{R}$ reachable from the identity for the system

$$\dot{X}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)X(t) \quad ; \quad y(t) = \mathscr{C}X(t)$$

is a group. Then the set of initial states which are indistinguishable from the identity is given by

$$\mathscr{P} = \{P : RPR^{-1} \in \mathscr{C} \quad \text{for all } R \in \mathscr{R}\}$$

$\mathscr{P}$ is a normal subgroup of $\mathscr{P}\mathscr{R}$ and a subgroup of $\mathscr{C}$.

Proof : Suppose that X is a starting state for the given equation which is indistinguishable from the identity. That means that for each R in $\mathscr{R}$ there is C(R) in $\mathscr{C}$ such that

$$C(R)RX = R$$

since $\mathscr{R}$ and $\mathscr{C}$ are groups we can take inverses to get

$$R\,X\,R^{-1} \in \mathscr{C}$$

Thus the set $\mathscr{P}$ is exactly those states indistinguishable from the identity. The remainder of the conclusions come from theorem 3.

Theorem 9 : Let $\mathscr{H}$ and $\mathscr{L}$ be Lie algebras in $\mathbb{R}^{n\times n}$ and suppose that all the points reachable from the identity for

$$\dot{X}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)X(t) \quad ; \quad y(t) = \{\exp\mathscr{H}\}_G X(t)$$

is $\{\exp\mathscr{L}\}_G$. Then the set of initial states $\mathscr{P}$ which are indistinguishable from the identity contains $\{\exp\mathscr{K}\}_G$ if and only if $\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A \subset \mathscr{H}$. Therefore a necessary condition for all states to be distinguishable from the identity is that $\mathscr{H}$ contains no subalgebra $\mathscr{K}$ such that $\{\mathrm{ad}_{\mathscr{L}}\mathscr{K}\}_A \subset \mathscr{H}$.

Proof : Theorem 8 gives a characterization of $\mathscr{P}$ which permits one to bring to bear theorem 4. Theorem 4 immediately gives the desired result.

One might be tempted to conclude that if there is no nontrivial algebra $\mathscr{K}$ meeting the requirements of theorem 9 then all initial states are distinguishable. This is not true because $\mathscr{P}$ can be a discrete subgroup and hence not trivial and yet not expressible as $\{\exp\mathscr{H}\}_G$ for any Lie algebra $\mathscr{K}$. The next example illustrates this.

Example 5. In the numerical integration of the equations of motion of a rigid body one usually avoids Euler angle representation and uses instead quaternion or direction cosine representations. As is well known, the group of unit quaternions covers $\mathscr{SO}(3)$ twice. This causes an ambiguity in going from $\mathscr{SO}(3)$ to the group of unit quaternions. This example illustrates this idea. Consider an equation in the group of unit quaternions $\mathscr{Q}$ which we parameterize in the usual way $(a^2+b^2+c^2+d^2 = 1.)$

$$\frac{d}{dt}\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} = \begin{bmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & +u_2 & -u_1 & 0 \end{bmatrix}\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

$$y(t) = \mathscr{C}\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

where $\mathscr{C}$ is the subgroup given by

$$\mathscr{C} = \{\exp\mathscr{H}\}_G;\ \mathscr{H} = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & -\alpha & 0 \end{bmatrix}$$

Now it is true that $\{\exp \mathscr{H}\}_G$ includes I and -I and it is also true that this pair of elements form a normal subgroup of $\mathscr{Q}$. Thus I as an initial state cannot be distinguished from -I. Yet there is no nontrivial Lie algebra $\mathscr{H}$ such that $\{\mathrm{ad}_{\mathscr{L}}\mathscr{H}\}_A \subset \mathscr{H}$.

6. Realization Theory

One of the central results in linear system theory is the fact that any two time invariant, controllable and observable realizations of a given time-invariant input-output map are related to each other in a very simple way. Our purpose here is to establish a similar theorem in this context.

Suppose we have two systems

$$\dot{X}(t) = \left(\sum_{i=1}^{m} u_i(t)B_i\right)X(t) \quad ; \quad y(t) = \mathscr{C}X(t)$$

$$\dot{Z}(t) = \left(\sum_{i=1}^{m} u_i(t)G_i\right)Z(t) \quad ; \quad y(t) = \mathscr{H}Z(t)$$

We assume that i) the systems are observable in the sense that no two initial states give rise to the same response y for all piecewise continuous inputs, and ii) that there exist one to one maps say $c(\cdot)$ and $h(\cdot)$ both mapping into a set S such that if each system starts at the identity state and if each system receives the same input, $c(\mathscr{C}X(t)) = h(\mathscr{H}z(t))$ for all future time. A pair of systems meeting these criteria will be said to be observable realizations of the same input-output map.
We emphasize that $X(t)$ and $Z(t)$ are square matrices but not necessarily of the same dimension.

Suppose we have two observable realizations of the same input-output map. Let $u(\cdot)$ be a nonzero piecewise constant control defined on $[0,1]$ which when applied to the X system takes the state $X(0) = I$ into the state $X(1) = I$. Then of course it must do the same for the Z system because they are observable realizations of the same input-output map. Thus we see that if $i_1, i_2, \ldots, i_q$ is a collection of integers with $1 \leq i_k \leq m$ and if α_i are any real numbers such that

$$e^{\alpha_1 B_{i_1}} e^{\alpha_2 B_{i_2}} \cdots e^{\alpha_q B_{i_q}} = I$$

then

$$e^{\alpha_1 G_{i_1}} e^{\alpha_2 G_{i_2}} \cdots e^{\alpha_q G_{i_q}} = I$$

Let $L_1, L_2, \ldots, L_r$ be a set of commutator expressions in $B_1, B_2, \ldots, B_m$ such that $\{L_i\}$ forms a basis for $\{B_i\}_A$. Let $K_1, K_2, \ldots, K_r$ be in an analogous expression obtained by replacing B_1 by G_1, B_2 by G_2, etc. Let S be an arbitrary commutator expression in $B_1, B_2, \ldots, B_m$ and let T be the analogous commutator expression in $G_1, G_2, \ldots, G_m$. Then in the notation of the proof of theorem 1, there exists differentiable functions $\alpha_i(\rho)$ such that for $|\rho|$ small

$$\text{EXP}\alpha_1(\rho)L_1 \text{ EXP } \alpha_2(\rho)L_2 \ldots \text{ EXP } \alpha_\nu(\rho)L_r = \text{EXP } \rho S$$

and

$$\text{EXP}\alpha_1(\rho)K_1 \text{ EXP}\alpha_2(\rho)K_2 \ldots \text{ EXP}\alpha(\rho)K_r = \text{EXP}\rho T$$

Since the α_i are differentiable we can write (prime denotes derivative)

$$I + \rho \sum_{i=1}^{m} \alpha_i'(0)L_i + o(\rho^2) = I + \rho S + o(\rho^2)$$

and

$$I + \rho \sum_{i=1}^{m} \alpha_i'(0)K_i + o(\rho^2) = I + \rho T + o(\rho^2)$$

Thus if

$$S = \sum_{i=1}^{r} \gamma_i L_i$$

then

$$T = \sum_{i=1}^{r} \gamma_i K_i$$

From this we see that the algebra $\{G_i\}_A$ is generated from $\{B_i\}$ in exactly the same way as the algebra $\{B_i\}_A$ is generated from $\{B_i\}$ and thus that the algebras are isomorphic. We summarize this discussion with a theorem.

<u>Theorem 10</u> : Consider the two systems

$$\dot{X}(t) = (\sum_{i=1}^{m} u_i(t)B_iX(t)) \quad ; \quad y(t) = \mathscr{C}X(t)$$

$$\dot{Z}(t) = (\sum_{i=1}^{m} u_i(t)G_iZ(t)) \quad ; \quad y(t) = \mathscr{H}Z(t)$$

where X and Z are n by n and q by q respectively. Suppose that these systems are observable realisations of the same input output map. Then $\{B_i\}_A$ and $\{G_i\}_A$ are isomorphic as Lie algebras and moreover if $L_1, L_2, \ldots, L_r$ are commutator expressions in $\{B_i\}$ which form a basis for $\{B_i\}_A$ and if $K_1, K_2, \ldots, K_r$ are the analogous expressions in G_i obtained by replacing B_i by G_i then $K_1, K_2, \ldots, K_r$ is a basis for $\{G\}_A$ and if

$$[L_i, L_j] = \sum_{k=1}^{r} \gamma_{ijk} L_k$$

Then

$$[K_i, K_j] = \sum_{k=1}^{r} \gamma_{ijk} K_k$$

Of course this does not mean that the reachable sets from I namely $\{\exp\{B_i\}_A\}_G$ and $\{\exp\{G_i\}_A\}_G$ are isomorphic as groups. For example the group of unit quaternions and the group of 3 by 3 orthogonal matrices have isomorphic Lie algebras yet they are not isomorphic as groups.

7. System Theory on Coset Spaces

In this section we reinterpret our results in a somewhat different way. This interpretation leads to some facts about systems on manifolds which do not admit a group structure. In particular we have in mind the n-sphere $S^n = \{x : x'x=1,\ x \in \mathbb{R}^{n+1}\}$ which, as is well known, does not admit a Lie group structure except for the cases n=1 and 3.

Let $M \subset \mathbb{R}^n$ be a manifold. Let $\mathcal{G}$ be a matrix group in $\mathbb{R}^{n\times n}$. We say that $\mathcal{G}$ acts on M if for every $x \in M$ and every $G \in \mathcal{G}$, Gx belongs to M. By the orbit of $\mathcal{G}$ through x we mean the set of points $\mathcal{G}x = \{y : y=Gx,\ G \in \mathcal{G}\}$. We say that $\mathcal{G}$ acts transitively on M if it acts on M and if for every pair of points x, y in M, there exists G in $\mathcal{G}$ such that $Gx = y$. If $\mathcal{G}$ acts transitively on M then at any point $x \in M$ there will be a subset $\mathcal{H}_x \subset \mathcal{G}$ such that for each $H \in \mathcal{H}_x$, $Hx = x$. Clearly if $H_1 \in \mathcal{H}_x$ and $H_2 \in \mathcal{H}_x$ then $H_1H_2x = H_1x = x$ and $H^{-1}x = x$ so that $\mathcal{H}_x$ is a subgroup. We call $\mathcal{H}_x$ the isotropy group at x. Notice that if $Gx = y$ then $y = G\mathcal{H}_x x = G\mathcal{H}_x G^{-1}y$ and thus $G\mathcal{H}_x G^{-1}$ is the isotropy group at y -- all isotropy groups are conjugate in $\mathcal{G}$. Now suppose M is a manifold for which there actually exists a group $\mathcal{G}$ acting transitively. Pick a point $x \in M$. Define in $\mathcal{G}$ an equivalence relation whereby $G_1 \sim G_2$ if and only if $G_1 = G_2H_x$ for some $H_x \in \mathcal{H}_x$. There is a one to one correspondence between this space of equivalence classes, $\mathcal{G}/\mathcal{H}_x$, and M. In this case we call M a coset space.

We study systems in which the state is represented as an n-vector and the evolution is governed by

$$\dot{x}(t) = (A + \sum_{i=1}^{\nu} u_i(t)B_i)x(t) \quad ; \quad y(t) = \mathscr{C}x(t) \qquad (*)$$

By $\mathscr{C}x(t)$ we mean an equivalence class of vectors, x_1 being equivalent to x_2 if and only if $Cx_1 = x_2$ for some C in $\mathscr{C}$.

Let $\mathscr{L}$ be the ... algebra generated by $\{A,B_i\}$ and let $M \subset \mathbb{R}^n$ be a manifold such that $\{\exp\mathscr{L}\}_G$ acts on it. Then the above equation can be thought of as evolving on the manifold $M \subset \mathbb{R}^n$ for if $x(0) \in M$ then regardless of the control, $x(t) \in M$ for all $t > 0$. If there exists a differentiable manifold $M \subset \mathbb{R}^n$ such that $\{\exp\mathscr{L}\}_G$ acts on M then we will say that (*) is <u>well posed</u> on M.

<u>Example 6</u> : Consider the n-sphere, S^n. Let $B_1,B_2,\ldots,B_m$ be n+1 by n+1 skew symmetric matrices. Clearly the system

$$\dot{x}(t) = [\sum_{i=1}^{m} u_i(t)B_i]x(t) \quad ; \quad y(t) = \mathscr{C}x(t)$$

is well posed on S^n since $\{\exp\mathscr{L}\}_G$ consists of orthogonal matrices and orthogonal transformations preserve norm. If we can observe only the first component of x then we should let $\mathscr{C}$ be the subsets of $\mathscr{S}o(n+1)$ consisting of those matrices which have a 1 in the first column and first row. That is

$$\mathscr{C} = \begin{bmatrix} 1 & 0 \\ 0 & \mathscr{S}o(n) \end{bmatrix}$$

With respect to controllability we can say given any two vectors x_1 and x_2 in $\mathscr{S}^n$ there exists a piecewise continuous control which steers the system from x_1 to x_2 if and only if $x_2 = Rx_1$ for some R in

$\{\exp \mathscr{L}\}_G$ where $\mathscr{L}$ is the Lie algebra generated by $\{A_i\}$. Also, an arbitrary point can be transferred to an arbitrary point if and only if $\{\exp \mathscr{L}\}_G$ acts transitively on S^n.

At the same time we might observe that any x_o such that $||x_o|| = 1$ can be transferred to any x_1 such that $||x_1|| = 1$ if and only if $\{\exp \mathscr{L}\}_G$ acts transitively on S^n. This second point of view is useful because it puts the problem of controllability on S^n in contact with standard results in geometry. In particular a great deal is known about Lie groups which act transivitely on S^n. [Samelson [17] page 26].

As for observability, we note that two initial states x_1 and x_2 in S^n give rire to the same y if and only if for all R in $\{\exp \mathscr{L}\}_G$ there exists C(R) in $\mathscr{C}$ such that $Rx_1 = C(R)Rx_2$ which is to say that $R^{-1}C(R)Rx_2 = x_1$.

We now abstract from this example the essential features and state formally a result which summarizes the development.

<u>Theorem 11</u> : Consider the dynamical system $(x(t) \in \mathbb{R}^n)$

$$\dot{x}(t) = (\sum_{i=1}^{q} u_i(t)B_i)x(t) \quad ; \quad y(t) = \{\exp \mathscr{H}\}_G x(t)$$

which is well posed on the manifold $M \subset \mathbb{R}^n$. Let $\mathscr{L}$ be the Lie algebra generated by $\{B_i\}$. A given state x_2 is reachable from x_1 if and only if $x_2 = Nx_1$ for some N in $\{\exp \mathscr{L}\}_G$. Let $\mathscr{P} = \{P: RPR^{-1} \in \{\exp \mathscr{H}\}_G$ all $R \in \exp \mathscr{L})\}$. Two states x_1 and x_2 are indistinguishable if and only if $x_2 = Px_1$ for some P in $\mathscr{P}$. In particular, two states x_1 and x_2 are indistinguishable if $x_2 = Px_1$ for P in $\{\exp \mathscr{K}\}_G$ with being any Lie algebra such that $\{\mathrm{ad}_{\mathscr{L}} \mathscr{K}\}_A \subset \mathscr{H}$.

<u>Example 7</u> : Consider the submanifold M of $\mathbb{R}^{n+1}$ consisting of those points whose last coordinate is 1. The evolution equation in M

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ 1 \end{bmatrix} = \begin{bmatrix} A & Bu(t) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ 1 \end{bmatrix} \; ; \quad y(t) = \left(\exp\begin{bmatrix} 0 & \mathrm{KerC} \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} x(t) \\ 1 \end{bmatrix}$$

corresponds to the more familiar $\dot{x}(t) = Ax(t) + Bu(t)$; $y(t) = Cx(t)$. Using theorem 7 we see that for the associated group equation

$$\dot{X}(t) = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \sum_{i=1}^{m} u_i(t) \begin{bmatrix} 0 & b_i \\ 0 & 0 \end{bmatrix} X(t)$$

the reachable set at time t consists of those matrices which can be written as

$$\mathscr{R} = \{X : X = \begin{bmatrix} e^{At} & x \\ 0 & 1 \end{bmatrix} \; ; \; x \in \mathrm{range}(B, AB, \ldots, A^{n-1}B)\}$$

Thus if $B, AB, \ldots, A^{n-1}B$ spans $\mathbb{R}^n$ then the reachable group acts transitively on M and we have controllability.

As for observability, we note that

$$\exp\begin{pmatrix} 0 & \mathrm{KerC} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & \mathrm{KerC} \\ 0 & 1 \end{pmatrix}$$

The subalgebras of $\mathscr{H}$ which are closed under commutation with $\mathscr{L}$ correspond to the linear subspaces of KerC which are invariant under A.

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The Control Problem $\dot{x} = (A(1-u)+Bu)x$: A Comment on an Article by J. Kucera

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In an article recently published in this journal ([1]), J. Kučera studied the control problem $\dot{x} = (A(1-u) + Bu)x$. The main results of [1] are that the set $\mathcal{A}(\omega,T)$ of points attainable at time $T > 0$ from a fixed point ω is an "integral manifold of the distribution $\mathcal{B}(A,B)$" ([1], Theorem 2.2), and that the set $\mathcal{A}'(\omega,T) = \bigcup\{\mathcal{A}(\omega,T) : 0 \leq t \leq T\}$ is an "integral manifold of the distribution $\mathcal{U}(A,B)$" ([1], Theorem 2.1). The purpose of this note is to show that Lemma 2.8 of [1], which is a fundamental step in the proof of Theorems 2.1 and 2.2, is false. The natural question to be asked now is whether these results are nevertheless valid; it will be shown in a forthcoming paper that they are. The proof, however, is based on a completely different technique.

We quote the statement of Lemma 2.8 of [1] :

"[Let] $T > 0$, $\delta \in (0, \frac{1}{2})$, $u \in M(\delta,1-\delta)$. Let the function u be not constant in $\langle 0,T\rangle$ (not equivalent with a constant function), then

$$\mathcal{V}(x(T,u)) \subset \bigcup_{r=1}^{\infty} r.K_u(T)."$$

The notations of the above statement have the following meaning:

a) "(α,β)" (resp. $\langle\alpha,\beta\rangle$) is the open (resp. closed) interval with endpoints α,β.

b) "$M(\alpha,\beta)$" is the set of all measurable functions in $(0,\infty)$ whose values lie in $\langle\alpha,\beta\rangle$, $\alpha < \beta$.

c) $t \to x(t,u)$ is the solution of the equation

$$\frac{d}{dt}x(t) = (A(1-u(t)) + Bu(t))x(t)$$

which satisfies $x(0) = \omega$. Here ω is a fixed element of R^n (n-dimensional

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real space), and $\underline{A}$, $\underline{B}$ are fixed elements of $\mathcal{M}_{\underline{n}}$ (the set of all $\underline{n}$ by $\underline{n}$ real matrices).

d) "$\mathcal{V}$" is the "distribution created by $\mathcal{B}$", i.e. the mapping that assigns to each $\underline{x} \in \underset{\sim}{R}^{\underline{n}}$ the set $\mathcal{V}(\underline{x})$ of all elements of $\underset{\sim}{R}^{\underline{n}}$ of the form $\underline{Px}$, $\underline{P} \in \mathcal{B}$.

e) "$\mathcal{B}$", or "$\mathcal{B}(\underline{A},\underline{B})$" is "the smallest linear space of $\underline{n}$ by $\underline{n}$ matrices which contains the matrix $\underline{C}[=\underline{B}-\underline{A}]$ and, with each $\underline{P} \in \mathcal{B}$, contains also both matrices $[\underline{A},\underline{P}]$ and $[\underline{B},\underline{P}]$" ([1], Def. 2.2; the notation "$[\underline{M},\underline{N}]$" means "$\underline{NM}-\underline{MN}$").

f) $\underline{X}_{\underline{u}}(\underline{T})$ is the set of all vectors $\underline{x}_1(\underline{T},\underline{v})$, $\underline{v} \in \underline{M}(-1,1)$, where

$$\underline{x}_1(\underline{T},\underline{v}) = \underline{X}(\underline{T})\left(\int_0^{\underline{T}} \underline{X}^{-1}(\underline{t})\underline{CX}(\underline{t})\underline{v}(\underline{t})\underline{dt}\right) \quad ,$$

and where

g) $\underline{t} \to \underline{X}(\underline{t})$ is the $\underline{n}$ by $\underline{n}$ matrix-valued solution of

$$\frac{d}{dt}\,\underline{X}(\underline{t}) = (\underline{A}(1-\underline{u}(\underline{t})) + \underline{Bu}(\underline{t}))\underline{X}(\underline{t})$$

which satisfies $\underline{X}(0) = \underline{I}$, ($\underline{I}$ is the $\underline{n}$ by $\underline{n}$ identity matrix).

We shall show that Lemma 2.8 is false by means of a counterexample. Consider the space $\mathcal{P}_0$ of all real polynomials in two <u>noncommuting</u> variables $\underline{y},\underline{z}$. Form the space $\mathcal{P}$ by equating to zero all the monomials of degree 5 or more (in other words $\mathcal{P}_0$ is the free algebra over the reals generated by $\underline{y}$ and $\underline{z}$, and $\mathcal{P}$ is the quotient of $\mathcal{P}_0$ by the ideal generated by all the monomials of degree 5). Thus, $\mathcal{P}$ is a 31-dimensional real vector space, and the monomials $1, \underline{y}, \underline{z}, \underline{y}^2, \underline{yz}, \underline{zy}, \underline{z}^2, \underline{y}^3, \underline{y}^2\underline{z}, \underline{yzy}, \underline{zy}^2, \underline{yz}^2, \underline{zyz}, \underline{z}^2\underline{y}, \underline{z}^3, \underline{y}^4, \underline{y}^3\underline{z}, \underline{y}^2\underline{zy}, \underline{yzy}^2, \underline{zy}^3, \underline{y}^2\underline{z}^2, \underline{yzyz}, \underline{zy}^2\underline{z}, \underline{yz}^2\underline{y}, \underline{zyzy}, \underline{z}^2\underline{y}^2, \underline{yz}^3, \underline{zyz}^2,$

z^2yz, z^3y, z^4 are a basis for $\mathcal{P}$. Moreover, $\mathcal{P}$ is an associative algebra over the reals, with the obvious multiplication table (for instance: $zy \cdot zy = zyzy$, $zy \cdot y^3 = 0$, etc.).

By means of this basis we can (and shall) identify $\mathcal{P}$ with $\mathbb{R}^{31}$. In $\mathcal{P}$, the mappings $p(y,z) \to yp(y,z)$ and $p(y,z) \to zp(y,z)$ are linear. Via the above mentioned identification, we obtain two 31 by 31 matrices M_y and M_z such that these mappings correspond to $x \to M_y x$ and $x \to M_z x$, respectively. We let $A = M_y$, $B = M_z + M_y$, so that $C = M_z$. To begin with, we compute the space $\mathcal{B}$. It is clear from the definition that $\mathcal{B}$ is the smallest linear space that contains C such that, if $P \in \mathcal{B}$, then $[A,P]$ and $[C,P]$ belong to $\mathcal{B}$. Thus $\mathcal{B}$ is the linear hull of the set $\mathcal{E}$ of all matrices $[Q_1,[Q_2,\ldots,[Q_{k-1},Q_k]\ldots]]$, where k is an integer > 0, and where $Q_i = A$ or $Q_i = C$ for $i = 1,\ldots,k-1$, $Q_k = C$ Using the facts that $[C,C] = 0$ and that $[A,[C,[A,C]]] = [C,[A,[A,C]]]$ (an immediate consequence of the equality $[P,P] = 0$ and of the Jacobi identity $[P,[Q,R]] = [[P,Q],R] + [Q,[P,R]]$) we see that the following are all the elements of $\mathcal{B}$ corresponding to $k \leq 4$:

$$M_1 = C,\ M_2 = [A,C],\ M_3 = [A,[A,C]],\ M_4 = [C,[A,C]],$$

$$M_5 = [A,[A,[A,C]]],\ M_6 = [C,[A,[A,C]]] \text{ and } M_7 = [C,[C,[A,C]]].$$

In addition, all the elements of $\mathcal{E}$ corresponding to $k \geq 5$ vanish. This is so because, via our identification of $\mathbb{R}^{31}$ with $\mathcal{P}$ (and of the corresponding identification of $\mathcal{M}_{31}$ with the set of endomorphisms of the vector space $\mathcal{P}$), every element $Q = [Q_1,[Q_2,\ldots,[Q_{k-1},Q_k]\ldots]]$ of $\mathcal{E}$ corresponds to the multiplication in $\mathscr{P}$ by a homogeneous polynomial $\underline{p}(\underline{Q})$ of degree $\underline{k}$ (for instance, if $\underline{Q} = [\underline{A},\underline{C}]$, then $\underline{p}(\underline{Q}) = \underline{zy}-\underline{yz}$). Since every homogeneous polynomial of degree ≥ 5 vanishes in $\mathscr{P}$, our assertion follows.

Thus $\mathscr{B}$ is the linear hull of $\underline{M}_1,\ldots,\underline{M}_7$. We show that these matrices are linearly independent. It is sufficient to prove that the multiplications by the corresponding polynomials $\underline{p}(\underline{M}_1),\ldots,\underline{p}(\underline{M}_7)$ are linearly independent. If these multiplications were not independent, then the images of the polynomial 1 would be dependent, i.e. the polynomials $\underline{p}(\underline{M}_1),\ldots,\underline{p}(\underline{M}_7)$ would be dependent. Thus, it is sufficient to show that these polynomials are independent. But $\underline{p}(\underline{M}_1),\ldots,\underline{p}(\underline{M}_7)$ are homogeneous polynomials both in $\underline{y}$ and $\underline{z}$, and no two of them have the same degrees both in $\underline{y}$ and $\underline{z}$. Hence they are independent.

We have shown that $\underline{M}_1,\ldots,\underline{M}_7$ form a basis for $\mathscr{B}$, so that $\mathscr{B}$ has dimension seven.

We shall take ω to be the element of $\underline{R}^{31}$ which corresponds to the polynomial 1 of $\mathscr{P}$. Thus, $\mathscr{V}(\omega)$ is the linear hull of $\underline{p}(\underline{M}_1),\ldots,\underline{p}(\underline{M}_7)$, and $\dim \mathscr{V}(\omega)=7$. We take δ to be an arbitrary element of $(0,\frac{1}{2})$, and define the control $\underline{u}$ by

$$\underline{u}(\underline{t}) = \delta + \underline{t} \qquad \text{for } 0 \leq \underline{t} \leq \underline{T},$$

where $\underline{T} = 1 - 2\delta$. Thus all the assumptions of Lemma 2.8 of [1] hold. We show that the dimension of $\mathscr{V}(\underline{x}(\underline{T},\underline{u}))$ is also seven. This is an immediate consequence of Lemma 2.11 of [1], or it can be proved directly as follows: the equation

$$\frac{d}{dt}\underline{x}(\underline{t},\underline{u}) = (\underline{A} + \underline{u}(\underline{t})\underline{C})\underline{x}(\underline{t},\underline{u})$$

implies the derivative of the polynomial $\underline{x}(\underline{t},\underline{u})$ is a polynomial in $\underline{y},\underline{z}$ without a constant term. This implies that the constant term of $\underline{x}(\underline{t},\underline{u})$ is 1 for all $\underline{t}$ (because $\underline{x}(0,\underline{u}) = \omega = 1$). From this it follows immediately that the seven polynomials $\underline{p}(\underline{M}_i)\underline{x}(\underline{t},\underline{u})$ are linearly independent.

We shall show that the dimension of the subspace $\bigcup_{\underline{r}=1}^{\infty}\underline{r}.\underline{K}_{\underline{u}}(\underline{T})$ is not greater than six. To begin with, $\bigcup_{\underline{r}=1}^{\infty}\underline{r}.\ \underline{K}_{\underline{u}}(\underline{T})$ is obviously the set of all elements of the form

$$\underline{X}(\underline{T})\left(\int_0^{\underline{T}} \underline{X}^{-1}(\underline{t})\underline{C}\underline{X}(\underline{t})\underline{v}(\underline{t})\underline{dt}\right)\omega \quad ,$$

where $\underline{v}$ is an arbitrary bounded measurable function in $<0,\underline{T}>$. We see immediately that this is the same as the linear hull $\underline{L}''$ of $\underline{X}(\underline{T})\underline{X}^{-1}(\underline{t})\underline{C}\underline{X}(\underline{t})\underline{\omega}$, $\underline{t} \in <0,\underline{T}>$. The dimension of $\underline{L}''$ is the same as that of the linear hull $\underline{L}'$ of all the elements $\underline{X}^{-1}(\underline{t})\underline{C}\underline{X}(\underline{t})\omega$ (because $\underline{X}(\underline{T})$ is nonsingular). Finally, this dimension is not greater than that of the linear hull $\underline{L}$ of all the matrices $\underline{X}^{-1}(\underline{t})\underline{C}\underline{X}(\underline{t})$, $\underline{t} \in <0,\underline{T}>$.

Thus, it is sufficient to show that dim $\underline{L} \leq 6$. Since $\underline{u}(\underline{t}) = \underline{t} + \delta$, the function $\underline{t} \to \underline{X}^{-1}(\underline{t})\underline{C}\underline{X}(\underline{t}) = \underline{y}(\underline{t})$ is analytic. Thus, $\underline{L}$ is the linear hull of the coefficients of the power series expansion of $\underline{y}$ in a neighborhood of $\underline{t} = 0$ or, equivalently, $\underline{L}$ is the linear hull of $\{\frac{d^n}{dt^n}\ \underline{y}(\underline{t})\big|_{\underline{t}=0} : \underline{n}=0,1,\ldots\}$.

Since $\frac{d}{dt}\underline{X}(\underline{t}) = (\underline{A}+\underline{u}(\underline{t})\underline{C})\underline{X}(\underline{t})$, we see that

$$\frac{d}{dt}\underline{X}^{-1}(\underline{t}) = -\underline{X}^{-1}(\underline{A}+\underline{u}(\underline{t})\underline{C}) \ .$$

If $\underline{M}(\underline{t})$ is any matrix-valued function, we have

$$\frac{d}{dt}(\underline{X}^{-1}(\underline{t})\underline{M}(\underline{t})\underline{X}(\underline{t})) = -\underline{X}^{-1}(\underline{t})(\underline{A}+\underline{u}(\underline{t})\underline{C})\underline{M}(\underline{t})\underline{X}(\underline{t})+\underline{X}^{-1}(\underline{t})\underline{M}(\underline{t})(\underline{A}+\underline{u}(\underline{t})\underline{C})\underline{X}(\underline{t})$$
$$+\underline{X}^{-1}(\underline{t})\frac{d}{dt}\underline{M}(\underline{t}) \cdot \underline{X}(\underline{t}) \ .$$

Applying this formula to the successive derivatives of Y we obtain

$$\frac{d^i}{dt^i}\, \underline{Y}(\underline{t}) = \underline{X}^{-1}(\underline{t})\underline{F}_i(\underline{t})\underline{X}(\underline{t}) \ ,$$

where

$$\underline{F}_0(\underline{t}) \equiv \underline{C}, \quad \underline{F}_1(\underline{t}) \equiv [\underline{A},\underline{C}], \quad \underline{F}_2(\underline{t}) = [\underline{A}+(\underline{t}+\delta)\underline{C},[\underline{A},\underline{C}]],$$

$$\underline{F}_3(\underline{t}) = [\underline{A}+(\underline{t}+\delta)\underline{C},[\underline{A}+(\underline{t}+\delta)\underline{C},[\underline{A},\underline{C}]]]+[\underline{C},[\underline{A},\underline{C}]],$$

$$\underline{F}_4(\underline{t}) = [\underline{C},[\underline{A}+(\underline{t}+\delta)\underline{C},[\underline{A},\underline{C}]]]+2[\underline{A}+(\underline{t}+\delta)\underline{C},[\underline{C},[\underline{A},\underline{C}]]],$$

$$\underline{F}_5(\underline{t}) \equiv 3[\underline{C},[\underline{C},[\underline{A},\underline{C}]]] \quad \text{and} \quad \underline{F}_6(\underline{t}) \equiv 0.$$

In the above computations we have used the fact that every element of $\mathcal{E}$ corresponding to $\underline{k} > 5$ vanishes.

Our computations show that, of all the matrices $\frac{d^n}{dt^n}\underline{Y}(\underline{t})\Big|_{\underline{t}=0}$, only those for $\underline{n}=0,\ldots,5$ are nonzero. Thus, the dimension of $\underline{L}$ is not greater than 6.

The preceeding remarks show that, in our example, the dimension of $\mathcal{E}$ $\mathcal{V}(\underline{X}(\underline{T},\underline{u}))$ is seven, while the dimension of $\bigcup_{\underline{r}=1}^{\infty} \underline{r}.\underline{K}_{\underline{u}}(\underline{T})$ is less than six. Therefore, the conclusion of Lemma 2.8 of [1] does not hold, even though all the assumptions are satisfied. Thus, Lemma 2.8 of [1] is false.

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THE BANG-BANG PROBLEM FOR CERTAIN CONTROL SYSTEMS IN GL(n, R) *

H. T. Sussmann**

ABSTRACT

We discuss the linear control problem $\dot{X}(t) = [A_0(t) + \sum_{i=1}^{m} u_i(t)A_i(t)] X(t)$, where $A_0, \ldots, A_m$ are $n \times n$ matrix-valued functions of time, and where $X(t) \in GL(n, R)$. We show that the set attainable from any element $M \in GL(n, R)$ at time t by "bang-bang" controls is closed, provided the following very strong assumption is satisfied: for all i, j and for all t', t'' such that $0 \leq t' \leq t$, $0 \leq t'' \leq t$, the matrices $A_i(t')$ and $A_j(t'')$ commute. We also show, by means of counterexamples, that these assumptions cannot be weakened.

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1. Introduction

Recently, interest has arisen in the study of the linear control problem in manifolds and Lie groups (Brockett [1], Haynes and Hermes [4], Kučera [5] and [6]). Such a control problem is of the form

$$\dot{\underline{x}}(\underline{t}) = \underline{X}_0(\underline{x}(\underline{t})) + \sum_{i=1}^{m} \underline{u}_i(\underline{t})\, \underline{X}_i(\underline{x}(\underline{t})) \quad ,$$

where $\dot{\underline{x}}(\underline{t})$ denotes the tangent vector to the curve $\tau \to \underline{x}(\tau)$ at $\tau = \underline{t}$, and where $\underline{X}_0, \ldots, \underline{X}_m$ are vector fields.

A particularly important case is that in which the manifold is a Lie group, and the vector fields are translation-invariant. When the Lie group is $\underline{GL}(\underline{n}, \underline{R})$, the problem takes the simple form

$$\dot{\underline{X}}(\underline{t}) = (A_0 + \sum_{i=1}^{m} \underline{u}_i(\underline{t})\, \underline{A}_i)\, \underline{X}(\underline{t}) \quad ,$$

where $\underline{A}_0, \ldots, \underline{A}_m$ are constant matrices.

The purpose of this article is to indicate what hopes there should be of building a reasonable "bang-bang" theory for this problem, and for the more general one in which the matrices $\underline{A}_i$ are time-dependent. The important issue is, as usual, to determine whether the attainable set at time $\underline{T}$ is closed. It might seem likely that the tool to be used should be some generalization of the well known theorem of Lyapounov, which has proved so fruitful for similar problems (Lyapounov [7], Halkin [2] and [3]). However, as we shall show, not much is to be expected in this direction. We shall prove, that, under certain <u>very restrictive</u> conditions, closedness of the attainable set follows by a straightforward application of Lyapounov's theorem. We shall also show that, if these conditions are weakened, it is possible to give examples of control systems for which the attainable set fails to be closed.

The new aspect that plays a fundamental role is that of the commutativity of the matrices $\underline{A}_i$. The very restrictive assumptions to which we have referred are the following:

a) The condition that $[\underline{A}_i(t), \underline{A}_j(t')]^{(1)}$ should vanish for all i, j, t, t'. This will guarantee that the set attainable at time T by "bang-bang" controls is closed, provided we define a "bang-bang" control as a measurable function with values in the set $\{-1, 1\}$.

b) The additional condition that the functions $\underline{A}_i(t)$ should be piecewise analytic. If this is true, we will be able to get closedness even if we restrict the class of "bang-bang" controls to piecewise constant functions with values in $\{-1, 1\}$. Of course, this covers the time-independent case in particular.

The main point of this paper is that these conditions cannot be weakened. This will be shown by giving three examples of non-closed attainable sets. These examples cover, in our opinion, the simplest possible conceivable departures from the commutativity condition. Thus, our results constitute a rather final answer to the closedness problem.

Our results also apply to systems in which $\underline{X}$ is a column vector in $\underline{R}^n$, rather than an $n \times n$ matrix (cf. Remark 1 of section 4).

2. Notations and Preliminary Lemmas

We shall consider the control problem

$$\dot{X}(t) = (A_0(t) + \sum_{i=1}^{m} u_i(t)\, A_i(t))\, X(t) \quad , \tag{1}$$

where $X(t)$ belongs to $GL(n, R)$ (the set of all nonsingular real $n \times n$ matrices). The functions $A_i(t)$ are supposed to be bounded and measurable, with values in the set $M(n, R)$ of all $n \times n$ real matrices. For $A \in M(n, R)$, define the norm of A (denoted by $\|A\|$) as the supremum of $\|Ax\|$, where x ranges through all the vectors in R^n such that $\|x\| = 1$, and where, for $x \in R^n$, $\|x\|$ denotes the Euclidean norm $(\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$.

We shall denote by $U(T)$, for each $T > 0$, the set of all measurable(2) functions defined in the closed interval $[0, T]$ with values in the cube $\{(u_1, \ldots, u_m): -1 \leq u_i \leq 1,\ i = 1, \ldots, m\}$. We shall denote by $UB(T)$ the subset of $U(T)$ whose elements are the "bang-bang" functions, i.e. the measurable functions $(u_1(t), \ldots, u_m(t))$ such that $u_i(t) = 1$ or $u_i(t) = -1$ for all $i = 1, \ldots, m$ and all $0 \leq t \leq T$. Finally, the set of all $u \in UB(T)$ that are piecewise constant will be denoted by $UBP(T)$.

It is clear that $U(T)$ is a bounded and weakly closed subset of $L_2[0, T]$, so that $U(T)$ is weakly compact. We also have:

LEMMA 1. $\underline{UBP}(\underline{T})$ is weakly dense in $\underline{U}(\underline{T})$.

Proof: It is clearly sufficient to assume $\underline{m} = 1$. Since every function in $\underline{U}(\underline{T})$ can be approximated in the $\underline{L}_2$-norm by piecewise constant functions, it follows that it will be sufficient to show that every constant function is a weak limit of elements of $\underline{UBP}(\underline{T})$.

Let $\underline{u}(\underline{t}) \equiv \underline{r} \leq 1$, for $0 \leq \underline{t} \leq \underline{T}$. We can assume $\underline{r} \geq 0$. For each interval $\underline{I} = [\underline{a}, \underline{b}]$, let the function $\underline{f}_{\underline{I}}$ be defined as follows:

$$\underline{f}_{\underline{I}}(\underline{t}) = -1 \qquad \text{for} \quad \underline{a} \leq \underline{t} \leq \underline{a} + \tfrac{1}{2}(1-\underline{r})(\underline{b}-\underline{a})$$

$$\underline{f}_{\underline{I}}(\underline{t}) = 1 \qquad \text{for} \quad \underline{a} + \tfrac{1}{2}(1-\underline{r})(\underline{b}-\underline{a}) < \underline{t} \leq \underline{b}.$$

Then, clearly, $\int_{\underline{a}}^{\underline{b}} \underline{f}_{\underline{I}}(\underline{t})\,\underline{dt} = \underline{r}(\underline{b}-\underline{a})$. Now define $\underline{u}_{\underline{k}}$ (for $\underline{k} = 1, 2, \ldots$) by partitioning the interval $[0, \underline{T}]$ into $\underline{k}$ intervals $\underline{I}_{\underline{k}1}, \ldots, \underline{I}_{\underline{kk}}$ of length $\underline{T}\underline{k}^{-1}$, and letting $u_k(\underline{t}) = \underline{f}_{\underline{I}_{ki}}(\underline{t})$ for each $\underline{t} \in \underline{I}_{\underline{ki}}$, $\underline{i} = 1, \ldots, \underline{k}$. It is now obvious that the functions $\underline{u}_{\underline{k}}$ belong to $\underline{UBP}(\underline{T})$ and that their weak limit is $\underline{u}$. The proof of our lemma is thus complete.

Let $\underline{u} \in \underline{U}(\underline{T})$. Let $\underline{X}(\underline{u}, \cdot)$ be the solution of equation (1) which satisfies the initial condition $\underline{X}(0) = \underline{I}$ ($\underline{I} = \underline{n} \times \underline{n}$ identity matrix). The set of all matrices $\underline{X}(\underline{u}, \underline{T})$, for $\underline{u} \in \underline{U}(\underline{T})$, is the attainable set at time $\underline{T}$, and we shall denote it by $\underline{S}(\underline{T})$. If we restrict ourselves to functions $\underline{u} \in \underline{UB}(\underline{T})$, (resp. $\underline{u} \in \underline{UBP}(\underline{T})$), we can similarly define the sets $\underline{SB}(\underline{T})$, $\underline{SBP}(\underline{T})$. The union of the sets $\underline{S}(\underline{t})$ for all $0 \leq \underline{t} \leq \underline{T}$ will be denoted by $\underline{S}'(\underline{T})$. In a similar way, we define the sets $\underline{SB}'(\underline{T})$, $\underline{SBP}'(\underline{T})$.

It is clear that no loss of generality is involved in limiting ourselves to the study of the sets attainable from the identity. Indeed, the set of matrices attainable from any other $\underline{M} \in \underline{GL}(\underline{n}, \underline{R})$ is just the set of all products $X(u,t)M$.

LEMMA 2. *Let the functions* u_k *converge weakly to* u. *Then* $\{X(u_k, t)\}$ *converges uniformly to* $X(u,t)$ *for* $0 \leq t \leq T$.

Proof: For each $v \in U(T)$, we have

$$X(v,t) = I + \int_0^t [A_0(\tau) + \sum_{i=1}^{m} v_i(\tau) A_i(\tau)] \, X(v,\tau) \, d\tau \qquad (2)$$

Since the functions A_i are bounded, and $|v_i(\tau)| \leq 1$, there is a constant $C > 0$ such that

$$\|X(v,t)\| \leq 1 + C \int_0^t \|X(v,\tau)\| \, d\tau$$

for all $v \in U(T)$, and all $0 \leq t \leq T$.

It follows by a well-known argument that

$$\|X(v,t)\| \leq \exp(Ct), \text{ for all } v, t.$$

In particular, we see that the functions $X(v, \cdot)(v \in U(T))$ are uniformly bounded. Equation (1) then implies that the derivatives of these functions are also uniformly bounded.

To show that $X(u_k, \cdot)$ converges uniformly to $X(u, \cdot)$, it is sufficient to show that every subsequence has a subsequence that converges uniformly to $X(u, \cdot)$. By the previous paragraph and the Ascoli-Arzelà theorem, every subsequence has a subsequence that converges uniformly to some function Thus, our lemma will be proved if we show that, if $\{v_k\}$ converges weakly to v, and if $X(v_k, \cdot)$ converges uniformly to $X(\cdot)$, then $X(\cdot) \equiv X(v, \cdot)$.

Equation (2) implies that

$$\underline{X}(\underline{v}_k, t) = \underline{I} + \int_0^t [\underline{A}_0(\tau) + \sum_{i=1}^m (\underline{v}_k)_i(\tau)\underline{A}_i(\tau)] [\underline{X}(\underline{v}_k, \tau) - \underline{X}(\tau)] \, d\tau$$

$$+ \int_0^t [\underline{A}_0(\tau) + \sum_{i=1}^m (\underline{v}_k)_i(\tau)\underline{A}_i(\tau)] \, \underline{X}(\tau) \, d\tau \ .$$

Using the weak convergence of $\underline{v}_k$ to $\underline{v}$, and the uniform convergence of $\underline{X}(\underline{v}_k, \cdot)$ to $\underline{X}(\cdot)$, it follows that

$$\underline{X}(t) = \underline{I} + \int_0^t [\underline{A}_0(\tau) + \sum_{i=1}^m \underline{v}_i(\tau)\, \underline{A}_i(\tau)] \, \underline{X}(\tau) \, d\tau \ .$$

Then, $\underline{X}(t) \equiv \underline{X}(\underline{v}, t)$, and our lemma is proved.

COROLLARY 1. *The mapping $\underline{u} \to \underline{X}(\underline{u}, \cdot)$ is continuous from* U(T) *with the weak topology into the space of continuous $\underline{M}(\underline{n}, \underline{R})$ -valued functions in $[0, \underline{T}]$ with the uniform topology*

COROLLARY 2. *The sets $\underline{S}(\underline{T})$, $\underline{S}'(\underline{T})$ are compact.*

COROLLARY 3. *The sets $\underline{SBP}(\underline{T})$, $\underline{SBP}'(\underline{T})$ are dense in $\underline{S}(\underline{T})$, $\underline{S}'(\underline{T})$, respectively.*

Proof. Corollary 1 is a restatement of Lemma 2. Cor. 2 follows from Cor. 1 and the fact that $\underline{U}(\underline{T})$ is weakly compact. Finally, Cor. 3 follows from Lemma 1 and Cor. 1.

3. Closedness of the "Bang-Bang"-Attainable Set

It is clear from the preceding section that closedness of the attainable set $\underline{SB}(\underline{T})$ (resp. $\underline{SBP}(\underline{T})$, $\underline{SB}'(\underline{T})$, $\underline{SBP}'(\underline{T})$) is equivalent to the identity $\underline{S}(\underline{T}) = \underline{SB}(\underline{T})$ (resp. $\underline{S}(\underline{T}) = \underline{SBP}(\underline{T})$, $\underline{S}'(\underline{T}) = \underline{SB}'(\underline{T})$, $\underline{S}'(\underline{T}) = \underline{SBP}'(\underline{T})$).

The following theorem is a positive result in this direction and, as we shall prove in the next section, it is the best possible result of that type.

THEOREM 1. *If all the brackets* $[\underline{A}_i(t), \underline{A}_j(t')]$ *vanish (for all i, j, t, t'), then* $\underline{SB}(\underline{T})$ *and* $\underline{SB}'(\underline{T})$ *are closed. If, in addition, the functions* $\underline{A}_i$ *are piecewise analytic, then the sets* $\underline{SBP}(\underline{T})$ *and* $\underline{SBP}'(\underline{T})$ *are also closed.*

Proof. If our assumption about the brackets holds, the solution of equation (1) is given by

$$\underline{X}(u,t) = \exp\left(\int_0^t \underline{A}_0(\tau)d\tau\right) \cdot \prod_{i=1}^{m} \exp\left(\int_0^t \underline{A}_i(\tau)\underline{u}_i(\tau)d\tau\right).$$

To verify this, notice that:(i) the derivative of $\exp(\underline{F}(\underline{t}))$ is $\underline{F}'(\underline{t})\exp(\underline{F}(\underline{t}))$, if $\underline{F}$ is a matrix-valued function *such that* $\underline{F}(\underline{t}_1)$ *and* $\underline{F}(\underline{t}_2)$ *commute* for all $\underline{t}_1, \underline{t}_2$, and that: (ii) $\exp(\underline{M} + \underline{N}) = \exp(\underline{M}) \cdot \exp(\underline{N})$ if $\underline{M}$ and $\underline{N}$ are two *commuting* matrices (these two facts are proved, using the power series expansion for the exponential, in exactly the same way as for the scalar case; the commutativity makes it possible to "rearrange" factors). From (i) it follows easily that $\underline{X}(\underline{u},\underline{t}) = \exp\left(\int^{t} (\underline{A}_0 + \Sigma\, \underline{u}_i\underline{A}_i)\right)$. The desired expression then results from (ii).

It follows from Lyapounov's theorem on the range of a vector-valued measure (Lyapounov [7], Halkin [2]), that the set of matrices $\int_0^t \underline{A}_i(\tau)\underline{u}(\tau)d\tau$, where $\underline{u}$ ranges over the set of all measurable functions with values in $\{-1, 1\}$, is compact for each $\underline{i}$. Thus, the first part of our statement is clear. The second part follows in a similar way: according to a theorem of Halkin (see[2] and [3]), the set of values $\int_0^t \underline{f}(\tau)\underline{u}(\tau)d\tau$, where $\underline{u}$ ranges over all piecewise constant $\{-1, 1\}$-valued functions in $[0, \underline{T}]$, and where $\underline{f}$ is a vector-valued piecewise analytic function, is compact.

4. Counterexamples

We now show that the assumptions of Theorem 1 cannot be weakened. Clearly, the simplest situations in which these assumptions do not hold are

a) the control problem

$$\dot{\underline{X}}(t) = (\underline{B} + u(t)\underline{C})\, \underline{X}(t), \qquad (3)$$

where $\underline{B}$ and $\underline{C}$ are constant matrices such that $\underline{BC} \neq \underline{CB}$,

b) the problem

$$\dot{\underline{X}}(t) = (\underline{u}(t)\underline{B} + \underline{v}(t)\underline{C})\,\underline{X}(t) \qquad (4)$$

where $\underline{B}$ and $\underline{C}$ are as in a), and

c) the problem

$$\dot{\underline{X}}(t) = \underline{v}(t)\,\underline{F}(t)\,\underline{X}(t), \qquad (5)$$

where $\underline{F}$ is a matrix-valued function such that $[\underline{F}(t), \underline{F}(t')] \neq 0$ for some t, t'.

THEOREM 2. *In each of the cases* a), b), c), *the set* $\underline{SB}(\underline{T})$ *need not be closed*.

Proof. We shall exhibit examples of problems where $\underline{SB}(\underline{T})$ fails to be closed. Our three examples will involve 4×4 matrices, i.e., we shall be working in $\underline{GL}(4,\underline{R})$. The examples for cases b) and c) will be derived from the example for case a).

We let

$$\underline{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \underline{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Let $\underline{X}(\underline{v}, t)$ be the solution of equation (3) whose value at $t = 0$ is the identity matrix.

It is possible to compute $\underline{X}(\underline{v}, t)$ explicitly. The result is

$$X(v,t) = \begin{bmatrix} 1 & f(t) & \frac{1}{2}[f(t)]^2 & g(t) \\ 0 & 1 & f(t) & h(t) \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where

$$f(t) = \int_0^t v(\tau)\, d\tau \quad ,$$

$$h(t) = \int_0^t \tau\, v(\tau)\, d\tau \quad ,$$

$$g(t) = \int_0^t v(\tau)\, h(\tau)\, d\tau$$

If $SB(T)$ were closed for some $T > 0$, it would follow from Cor. 3 that $SB(T) = S(I)$. In particular, the matrix $X(0,t)$ would belong to $SB(T)$. Thus, there would exist a "bang-bang" control v for which

$$0 = \int_0^T v(\tau)\, d\tau = \int_0^T \tau\, v(\tau)\, d\tau = \int_0^T v(\tau) \int_0^\tau \lambda\, v(\lambda)\, d\lambda\, d\tau. \qquad (6)$$

We shall show that this is impossible. Indeed, by repeated integrations by parts we get:

$$\int_0^T v(\tau) \int_0^\tau \lambda\, v(\lambda)\, d\lambda\, d\tau = \int_0^T f'(\tau)\, h(\tau)\, d\tau$$

$$= f(T)\, h(T) - \int_0^T f(\tau)\, h'(\tau)\, d\tau$$

$$= f(T)\, h(T) - \int_0^T f(\tau)\, f'(\tau)\, \tau\, d\tau$$

$$= f(T)\, h(T) - \tfrac{1}{2} \int_0^T \frac{df^2}{d\tau}(\tau)\, \tau \, d\tau$$

$$= f(T) h(T) - \tfrac{1}{2}\ T\, f^2(T) + \tfrac{1}{2} \int_0^T [f(\tau)]^2 \, d\tau$$

If a control v satisfies (6), it follows that $\int_0^T [f(\tau)]^2 \, d\tau = 0$. This implies that $f(\tau) = 0$ almost everywhere. Since v is the derivative of f a.e., we must have that $v = 0$ a.e., so that v cannot be "bang-bang". This completes the proof that $SB(T)$ is not closed for any $T > 0$.

Turning now to case b), we shall use the same matrices B and C as before If $X(u, v, \cdot)$ denotes the solution of equation (4) whose value at $t = 0$ is I, it is clear that $X(1, v, \cdot)$ is the function $X(v, \cdot)$ of the previous paragraph. We claim that $X(1, 0, T)$ cannot be attained by "bang-bang" controls u, v. In view of what we have proved above, it is sufficient to show that, if $X(u, v, T) = X(1, 0, T)$, then $u \equiv 1$. But this can be seen easily as follows: we can compute $X(u, v, T)$ explicitly and obtain $\int_0^T u(\tau)\, d\tau$ as the value of the entry in the third row, fourth column. Since this entry has to be equal to T for $X(1, 0, T)$, and since $u(\tau) \leq 1$ for all τ, u must be 1 almost everywhere.

Finally, we consider case c). Here, we define

$$F(t) = \exp(-Bt) .\ C .\ \exp(Bt),$$

where B and C are the same matrices that have been used for the other two cases. Let $Y(v, \cdot)$ be the solution of equation (5) whose value at $t = 0$ is I. It is seen immediately that

$$Y(v, t) = \exp(-Bt) \cdot X(v, t) \ .$$

C2

Since we know that there does not exist a "bang-bang" control v such that $X(v, T) = X(0, T)$, it follows that $\dot{Y}(0, T)$ is not attainable at time T by "bang-bang" controls. Our proof is thus complete.

Remarks:

1) It is clear that all our results are equally valid for control problems of the type

$$\dot{x}(t) = [A_0(t) + \sum_{i=1}^{m} u_i(t) A_i(t)] x(t) , \qquad (7)$$

where $x(t)$ is a column vector in R^n and $A_0, \ldots, A_m$ are matrix-valued functions[3]. This is obvious for all our positive results, because the solution of equation (7) with initial condition $x(0) = x_0$ is just $X(u, \cdot) x_0$. As for Theorem 2, we need only observe that the problem considered in case a) is equivalent to the "vector" problem

$$\dot{x}(t) = (\bar{B} + v(t) \bar{C}) x(t) ,$$

where $x(t)$ is a 4×4 matrix, considered as a vector in 16-dimensional space, and where $\bar{B}$ and $\bar{C}$ are suitable 16×16 matrices (the matrices of the linear transformations $x \to Bx$ and $x \to Cx$, respectively, with respect to an appropriate basis of R^{16}). The sets attainable from the "vector" $x_0 = I$ will coincide with the sets $S(\dot{T})$, $SB(T)$, and in this way we get a counterexample for case a) of the "vector" problem Obviously, similar considerations apply to the other cases.

2) For completeness, we should give an example of a situation in which all the commutativity assumptions of Theorem 1 hold, but $SBP(T)$ fails to be closed because of non-analyticity of the functions A_i. It is well-known (and easy to prove) that the set of numbers $\int_0^T f(t) u(t) dt$, where

$f(t) = \sin(1/t)$, and where u ranges over all piecewise constant $\{-1, 1\}$-valued functions in $[0, T]$, is not closed for any $T > 0$. Let this be denoted by A_T. The set $B_T = \{e^x : x \in A\}$ is therefore not closed. But B_T is the set of points attainable from $x = 1$ at time T by "bang-bang" controls, for the system

$$\dot{x}(t) = f(t)\, u(t)\, x(t)$$

(which is of the form that we are considering, with $n = 1$). Moreover, by multiplying the function f by a smooth function that vanishes at the origin to a sufficiently high order, we can modify our counterexample so that f will be as smooth as desired and even C^{∞}).

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Footnotes

1. We are using here the standard notation $[\underline{M}, \underline{N}] = \underline{MN} - \underline{NM}$. Thus "$[\underline{M}, \underline{N}] = 0$" is another way of saying that the matrices $\underline{M}$ and $\underline{N}$ commute.

2. We follow the standard convention of identifying functions that are equal almost everywhere.

3. This is the problem studied by Kučera in [5] and [6].

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CONTROLLABILITY OF NON-LINEAR SYSTEM*

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1. Introduction

In this article we study the controllability of non-linear systems of the form

$$\frac{dx}{dt} = F(x, u). \qquad (*)$$

Our objective is to establish criteria in terms of $\underline{F}$ and its derivatives at a point $\underline{x}$ which will give qualitative information about the sets attainable from $\underline{x}$. The study is based primarily on the work of Chow [4] and Lobry [16], although it is similar in its approach to works by other authors in that it makes systematic use of differential geometry (for instance, see Hermann [8], [9], Haynes & Hermes [6], Brockett [2], etc.).

The state variable $\underline{x}$ is assumed to take values in an arbitrary real analytic manifold $\underline{M}$, rather than in $\underset{\sim}{R}^n$. We chose this generalization

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because it creates no essential new difficulties while, on the other hand, it allows for certain applications which are not commonly treated in control theory. For instance, when $\underline{M}$ is a Lie group, then the present results can be specialized to obtain more detailed controllability criteria. Control problems on Lie groups were first considered by R. W. Brockett in [2], and will be treated in a forthcoming paper by the authors.

Most of the recent studies on controllability of non-linear systems have essentially dealt with symmetric systems, i.e., systems of the form (*) with the property that $F(x,-u) = -F(x,u)$ (Hermann [9], Haynes and Hermes [6], Lobry [16]). As remarked by Lobry in [16], the consideration of symmetric systems often excludes interesting situations arising from mechanics. In these cases the system is of the form

$$\frac{dx}{dt} = A(x) + H(x)\cdot u.$$

A notable exception is the work by Lobry [17]. Lobry stated (and proved for the case of two vector fields in $\underset{\sim}{R}^3$) the result for non-symmetric systems that appears here as Theorem 3.1.

Our results apply to non-symmetric systems. We obtain some general information about the geometric structure of the attainable sets showing that they "practically" are submanifolds (see Theorems 4.4 and 4.5 for the precise statements). This information yields a complete answer to the problem of deciding when the sets attainable from a poi: $\underline{x}$ have a non-empty interior. The criteria obtained involve purely <u>algebraic</u> manipulations of $\underline{F}$ and its derivatives (of all orders) <u>at the point x</u> (see the Remark below).

In particular, our results contain those of Kučera [14]. In this connection we observe that our proofs are of interest even for the case treated by Kucera (see Sussman [21]).

We have omitted the consideration of non-autonomous systems; they can be treated analogously by the familiar procedure of reduction to an autonomous system (i.e., by considering the state variable to be defined in $\underline{M} \times \underset{\sim}{R}$).

The organization of the article is as follows: in section 2 we introduce notations and basic concepts; in addition, we quote some well-known basic results which will be used later. In section 3 we prove our main results in differential geometric terminology.

In section 4, we apply these results to control systems. We derive the algebraic criteria mentioned above (Corollaries 4.6 and 4.7) and we prove two "global results: we show that, for a large class of manifolds, accessibility (i.e. the property that, for any given $\underline{x}$, the set of points attainable from $\underline{x}$ has a nonempty interior) implies strong accessibility (i.e. that, for any given $\underline{x}$ and any given fixed positive $\underline{t}$, the set of points attainable from $\underline{x}$ at time $\underline{t}$ has a nonempty interior). We also show that, for a still larger class, including the Euclidean spaces, controllability implies strong accessibility.

Finally, section 5 contains examples. We show how our results can be used to derive the classical controllability criteria for the system

$$\frac{dx}{dt} = Ax + Bu .$$

We also derive the results of Kučera and indicate some generalizations.

Remark. An assumption that is made throughout the article is that F is an analytic function of x. This guarantees that all the information about the system is actually contained in F and its derivatives (of all orders) at a given point x. The analyticity assumption cannot be relaxed without destroying the theory (cf. Example 5. 3).

Another assumption that we make is that the trajectories of the system are everywhere defined. As opposed to the previous one, this assumption is not essential (except for the "global" Theorems 4. 9 and 4. 10). We use it, however, because it considerably simplifies all the proofs.

2. Preliminaries

We shall assume that the reader is familiar with the fundamental notions of differential geometry. All the definitions and basic concepts utilized in this paper can be found in standard books, (for instance, [1], [3], [7], [13] and [19]).

The following notations will be used throughout:

R--the set of real numbers.

R^n--n-dimensional Euclidean space.

M_x--the tangent space to the manifold M at the point x.

TM--the tangent bundle of the manifold M.

$V(M)$--the set of all analytic vector fields on the analytic manifold M.

We will regard $V(M)$ as a Lie algebra over the reals. For any X and Y in $V(M)$, we will denote the Lie product by

$[X, Y]$(i.e., $[X, Y] = XY - YX$). All the manifolds will be assumed to be paracompact. Recall that a submanifold of a paracompact manifold is paracompact. Also, a connected paracompact manifold is a countable union of compact sets. These facts imply (cf. Lobry [16], p 589):

LEMMA 2.1. Let M be a (paracompact) manifold of dimension n. Let S be a k-dimensional connected submanifold of M. If $k < n$, then the set of points of S has an empty interior in M.

A subset D of $V(M)$ will be called _involutive_ if, whenever X and Y belong to D, then $[X, Y]$ also belongs to D. A subalgebra of $V(M)$ is an involutive subspace. Let $D \subset V(M)$. An _integral manifold_ of D is a connected submanifold S of M with the property that $S_x = \mathscr{L}(D(x))$ for every $x \in S$, where $D(x) = \{X(x) : X \in D\}$, and where $\mathscr{L}(D(x))$ is the subspace of M_x spanned by D(x). We state the following basic results about integral manifolds:

LEMMA 2.2. Let D be an involutive subset of $V(M)$, and let $x \in M$. Then x is contained in a unique maximal integral manifold of D (here "maximal" means "maximal with respect to inclusion").

This result is classical if the dimension of $\mathscr{L}(D(x))$ is the same for each $x \in M$(Chevalley [3]). For a proof in the general case, see Lobry [16].

If $D \subset V(M)$, we denote the smallest subalgebra of $V(M)$ which contains D by $\mathscr{J}(D)$, and the maximal integral manifold of [illegible])) through x by $I(D, x)$. Recall that, if X is a vector field on M, then [illegible] an _integral curve_ of X if α is a smooth mapping from a closed interval I, $I \subset R$, into M such that

$\frac{d\alpha(t)}{dt} = X(\alpha(t))$ for all $t \in I$.

DEFINITION 2.3. If D is a subset of $V(M)$, then an integral curve of D is a mapping α from a real interval $[t, t']$ into M such that there exist $t = t_0 < t_1 < \ldots < t_k = t$, and elements $X_1, \ldots, X_k$ of D with the property that the restriction of α to $[t_{i-1}, t_i]$ is an integral curve of X_i for each $i = 1, 2, \ldots, k$. We have the following elementary fact:

LEMMA 2.4. Let $D \subset V(M)$. Let $\alpha: [t_0, t_1] \to M$ be an integral curve of D, and let $a(t) = x$ for some $t \in [t_0, t_1]$. Then $\alpha(s) \in I(D, x)$ for all $s \in [t_0, t_1]$.

Proof. It is sufficient to consider the case when α is an integral curve of X, $X \in D$. For each maximal integral manifold S of $\mathscr{I}(D)$, let $J(S)$ be the set of all $s \in [t_0, t_1]$ such that $\alpha(s) \in S$. From the local existence and uniqueness of solutions of ordinary differential equations it follows that, if $s \in J(S)$, then there exists $r > 0$ such that $(s-r, s+r) \cap [t_0, t_1] \subset J(S)$. Thus, $J(S)$ is open relative to $[t_0, t_1]$. Since the maximal integral manifolds of $\mathscr{I}(D)$ are disjoint, we have that, for some maximal integral manifold S, $[t_0, t_1] \subset J(S)$. But $\alpha(t) \in I(D, x)$; therefore, our proof is complete.

Chow's theorem provides a partial converse to the above lemma. If $D \subset V(M)$, then D is symmetric if, whenever $X \in D$, $-X$ also belongs to D. We can now state Chow's theorem as follows:

LEMMA 2.5. Let $D \subset V(M)$ be symmetric, and let $x \in M$. Then, for every $y \in I(D, x)$ there exists an integral curve $\alpha: [0, T] \to M$ of D, with $T \geq 0$, such that $\alpha(0) = x$ and $\alpha(T) = y$.

In other words, every point of the maximal integral manifold of $\mathscr{T}(\underline{D})$ through $\underline{x}$ can be reached in positive time by following an integral curve of $\underline{D}$ having $\underline{x}$ as its initial point.

DEFINITION 2.6. Let $\underline{D} \subset \underline{V}(\underline{M})$, and let $\underline{x} \in \underline{M}$. If $T \geq 0$, then, for any $\underline{y} \in \underline{M}$, $\underline{y}$ is D-reachable from x at time $\underline{T}$ if there exists an integral curve α of $\underline{D}$ defined on $[0, T]$ such that $\alpha(0) = x$ and $\alpha(T) = y$. The set of all $\underline{D}$-reachable points from $\underline{x}$ at time $\underline{T}$ is denoted by $L_x(D, T)$. The union of $L_x(D, t)$ for $0 \leq t < \infty$ (respectively for $0 \leq t \leq T$) is denoted by $\underset{\sim}{L}_x(D)$ (respectively $\underset{\sim}{L}_x(D, T)$).

3. Integrability of Families of Analytic Vector Fields

As an introduction to the general situation, we first considered the case when $\underline{D}$ is a symmetric subset of $\underline{V}(\underline{M})$. Chow's theorem can be utilized to obtain a necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$. Let $\underline{n} = \dim \underline{M} = \dim \mathscr{T}(\underline{D})(\underline{x})$. Then $\underline{I}(\underline{D}, \underline{x})$ is an $\underline{n}$-dimensional submanifold of $\underline{M}$, and hence is open in $\underline{M}$. By Chow's theorem we have that $\underset{\sim}{L}_x(D) = I(D, x)$. We conclude that $\underset{\sim}{L}_x(D)$ is open in $\underline{M}$. Conversely (and without invoking the symmetry of $\underline{D}$) if $\dim \mathscr{T}(\underline{D})(\underline{x}) < \underline{n}$, then $\underline{I}(\underline{D}), \underline{x})$ is a connected submanifold of $\underline{M}$ of dimension less than $\underline{n}$; then from Lemma 2.1 it follows directly that $\underline{I}(\underline{D}, \underline{x})$ has an empty interior in $\underline{M}$. Since $\underset{\sim}{L}_x(D) \subset I(D, x)$, $\underset{\sim}{L}_x(D)$ also has an empty interior. Thus, if $\underline{D}$ is symmetric, a necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$ is that $\dim \mathscr{T}(\underline{D})(\underline{x}) = \dim \underline{M}$. Moreover, this condition is necessary even in the non-symmetric case (Lobry [16]). We shall show that it is also sufficient. For this purpose we shall assume that the elements of $\underline{D}$ are complete--recall that a vector field $\underline{X}$ is complete the integral curves of $\underline{X}$ are defined for all real $\underline{t}$ (cf. [13], p. 13).

THEOREM 3.1. Let $\underline{M}$ be an $\underline{n}$-dimensional analytic manifold, and let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vector fields. A necessary and sufficient condition for $\underset{\sim}{L}_x(D)$ to have a non-empty interior in $\underline{M}$ is that $\dim \mathscr{T}(\underline{D})(\underline{x}) = \underline{n}$. Moreover, if this condition is satisfied, then for each $\underline{T} > 0$, the interior of $\underset{\sim}{L}_x(D,T)$ is dense in $\underset{\sim}{L}_x(D,T)$ (thus, in particular, $\underset{\sim}{L}_x(D,T)$ has a non-empty interior).

Proof. We already know that the condition of the theorem is necessary. So we assume that $\dim \mathscr{T}(\underline{D})(\underline{x}) = \underline{n}$, and we prove the second statement. Clearly, this will imply that $\underset{\sim}{L}_x(D)$ has a non-empty interior in $\underline{M}$. Without loss of generality we can assume that $\underline{D}$ is finite. Let $D = \{X_1, \ldots, X_k\}$. For each $i = 1, 2, \ldots, k$, let $\Phi_i(t, \cdot)$ be the one-parameter group of diffeomophisms induced by X_i(i.e., $t \to \Phi_i(t,y)$ is the integral curve of X_i which passes through $\underline{y}$ at $\underline{t} = 0$; the fact that it is defined for all real $\underline{t}$ follows from the completeness of X_i). If $\underline{m}$ is a natural number $\underset{\sim}{t} = (t_1, \ldots, t_m)$ is an element of $\underset{\sim}{R}^m$, and $\underset{\sim}{i} = (i_1, \ldots, i_m)$ is an $\underline{m}$-tuple of natural numbers between 1 and $\underline{k}$, then we denote the element $\Phi_{i_1}(t_1, \Phi_{i_2}(t_2, \ldots, \Phi_{i_m}(t_m, x)\ldots))$ by $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$. Let $\underline{\pm}\,\underline{D}$ be the family of vector fields obtained from $\underline{D}$ by adjoining the vector fields $-X_1, \ldots, -X_k$ to $\underline{D}$. Then, $\underline{\pm}\,\underline{D}$ is symmetric, and $\dim \mathscr{T}(\underline{\pm D})(x) = n$. From Chow's theorem we conclude that $\underset{\sim}{L}_x(\underline{\pm}\,\underline{D})$ is open in $\underline{M}$. Clearly, the elements of $\underset{\sim}{L}_x(\underline{\pm D})$ are exactly those elements of $\underline{M}$ which are of the form $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$ for some $\underline{m}$, some $\underline{m}$-tuple $\underset{\sim}{i}$, and some $\underset{\sim}{t} \in \underset{\sim}{R}^m$. For each $\underset{\sim}{i}$, and for each natural number $\underline{N} > 0$, let $\underline{A}(\underset{\sim}{i}, \underline{N})$ be the set of all points of $\underline{M}$ of the form $\Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, where $\|\underset{\sim}{t}\| \le \underline{N}$ (here, $\|\underset{\sim}{t}\| = |t_1| + \ldots + |t_m|$). Since $\underline{A}(\underset{\sim}{i}, \underline{N})$ is the image of the compact set $\{\underset{\sim}{t} : \|\underset{\sim}{t}\| \le N\}$ under the continuous mapping $\underset{\sim}{t} \to \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, we have that $A(\underset{\sim}{i}, N)$ is compact. Also, since $\underset{\sim}{L}_x(\pm D)$ is the union of the sets $A(\underset{\sim}{i}, N)$ (taken over m, $\underset{\sim}{i}$ and N), it follows from the category theorem that, for some $\underset{\sim}{i}$ and N, the set $A(\underset{\sim}{i}, N)$ has a non-empty interior in M. For such an $\underset{\sim}{i}$, let $F:\underset{\sim}{R}^m \to M$ be defined by $F(\underset{\sim}{t}) = \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$. Then F is an analytic mapping whose image has a non-empty interior in M. By Sard's theorem (Sternberg [19]), the differential $dF_{\underset{\sim}{t}}$ of F at $\underset{\sim}{t}$ must have rank n for some $\underset{\sim}{t} \in \underset{\sim}{R}^m$. Since $dF_{\underset{\sim}{t}}$ depends analytically on $\underset{\sim}{t}$, it follows that the set $\Omega^{\#} = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^m, \text{ rank } dF_{\underset{\sim}{t}} < n\}$ has an empty interior. Let $\Omega = \underset{\sim}{R}^m - \Omega^{\#}$. Then Ω is open and dense in $\underset{\sim}{R}^m$.

Let $T > 0$, and let $y \in \underset{\sim}{L}_x(D, T)$. We now show that y is in the closure of the interior of $\underset{\sim}{L}_x(D, T)$. It is clearly sufficient to assume that $y \in L_x(D, t)$, where $0 \le t < T$ (for each point of $\underset{\sim}{L}_x(D, T)$ is in the closure of $\bigcup \{L_x(D, t) : 0 \le t < T\}$). Let $y = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, x)$ where $\underset{\sim}{j} = (j_1, \ldots, j_p)$, $\underset{\sim}{s} = (s_1, \ldots, s_p)$, $s_1 > 0, \ldots, s_p > 0$, and $s_1 + \ldots + s_p = t$. Let $U = \Omega \cap \{\underset{\sim}{t} : \|\underset{\sim}{t}\| < T - t\} \cap \{\underset{\sim}{t} : t_1 > 0, \ldots, t_m > 0\}$. U is open, and its closure contains the original $\underset{\sim}{0}$ of $\underset{\sim}{R}^m$. Since $dF_{\underset{\sim}{t}}$ has rank n at each point $\underset{\sim}{t} \in U$, it follows that $F(U)$ is open. Let $V = \{\Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{t})) : \underset{\sim}{t} \in U\}$. V is the image of $F(U)$ under the diffeomorphism $z \to \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, z)$; therefore, V is open in M and, moreover, every element of V is D-reachable from x at time $\|\underset{\sim}{s}\| + \|\underset{\sim}{t}\| = t + \|\underset{\sim}{t}\| < T$ (here we use essentially the fact that $t_1, \ldots, t_m$ are non-negative). It remains to be shown that y belongs to the closure of V. Let $\{\underset{\sim}{t}_q\}$ be a sequence of elements of U which converges to $\underset{\sim}{0}$. Then

$$\lim \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{t}_q)) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, F(\underset{\sim}{0})) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{s}, x) = y.$$

This completes the proof of the theorem.

We now want to state an analogous theorem for the sets $L_x(D,T)$. For this purpose, we shall introduce a Lie subalgebra $\mathscr{T}_0(D)$ of $\mathscr{T}(D)$ which will be related to these sets in the same way as $\mathscr{T}(D)$ is related to the sets $\underset{\sim}{L}_x(D,T)$. The aim of the following informal remarks is to motivate our definition of $\mathscr{T}_0(D)$. We shall ignore the fact that time has to be positive. Moreover, we shall assume, for simplicity, that $\underline{D}$ consists of three vector fields X_1, X_2 and X_3. Let Φ_1, Φ_2 and Φ_3 be the corresponding one-parameter groups. It is clear that $\mathscr{T}(D)$ has the following "geometric interpretation": $\mathscr{T}(D)(x)$ is, for each $x \in M$, the set of all limiting directions of curves through $\underline{x}$ that are entirely contained in $\underset{\sim}{L}_x(D)$. Thus, for instance, if $i=1,2,3$, then all the points in the curve $t \to \Phi_i(t,x)$ are attainable from $\underline{x}$ (recall that we are forgetting about positivity), and this is reflected in the fact that $X_i(x)$ belongs to $\mathscr{T}(D)(x)$. Similarly, the curves $\alpha_{ij}(t) = \Phi_i(-t, \Phi_j(-t, \Phi_i(t, \Phi_j(t,x))))$ are also contained in $\underset{\sim}{L}_x(D)$. By the well known geometric interpretation of the Lie bracket (cf. Helgason [7], p. 97), the limiting direction of α_{ij} is $[X_i, X_j](x)$ (after a reparametrization). Thus, it is clear why $[X_i, X_j]$ belongs to $\mathscr{T}(D)$. Obviously, a similar argument works for the brackets of higher order. The geometrical meaning of $\mathscr{T}(D)$ is now obvious.

If $\mathscr{T}_0(D)$ is going to play the desired role it is clear that $\mathscr{T}_0(D)(x)$ will have to be the set of all limiting directions of curves γ through $\underline{x}$ such that $\gamma(t)$ is "attainable from $\underline{x}$ in zero units of time" for all $\underline{t}$. Notice that the curves $\alpha_{ij}(t)$ of the preceding paragraph have this property. Indeed, $\alpha(t)$ can be reached from $\underline{x}$ by "moving forward" in time 2t units, and then "backward" another 2t units. This shows that the vector fields $[X_i, X_j]$ are reasonable candidates for membership in $\mathscr{T}_0(D)$. A similar argument applies to higher order brackets, such as $[X_i, [X_j, X_k]]$, etc. On the other hand, a vector field such as X_i should not be included in $\mathscr{T}_0(D)$ by definition, because we do not know whether the points $\Phi_i(t, x)$, $t \neq 0$, can be reached from $\underline{x}$ in 0 units of time (but, of course, it may happen that some X_i will belong to $\mathscr{T}_0(D)$ anyhow; for instance, we could have $X_1 = [X_2, X_3]$). However, the vector fields $X_i - X_j$ will have to be included, because $(X_i - X_j)(x)$ is the limiting direction of the curve $t \to \Phi_j(-t, \Phi_i(t, x))$. In other words, the subspace generated by the differences $X_i - X_j$ will have to be included in $\mathscr{T}_0(D)$. This subspace can also be defined as the set of all linear combinations $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ (that all the differences $X_i - X_j$ are linear combinations of this type is trivial; conversely, if $Y = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$, then $Y = \lambda_1 X_1 + \lambda_2 X_2 + (-\lambda_1 - \lambda_2) X_3$, i.e., $Y = \lambda_1 (X_1 - X_3) + \lambda_2 (X_2 - X_3)$).

We conclude that the reasonable candidates for membership in $\mathscr{T}_0(D)$ are: (i) all the brackets $[X_i, X_j]$, $[X_i, [X_j, X_k]]$, etc., and (ii) all the sums $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, where $\Sigma \lambda_i = 0$. Notice that the subset generated by (i) is clearly the derived algebra of $\mathscr{T}(D)$ (by definition, the derived algebra of a Lie algebra $\underline{L}$ is the subalgebra $\underline{L}'$ of $\underline{L}$ generated by all the brackets $[X, Y]$, $X \in L$, $Y \in L$; it is easy to check that L' is in fact an ideal of $\underline{L}$; cf. Helgason [7], p. 133.

We now return to our formal development. Let $\mathscr{T}'(D)$ denote the derived algebra of $\mathscr{T}(D)$. Motivated by the previous remarks, we define $\mathscr{T}_0(D)$ to be the set of all sums $X + Y$, where X is a linear combination

$$\sum_{i=1}^{p} \lambda_i X_i \text{ with } X_1, \ldots, X_p \in D \text{ and } \Sigma \lambda_i = 0,$$

and where $Y \in \mathscr{T}'(D)$. It is obvious that $\mathscr{T}_0(D)$ is an ideal of $\mathscr{T}(D)$. One shows easily that $\mathscr{T}(D)$ is the set of all vector fields of the form

$$\sum_{i=1}^{p} \lambda_i X_i + Y$$

where $X_1, \ldots, X_p$ belong to D, Y belongs to $\mathscr{T}'(D)$, and $\lambda_1, \ldots, \lambda_p$ are reals (but $\lambda_1 + \ldots + \lambda_p$ need not be zero). From this it follows immediately that $\mathscr{T}_0(D)$ is a subspace of $\mathscr{T}(D)$ of codimension zero or one. The codimension will be zero if and only if some $X \in D$ belongs to $\mathscr{T}_0(D)$ (in which case every $X \in D$ will belong to $\mathscr{T}_0(D)$). Similarly for each $x \in M$, if $k = \dim \mathscr{T}(D)(x)$, then the dimension of $\mathscr{T}_0(D)(x)$ will either be k or k-1.

We shall also be interested in associating to each $D \subset V(M)$, a set D^* of vector fields in the manifold $M \times \underset{\sim}{R}$. Recall that the tangent space to $M \times R$ at a point (x, r) ($x \in M$, $r \in \underset{\sim}{R}$) is identified, in a natural way, to the direct sum $M_x \oplus \underset{\sim}{R}_r$. If $x \in V(M)$, $Y \in V(\underset{\sim}{R})$, we define the vector field $X \oplus Y \in V(M \times \underset{\sim}{R})$ by

$$(X \oplus Y)(x, r) = (X(x), Y(r)).$$

The set D^* is defined to be the set of all vector fields $X \oplus \frac{\partial}{\partial t}$, where $X \in D$, and where $\frac{\partial}{\partial t}$ is the "canonical" vector field on $\underset{\sim}{R}$ $((\frac{\partial}{\partial t} f)(r) = \frac{df}{dt}(r))$. Using the identity $[X \oplus X', Y \oplus Y'] = [X, Y] \oplus [X', Y']$, one shows easily that $\mathscr{T}'(D^*)$ is the set of all vector fields of the form $X \oplus \underset{\sim}{0}$, where $X \in \mathcal{J}'(D)$ and $\underset{\sim}{0}$ is the zero vector field. Therefore, $\mathcal{J}(D^*)$ is the set of vector fields of the form

$$\sum_{i=1}^{p} \lambda_i (X_i \oplus \frac{\partial}{\partial t}) + Y \oplus \underset{\sim}{0} \qquad (\#)$$

where $X_1, \ldots, X_p$ belong to D, $Y \in \mathcal{J}'(D)$, and $\lambda_1, \ldots, \lambda_p$ are scalars.

THEOREM 3.2. Let $\underline{M}$ be an analytic $\underline{n}$-dimensional manifold, and let $\underline{D}$ be a family of complete analytic vector fields on M. Let $\underline{x} \in \underline{M}$, and let $\underline{T} > 0$. Then $L_x(D, T)$ has a non-empty interior in $\underline{M}$ if and only if $\dim \mathcal{J}_0(D)(x) = n$. Moreover, in this case, the interior of $L_x(D, T)$ is dense in $L_x(D, T)$.

Proof. The main idea in this proof is to modify our problem so that we can "keep track" of the time elapsed while we move along an integral curve of $\underline{D}$. We shall then apply Theorem 3.1 to the modified system. We shall work in the manifold $\underline{M} \times \underset{\sim}{R}$. As in the preceding paragraphs, we let the family $\underline{D}^*$ of vector fields on $\underline{M} \times \underset{\sim}{R}$ be defined by $D^* = \{X \oplus \frac{\partial}{\partial t} : X \in L\}$. It is clear that there is a one-to-one correspondence between integral curves α of $\underline{D}$ such that $\alpha(0) = x$, and integral curves $\beta(\underline{D}^*)$ such that $\beta(0) = (x, 0)$. This correspondence is given by assigning to each curve α the curve $t \to (\alpha(t), t)$. It follows that $y \in L_x(D, T)$ if and only if $(y, T) \in \underset{\sim}{L}_{(x,0)}(D^*, T)$. We show that $L_x(D, T)$ has a non-empty interior in $\underline{M}$ if and only if $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $\underline{M} \times \underset{\sim}{R}$. Assume that $L_x(D, T)$ has a non-empty interior in $\underline{M}$, and let $\underline{V}$ be a non-empty open set such that $V \subset L_x(D, T)$. Let $\underline{X} \in \underline{D}$, and let Φ be the one-parameter group of diffeomorphisms of $\underline{M}$ generated by $\underline{X}$. Consider the mapping $F : V \times \underset{\sim}{R} \to M \times \underset{\sim}{R}$ defined by $F(v, t) = (\Phi(t, v), T + t)$. It is immediate that the differential of $\underline{F}$ has rank $\underline{r} + 1$ everywhere. Therefore $\underline{F}$ maps open sets onto open sets.

Since $F(V X(0,\infty)) \subset \underset{\sim}{L}_{(x,0)}(D^*)$, we conclude that $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $M X \underset{\sim}{R}$.

To prove the converse, assume that $\underset{\sim}{L}_{(x,0)}(D^*)$ has a non-empty interior in $\underline{M} X \underset{\sim}{R}$. By Theorem 3.1, for each $\underline{t}$ with $0 < \underline{t} < \underline{T}$, $\underset{\sim}{L}_{(x,0)}(D^*,t)$ has a non-empty interior in $\underline{M} X \underset{\sim}{R}$. Let $\underline{V}$ be a non-empty open subset of $\underline{M}$, and let $\underline{W}$ be a non-empty open subset of $\underset{\sim}{R}$ such that $\underline{V} X \underline{W} \subset \underset{\sim}{L}_{(x,0)}(D^*,t)$. Let $\underline{s} \in \underline{W}$. Since $\underline{V} X \{\underline{s}\} \subset \underset{\sim}{L}_{(x,0)}(D^*,t)$, we conclude that $V \subset L_x(D,s)$. Let $\underline{X} \in \underline{D}$, and let Φ be the corresponding one-parameter group on $\underline{M}$. Denote the mapping $y \to \Phi(T-s,y)$ by $\underline{G}$. Then $\underline{G}(\underline{V})$ is open. Since $\underline{G}(\underline{V})$ is contained in $L_x(D,T)$, it follows that $L_x(D,T)$ has a non-empty interior.

We conclude from Theorem 3.1 that $L_x(D,T)$ has a nonempty interior if and only if $\dim \mathscr{T}(D^*)(x,0) = n+1$. To complete the proof of the first part of our statement, we must show that this last condition holds if and only if $\dim \mathscr{T}_0(D)(x) = \underline{n}$. We recall, from the remarks preceding this proof, the fact that every $X^* \in \mathscr{T}(D^*)$ can be expressed as

(#) $X^* = \sum_{i=1}^{p} \lambda_i (X_i \oplus \frac{\partial}{\partial t}) + Y \oplus \underset{\sim}{0}$ where $X_1, \ldots, X_p$ belong to $\underline{D}$

and $Y \in \mathscr{T}'(D)$. Now assume that $\dim \mathscr{T}(D^*)(x,0) = n+1$. Let $v \in M_x$. Then $(v,0)$ must belong to $\mathscr{T}(D^*)(x,0)$, so that $(v,0) = X^*(x,0)$, where $X^* \in \mathscr{T}(D^*)$. Then formula (#) holds for suitable λ_i, X_i, Y. Therefore

$$v = (\Sigma \lambda_i X_i + Y)(x),$$

and

$$\underset{\sim}{0} = \Sigma \lambda_i \frac{\partial}{\partial t}(0).$$

The last equality implies that $\Sigma\lambda_i = 0$, so that the vector field $\Sigma\lambda_i X_i + Y$ belongs to $\mathscr{T}_0(D)$. Thus $v \in \mathscr{T}_0(D)(x)$. We have shown that $M_x \subseteq \mathscr{T}_0(D)(x)$. Therefore the dimension of $\mathscr{T}_0(D)(x)$ is $\underline{n}$. Conversely, let $\dim \mathscr{T}_0(D)(x) = n$. Let $v \in M_x$. Then $v \in \mathscr{T}_0(D)(x)$, so that

$$v = (\Sigma\lambda_i X_i + Y)(x) ,$$

where the X_i belong to D, $Y \in \mathscr{T}'(D)$ and $\Sigma\lambda_i = 0$. Therefore,

$$\begin{aligned}(v, 0) &= ((\Sigma\lambda_i X_i + Y) \oplus (\Sigma\lambda_i) \frac{\partial}{\partial t})(x, 0) \\ &= (\Sigma\lambda_i (X_i \oplus \frac{\partial}{\partial t}) + Y \oplus \underset{\sim}{0})(x, 0).\end{aligned}$$

This shows that $(v, 0)$ belongs to $\mathscr{T}(D^*)(x, 0)$. Pick an $X \in D$. Then $X \ominus \frac{\partial}{\partial t}(x, 0)$ belongs to $D^*(x, 0)$ by definition, and $X \oplus \underset{\sim}{0}$ $(x, 0)$ belongs to $\mathscr{T}(D^*)(x, 0)$ by the previous remarks. Therefore $(\underset{\sim}{0}, \frac{\partial}{\partial t}(0))$ belongs to $\mathscr{T}(D^*)(x, 0)$. We have thus shown that $\mathscr{T}(D^*)(x, 0)$ contains all the vectors $(v, 0)$, $v \in M_x$, and also the vector $(\underset{\sim}{0}, \frac{\partial}{\partial t}(0))$. Therefore $\mathscr{T}(D^*)(x, 0) = (M \times \underset{\sim}{R})_{(x, 0)}$, so that $\dim \mathscr{T}(D^*)(x, 0) = n + 1$ as stated.

We now prove the second part of the theorem. As we remarked earlier, there is no loss of generality in assuming that $\underline{D}$ is finite. Let $y \in L_x(D, T)$. Using the notations of the proof of Theorem 3.1, let $y = \Phi_{\underset{\sim}{i}}(\underset{\sim}{t}, x)$, where $\underset{\sim}{i} = (i_1, \ldots, i_m)$, and where $\underset{\sim}{t} \in \underset{\sim}{R}^m$ is such that $t_i > 0$ for $i = 1, \ldots, m$ and $\|\underset{\sim}{t}\| = T$. Let $\{s_k\} \subset (0, t_m)$ be such that $\lim_{k \to \infty} s_k = 0$. Since our condition for $L_x(D, T)$ to have a non-empty interior is independent of $\underline{T}$, we conclude that $L_x(D, t)$ has a non-empty interior for all $\underline{t} > 0$. In particular, for each $\underline{k} > 0$, there exists x_k which belongs to the interior of $L_x(D, s_k)$. Let $\underset{\sim}{t}_k = (t_1, \ldots, t_{m-1},$

$t_m - s_k$), and let $y_k = \underset{\sim}{\Phi_i}(\underset{\sim}{t}_k, x_k)$. For each $\underline{k} > 0$, y_k belongs to $L_x(D, T)$; since $\underset{\sim}{\Phi_i}$ is a diffeomorphism, y_k is the interior of $L_x(D, T)$. Also, $x_k \to x$ as $k \to \infty$ because $\underline{D}$ is finite and $s_k \to 0$. Since $\underset{\sim}{\Phi_i}$ is continuous in both variables, and since $\underset{\sim}{t}_k \to \underset{\sim}{t}$, we have that $y_k \to y$, and our theorem is proved.

The results of the previous theorems can be utilized to obtain information about the sets $\underset{\sim}{L}_x(D, T)$ and $L_x(D, T)$, even when $\dim \mathscr{T}(\underline{D})(\underline{x}) < \underline{n}$.

THEOREM 3.3. Let $D \subset V(M)$ be a family of complete vector fields. Then, for each $T > 0$, the set $\underset{\sim}{L}_x(D, T)$ is contained in $I(D, x)$. Moreover, in the topology of $I(D, x)$, the interior of $\underset{\sim}{L}_x(D, T)$ is dense in $\underset{\sim}{L}(D, T)$. $L_x(D, T)$ has a non-empty interior in $I(D, x)$ if and only if $\dim \mathscr{T}_0(D(x)) = \dim \mathscr{T}(D)(x)$ and, in this case, the interior of $L_x(D, T)$ is dense in $L_x(D, T)$.

Proof. If $\underline{X} \in \mathscr{T}(D)$, then $\underline{X}$ is tangent to $I(D, x)$. Thus, there is a well-defined restriction $\underline{X}\#$ of $\underline{X}$ to $\underline{I}(\underline{D}, \underline{x})$. We denote the set of all such restrictions of elements of D by $D\#$. Since $[\underline{X}, \underline{Y}]\# = [\underline{X}\#, \underline{Y}\#]$, it follows that $\mathscr{T}(\underline{D})\# = \mathscr{T}(\underline{D}\#)$. Analogously, we have that $\mathscr{T}_0(D)\# = \mathscr{T}_0(D\#)$. If we now apply the previous theorems to the family $D\#$ of vector fields in $I(D, x)$, we get all the conclusions of the theorem.

COROLLARY 3.4. Let $\underline{S}$ be a maximal integral manifold of $\mathscr{T}(\underline{D})$. Then the dimension of $\mathscr{T}_0(\underline{D})(\underline{x})$ is the same for all $\underline{x} \in \underline{S}$.

Proof. If $\dim \mathscr{T}(D)(x) = k$ then, for each $\underline{x} \in \underline{S}$, the dimension of $\mathscr{T}_0(D)(x)$ is either $\underline{k}$ or $\underline{k}-1$. We show that, if $\dim \mathscr{T}_0(D)(y) = k-1$ for some $\underline{x} \in \underline{S}$, then $\dim \mathscr{T}_0(D)(y) = k-1$ for $\underline{\text{all}}$ $\underline{y} \in \underline{S}$. Let Ω be a non-empty, open (relative to $\underline{S}$) subset of $\underset{\sim}{L}_x(D)$ (this is possible by Theorem 3.3). We first show that, if $\underline{y} \in \Omega$, then $\dim \mathscr{T}_0(D)(y) = k-1$. If this were not the case, then necessarily $\dim \mathscr{T}_0(D)(y) = k$. Then $L_y(D, t)$ would have a non-empty interior in $\underline{S}$ for all $\underline{t} > 0$. This would imply that $L_x(D,t)$ has a non-empty interior in $\underline{S}$. But by our assumption this is impossible. Thus, $\dim \mathscr{T}_0(D)(y) = \underline{k} - 1$ for all $\underline{y} \in \Omega$. Since $\underline{S}$ is connected, and Ω is open in $\underline{S}$, we have that $\dim \mathscr{T}_0(D)(y) = k-1$ for all $\underline{y} \in \underline{S}$; therefore, our statement is proved.

We now proceed to study the case when $\dim \mathscr{T}_0(D)(x) = \dim \mathscr{T}(D)(x) - 1$. We begin by proving some preliminary lemmas.

LEMMA 3.5. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vector fields. If $\underline{X} \in \underline{D}$, let $\{\Phi_t\}$ be the one-parameter group generated by $\underline{X}$. Then, for every $\underline{x} \in \underline{M}$, and every $\underline{t} \in \underset{\sim}{R}$ the differential $d\Phi_t$ maps $\mathscr{T}_0(D)(x)$ onto $\mathscr{T}_0(D)(\Phi_t(x))$.

Proof. We first show that for every $\underline{y} \in \underline{M}$ there is an $\underline{r} > 0$ such that, if $v \in \mathscr{T}_0(D)(y)$, then $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for all $\underline{t}$ with $|t| < r$. It is sufficient to show that for every $\underline{y} \in \underline{M}$ and every $v \in \mathscr{T}_0(D)(y)$ there exists an $\underline{r} > 0$ such that $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for all t with $|t| < r$. Let $\underline{y} \in \underline{M}$, and let $v \in \mathscr{T}_0(D)(y)$. If $v = Y(y)$ for some $Y \in \mathscr{T}_0(D)$, then an easy computation shows that there exists a neighborhood of $\underline{t} = 0$ such that $d\Phi_t(v) = \Sigma_{i=1}^{\infty} \frac{(-1)^i}{i!} [X^{(i)}, Y](\Phi_t(y))t^i$ for all $\underline{t}$ in this neighborhood, where $[X^{(0)}, Y] = Y$, and $[X^{(n)}, Y] = [X,[X^{(n-1)}, Y]]$ for $n = 1, 2, \ldots$ Since each term of the above series belongs to $\mathscr{T}_0(D)(\Phi_t(y))$, we have that $d\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(y))$ for $\underline{t}$ sufficiently small. Also, for such $\underline{t}$ we have that $d\Phi_t(\mathscr{T}_0(D)(y)) = \mathscr{T}_0(D)(\Phi_t(y))$; this is so because $d\Phi_t$ is one-to-one, and $\dim \mathscr{T}_0(D)(y) = \dim \mathscr{T}_0(D)(\Phi_t(y))$ (Corollary 3.4). It follows easily that the set of all $\underline{t}$ such that $d\Phi_t(\mathscr{T}_0(D)(x)) = \mathscr{T}_0(D)(\Phi_t(x))$ is both open and closed. If $v \in \mathscr{T}_0(D)(x)$, we can conclude that $\Phi_t(v) \in \mathscr{T}_0(D)(\Phi_t(v))$ for all t. This completes our proof.

As we remarked earlier, $\mathcal{T}_0(\underline{D})$ is a subalgebra of $\mathcal{T}(D)$. We will denote the maximal integral manifold of $\mathcal{T}_0(\underline{D})$ through $\underline{x}$ by $I_0(D, x)$. If $\underline{X} \in \underline{D}$ then, by the previous lemma, $\Phi_t(I_0(D, x))$ is a maximal integral manifold of $\mathcal{T}_0(D)$.

LEMMA 3.6. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a family of complete vecto. fields. Let $\underline{X}$ and $\underline{Y}$ be elements of $\underline{D}$, and let $\{\Phi_t\}$ and $\{\Psi_t\}$ be their corresponding one-parameter groups. If $\underline{S}$ is a maximal integral manifold of $\mathcal{T}_0(D)$ then, for any $\underline{t} \in \underset{\sim}{R}$, $\Phi_t(S) = \Psi_t(S)$.

Proof. Let $\underline{X}, \underline{Y}$, Φ_t, Ψ_t and $\underline{S}$ satisfy the conditions of the lemma. Let P be the maximal integral manifold of $\mathcal{T}(D)$ which contains $\underline{S}$. If $\dim \underline{P} = \dim \underline{S}$, then $\underline{S} = \underline{P}$, and $\Phi_t(S) = S = \Psi_t(S)$. Assume that $\dim S = k = \dim(P) - 1$. We first show that there is an $\underline{r} > 0$ such that $\Phi_t(S) = \Psi_t(S)$ whenever $|t| < r$. Let $\underline{x} \in \underline{S}$. The mapping $(s, t) \to \Phi_t(s)$ has rank $\underline{k} + 1$ at $(\underline{x}, 0)$. Let Ω be a neighborhood of $\underline{x}$ in $\underline{S}$, and let $\delta > 0$ be such that this mapping, restricted to $\Omega \times (-\delta, \delta)$ is a diffeomorphism onto an open subset $\Omega\#$ of $\underline{P}$. If $\underline{y} \in \Omega\#$, let $\underline{s}(\underline{y})$ and $\underline{f}(\underline{y})$ be such that $\Phi_{f(y)}(s(y)) = y$. Clearly, $\underline{f}$ is analytic in $\Omega\#$, and $f(y) = 0$ if and only if $\underline{y} \in \Omega$. Moreover, $Xf \equiv 1$ in $\Omega\#$. For every $\underline{t}$ such that $|t| < \delta$, the set $\Phi_t(\Omega)$ is an integral manifold of $\mathcal{T}_0(\underline{D})$. The vector field $\underline{Y} - \underline{X}$ is tangent to $\Phi_t(\Omega)$ and, since $\underline{f}$ is constant on $\Phi_t(\Omega)$, it follows that $Yf \equiv Xf$ on $\Phi_t(\Omega)$. Since $\Omega\#$ is the union of the sets $\Phi_t(\Omega)$ over $-\delta < t < \delta$, we conclude that $Yf \equiv Xf \equiv 1$ on $\Omega\#$. Let $r > 0$ be such that the curve $t \to \Phi_{-t}(\Psi_t(x))$, defined on $-r < t < r$, is contained in $\Omega\#$. In addition, let $r < \delta$. Let $g(t) = f(\Phi_{-t}(\Psi_t(x)))$. Then $\underline{g}$ is analytic in $(-r, r)$, and moreover $\underline{g}(t) = f(\Psi_t(x)) - t$. We have that $q' = (Yf)(\Psi_t(x)) - 1 \equiv 0$ and, since $g(0) = 0$ it follows that $g \equiv 0$ on $(-r, r)$. But this means that $\Phi_{-t}(\Psi_t(x)) \in \Omega$ for all $t \in (-r, r)$. Hence, if $|t| < r$, the manifold

$\Phi_{-t}(\Psi_t(S))$ intersects $\underline{S}$. Since $\Phi_{-t}(\Psi_t(S))$ is a maximal integral manifold of $\mathscr{T}_0(D)$, it follows that $\Phi_{-t}(\Psi_t(S)) = S$, and that $\Psi_t(S) = \Phi_t(S)$. Let $\underline{A}$ be the set of all $\underline{t}$ such that $\Phi_\tau(S) = \Psi_\tau(S)$ for all τ in a neighborhood of $\underline{t}$. Then $\underline{A}$ is obviously open, and we have shown that $0 \in \underline{A}$. It follows easily from the preceding argument that $\underline{A}$ is closed. Therefore, $\Phi_t(S) = \Psi_t(S)$ for all real $\underline{t}$, and our proof is complete.

According to the above lemma, if $\underline{D} \subset \underline{V}(\underline{M})$ and if $\underline{x} \in \underline{M}$, then the manifold $\Phi_t(I_0(D,x))$ depends only on $\underline{t}$, and not on the particular choice of $\underline{X}$. We shall denote this manifold by $I_0^t(D,x)$. It is clear that $I_0^t(D,x)$ could be defined as the maximal integral manifold of $\mathscr{T}_0(D)$ passing through y, where y is an arbitrary point of $L_x(D,t)$.

Finally, we prove a factorization property of maps that will be utilized several times.

LEMMA 3.7. Let $\underline{E}$ be a locally convex vector space, let $\underline{K} \subset \underline{E}$, and let $\underline{C}$ be a convex dense subset of $\underline{K}$. Let $F : K \to I(D,x)$ be a continuous mapping such that $\underline{F}(\underline{C})$ is contained in a maximal integral manifold $\underline{S}$ of $\mathscr{T}_0(\underline{D})$. Then $\underline{F}(\underline{K})$ is contained in $\underline{S}$, and $\underline{F}$, as a mapping from $\underline{K}$ into $\underline{S}$, is continuous.

Proof. If dim S = dim I(D, x), then S = I(D, x), and the conclusion follows trivially. Therefore, we shall assume that dim S = dim I(D, x) - 1.

Let $\underline{k} \in \underline{K}$, let $\underline{X} \in \underline{D}$, and let $\{\Phi_t\}$ be the one-parameter group induced by $\underline{X}$. Then, as in the proof of Lemma 3.6, we can find a neighborhood Ω of $\underline{F}(\underline{k})$ in $I_0(D, \underline{F}(\underline{k}))$, and a positive number δ, such that the mapping $(s,t) \to \Phi_t(s)$ is a diffeomorphism of $\Omega \times (-\delta, \delta)$ onto an open subset $\Omega^{\#}$ of $I(D,x)$. Let $\underline{U}$ be an open convex neighborhood of $\underline{k}$ such that $\underline{F}(\underline{U} \cap \underline{K}) \subset \Omega^{\#}$. For each $\underline{t} \in (-\delta, \delta)$, the set $\Phi_t(\Omega)$ is an integral manifold of $\mathcal{J}_0(D)$; therefore, if $\Phi_t(\Omega)$ intersects $\underline{S}$, then $\Phi_t(\Omega)$ is contained and open in $\underline{S}$. Let $A = \{t : |t| < \delta,\ \Phi_t(\Omega) \subset S\}$. It follows that $\underline{S} \cap \Omega^{\#}$ is the union of the sets $\Phi_t(\Omega)$, $\underline{t} \in \underline{A}$. These sets are mutually disjoint and, since $\underline{S}$ is separable, it follows that A is at most countable. Let $y \to (s(y), f(y))$ be the inverse in $\Omega^{\#}$ of the map $(s, t) \to \Phi_t(s)$. Then the function $\underline{g}$ defined in $\underline{U} \cap \underline{K}$ by $g(m) = f(F(m))$ is continuous. Since $F(U \cap C) \subset S \cap \Omega^{\#}$, we conclude that $\underline{g}(\underline{m}) \in \underline{A}$ for all $\underline{m} \in \underline{U} \cap \underline{C}$. But $\underline{A}$ is at most countable, and $\underline{U} \cap \underline{C}$ is convex; therefore $\underline{g}$ is constant on $\underline{U} \cap \underline{C}$. Since $\underline{U} \cap \underline{C}$ is dense in $\underline{U} \cap \underline{K}$, we have that $\underline{g}$ is constant on $\underline{U} \cap \underline{K}$. Obviously $g(k) = 0$, and therefore $g(m) = 0$ for all $\underline{m} \in \underline{U} \cap \underline{K}$; thus $\underline{F}(\underline{m}) \in \Omega$. This shows that Ω contains a point of $\underline{S}$; hence $\Omega \subset \underline{S}$, and $\underline{F}(\underline{k}) \in \underline{S}$. This proves the first part of the lemma.

To prove the second part, let $\{k_n\} \subset K$ converge to $\underline{k}$. Since $\underline{F}$ is continuous, $F(k_n) \to F(k)$. For large $\underline{n}$, $s(F(k_n))$ is defined. Since $\underline{s}$ is continuous, $s(F(k_n))$ converges to $s(F(k))$ in $\underline{S}$. But $g(k_n) = 0$, and therefore $s(F(k_n)) = F(k_n)$. Similarly, $s(F(k)) = F(k)$. We have thus shown that $F(k_n)$ converges to $F(k)$ in $\underline{S}$, and our proof is complete.

Remark 3.8. It is clear that the preceding lemma is valid under weaker assumptions about $\underline{C}$ and $\underline{K}$. For instance, it is sufficient to assume that, for every $\underline{k} \in \underline{K}$ and for every neighborhood $\underline{U}$ of $\underline{k}$, there exists a neighborhood $\underline{V}$ of $\underline{k}$ such that $\underline{V} \subset \underline{U}$ and $\underline{V} \cap \underline{C}$ is connected.

We now state and prove the theorem towards which we have been aiming.

THEOREM 3.9. Let $\underline{D} \subset \underline{V}(\underline{M})$ be a set of complete vector fields, and let $\underline{x} \in \underline{M}$. Then, for each $\underline{T} > 0$, $L_x(D,T) \subset I_0^t(D,x)$ and, moreover, the interior of $L_x(D,T)$(relative to $I_0^t(D,x)$) is dense in $L_x(D,T)$ (and is, in particular, non-empty).

Proof. If $\dim \mathscr{T}_0(D)(x) = \dim \mathscr{T}(D)(x)$; then we have from Corollary 3.4 that $\mathscr{T}_0(D)(y) = \mathscr{T}(D)(y)$ for all $y \in I(D,x)$. Therefore, $I_0(D,x) = I_0^t(D,x)$ and our conclusion follows from Theorem 3.3. Assume that $\dim \mathscr{T}_0(D)(x) = k = \dim \mathscr{T}(D)(x) - 1$. It is clear from Lemma 3.6 that, if α is an integral curve of $\underline{D}$ such that $\alpha(0) = x$, then $\alpha(T) \in I_0^T(D,x)$; hence, $L_x(D,T) \subset I_0^T(D,x)$.

We now show that, if $y \in L_x(D,T)$, then $\underline{y}$ is the limit of points which belong to the interior of $L_x(D,T)$. Let $D = \{X_1,\ldots,X_k\}$ and let $y = \Phi_{\underset{\sim}{i}}(\underset{\sim}{T},x)$, where $\|\underset{\sim}{T}\| = T$, and $T_i > 0$ for $i = 1,2,\ldots,m$ (the notations here are the same as in the proof of Theorem 3.1). Let $\underset{\sim}{j} = (j_1,\ldots,j_s)$ be an $\underline{s}$-tuple of integers between 1 and $\underline{k}$ such that the rank of $\underset{\sim}{t} \to \Phi_{\underset{\sim}{j}}(\underset{\sim}{t},x)$ is equal to $\dim \mathscr{T}(D)(x)$ for all $\underline{t}$ in an open dense subset Ω of $\underset{\sim}{R}^{\tilde{s}}$. Let $\Omega' = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^s,\ t_i > 0 \text{ for } i = 1,\ldots,s\} \cap \Omega$. Let $\{\underset{\sim}{t}_p\} \subset \Omega$ be a sequence that converges to 0, and let $\underset{\sim}{T}_p = (T_1,\ldots, T_{m-1}, T_m - \|\underset{\sim}{t}_p\|)$. We can assume that $\|t_p\| < T_m$ for all $p > 0$. If we let $y_p = \Phi_{\underset{\sim}{i}}(\underset{\sim}{T}_p, \Phi_{\underset{\sim}{j}}(\underset{\sim}{t}_p,x))$, then $y_p \in L_x(D,T)$. We next show that y_p is in the interior of $L_x(D,T)$ relative to $I_0^T(D,x)$. Since the mapping $z \to \Phi_{\underset{\sim}{i}}(\underset{\sim}{T}_p, z)$ is a diffeomorphism from $I_0^{\|\underset{\sim}{t}_p\|}(D,x)$ onto $I_0^T(D,x)$, it suffices to show that $\Phi_{\underset{\sim}{j}}(\underset{\sim}{t}_p,$ .s in the interior of $L_x(D,\|\underset{\sim}{t}_p\|)$. Let $V_p = \{\underset{\sim}{t} : \underset{\sim}{t} \in \underset{\sim}{R}^s,\ t_1 > 0,\ldots,t_s > 0,\ \|\underset{\sim}{t}\| = \|\underset{\sim}{t}_p\|\}$. Clearly, if $\underset{\sim}{t} \in V_p$, then $\Phi_{\underset{\sim}{j}}(\underset{\sim}{t},x) \in L_x(D,\|\underset{\sim}{t}_p\|)$. Let $F_p : V_p \to I_0^{\|\underset{\sim}{t}_p\|}(D,x)$ be defined by $F_p(\tilde{\underset{\sim}{t}}) = \Phi_{\underset{\sim}{j}}(\underset{\sim}{t},x)$. We show that F_p is analytic. Since F_p

is analytic as a map from V_p into $I(D, x)$, it suffices to show that it is continuous. But this follows from the previous lemma, because V_p is convex. The rank of $\underset{\sim}{t} \to \Phi_j(\underset{\sim}{t}, x)$ is equal to $\dim \mathscr{T}(D)(x)$ at $\underset{\sim}{t} = \underset{\sim}{t}_p$. Since V_p is a submanifold of $\underset{\sim}{R}^s$ of codimension 1, it follows that the rank of F_p at $\underset{\sim}{t}_p$ is equal to the dimension of $I_0^{\|\underset{\sim}{t}_p\|}(D, x)$. Thus, $F_p(V_p)$ contains a neighborhood of $F_p(\underset{\sim}{t}_p)$ in $I_0^{\|\underset{\sim}{t}_p\|}(D, x)$. It follows that $\underset{\sim}{\Phi}_j(\underset{\sim}{t}_p, x)$ is in the interior of $L_x(D, \|\underset{\sim}{t}_p\|)$. By the previous remark we conclude that y_p is interior to $L_x(D, T)$ in $I_0^T(D, x)$. There remains to be shown that y_p converges to y in $I_0^T(D, x)$. The mapping $(\underset{\sim}{t}, \underset{\sim}{s}) \to \underset{\sim}{\Phi}_i(\underset{\sim}{t}, \underset{\sim}{\Phi}_j(s, x))$ is continuous as a map from $\underset{\sim}{R}^m X \underset{\sim}{R}^s$ into $I(D, x)$. The set $V = \{ (\underset{\sim}{t}, \underset{\sim}{s}) : t_i > 0,\ s_j > 0, i = 1, \ldots, m,\ j = 1, \ldots, s,\ \|\underset{\sim}{t}\| + \|s\| = T\}$ is convex, and is mapped into $I_0^T(D, x)$. Therefore, the previous lemma is applicable, and we conclude that $y_p \to y$ in $I_0^T(D, x)$. This proves our theorem.

4. Applications to Control Systems

We shall consider systems of the form

$$\frac{dx(t)}{dt} = F(x(t), u(t))$$

defined on an analytic manifold $\underline{M}$. The functions $\underline{u}$ belong to a class $\mathscr{U}$ of "admissible controls". We make the following assumptions about $\mathscr{U}$ and the system function $\underline{F}$:

(i) <u>The elements of $\mathscr{U}$ are piecewise continuous functions defined in</u> $[0, \infty)$, <u>having values in a locally path connected set</u> Ω. $\Omega \subset \underset{\sim}{R}^m$ (Ω is locally path connected if, for every $\omega \in \Omega$ and every neighborhood $\underline{V}$ of ω, there exists a neighborhood $\underline{U}$ of ω such that $\underline{U} \subset \underline{V}$, and $\underline{U} \cap \Omega$ is path connected). In addition, we assume that $\mathscr{U}$ <u>contains</u>

all the piecewise constant functions with values in Ω, and that every element of $\mathcal{U}$ has finite one-side limits in each point of discontinuity. $\mathcal{U}$ is endowed with the topology of uniform convergence on compact intervals.

(ii) $F: M \times \Omega \to TM$ is jointly continuously differentiable. For each $u \in \Omega$, $F(\cdot, u)$ is a complete analytic vector field on M. We know that for each $x \in M$, $u \in \mathcal{U}$, the differential equation

$$\frac{dx(t)}{dt} = F(x(t), u(t)) \qquad x(0) = x, \tag{1}$$

has a solution defined for all $t \in [0, \delta)$, where $\delta > 0$. We denote such a solution by $\pi(x, u, \cdot)$, and we assume that $\pi(x, u, t)$ is defined for all $t \in [0, \infty)$.

For the above defined control system, we now state the basic controllability concepts. We say that $y \in M$ is attainable from $x \in M$ at time t $(t \geq 0)$, if there exists $u \in \mathcal{U}$ such that $\pi(x, u, t) = y$. For each $x \in M$, we let $A(x, t)$ denote the set of all points attainable from x at time t. If $0 \leq t < \infty$, we define $\underset{\sim}{A}(x, t) = \bigcup_{s \leq t} A(x, s)$ and $\underset{\sim}{A}(x) = \bigcup_{t \geq 0} \underset{\sim}{A}(x, s)$. We say that the system is controllable from x if $\underset{\sim}{A}(x) = M$, and that it is controllable if it is controllable from every $x \in M$. We say that the system has the accessibility property from x if $\underset{\sim}{A}(x)$ has a non-empty interior, and that is has the accessibility property if it has the accessibility property from every $x \in M$. Finally, we shall say that the system has the strong accessibility property from x if $A(x, t)$ has a non-empty interior for some $t > 0$, and that it has the strong accessibility property if it has the strong accessibility property from x for every $x \in M$.

For $\omega \in \Omega$, let $X_\omega = F(\cdot, \omega)$; from assumption (ii) it follows that X_ω is a complete analytic vector field on $\underline{M}$. Throughout the remaining part of this article we let $D = \{X_\omega : \omega \in \Omega\}$.

LEMMA 4.1. For each $\underline{x} \in \underline{M}$, $\underset{\sim}{A}(x)$ is contained in $I(D, x)$.

The proof is identical to that of Lemma 2.4, and will, therefore, be omitted.

Remark 4.2. It is easy to see that the control system defined by restricting $\underline{F}$ to $\underline{I}(\underline{D}, \underline{x})$ satisfies the same assumptions as the original system. Since it can be readily verified that the map $u \to \Pi(x, u, t)$ is continuous as a map from $\mathcal{U}$ into $\underline{M}$, it follows that this map is also continuous as a map from $\mathcal{U}$ into $I(D, x)$.

We now want to obtain a result for $\underline{A}(\underline{x}, \underline{t})$ which is similar to that of Lemma 4.1. It is here that the assumption about Ω will be utilized. Let $\mathcal{P}$ be the class of piecewise constant Ω-valued functions defined on $[0, \infty)$. Clearly, $\mathcal{P}$ is dense in $\mathcal{U}$. Moreover, the local connectedness of Ω implies that the condition of Remark 3.8 is satisfied (this can be easily verified, and we omit the proof). Thus, we can apply Lemma 3.7, with $C = \mathcal{P}$ and $K = \mathcal{U}$, to obtain the following result:

LEMMA 4.3. Let $\underline{x} \in \underline{M}$. For each $t \geq 0$, $A(x, t) \subset I_0^t(D, x)$.

Proof. Since $\mathcal{U}$ contains $\mathcal{P}$, we have that $L_x(D, t) \subset A(x, t)$. Let $G : \mathcal{U} \to I(D, x)$ be defined by $G(u) = \Pi(x, u, t)$. We have that $G(\mathcal{P}) = L_x(D, t)$ and by Theorem 3.9, $G(\mathcal{P}) \subset I_0^t(D, x)$. Now our conclusion follows immediately from Lemma 3.7, and the proof is complete.

The above lemmas combined with the theorems of the preceding section yield the following results:

THEOREM 4.4. Let $\underline{x} \in \underline{M}$. Then $\underset{\sim}{A}(x) \subset I(D, x)$. Moreover, for every $\underline{T} > 0$, the interior of $\underset{\sim}{A}(x, T)$ relative to $I(D, x)$ is dense in

$\underset{\sim}{A}(x, T)$(and, in particular, is non-empty).

Proof. The first part is just the statement of Lemma 4. 1. To prove the second part, we can assume that $I(\dot{D}, x) = M$ (if not, replace the original system by its restriction to $I(D, x)$, cf. Remark 4. 2). Since $\underset{\sim}{L}_x(D, T)$ is dense in $\underset{\sim}{A}(x, T)$, our conclusion follows immediately from Theorem 3. 1.

THEOREM 4. 5. Let $\underline{x} \in \underline{M}$. Then, for each $t > 0$, $A(x,t) \subset I_0^t(D, x)$ and, moreover, the interior of $A(x, t)$ relative to $I_0^t(D, x)$ is dense in $A(x, t)$ (and, in particular, is non-empty).

Proof. The first part is just the statement of Lemma 4. 3. To prove the second part, we apply Lemma 3. 7 to the function $\underline{G}$ of Lemma 4. 3, and we get that $\underline{G}$ is continuous as a map into $I_0^t(D, x)$; therefore, $L_x(D, t)$ is dense in $A(x, t)$ relative to $I_0^t(D, x)$. Our conclusion now follows immediately from Theorem 3. 9, and the proof is complete.

The following two controllability criteria follow immediately from the Theorems 4. 4 and 4. 5, and from Lemma 2. 1:

COROLLARY 4. 6. The system has the accessibility property from $\underline{x}$ if and only if $\dim \mathscr{T}(D)(x) = \dim M$. In this case $\underset{\sim}{A}(x, T)$ has a non-empty interior for every $T > 0$.

COROLLARY 4. 7. The system has the strong accessibility property from $\underline{x}$ if and only if $\mathscr{T}_0(D)(x) = \dim M$. In this case $A(x, T)$ has a non-empty interior for every $T > 0$.

The preceding results can be utilized to derive relationships between accessibility and strong accessibility. Even though the latter property seems much stronger than the former, we show that, for a very large class of manifolds (including the spheres S^n for $n > 1$, and all compact semisimple Lie groups, but not $\underset{\sim}{R}^n$), it is in fact implied by it.

On the other hand for a still larger class of manifolds (including $\underset{\sim}{R}^n$) controlability (which trivially implies accessibility), is sufficient to guarantee strong accessibility (the fact that controllability implies that $\dim \mathscr{T}(D^*)(x) = n + 1$ for all $\underline{x}$ was proved by Elliott in [5]).

Consider a system on a connected n-dimensional analytic manifold $\underline{M}$, having the accessibility property but not having the strong accessibility property. Let $\underline{D}$ be the family of associated vector fields. By Corollary 4.6, $\dim \mathscr{T}(D)(x) = n$ for all $x \in M$. By Corollary 3.4 the number $\dim \mathscr{T}_0(D)(x)$ is independent of $\underline{x}$. Since this number is either $\underline{n}$ or $\underline{n}$ -1, Corollary 4.7 implies that $\dim \mathscr{T}_0(D)(x) = n - 1$ for all $x \in M$. Choose a fixed $X \in D$, and use Φ_t to denote the one-parameter group generated by X(i.e., for every $y \in M$, the integral curve of X that passes through y at $t = 0$ is the curve $t \to \Phi_t(y)$). Define a mapping F from the manifold $S \times \underset{\sim}{R}$ into M by

$$F(s,t) = \Phi_t(s).$$

One shows easily that $\underline{F}$ is a local diffeomorphism onto $\underline{M}$. Moreover, $S \times \underset{\sim}{R}$ is connected. In fact, we have (cf. [18], Ch. 2, for the definition of a covering projection):

LEMMA 4.8. The map $\underline{F}$ is a covering projection.

Before we prove Lemma 4.8, we show how the results mentioned above follow from it.

THEOREM 4.9. Let $\underline{M}$ be a manifold whose universal covering space (cf. [18]) is compact. Then every system having the accessibility property has the strong accessibility property.

Proof. If the universal covering space of $\underline{M}$ is compact, then every covering space of $\underline{M}$ is compact. If it were possible to have a system on $\underline{M}$ having the accessibility property but not the strong accessibility property, we could define, for such a system, $\underline{S}$ and $\underline{F}$ as above. It would follow that $S \times \underset{\sim}{R}$ is compact, which is clearly a contradiction.

Remark. If $n > 1$, the sphere S^n is simply connected (and compact). Therefore Theorem 4.9 applies. Also, if $\underline{M}$ is a connected compact semisimple Lie group (for instance $SO(n)$, if $n > 2$), the universal covering group of $\underline{M}$ is also compact (cf. [7], p. 123) and, therefore, Theorem 4.9 applies in this case as well.

THEOREM 4.10. Let $\underline{M}$ be a manifold whose fundamental group has no elements of infinite order. Then every controllable system on $\underline{M}$ has the strong accessibility property.

Proof. A controllable system obviously has the accessibility property. Assume it does not have the strong accessibility property. Define $\underline{S}$ and $\underline{F}$ as before. We show that $\underline{F}$ is one-to-one. Otherwise, there would exist $s_0, s_0' \in S$ and a $T \neq 0$ such that $F(T, s_0') = \Phi_T(s_0') = F(0, s_0) = s_0$. Therefore $\Phi_T(S) = S$. Define $H: S \times \underset{\sim}{R} \to S \times \underset{\sim}{R}$ by $H(s,t) = (\Phi_T(s), t-T)$. Then $\underline{H}$ is well defined, because $\Phi_T(S) = S$, and is a homeomorphism. Moreover, if $(s,t) \in S \times \underset{\sim}{R}$

$$F(H(x,t)) = \Phi_{t-T}(\Phi_T(s)) = \Phi_t(s) = F(s,t).$$

Therefore $\underline{H}$ is a covering transformation (cf. [18], Ch. 2). Moreover, H has infinite order, because $H^m(s,t) = (\Phi_{mT}(s), t-mT)$, so that H^m is not the identity map if $m \neq 0$. We know from [18] Ch. 2 that the group of covering transformations of the covering space $(S \times \underset{\sim}{R}, F)$ is isomorphic to a subgroup of the fundamental group π of M.

If π has no elements of infinite order, then this is a contradiction. Therefore F must be one-to-one. On the other hand, the points that are attainable from x_0 must belong to $S_t (= \Phi_t(S))$ for some nonnegative t (cf. Theorem 4.5). Therefore the points in S_{-t} are not attainable, if $t > 0$. Thus, the system is not controllable, and we have reached a contradiction.

Remark. Theorem 4.10 applies, in particular, to any simply connected manifold, such as $\underset{\sim}{R}^n$.

Proof of Lemma 4.8. We must show that every point of $\underline{M}$ has a neighborhood that is evenly covered by $\underline{F}$. Let $m \in M$. Since $\underline{F}$ is a local diffeomorphism onto, there exist $\underline{s} \in \underline{S}$, $t \in \underset{\sim}{R}$, $\epsilon > 0$ and a connected neighborhood $\underline{U}$ of $\underline{s}$ in $\underline{S}$ such that $F(s,t) = m$ and that the restriction of F to $UX(t-\epsilon, t+\epsilon)$ is a diffeomorphism onto an open subset $\underline{V}$ of $\underline{M}$. We claim that $\underline{V}$ is evenly covered. Let $A = \{\tau : \Phi_\tau(S) = S\}$. For each $\tau \in A$, let $U_\tau = \Phi_\tau(U)$. Since $\Phi_\tau : S \to S$ is a diffeomorphism, it follows that U_τ is open in $\underline{S}$ and connected for each $\tau \in A$. We first show that, if $0 < |\tau - \eta| < 2\epsilon$, $\tau \in A$, $\eta \in A$, then U_τ and U_η are disjoint. Assume they are not. Then $\Phi_T(U_\tau)$ and $\Phi_T(U_\eta)$ are not disjoint, for any $\underline{T}$. Choose $\underline{T}$ such that both $\underline{T} + \tau$ and $\underline{T} + \eta$ belong to $(t-\epsilon, t+\epsilon)$. Let $\underline{u} = \Phi_{T+\tau}(u_1) = \Phi_{T+\eta}(u_2)$ be a common element, where u_1 and u_2 belong to $\underline{U}$. Then the points $(u_1 T + \tau)$ and $(u_2, T + \eta)$ belong to $UX(t-\epsilon, t+\epsilon)$. Since the restriction of $\underline{F}$ to this set is one-to-one, it follows that $\tau = \eta$, which is a contradiction. For each $\tau \in A$, let $W_\tau = U_\tau \times (t-\tau-\epsilon, t-\tau+\epsilon)$. We shall conclude our proof that $\underline{V}$ is evenly covered by showing:

(a) the sets W_τ are open, connected and pairwise disjoint,

(b) for each $\tau \in A$, $\underline{F}$ maps W_τ diffeomorphically onto $\underline{V}$, and

(c) the inverse image of $\underline{V}$ under $\underline{F}$ is the union of the sets W_τ.

The first two assertions of (a) are obvious. If τ and η belong to A, and $\tau \neq \eta$, then either $|\tau - \eta| < 2\varepsilon$ or $|\tau-\eta| \geq 2\varepsilon$. In the first case W_τ and W_η must be disjoint, because U_τ and U_η are disjoint. In the second case, W_τ and W_η are also disjoint, because the intervals $(t-\tau-\varepsilon, t-\tau+\varepsilon)$ and $(t-\eta-\varepsilon, t-\eta+\varepsilon)$ cannot have a point in common.

To prove (b), take $\tau \in A$. Define $G : U \times (t-\varepsilon, t+\varepsilon) \to W_\tau$ by $G(u,\sigma) = (\Phi_\tau(u), \sigma - \tau)$. Clearly, G is a diffeomorphism from $U\times(t-\varepsilon, t+\varepsilon)$ onto W_τ. Moreover if $u \in U$, $t-\varepsilon < \sigma < t+\varepsilon$, then $F(G(u,\sigma)) = \Phi_{\sigma-\tau}(\Phi_\tau(u)) = \Phi_\sigma(u) = F(u,\sigma)$. Since the restriction of $\underline{F}$ to $U\times(t-\varepsilon, t+\varepsilon)$ is a diffeomorphism onto $\underline{V}$, the same must be true for the restriction of $\underline{F}$ to W_τ.

Finally, we prove (c). Let $u \in S$, $\sigma \in \underset{\sim}{R}$ be such that $F(u,\sigma) \in \underline{V}$. Then there exist $u' \in U$, $\sigma' \in (t-\varepsilon, t+\varepsilon)$ such that $F(u',\sigma') = F(u,\sigma)$. Therefore $u = \Phi_{\sigma'-\sigma}(u')$. This implies, in particular, that $\tau = \sigma' - \sigma$ belongs to $\underline{A}$, and that $u \in U_\tau$. Moreover, since $t-\varepsilon < \sigma' < t+\varepsilon$, it follows that $t-\tau-\varepsilon < \sigma < t-\tau+\varepsilon$. Therefore $(u,\sigma) \in W_\tau$.

The proof Lemma 4.8 is now complete.

5. Examples.

Example 5.1. Let $M = \underset{\sim}{R}^n$, $\Omega = \underset{\sim}{R}^m$, and let $F : M \times \Omega \to \underline{\underline{TM}}$ be defined by $F(x,u) = Ax + Bu$, where $\underline{A}$ and $\underline{B}$ are, respectively, $\underline{n} \times \underline{n}$ and $\underline{n} \times \underline{m}$ real matrices. We have that $D = \{A(\cdot) + Bu : u \in \underset{\sim}{R}^m\}$. Let b_i denote the $\underline{i}$-th column of $\underline{B}$. Then, as shown by Lobry [16], $\mathscr{T}(D)(x)$ contains the vectors:

$$Ax \pm b_i \, , \pm Ab_i, \ldots, \pm A^{n-1}b_i \quad i = 1, \ldots, m.$$

It is not difficult to see that the above set of vectors forms a system of generators for $\mathscr{T}(\underline{D})(\underline{x})$. From Corollary 4.6 we get that $\underset{\sim}{A}(\underset{\sim}{0}, t)$ has a non-empty interior in $\underset{\sim}{R}^n$ if and only if $\{\pm b_i, \pm Ab_i, \ldots, \pm A^{n-1}b_i, i = 1, 2, \ldots, m\}$ has rank $\underline{n}$; equivalently, $\underset{\sim}{A}(\underset{\sim}{0}, t)$ has a non-empty interior in $\underset{\sim}{R}^n$ if and only if rank $[B, AB, \ldots, A^{n-1}B] = n$.

Since, obviously, $\mathscr{T}_0(D)(\underset{\sim}{0}) = \mathscr{T}(D)(\underset{\sim}{0})$, we conclude that $A(\underset{\sim}{0}, t)$ has a non-empty interior whenever $\underset{\sim}{A}(\underset{\sim}{0}, t)$ does. The above statements, along with the fact that $A(\underset{\sim}{0}, t)$ and $\underset{\sim}{A}(\underset{\sim}{0}, t)$ are linear subspaces of $\underset{\sim}{R}^n$, imply that, if rank $[B, AB, \ldots, A^{n-1}B] = n$, then for each $t > 0$ $A(\underset{\sim}{0}, t) = \underset{\sim}{A}(\underset{\sim}{0}, t) = \underset{\sim}{A}(\underset{\sim}{0}) = \underset{\sim}{R}^n$ (Kalman [12]). Thus, in this example, the accessibility property is equivalent to controllability. This is, of course, not true in general.

Example 5.2. Let $M = \underset{\sim}{R}^n$, $\Omega = \{u \in \underset{\sim}{R}^m : 0 \le u_i \le 1, \ i = 1, \ldots, m\}$, and let $F(x, u) = (A_0 + \Sigma_{i=1}^m A_i u_i)x$ for all $(\underline{x}, \underline{u}) \in \underset{\sim}{R}^n X \, \Omega$, where $A_0, \ldots, A_m$ are nXn real matrices. Then $\underline{D}$ is the set of all vector fields X_u where $X_u(x) = (A_0 + \Sigma_{i=1}^m u_i A_i) \, x$. The set $\underset{\sim}{M}^n$ of all nXn real matrices is a Lie algebra, with the bracket defined by $[P, Q] = PQ - QP$. To each matrix $\underline{P}$ there corresponds a vector field $\underline{V}(\underline{P})$ defined by $V(P)(x) = Px$. It is easy to check that $V([P, Q]) = [V(P), V(Q)]$. Using this fact, the spaces $\mathscr{T}(D)(x)$ and $\mathscr{T}_0(D)(x)$ can be readily computed:

$$\mathscr{T}(D)(x) = \{Px : P \in \underset{\sim}{L}\},$$

and

$$\mathscr{T}_0(D)(x) = \{Px : P \in L\}$$

where $\underset{\sim}{L}$ is the Lie algebra spanned by $A_0, \ldots, A_m$, and $\underline{L}$ is the ideal of $\underset{\sim}{L}$ spanned by $A_1, \ldots, A_m$. We remark that for this example the theory of Section 4 is valid even if $\mathscr{U}$ is the set of all bounded and measurable Ω -valued functions. This is so because the only properties of the class of admissible controls that were utilized in Section 4 were: (a) that the class of piecewise constant controls is dense in $\mathscr{U}$ (in the topology of uniform convergence), and (b) that, if $\{\underline{u}_\alpha\}$ are elements of $\mathscr{U}$ that converge uniformly to $\underline{u}$, then $\Pi(\underline{u}_\alpha, \underline{x}, \underline{t})$ converges to $\Pi(\underline{u}, \underline{x}, \underline{t})$.

In our example, both (a) and (b) remain valid if the topology of uniform convergence is replaced by that of weak convergence. This is easy to verify, and we shall not do it here (see Kučera [14]). Moreover, the set of Ω-valued measurable functions defined in $[0, \underline{T}]$ is weakly compact. It follows that the sets $\underset{\sim}{A}(x, T)$, $A(x, T)$ are compact for each $\underline{T} > 0$. Denote their interiors (relative to $I(D, x)$ and $I_0^T(D, x)$ respectively) by Int $\underset{\sim}{A}(x, T)$, Int $A(x, T)$. It follows that $\underset{\sim}{A}(x, T)$ is the closure of Int $\underset{\sim}{A}(x, T)$, and that $A(x, T)$ is the closure of Int $A(x, T)$. Therefore, our results contain those of Kučera (in this connection, see also Sussmann [21]).

Remark. The result of the preceding example is a particular case of a more general situation. Let $\underline{G}$ be a Lie group, and let $\underline{M}$ be an analytic manifold on which $\underline{G}$ acts analytically to the left. Then there is a homomorphism λ from the Lie algebra of $\underline{G}$ into $\underline{V}(\underline{M})$, defined by

$$\lambda(X)(m) = \frac{d}{dt} (\exp(tX) \cdot m),$$

the derivative being evaluated at $\underline{t} = 0$. If $X_0, \ldots, X_k$ belong to the Lie algebra of $\underline{G}$, we can consider the control problem

$$\frac{dx}{dt} = X'_0(x) + \sum_{i=1}^{k} u_i X'_i(x),$$

where $X'_i = \lambda(X_i)$. Example 5.2 results by letting $G = GL(n, \underset{\sim}{R})$ and $\underline{M} = \underset{\sim}{R}^n$.

Example 5.3. This example shows that the analyticity assumptions are essential. Consider the following two systems defined in the $(\underline{x}, \underline{y})$ plane:

$$(S_1) \quad \dot{x} = f_1(x, y, u)$$
$$\dot{y} = g_1(x, y, u)$$

and

$$(S_2) \quad \dot{x} = f_2(x, y, u)$$
$$\dot{y} = g_2(x, y, u)$$

Let $f_1 \equiv f_2 \equiv 1$, $g_1 \equiv 0$, and $g_2(x, y, u) = \varphi(x)$ where φ is a C^∞ function which vanishes for $-\infty < \underline{x} < 1$, and which is equal to 1 for $\underline{x} > 2$. It is clear that for (S_1) the set $\underset{\sim}{A}((0, 0))$ is the half line $\{(x, y) : y = 0, x \geq 0\}$ while, for (S_2), $\underset{\sim}{A}((0, 0))$ has a non-empty interior. However, both systems are identical in a neighborhood of $(0, 0)$.

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CONTROL SYSTEMS ON LIE GROUPS*

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1. Introduction

In this article we study the controllability properties of systems which are described by an evolution equation in a Lie group $\underline{G}$ of the form:

$$\frac{dx}{dt}(t) = \underline{X}_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, \underline{X}_i(x(t)) \tag{+}$$

where $\underline{X}_0, \ldots, \underline{X}_m$ are right-invariant vector fields on $\underline{G}$. Systems described by (+) we term right-invariant. This study is based on the results of [11], and is related to the work of R. W. Brockett [1]. As remarked by Brockett, there are many important applications in engineering and in physics which are not treated by classical control theory because of the assumption that the state space is a vector space. In particular, when controlling the orientation of a rigid body relative to some fixed set of axes, the state space can be naturally regarded as

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$\underline{SO}(3)$, the group of orthogonal 3 x 3 real matrices with positive determinant. The evolution equation describing this system is of the form given by (+) ([1]). Instead of restricting our study to groups of matrices, we consider systems described in an abstract Lie group $\underline{G}$. This generalization in no essential way affects the nature of the problem.

From the theoretical point of view a study of systems of the form (+) appears natural for several reasons. For instance, the algebraic criteria developed in [11] can be used to obtain global results by exploiting the algebraic structure of the state space and the sets attainable from the identity. In this regard, the analogy with the controllability of linear systems is striking.

In this article we shall look for necessary and sufficient conditions for a right-invariant system to be controllable. A necessary condition is that the system have the "accessibility property" (cf. [11]). We show that this condition is also sufficient if $\underline{G}$ is connected and if either (a) the system is homogeneous (i. e. $\underline{X}_0 = 0$) or (b) $\underline{G}$ is compact. When neither (a) nor (b) hold, accessibility (plus the connectedness of $\underline{G}$) is not sufficient for controllability. In this case we give some sufficient conditions, and a necessary condition, and we single out a particular situation in which a necessary and sufficient condition can be obtained.

An obvious necessary condition for controllability is that the set $\utilde{A}(\underline{e})$ of points reachable from the identity of $\underline{G}$ be a subgroup of $\underline{G}$. Thus, the controllability problem reduces to the following two:

(a) when is $\utilde{A}(\underline{e})$ is a subgroup? , and

(b) if $\utilde{A}(\underline{e})$ is a subgroup, when is $\utilde{A}(\underline{e}) = \underline{G}$? Question (b) is much easier to answer than question (a). In Theorem 4. 6 we show that if $\utilde{A}(\underline{e})$

is a subgroup then, necessarily, this subgroup is the connected Lie subgroup $\underset{\sim}{S}$ of $\underline{G}$ whose Lie algebra is the subalgebra $\underset{\sim}{L}$ generated by $\underline{X}_0, \ldots, \underline{X}_m$. From this it follows that the system will be controllable if and only if (i) $\underset{\sim}{A}(\underline{e})$ is a subgroup, (ii) $\underline{G}$ is connected, and (iii) $\underset{\sim}{L}$ is the Lie algebra of $\underline{G}$. This shows that the crucial question is that of determining when $\underset{\sim}{A}(\underline{e})$ is a subgroup.

This question is (partially) answered in Sections 5 and 6.

The organization of the article is as follows: In Section 2 we introduce notation and basic concepts: in addition, we quote a result about Lie groups which will be used later. In Section 3 we single out the relevant Lie algebras induced by a right invariant system. In Section 4 we derive the basic properties of attainable sets. In Section 5 we study the homogeneous case, and in Section 6 we study the general case. In Section 7 we interpret our results in terms of controllability.

Finally, Section 8 contains examples.

2. Preliminaries

We shall assume that the reader is familiar with the basic facts about Lie groups (cf. [2], or [4], or [5]).

Throughout this paper, $\underline{G}$ will denote a Lie group, and $\underline{L}(\underline{G})$ will denote the Lie algebra of $\underline{G}$. We shall think of $\underline{L}(\underline{G})$ as the set of vector fields on $\underline{G}$ that are invariant under <u>right</u> translations. It is known that every $\underline{X} \in \underline{L}(\underline{G})$ is analytic, and that $\underline{L}(\underline{G})$ is a Lie algebra with the obvious vector operations, and with the Lie product defined by

$$[\underline{X}, \underline{Y}] = \underline{X}\,\underline{Y} - \underline{Y}\,\underline{X}$$

The exponential map from $\underline{L}(\underline{G})$ into $\underline{G}$ is denoted by exp. Recall that $\exp(0) = \underline{e}$ (the identity element of $\underline{G}$), and that, for each $\underline{X} \in \underline{L}(\underline{G})$, the curve $\underline{t} \to \exp(\underline{t}\,\underline{X})$ is an integral curve of $\underline{X}$.

We recall that there is a one-to-one correspondence between Lie subalgebras of $L(G)$ and connected Lie subgroups of G. If H is a connected Lie subgroup of G, the Lie algebra $L(H)$ is naturally identified with a subalgebra of $L(G)$. We shall also denote this subalgebra by $L(H)$.

Let $X_0, \ldots, X_m$ be elements of $L(G)$. We shall consider the following control system defined on G:

$$(1) \quad \dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, X_i(x(t))$$

where $u = (u_1, \ldots, u_m)$ belongs to the class of admissible controls U. Throughout the article we shall assume that U is one of the classes U_u, U_r or U_b, defined as follows:

(i) U_u is the class of all locally bounded and measurable functions defined on the interval $[0,\infty)$ having values in R^m.

(ii) U_r is the subset of U_u consisting of all elements which take values in the cube $\{x \in R^m : |x_i| \le 1,\ i = 1, \ldots, m\}$.

(iii) U_b is the class of all piecewise constant functions defined on $[0,\infty)$ with values in R^m such that the components of its elements only take values 1 and -1.

We will refer to U_u, U_r and U_b as the class of unrestricted, restricted and "bang-bang" controls, respectively.

If $X = (X_0, \ldots, X_m)$ is an $m + 1$ - tuple of elements of $L(G)$, and if U is a class of admissible controls, then the system described by equation (1) will be termed right-invariant. For notational convenience, we will denote such a system by (X, U). We will also adopt the convention that if in a particular statement U is not specified explicitly, we will mean that such a statement is true for any class of admissible controls

(i. e., $\underline{U}_u$, $\underline{U}_r$ or $\underline{U}_b$).

We have the following basic fact:

LEMMA 2.1. Let $(\underline{X}, \underline{U})$ be a right-invariant system on G, and let $\underline{u} \in \underline{U}$. Then for every $\underline{g} \in \underline{G}$, there exists a unique solution[1] $\underline{x}$ of (1) defined for $0 \leq \underline{t} < \infty$, such that $\underline{x}(0) = \underline{g}$.

Proof. Uniqueness and local existence follow from the standard results on ordinary differential equation. Moreover, we know from these results that there is a maximal interval $[0, \underline{T})$ $(\underline{T} > 0)$ on which there exists a solution $\underline{x}$ of (1) with $\underline{x}(0) = \underline{g}$. We show that $\underline{T} = \infty$. Assume $\underline{T} < \infty$. Let $\underline{y}(\underline{t})$ be a solution of (1) defined for $\underline{T} - \delta < \underline{t} < \underline{T} + \delta$, where $\delta > 0$, and such that $\underline{y}(\underline{T}) = \underline{e}$. Let $\underline{g}' = \underline{y}(\underline{T} - \frac{1}{2}\delta)$, $\underline{g}'' = \underline{x}(\underline{T} - \frac{1}{2}\delta)$.
Let $\underline{z}(\underline{t})$ be defined by

$$\underline{z}(\underline{t}) = \underline{x}(\underline{t}) \text{ for } 0 \leq \underline{t} \leq \underline{T} - \frac{1}{2}\delta$$

$$\underline{z}(\underline{t}) = \underline{y}(\underline{t})\, \underline{g}'^{-1}\, \underline{g}'' \text{ for } \underline{T} - \frac{1}{2}\delta < \underline{t} \leq \underline{T} + \delta .$$

Than $\underline{z}(\underline{t})$ is a solution of (1) which satisfies $\underline{z}(0) = \underline{g}$ and is defined for $0 \leq t < \underline{T} + \delta$. This contradicts the maximality of the interval $[0, \underline{T})$. Therefore $\underline{T} = \infty$ and our proof is complete.

If $\underline{u} \in \underline{U}$ and $\underline{g} \in \underline{G}$ we will denote the solution $\underline{x}$ of (1) which satisfies $\underline{x}(0) = \underline{g}$ by $\pi(\underline{g}, \underline{u}, \cdot)$; i.e., $\underline{x}(\underline{t}) = \pi(\underline{g}, \underline{u}, \underline{t})$ for all $\underline{t} \in [0, \infty)$. If, for some $\underline{t} \geq 0$, $\pi(\underline{g}, \underline{u}, \underline{t}) = \underline{g}'$, we say that the control $\underline{u}$ steers $\underline{g}$ into $\underline{g}'$ in $\underline{t}$ units of time. If there exists $\underline{u} \in \underline{U}$ which steers $\underline{g}$ into $\underline{g}'$ in $\underline{t}$ units of time, we say that $\underline{g}'$ is attainable (or reachable) from $\underline{g}$ at time $\underline{t}$. The set of all $\underline{g}' \in \underline{G}$ which are attainable from $\underline{g}$ at time $\underline{t}$ will be denoted by $\underline{A}(\underline{g}, \underline{t})$. We shall also use the notations

$$\mathbf{A}(g, T) = \bigcup_{0 \le t \le T} A(g, T)$$

$$\mathbf{A}(g) = \bigcup_{0 \le t < \infty} A(g, T)$$

We shall refer to $\mathbf{A}(g)$ as the *set attainable from* g.

From the right invariance of our control systems it follows trivially that $A(g, T) = A(e, T)g$, $\mathbf{A}(g, T) = \mathbf{A}(e, T)g$, and $\mathbf{A}(g) = \mathbf{A}(e)g$[2]. Therefore, without loss of generality, *we can limit ourselves to the study of the sets attainable from the identity*.

We finish this section by quoting a result about Lie groups whose proof can be found in [12] (cf. - also [5], pp. 275).

THEOREM 2.2. *Let G be a Lie group, and let H be a path-connected subgroup of G. Then H is a Lie subgroup of G.*

3. The Associated Lie Subalgebras.

To every right-invariant control system $(\mathbf{X}, U)$ on a Lie group G, we shall associate the following three Lie subalgebras of $L(G)$:

(1) The subalgebra $\mathbf{L}$ generated by $X_0, \ldots, X_m$,

(2) The ideal of $\mathbf{L}$ generated by $X_1, \ldots, X_m$. This ideal will be denoted by $\mathbf{L}_0$.

(3) The subalgebra L of $L(G)$ generated by $X_1, \ldots, X_m$.

We denote the corresponding connected Lie subgroups by $\mathbf{S}$, $\mathbf{S}_0$ and S. We have:

LEMMA 3.1.

(i) $\underline{L} \subset \underset{\sim}{L}_0 \subset \underset{\sim}{L}$ and $\underline{S} \subset \underset{\sim}{S}_0 \subset \underset{\sim}{S}$,

(ii) $\underset{\sim}{L}_0$ is a subspace of $\underset{\sim}{L}$ of codimension zero or one,

(iii) $\underset{\sim}{S}_0$ is a normal subgroup of $\underset{\sim}{S}$.

Proof. (i) and (ii) are trivial. (iii) follows from the fact that a connected Lie subgroup $\underline{H}$ of a connected Lie group $\underline{K}$ is a normal subgroup of $\underline{K}$ if and only if $\underline{L}(\underline{H})$ is an ideal of $\underline{L}(\underline{K})$ (cf. [2], p. 124).

We shall use the notation $\underset{\sim}{S}_0^{\,t}$ for the coset of $\underset{\sim}{S}$ modulo $\underset{\sim}{S}_0$ which contains exp $(\underline{t}\ \underline{X}_0)$.

4. Elementary Properties of the Attainable Sets.

If $(\underset{\sim}{X}, \underline{U})$ is a right-invariant control system on $\underline{G}$, then the vector fields $\underline{X}_0, \ldots, \underline{X}_m$ belong to the Lie algebra of $\underset{\sim}{S}$. Therefore, we can consider $(\underset{\sim}{X}, \underline{U})$ as a right-invariant control system on $\underset{\sim}{S}$, and Lemma 2.1 will be valid if $\underline{G}$ is replaced by $\underset{\sim}{S}$. This gives

LEMMA 4.1. If $(\underset{\sim}{X}, \underline{U})$ is a right-invariant system on $\underline{G}$, then $\underset{\sim}{A}(\underline{e})$ is considered in $\underset{\sim}{S}$.

The following lemma states a similar result for the sets $\underline{A}(\underline{e}, \underline{t})$.

LEMMA 4.2. If $(\underset{\sim}{X}, \underline{U})$ is a right-invariant system on $\underline{G}$, then for each $\underline{t} \geq 0$ $\underline{A}(\underline{e}, t)$ is contained in $\underline{S}_0^{\,t}$.

It would be easy to prove this lemma directly, but since this result is included in that of Lemma 6.1 we omit the proof.

We next derive some elementary topological properties of the attainable sets. If $\underline{T} \geq 0$, we will denote the set of all restrictions of elements of $\underline{U}$ to $[0, \underline{T}]$ by $\underline{U}(\underline{T})$.

LEMMA 4.3. Let $(\underset{\sim}{X}, \underline{U})$ be a right-invariant control system on $\underline{G}$. The mapping $(\underline{u}, \underline{t}) \to \pi(\underline{g}, \underline{u}, \underline{t})$ from $\underline{U}(\underline{T}) \times [0, \underline{T}]$ into $\underline{G}$ is continuous for each $\underline{g}$ and each $\underline{T} \geq 0$, if $U(\underline{T})$ is given the topology of weak convergence. The proof of this result appears in [10], and therefore we will omit it.(3)

From this we obtain:

LEMMA 4.4. *Let $(\tilde{X}, U)$ be a right-invariant control system on G.*

(i) The sets $\tilde{A}(e, T)$, $\tilde{A}(e)$, $A(e, T)$ are path-connected, for each $T \geq 0$.

(ii) If $U = U_r$ then $\tilde{A}(e, T)$ and $A(e, T)$ are compact.

Proof. (i) will be an immediate consequence of the fact that $U(T)$ is path-connected and of Lemma 4.3. The path-connectedness of $U(T)$ is trivial in the unrestricted and in the restricted case. In the "bang-bang" case, let u and v belong to $U(T)$.
For each t such that $0 \leq t \leq T$, let w_t be defined by

$$w_t(\tau) = v(\tau) \quad \text{if } 0 \leq \tau \leq t$$

$$w_t(\tau) = u(\tau) \quad \text{if } t < \tau \leq T$$

Then $w_t \in U(T)$. Moreover, $w_0 = u$ and $w_T = v$. Since $t \to w_t$ is a continuous path in $U(T)$, it follows that $U(T)$ is indeed path-connected.

To prove (ii) we remark that, if U is the class of restricted controls, then $U(T)$ is compact in the weak topology. The proof is now complete.

In regard to the algebraic properties of the attainable sets we have the following:

LEMMA 4.5. *Let $(\tilde{X}, U)$ be a right-invariant control system on G.*

hen the set $\underset{\sim}{A}(e)$ is a semi-group.

Proof. Let g and g belong to $\underset{\sim}{A}(e)$. Let $g = \pi(e, u, t)$, $g' = \pi(e, u', t')$. Let the control v be defined by

$$v(\tau) = u(\tau) \qquad \text{for} \qquad 0 \leq \tau \leq t$$

$$v(\tau) = u'(\tau - t) \qquad \text{for} \qquad t \leq t + t'$$

Then $\pi(e, v, t+t') = g'g$, and therefore, $g'g \in \underset{\sim}{A}(e)$. The proof is then complete.

We cannot assert, in general, that $\underset{\sim}{A}(e)$ is a group. However, the following theorem tells us that, if $\underset{\sim}{A}(e)$ is a group, then it must be the group $\underset{\sim}{S}$.

THEOREM 4.6. Let $(\underset{\sim}{X}, U)$ be a right-invariant control system on G. If $\underset{\sim}{A}(e)$ is a subgroup of G, then $\underset{\sim}{A}(e) = \underset{\sim}{S}$.

Proof. We know that $\underset{\sim}{A}(e)$ is path-connected. If $\underset{\sim}{A}(e)$ is a subgroup, it follows from Theorem 2.2 that it is a Lie subgroup of G. Let Λ be its Lie algebra. Then $\Lambda \subset \underset{\sim}{L}$, because $\underset{\sim}{A}(e) \subset \underset{\sim}{S}$ (Lemma 4.1). On the other hand, let $\underset{\sim}{a} = (a_1, \ldots, a_m)$ be an m-tuple such that each a_i is ± 1.

Let $\underline{u}$ be the constant control $\underline{u} \equiv (\underline{a}_1, \ldots, \underline{a}_m)$. Then $\underline{u} \in \underline{U}$ and, therefore, the curve $\underline{t} \to \pi(\underline{e}, \underline{u}, \underline{t})$ $(\underline{0} \le \underline{t} \le \infty)$ is contained in $\underset{\sim}{A}(\underline{e})$. In other words, if we let

$$\underline{X}(\underset{\sim}{a}) = \underline{X}_0 + \Sigma_{\underline{i}=1}^{m} \underline{a}_{\underline{i}} \underline{X}_{\underline{i}} ,$$

it follows that $\exp(\underline{t}\ \underline{X}(\underset{\sim}{a}))$ belongs to $\underset{\sim}{A}(\underline{e})$ for all $\underline{t} \ge 0$. Since $\underset{\sim}{A}(\underline{e})$ is a subgroup, this will be true for all real $\underline{t}$. Therefore, (cf. [4], p. 94), we can conclude that $\underline{X}(\underset{\sim}{a})$ belongs to Λ. Since the elements $\underline{X}(\underset{\sim}{a})$ form a system of generators of $\underset{\sim}{L}$, we conclude that $\underset{\sim}{L} \subset \Lambda$ and, therefore, $\underset{\sim}{L} = \Lambda$ and $\underset{\sim}{A}(\underline{e}) = \underset{\sim}{S}$.

5. The Homogeneous Case

A right-invarient control system $(\underset{\sim}{X}, \underline{U})$ is <u>homogeneous</u> if $\underline{X}_0 = 0$. As an introduction to the general case, we consider these systems first.

The result stated in the next theorem appeared first in a study by R. W. Brockett [1].

THEOREM 5. 1. <u>Let $(\underset{\sim}{X}, \underline{U})$ be a homogeneous right-invariant control system on $\underline{G}$. Then the set attainable from the identity is the subgroup $\underset{\sim}{S}$. Moreover, if $\underline{U}$ is unrestricted then, for each $\underline{T} > 0$,</u> $\underline{A}(\underline{e}, \underline{T}) = \underset{\sim}{A}(\underline{e}) = \underset{\sim}{S}$.

<u>Proof</u>. To prove the first statement it is sufficient, in view of Theorem 4. 6, to show that $\underset{\sim}{A}(\underline{e})$ is a subgroup. We know that $\underset{\sim}{A}(\underline{e})$ is a semigroup. It remains to be shown that, if $\underline{g} \in \underset{\sim}{A}(\underline{e})$, then $\underline{g}^{-1} \in \underset{\sim}{A}(\underline{e})$. Let $\pi(\underline{e}, \underline{u}, \underline{t}) = \underline{g}$, where $\underline{u} \in \underline{U}$, $\underline{t} \ge 0$. Let

$$\underline{v}(\underline{s}) = -\underline{u}(\underline{t} - \underline{s}) \quad \text{for } 0 \le \underline{s} \le \underline{t},$$

$$\underline{v}(\underline{s}) = \underline{u}(\underline{s}) \quad \text{for } \underline{s} > \underline{t}.$$

Obviously, $\underline{v} \in \underline{U}$. Let

$$\underline{f}(\underline{s}) = \pi(\underline{e}, \underline{u}, \underline{t} - \underline{s}) .$$

Then

$$\underline{f}(\underline{s}) = \Sigma_{\underline{i}=1}^{m} \underline{v}_{\underline{i}}(s) \underline{X}_{\underline{i}}(\underline{f}(\underline{s})) .$$

Therefore, $\underline{f}$ is a solution of the evolution equation corresponding to the control $\underline{v}$. By the right-invariance we must have $\underline{f}(\underline{s}) = \pi(\underline{e}, \underline{v}, \underline{s})\, \underline{h}$, where $\underline{h} = \underline{f}(0) = \underline{g}$. But $\underline{f}(\underline{t}) = \pi(\underline{e}, \underline{u}, 0) = \underline{e}$. Therefore, $\pi(\underline{e}, \underline{v}, \underline{t}) = \underline{g}^{-1}$, and we have shown that $\underline{g}^{-1} \in \underset{\sim}{A}(\underline{e})$.

To prove the second statement, assume that $\underline{U}$ is unrestricted. Let $\underline{g} = \pi(\underline{e}, \underline{u}, \underline{t})$ for some $\underline{u} \in \underline{U}$ and $\underline{t} > 0$. Let $\underline{s} > 0$, and define a control $\underline{v}$ by

$$\underline{v}(\tau) = (\underline{t}/\underline{s})\, \underline{u}(\tau \underline{t}/\underline{s}) \text{ for } 0 \leq \tau < \infty .$$

An easy computation shows that $\pi(\underline{e}, \underline{v}, \underline{s}) = \underline{g}$. We have therefore shown that $\underline{A}(\underline{e}, \underline{t}) \subset \underline{A}(\underline{e}, \underline{s})$. Similarly, $\underline{A}(\underline{e}, \underline{s}) \subset \underline{A}(\underline{e}, \underline{t})$. Thus, $\underline{A}(\underline{e}, \underline{s}) = \underline{A}(\underline{e}, \underline{t})$ for all $\underline{t}, \underline{s}$ such that $0 < \underline{t}, 0 < \underline{s}$ such that $0 < \underline{t}$, $0 < \underline{s}$. Our proof is then complete.

<u>Remark</u>. The previous theorem implies that, for a homogeneous system

(a) The attainable set $\underset{\sim}{A}(\underline{e})$ is a subgroup of $\underline{G}$.

(b) The set $\underset{\sim}{A}(\underline{e})$ is the same for the three classes of controls, so that, in particular, every $\underline{g} \in \underline{G}$ that can be reached from the identity by means of an arbitrary control, can also be reached by means of a "bang-bang" control (possibly at a later time).

(c) If $\underline{U} = \underline{U}_u$, then every $\underline{g} \in \underline{G}$ that can be reached from the identity can in fact be reached in an arbitrarily short time.

We shall see later that neither (a), nor (b), nor (c) need be true in the non-homogeneous case.

6. The general Case.

Our subsequent study will be based on the following lemma.

LEMMA 6.1. Let $(\underset{\sim}{X}, \underline{U})$ be a right-invariant control system on $\underline{G}$. Then for each $\underline{T} > 0$,

(i) $\underset{\sim}{A}(\underline{e}, \underline{T})$ is contained in $\underset{\sim}{S}$, and the interior of $\underset{\sim}{A}(\underline{e}, \underline{T})$ is dense (in the topology of $\underset{\sim}{S}$) in $\underset{\sim}{A}(\underline{e}, \underline{T})$.

(ii) $\underline{A}(\underline{e}, \underline{T})$ is contained in $\underline{S}_0^T$, and the interior of $\underline{A}(\underline{e}, \underline{T})$ is dense (in the topology of S_0^T) in $\underline{A}(\underline{e}, \underline{T})$.

Proof. We shall use the results of [11]. Our system is of the form considered in the Remark following Example 5.2 of [11], with $\underline{M} = \underline{G}$, and with $\underline{G}$ acting on $\underline{G}$ by left translations. In the notations of [11], we have $\Omega = \underset{\sim}{R}^m$, or $\Omega = \underset{\sim}{C}^m$ (the unit cube in $\underset{\sim}{R}^m$), or $\Omega = \underset{\sim}{V}^m$ (the set of vertices of $\underset{\sim}{C}^m$) in the unrestricted, restricted and "bang-bang" cases, respectifely. In each of the three cases, the assumptions of [11] are satisfied, and an easy computation shows that $\mathscr{I}(\underline{D}) = \underset{\sim}{L}$ and that $\mathscr{I}_0(\underline{D}) = \underset{\sim}{L}_0$.

Since $\underset{\sim}{S}$ is the integral manifold of $\underset{\sim}{L}$ through $\underline{e}$ (cf. [2] p. 108), our first statement follows from [11]. Similarly, it is easy to verify that $\underset{\sim}{S}_0^T$ is precisely the submanifold $\underline{I}_0^T(\underline{D}, \underline{e})$ of [11], and the second part of our lemma follows.

In particular, it follows from Lemma 6.1 that the interior of $\underset{\sim}{A}(\underline{e})$ relative to $\underset{\sim}{S}$ is nonempty.

We shall also need the following:

LEMMA 6.2. *Let H be a connected Lie group, and let $L_1, \ldots, L_n$ be elements of $L(H)$ that generate $L(H)$. Then every $h \in H$ is a finite product of elements of the form $\exp(t\, L_i)$, where t is real and $i = 1, \ldots, n$.*

Proof. The set H' of all finite products of elements of the form $\exp(t\, L_i)$ is obviously a path-connected subgroup of H. Therefore, H' is a connected Lie subgroup of H(cf. Theorem 2.2). Obviously, H' contains the one-parameter subgroups generated by $L_1, \ldots, L_n$. Therefore, (cf. [4], p. 94) $L_1, \ldots, L_n$ belong to $L(H')$. Then, $H' = H$, and our proof is complete.

LEMMA 6.3. *Let (X, U) be a right-invariant control system on G. If the set attainable from the identity is dense in S, then it is equal to S.*

Proof. Let $g \in A(e)$ belong to the interior of $A(e)$ relative to S (cf. Lemma 6.1). Let $V \subset A(e)$ be relatively open in S and such that $g \in V$. Let $W = \{h^{-1} : h \in V\}$. Then W is a nonempty relatively open subset of S. Our assumption implies that W contains an element h of $A(e)$; then the set $V\,h$ (cf. footnote (2)) is relatively open in S, and is contained in $A(e)$. Moreover, $V\,h$ contains the identity. Therefore, the semigroup $A(e)$ contains a neighborhood of the identity in S. Since S is connected, we have that $A(e) = S$, and our proof is complete.

LEMMA 6.4. *Let $(\underset{\sim}{X}, U)$ be a right-invariant control system on G with $U = U_u$. Then $\overline{S} \subset \overline{A(e)}$ (the closure is taken relative to $\underset{\sim}{S}$).*

Proof. By Lemma 6.2, every element of S is a product of elements of the form $\exp(t X_i)$ $(-\infty < t < \infty,\ i = 1, \dots, m)$. We show that $\exp(t X_i)$ belongs to $\overline{\underset{\sim}{A}(e)}$ for every real t and for every $i = 1, \dots, m$. Since $\overline{\underset{\sim}{A}(e)}$ is a semigroup, this will imply that $S \subset \overline{\underset{\sim}{A}(e)}$, and the desired conclusion will follow immediately.

Let t be a real number, and let $1 \leq i \leq m$. Let u_n be the constant control $(0, \dots, 0, n, 0, \dots 0)$ where n appears in the i-th position. Then $u_n \in U$ for each $n > 0$. We have

$$\pi(e, u_n, t/n) = \exp(X_0 + n X_i)(t/n)$$
$$= \exp((t/n) X_0 + t X_i).$$

Letting $n \to \infty$, we conclude that $\exp(t X_i) \in \overline{\underset{\sim}{A}(e)}$, and our proof is complete.

Remark. If U is not unrestricted, then $\overline{S}$ need not be contained in $\overline{\underset{\sim}{A}(e)}$ (cf. Example 8.4).

We can now prove:

THEOREM 6.5. *Let $(\underset{\sim}{X}, U)$ be a right-invariant control system on G. Assume that the subgroup $\underset{\sim}{S}$ is compact. Then*

(i) $\underset{\sim}{A}(e) = \underset{\sim}{S}$.

(ii) *There exist a $T > 0$ such that $\underset{\sim}{A}(e, T) = \underset{\sim}{A}(e)$.*

Proof. Let H be the closure of $\underset{\sim}{A}(e)$ relative to $\underset{\sim}{S}$. Then H is a semigroup. We show that H is a group. Let $h \in H$. Then, for every positive integer n, $h^n \in H$. The sequence $\{h^n\}_{n=1,\dots}$ has a convergent subsequence $\{h^{n(k)}\}_{k=1,\ldots}$, and we can assume that $n(k) < n(k+1)$ for all k. Now, as $k \to \infty$, $h^{-1} = \lim h^{n(k+1)-n(k)-1} = \lim h_k$. Since $n(k+1)-n(k)-1$ is non-negative, it follows that h_k belongs to H for each k. Since H is closed, $h^{-1} \in H$. Therefore, H is a group. Since $A(e) \subset H$ and $A(e)$ has a non-empty interior relative to S, the same is true for H. Since H is a group and S is connected, we conclude that $H = S$. Therefore, $A(e)$ is dense in S, and (i) follows from Lemma 6.3.

To prove (ii) we let $W(t)$ denote, for each $t > 0$, the interior, relative to S, of $A(e,t)$. It is easy to see that the union of all the sets $W(t)$ is S (if $g \in S$, let $g \in A(e,T)$; let h be interior to $A(e,T')$ and let $h^{-1} \in A(e,T'')$; then $g \in W(T + T' + T'')$).

Since the sets $W(t)$ are increasing, it follows that $W(t) = S$ for sufficiently large T, and our proof is complete.

Remark. Theorem 6.5 shows that, if S is compact, then conditions (a) and (b) of the remark following Theorem 5.1 are satisfied. However, in this case condition (c) need not be satisfied. Even if U is unrestricted, it may not be possible to reach every element of S in an arbitrarily small time (cf. Example 8.1).

If S is not compact, then $A(e)$ need not be equal to S. The following theorem gives a sufficient condition under which $A(e) = S$; we do not know if this condition is also necessary.

THEOREM 6.6. *Let (X, U) be a right-invariant control system on G with $U = U_u$. If there exists a constant control u and a sequence of positive numbers $\{t_n\}$ with $t_n > \varepsilon > 0$ for some ε, with the property that $\lim \pi(e,u,t_n)$ exists and belongs to S (the closure is relative to S), then $A(e) = S$.*

Proof. Let u and $\{t_n\}$ satisfy the conditions of the theorem, and let $\lim \pi(e, u, t_n) = x$. If $X = X_0 + \sum_{i=1}^{m} u_i X_i$ then, since u is constant, $\pi(e, u, t) = \exp(t X)$. We first show that $\exp(t X) \in \overline{A(e)}$ for every real number t. If $\{t_n\}$ is bounded, then there exists a positive number T such that $\exp(T X) \in \overline{S}$. Let t be any real number, and let n be a natural number with $n T + t > 0$. Since $\overline{S}$ is a group we have that $\exp(-T X) \in \overline{S}$, and hence, $\exp(-T n X) \in \overline{S}$. By Lemma 6.4, it follows that $\exp(-T n X) \in \bar{A}(e)$. Since, obviously $\exp((n T + t) X) \in A(e)$, we have that $\exp(t X) = \exp(-T n X) \cdot \exp((-T n + t) X)$, and hence, $\exp(t X) \in \overline{A(e)}$. If $\{t_n\}$ is unbounded, let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ with $t_{n_{k+1}} - t_{n_k} > k$, and let $\tau_k = t_{n_{k+1}} - t_{n_k}$. We have that $\tau_k \to \infty$ and $\exp \tau_k X \to e$ as $k \to \infty$. Thus, for any real number t, $\exp(t X) = \lim_{k \to \infty} \exp((t + \tau_k) X)$.

If k is sufficiently large, then $t + \tau_k$ is positive. Therefore, $\exp((t + \tau_k) X)$ belongs to $A(e)$ for k large. It follows that $\exp(t X) \in \overline{A(e)}$.

By Lemma 6.4, $\exp(t X_i)$ belongs to $\overline{A(e)}$ for every real t and every $i = 1, \ldots, m$. Since $\overline{A(e)}$ is a semigroup, it follows that every product of elements of the form $\exp(t Y)$ (t real, $Y \in \{X, X_1, \ldots, X_m\}$) belongs to $\overline{A(e)}$. Clearly, the elements $X, X_1, \ldots, X_m$ generate L. By Lemma 6.2, $A(e) = S$. This completes the proof.

The following corollary is immediate:

COROLLARY 6.7. *Let (X, u) be a right-invariant control system on G with $U = U_u$. If there exists a constant control u such that $t \to \pi(e, u, t)$ is periodic, then $A(e) = S$.*

The following lemma gives a necessary condition for $\underset{\sim}{A}(\underline{e})$ to be equal to $\underset{\sim}{S}$; however, this condition is not sufficient (see Example 8.3).

LEMMA 6.8. Let $(\underset{\sim}{X}, \underline{U})$ be a right-invariant control system on $\underline{G}$, and let $\underset{\sim}{A}(\underline{e}) = \underset{\sim}{S}$. Then, there exists a non-zero number T such that $\exp(\underline{X}_0 \underline{T}) \in \underset{\sim}{S}_0$.

Proof. Our assumption implies that $\exp(-\underline{X}_0)$ belongs to $\underline{A}(\underline{e}, \underline{t})$ for some $\underline{t} \geq 0$. Therefore, by Lemma 4.2, $\exp(-\underline{X}_0) = \exp(\underline{t}\ \underline{X}_0)\underline{g}$ where $\underline{g} \in \underset{\sim}{S}_0$. To complete the proof, take $\underline{T} = -1 - \underline{t}$.

There is one important case when Theorem 6.6 and Lemma 6.8 yield a necessary and sufficient condition for $\underset{\sim}{A}(\underline{e}) = \underset{\sim}{S}$, namely when $\underline{S} = \underset{\sim}{S}_0$. This will happen if and only if $\underline{L} = \underset{\sim}{L}_0$. It is easy to check that this equality holds if and only if all the brackets $[\underline{X}_0, \underline{X}_i]$ belong to $\underline{L}(i=1,\ldots \underline{m})$.

THEOREM 6.9. Let $(\underset{\sim}{X}, \underline{U})$ be a right-invariant control system on $\underline{G}$ with $\underline{U} = \underline{U}_u$. If $\underline{L} = \underset{\sim}{L}_0$, then a necessary and sufficient condition for $\underset{\sim}{A}(\underline{e})$ to be equal to $\underset{\sim}{S}$ is that there exist a number $\underline{T}$, $\underline{T} \neq 0$, such that $\exp(\underline{T}\ \underline{X}_0)$ belongs to $\underline{S}$.

Remark. The condition $\underline{L} = \underset{\sim}{L}_0$ holds, in particular, when $[\underline{X}_0, \underline{X}_i] = 0$ $(\underline{i}=1,\ldots,\underline{m})$ i.e., when $\exp(\underline{t}\ \underline{X}_0)$ commutes with the elements of $\underline{S}$.

7. Controllability

Let $(\underset{\sim}{X}, \underline{U})$ be a right invariant control system on $\underline{G}$, and let $\underline{g} \in \underline{G}$. We say that $(\underset{\sim}{X}, \underline{U})$ is controllable from $\underline{g}$ if $\underset{\sim}{A}(\underline{g}) = \underline{G}$. We say that $(\underset{\sim}{X}, \underline{U})$ is controllable if it is controllable from every $\underline{g} \in \underline{G}$.

THEOREM 7.1 A necessary condition for $(\underset{\sim}{X}, \underline{U})$ to be controllable is that $\underline{G}$ be connected and that $\underset{\sim}{L} = \underline{L}(\underline{G})$. If $\underline{G}$ is compact, or if the system is homogeneous, the condition is also sufficient.

Proof. The condition of the theorem holds if and only if $\underline{G} = \underset{\sim}{S}$.

By Lemma 4.1, the condition is necessary. The second part of the statement follows from Theorems 5.1 and 6.5 (and from the obvious fact that, if $A(e) = G$, then $A(g) = G$ for every g).

In the compact case, we can prove stronger controllability properties.

THEOREM 7.2. *Let G be compact, and let (X, U) be controllable. Then there exists $T > 0$ such that, for every $g \in G$, $g' \in G$, there is a control that steers g into g' in less than T units of time. If G is semisimple, then there exists $T > 0$ such that, for every $g \in G$, $g' \in G$, there is a control that steers g into g' in exactly T units of time.*

Proof. The first statement follows from Theorem 6.5 (ii). To prove the second statement, we observe that, if G is semisimple, then (X, U) has the "strong accessibility property", i.e. the set $A(e, t)$ has a non-empty interior for every $t > 0$ (for a proof of this, see [11]). From this fact the conclusion follows as in the proof of Theorem 6.5 (ii).

Finally, Theorem 6.9 can also be interpreted as a controllability result.

THEOREM 7.3. *Assume that the necessary conditions of Theorem 7.1 hold, and that (i) $U = U_u$, and (ii) $L = L_0$ (or, equivalently, L is an ideal of L). Then (X, U) is controllable if and only if $\exp(T X_0)$ belongs to S ($=S_C$) for some $T \neq 0$.*

COROLLARY 7.4. *If G is connected, $L = L(G)$, $U = U_u$ and X_0 belongs to the Lie algebra generated by $X_1, \ldots, X_m$, then (X, U) is controllable.*

8. Examples

In most of the following examples, we shall work with groups of matrices. Our groups will be Lie subgroups of $GL(n, R)$, the group of all $n \times n$ non-singular real matrices. Recall that $GL(n, R)$ is an open subset of $M(n, R)$ (the set of all $n \times n$ real matrices). Since $M(n, R)$ is a vector space, we can identify the tangent space to $GL(n, R)$ at each point with $M(n, R)$. With this identification, a right-invariant vector field corresponds to a mapping $X \to A X$ from $GL(n, R)$ into $M(n, R)$, where A is a fixed matrix. If $X_0, \ldots, X_m$ are right-invariant vector fields, given by $X \to A_i X (i=0, \ldots, m)$, then the evolution equation becomes

$$\dot{X}(t) = (A_0 + \Sigma_{i=1}^{m} u_i(t) A_i) X(t) .$$

Example 8.1. Let $G = SO(3)$, the set of all 3 x 3 real orthogonal matrices with positive determinant. The algebra $L(G)$ is the set of all 3 x 3 antisymmetric matrices. A basis for $L(G)$ is given by the matrices

$$K_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$K_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

It is easy to check that $[K_1, K_2] = K_3$, $[K_2, K_3] = K_1$ and $[K_3, K_1] = K_2$. Thus $L(G)$ is isomorphic to three-dimensional real space, with the Lie bracket corresponding to the vector product. Using this correspondence, it is obvious that, if A and B are any two linearly independent elements of $L(G)$, then $\{A, B, [A, B]\}$ is a basis for $L(G)$.

Let A and B be any linearly independent 3 x 3 anti-symmetric matrices, and let our right-invariant control system on $SO(3)$ be described by

$$\dot{X}(t) = (A + uB)X(t)$$

where u belongs to any class of admissible controls. Since $SO(3)$ is compact and connected, Theorem 6.5 applies, and our system is controllable. Moreover, there is a $T > 0$ such that, given any two elements P, Q of $SO(3)$ there is a "bang-bang" control u that steers P into Q in less than T units of time. In this connection, it is interesting to observe that, in general, there may not exist arbitrarily small numbers T with the above property, even if the control u is completely unrestricted. Take, for instance, $A = K_1$ and $B = K_2$. If u is an arbitrary control, and if $X(t)$ is the solution of the evolution equation corresponding to u with initial condition $X(0) = I$, write $X = (x_{ij})_{i,j=1,2,3}$. Then we have

$$\dot{x}_{12} = x_{22} + u x_{32}$$

and

$$\dot{x}_{32} = - u x_{12}$$

Multiplying the first equation by x_{12}, the second equation by x_{32} and adding, we get

$$\frac{1}{2} \frac{d}{dt} (x_{12}^2 + x_{32}^2) = x_{22} x_{12}$$

Since $x_{12}^2 + x_{32}^2$ vanishes at $t = 0$, we have:

$$(x_{12}^2 + x_{32}^2)(t) = 2 \int_0^t x_{22}(\tau)\, x_{12}(\tau)\, d\tau$$

But $x_{22}(\tau)$ and $x_{12}(\tau)$ are entries of orthogonal matrices. Hence, they are bounded in absolute value by 1. Therefore, we conclude that

$$(x_{12}^2 + x_{32}^2)(t) \leq 2t .$$

This shows that a matrix (a_{ij}) for which $a_{12}^2 + a_{32}^2 = 1$ cannot be reached from the identity in less than $\frac{1}{2}$ units of time.

Example 8.2. The considerations of the previous example can be generalized to $G = SO(n)$. In this case the Lie algebra of G is the set of all $n \times n$ anti-symmetric matrices.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be matrices defined as follows: $a_{i,i+1} = 1$ for $i = 1, \ldots, n-2$, $a_{i,i-1} = -1$ for $i = 2, \ldots, n-1$, $a_{ij} = 0$ otherwise, and let $b_{n-1,n} = 1$, $b_{n,n-1} = -1$, $b_{ij} = 0$ otherwise. It is easy to show that the smallest subalgebr that contains A and B is exactly $L(G)$. Thus, even though $SO(n)$ is $\frac{1}{2} n(n-1)$ - dimensional, the system $\dot{X} = (A + u B) X$, in which only one control is involved, is controllable. Moreover, as before, we can limit u to be "bang-bang". An easy argument shows that this fact, which has been shown to be true for the particular matrices A and B defined above, is in fact true for "almost all" pairs $(A, B) \in L(G) \times L(G)$. Precisely, the set of pairs (A, B) such that A and B generate $L(G)$ is open and dense in $L(G) \times L(G)$.

Remark. If G is an arbitrary connected Lie group such that $L(G)$ is generated by two elements, then Theorem 5.1 enables us to conclude, in a way similar to that of the previous examples that the homogeneous system on G of the form $\dot{X}(t) = (u A + v B) X(t)$ is controllable for "almost all" pairs $(A, B) \in L(G) \times L(G)$. This result holds even if we restrict u and v to be "bang-bang".

The previous statement holds, in particular, when $G = SL(n, \mathbb{R})$, the set of all $n \times n$ real matrices whose determinant is 1, or when $G = G_+ L(n, \mathbb{R})$, the set of all $n \times n$ real matrices whose determinant is greater than 0.

C. 3.

Example 8.3. We show that, if A, B generate $L(G)$, and if G is connected and not compact, then the system $\dot{X} = (A + u\,B)X$ need not be controllable, even if $L(G)$ is a simple Lie algebra. In particular, this will show that the necessary condition of Lemma 6.8 is not sufficient. Take $G = SL(2, R)$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that A and B generate $L(G)$ and that $L(G)$ is simple. Let u be an arbitrary control, and let $X(t)$ be the solution of the evolution equation corresponding to u, with initial condition $X(0) = I$. Let $X = (x_{ij})_{i,j=1,2}$. Then $\dot{x}_{11} = x_{11} + u\,x_{21}$ and $\dot{x}_{21} = u\,x_{11} - x_{21}$. Multiplying the first equation by x_{11}, the second one by x_{21} and subtracting, we get

$$\frac{1}{2}\,\frac{d}{dt}\,(x_{11}^2 - x_{21}^2) = x_{11}^2 + x_{21}^2$$

Thus, the function $x_{11}^2(t) - x_{21}^2(t)$ is non-decreasing for every trajectory of our system. Since its value for $t = 0$ is 1, it follows that every element of $SL(2, R)$ that can be reached from the identity in positive time satisfies the inequality $x_{11}^2 \geq x_{21}^2 + 1$. Hence, the system is not controllable. In the notations of Section 3, it is clear that $L = L(G)$. Thus, we have shown that $A(e)$ is not a group. However, $L_0 = L(G)$ (because $L(G)$ is simple), and hence $S_0 = G$. Therefore, $\exp(t\,A)$ belongs to S_0 for all $t \geq 0$. This shows that the condition of Lemma 6.8 is satisfied.

Example 8.4. In this example we show that Lemma 6.4 and Theorem 6.6 need not be valid if U is not assumed to be unrestricted. Let $G = R \times S^1$, the product of the real line and the unit circle. Let X_0 be the generator of the one parameter group $t \to (t, e^{2\pi it})$, and let X_1 be the generator of the one parameter group $t \to (t, 1)$. Let $U = U_r$, or $U = U_b$. Then $A(e) = [0,\infty) \times S^1$, which is not a group. But if $u = 0$, then $\pi(e, u, 1) = \exp X_0 = (1, 1)$ which belongs to S. Thus, Theorem 6.6 does not hold. As for Lemma 6.4 it is clear that S and $A(e)$ are closed, but $S \not\subset A(e)$.

Example 8.5 In view of Theorem 6.6 it might seem that a necessary condition for a right-invariant system to be controllable is that $\exp t X_0$ "gets arbitrarily close" to S for some non-zero values of t. This example shows that such a statement is false.

Let $G = SL(2, R)$, and let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Consider the system $X = (A + B u) X$ where u belongs to the class of unbounded controls.

Let u be the constant control $u = 1$. Then the trajectory $t \to \pi(i, u, t)$ is the curve $t \to e^{t(A + B)}$, which is periodic with period 2π. By Corollary 6.7, the system is controllab.e

We have that $$e^{tA} \begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}, \text{ and } e^{\tau B} = \begin{pmatrix} 1-\tau & \tau \\ -\tau & 1+\tau \end{pmatrix}$$

Now, it is obvious that e^{tA} stays away from S for all positive values of t.

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FOOTNOTES

1. A solution of (*) is an absolutely continuous G-valued function of the real variable t, with the property that (*) is satisfied for almost every t.

2. If A is a subset of G, and $g \in G$, we use $A\,g$ to denote the set of all products $a\,g$, where $a \in A$.

3. The result is proved in [10] for groups of matrices, but the proof is valid for arbitrary Lie groups. Alternatively, one could use Ado's Theorem (cf. [4]) to go from the result of [10] to a "local" version of Lemma 4. 3, and then deduce the general result.

LIE ALGEBRAS AND LINEAR DIFFERENTIAL EQUATIONS

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1. Differential Equations

In this paper we study certain symmetry properties possessed by the solutions of linear differential equations. This is accomplished by use of some basic ideas from the theory of finite dimensional linear systems together with the work of Wei and Norman [1] on the use of Lie algebraic methods in differential equation theory. Our study is also strongly motivated by the results of reference [2] which provided a link between the present paper and a number of questions about the controllability of systems for which the control enters multiplicatively.

Let $\mathbb{R}^{n\times m}$ denote the set of real n by m matrices. By a Lie algebra $\mathscr{L}$ in $\mathbb{R}^{n\times n}$ we understand a subset of $\mathbb{R}^{n\times n}$ which is a real vector space having the property that if A and B belong to $\mathscr{L}$ then so does $[A,B] = AB - BA$. Given an arbitrary subset $\mathscr{N}$ of $\mathbb{R}^{n\times n}$ we denote by $\{\mathscr{N}\}_A$ the smallest Lie algebra which contains $\mathscr{N}$. We denote the identity matrix by I and introduce the square matrix

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$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Letting prime denote transpose, we say that a matrix A is <u>Hamiltonian</u> if

$$JA = (JA)' .$$

We call a matrix P <u>symplectic</u> if $P'JP = J$.

Wei and Norman [1] have observed that for $|t|$ small the solution of differential equations of the type

$$\dot{x}(t) = \left(\sum_{i=1}^{\nu} a_i(t)A_i\right)x(t)$$

can be expressed as

$$x(t) = e^{H_1 g_1(t)} e^{H_2 g_2(t)} \cdots e^{H_\mu g_\mu(t)} x(0)$$

where $\{H_i\}$ is a basis for the Lie algebra of n by n matrices generated by $\{A_i\}$ and where the g_i satisfy a set of nonlinear differential equations. In this paper we investigate some aspects of this theory in the special case where $\nu = 2$.

2. <u>Lie Algebras and Rational Functions</u>

We begin by establishing two results on the Lie algebra generated by a pair of n by n matrices.

In order to avoid undue repetition let us agree to call a matrix of rational functions $G(s)$ <u>regular</u> if it is square and approaches zero as $|s|$ approached infinity. Our first point is that it is possible to associate a Lie algebra with each regular matrix of rational functions in a natural way. This correspondence goes as follows. It is well known [3, 4] that every regular matrix of rational functions can be expressed as

$$G(s) = C(Is - A)^{-1}B$$

with $C \in \mathbb{R}^{m\ n}$, $A \in \mathbb{R}^{n\ n}$, $B \in \mathbb{R}^{n\ m}$. Moreover it is always possible to pick A, B and C such that

$$\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = \operatorname{rank}(C; CA; \ldots; CA^{n-1}) = n$$

where $(\cdot,\ )$ denotes a column partition and $(\ ;\ )$ a row repetition. In this case we say that the triple $[A,B,C]$ is a minimal realization of $G(s)$. Now minimal realizations are not uniquely determined by $G(s)$, but if $[A,B,C]$ and $[F,G,H]$ are two minimal realizations then there exists a unique nonsingular P such that $PAP^{-1} = F$, $PB = G$ and $CP^{-1} = H$. This result first stated by Kalman [3], is known in system theory as the state space isomorphism theorem (see e.g. [5] for an introductory account in the present notation). We now come to the Lie algebra. Given a regular matrix $G(s)$ we find a minimal realization $[A,B,C]$, and construct $\{A,BC\}_A$, the Lie algebra of n by n matrices generated by A and BC. This collection depends on the particular realization but if $[F,G,H]$ is a second minimal realization of $G(s)$ then $F = PAP^{-1}$ and $GH = PBCP^{-1}$ so that the Lie algebras are isomorphic. That is $\{A,BC\}_A$ and $\{F,GH\}_A$ are matrix representations of the same abstract Lie algebra. We call this abstract algebra the Lie algebra associated with $G(s)$. This Lie albegra reflect the symmetry properties of $G(s)$ as the following theorems make clear.

THEOREM 1. Let A, B and C belong to $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{m\times n}$ respectively. Suppose $[A,B,C]$ is a minimal realization of $G(s)$ and suppose B and C are of rank m. Then:

i) There exists a nonsingular matrix P such that PAP^{-1} and $PBCP^{-1}$ are both Hamiltonian if and only if there exists a nonsingular symmetric matrix T such that

$$TG(s) = G'(-s)T\ .$$

ii) There exists a nonsingular P such that PAP^{-1} and $PBCP^{-1}$ are both skew-symmetric if and only if there exists a nonsingular skew-symmetric T such that $TG(s) = G'(-s)T$ and

$$TG(s) = \sum_{i=1}^{\nu} R_i \frac{s}{s^2+\lambda_i^2}\ ; \qquad R_i = R_i' \geqslant 0\ .$$

Proof. (Hamiltonian Case) Suppose that A and BC are Hamiltonian. Then we have $JBC = C'B'J'$ and in view of the rank conditions $JB = C'T$ for $T = B'J'C'(CC')^{-1}$. Note T is nonsingular. Clearly $C'TC = C'T'C$ so T is symmetric. Thus (recall $J^2 = -I$)

$$\begin{aligned} TG(s) &= TC(Is-A)^{-1}B \\ &= TCJ'(Is-JAJ')^{-1}JB \\ &= B'(-Is-A')^{-1}C'T \\ &= G'(-s)T\ . \end{aligned}$$

On the other hand, suppose that for some symmetric nonsingular T we have $TG(s) = G'(-s)T$. Thus

$$\begin{aligned} TC(Is-A)^{-1}B &= B'(-Is-A')^{-1}C'T \\ &= -B'(Is+A')^{-1}C'T\ . \end{aligned}$$

Since both sides are minimal realizations it follows from the state space isomorphism theorem referred to above, that there exists a nonsingular matrix P such that

$$PAP^{-1} = -A'\ ;\ PB = C'T\ ;\ TCP^{-1} = -B'\ ;$$

thus upon transposition and rearrangement we get

$$P'AP'^{-1} = -A'\ ;\ P'B = -C'T\ ;\ CP'^{-1} = -B'\ .$$

Now by uniqueness of P (compare with [4,6]) we see that $P = -P'$. Thus there exists a nonsingular Q such that $Q'JQ = P$. Finally we see that $[QAQ^{-1}, QB, CQ^{-1}]$ is a realization such that QAQ^{-1} and $QBCQ^{-1}$ are Hamiltonian. See references [4] and [6] for additional insight into arguments of this type.

(Skew-Symmetric Case) Suppose that A and BC are skew-symmetric. Then we have $BC = -C'B'$ and in view of the rank conditions $B = C'T$ for $T = (B'C')(CC')^{-1}$. Note T is nonsingular. Clearly $C'TC = -C'T'C$ and so T is skew-symmetric. Thus

$$\begin{aligned} TG(s) &= TC(Is-A)^{-1}B \\ &= B'(Is-A)^{-1}B \\ &= B'(Is+A')^{-1}B \\ &= B'(-Is-A')^{-1}C'T \\ &= G'(-s)T\ . \end{aligned}$$

All zeros of $\det(Is - A)^{-1}$ are on the imaginary axis since $A = -A'$. The partial fraction expansion of $(Is - A)^{-1}$ has only terms of multiplicity one since A is normal. Clearly the residues of $TG(s) = TC(Is - A)^{-1}C'T'$ at these poles are symmetric and nonnegative definite.

On the other hand, suppose that for some nonsingular skew-symmetric T we have $TG(s) = -G'(-s)T$, with $TG(s)$ given by the partial fraction expansion displayed in the theorem statement. Expand each R_i as the sum of dyads and renumber (if necessary) the λ's so that

$$TG(s) = \sum_{i=1}^{\mu} b_i b_i' \frac{s}{s^2+\lambda_i^2}$$

with each b_i being an m by m vector. Now let A, B, and C be given by

$$A = \begin{bmatrix} 0 & \lambda_1 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ 0 & 0 & -\lambda_2 & 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot \end{bmatrix} \; ; \quad B = \begin{bmatrix} b_1' \\ 0 \\ b_2' \\ 0 \\ \vdots \end{bmatrix} \; ; \quad TC = [\, b_1 \;\; 0 \;\; b_2 \;\; 0 \;\; \cdots] \, .$$

Then $TC(Is - A)^{-1}B = TG(s)$, $A = -A'$ and $BC = -C'B'$. (Compare with Theorem 2 of [7] from which one can see a relationship between this result and the structure of lossless electrical networks.) ∎

We now characterize the conditions under which the representation of the Lie algebras obtained this way are irreducible. We call a set of matrices $\{A_1, A_2, \ldots, A_n\}$ <u>irreducible</u> if there exists no nonsingular P such that all the PA_iP^{-1} are in block triangular form:

$$PA_iP = \begin{bmatrix} F_i & G_i \\ 0 & H_i \end{bmatrix} \; ; \quad F_i = \nu \text{ by } \nu \; ; \quad H_i = \mu \text{ by } \mu \, .$$

We recall the matrix form of Schur's lemma which says that a set of n by n matrices are irreducible if and only if there exists no nonsingular matrix which is not a scalar multiple of the identity and which commutes with all the matrices in the set.

THEOREM 2. Let $G(s)$ be a given regular matrix of rational functions and let $[A,B,C]$ be a minimal realization of $G(s)$. Let A belong to $R^{n\times n}$ and let B and C belong to $R^{n\times m}$ and $R^{m\times n}$ respectively. Suppose B and C are of rank m. Then the Lie algebra $\{A,BC\}_A$ is irreducible if and only if the set of m by m matrices $G(C)$ is irreducible (C is the field of complex numbers and $G(C)$ is its image under $G(\cdot)$).

Proof. Suppose that $TG(s) = G(s)T$ for some constant matrix T which is invertible and not a multiple of the identity. Let $[A,B,C]$ be a minimal realization of $G(s)$. Then since

$$C(Is-A)^{-1}B = T^{-1}C(Is-A)^{-1}BT$$

we see that $[A,BT,T^{-1}C]$ is also a minimal realization of $G(s)$. By the state space isomorphism theorem we know there exists P such that

$$PB = BT$$
$$CP^{-1} = T^{-1}C$$
$$PAP^{-1} = A\ .$$

Since B is of full rank P cannot be a multiple of the identity if T is not.

On the other hand, if $\{A,BC\}$ is reducible then there exists a nonsingular P, unequal to a multiple of the identity, such that $PAP^{-1} = A$ and $PBCP^{-1} = BC$. However, since B is one to one and C is onto, it follows that $PB = BT$ for some nonsingular T and $CP^{-1} = RC$ for some nonsingular R. Thus $BTRC = BC$ and since B and C have left and right inverses respectively, we see that $T = R^{-1}$ and thus $TG(s) = G(s)T$. Now P is not a multiple of the identity, and so $PB \neq \alpha B$ (this would violate uniqueness of P in the state space isomorphism theorem). Thus T is not a multiple of the identity and $G(C)$ is reducible. ∎

We note that in particular, if BC is a dyad and $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = \operatorname{rank}(C; CA; \ldots; CA^{n-1}) = n$, then the representation $\{A,BC\}_A$ is irreducible and it is equivalent to a Hamiltonian algebra if and only if $g(s) = g(-s)$. In particular, the algebra associated with $1/s^n$ depends on whether n is even or odd. It has been shown by direct construction in [8] that it is the full $n(n+1)/2$ dimensional Hamiltonian algebra if n is even. The Lie algebra associated with $s^{n-1}/(s^n+1)$ is the full n^2-1 dimensional algebra of zero trace matrices for every n (see [8]). We observe that to generate skew-symmetric algebras we can use a $G(s)$ of the form

$$G_n(s) = \frac{1}{s}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} + \frac{1}{s^2+1}\begin{bmatrix} 1 & s \\ -s & 1 \end{bmatrix} + \cdots + \frac{1}{s^2+n^2}\begin{bmatrix} n & s \\ -s & n \end{bmatrix}$$

for the odd dimensional case and

$$G_n(s) = \frac{1}{s^2+1}\begin{bmatrix} 1 & s \\ -s & 1 \end{bmatrix} + \frac{1}{s^2+4}\begin{bmatrix} 2 & s \\ -s & 2 \end{bmatrix} + \cdots + \frac{1}{s^2+n^2}\begin{bmatrix} n & s \\ -s & n \end{bmatrix}$$

for n even.

3. An Application to Stability

As is well known, the symplectic matrices form a group and the eigenvalues of symplectic matrices occur in reciprocal pairs. That is to say, if λ is an eigenvalue of a symplectic matrix then so is $1/\lambda$. This observation together with the basic ideas of Floquet theory enables one to show that for $0 \leq t < \infty$ all solutions of

$$\dot{x}(t) = (A(t) + \varepsilon B(t))x(t) \ ; \ A(t+T) = A(t) \ ; \ B(t+T) = B(t)$$

are bounded for ε sufficiently small provided $A(t)$ and $B(t)$ are Hamiltonian and the solution of the equation

$$\dot{x}(t) = A(t)x(t)$$

has distinct characteristic multipliers all lying on the unit circle (see reference [9]). This together with Theorem 1 yields the following theorem.

THEOREM 3. Let p and q be polynomials with the degree of p larger than that of q. Suppose $k(t)$ is periodic with period T. Then there exists $\varepsilon > 0$ such that for $|k(t)| < \varepsilon$ all solutions of

$$p(D)x(t) + k(t)q(D)x(t) = 0 \; ; \quad D = \frac{d}{dt}$$

are bounded provided i) $q(s)/p(s) = q(-s)/p(-s)$ where ii) $p(s) = (s^2+\lambda_1^2)(s^2+\lambda_2^2)\ldots(s^2+\lambda_n^2)$ with λ_j all real and nonzero mod $2\pi/T$ with $(\lambda_i-\lambda_j)$ nonzero mod $2\pi/T$.

Proof. Under the given hypothesis there exists a realization of $[A,B,C]$ of $q(s)/p(s)$ such that A and BC are Hamiltonian. Thus we can express the evolution equations in first order form as

$$\dot{x}(t) = (A + k(t)D)x(t)$$

with A and D Hamiltonian. By hypothesis e^{AT} has all its eigenvalues on the unit circle, and none are repeated. Thus by the perturbation result quoted, there exists $\varepsilon > 0$ such that if $|k(t)| < \varepsilon$ for all t and $k(t+T) = k(t)$ then we have stability.

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