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ON THE ANALYTIC LUNAR AND SOLAR PERTURBATIONS OF A NEAR EARTH SATELLITE

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ABSTRACT

The disturbing function of the moon (sun) is expanded as a sum of products of two harmonic functions, one depending on the position of the satellite and the other on the position of the moon (sun). The harmonic functions depending on the position of the perturbing body are developed into trigonometric series with the ecliptic elements ℓ, ℓ', F, D and Γ of the lunar theory which are nearly linear with respect to time. Perturbation of elements are in the form of trigonometric series with the ecliptic lunar elements and the equatorial elements ω and Ω of the satellite so that analytic integration is simple and the results accurate over a long period of time.

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NOTATION

G - the gravitational constant

M - the mass of the earth

m' - the mass of the moon

m'' - the mass of the sun

$\bar{r}$ - the geocentric position vector of the satellite

$r = |\bar{r}|$,

$\hat{r} = \bar{r}/r$

$\bar{r}'$ - the geocentric position vector of the moon

$r' = |\bar{r}'|$

$\hat{r}' = \bar{r}'/r'$

$\bar{r}''$ - the geocentric position vector of the sun

$r'' = |\bar{r}''|$

$\hat{r}'' = \bar{r}''/r''$

a' - the mean distance of the moon from the earth, defined in such a manner that the constant part in the expansion of the lunar parallax equals unity

a'' - the mean geocentric distance of the sun

γ' - the angle between $\bar{r}$ and $\bar{r}'$

γ'' - the angle between $\bar{r}$ and $\bar{r}''$

$\lambda, \mu, \nu, \lambda', \mu', \nu', \lambda'', \mu'', \nu''$ - the rectangular components of $\hat{r}$, $\hat{r}'$ and $\hat{r}''$ respectively in the geocentric equatorial coordinate system.

R_e - the equatorial radius of the earth

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g - the mean anomaly of the satellite

n - the mean motion of the satellite

f - the true anomaly of the satellite

e - the eccentricity of the satellite orbit

i - the inclination of the satellite orbit

Ω - the longitude of the ascending node of the satellite

ω - the argument of perigee of the satellite

$\tilde{\omega} = \omega + \Omega$

a - the semi-major axis of the satellite orbit

n - the mean motion of the satellite

δa - the perturbations in a caused by the moon (primed)/sun (double primed)

δe - the perturbations in e caused by the moon (primed)/sun (double primed)

δi - the perturbations in i caused by the moon (primed)/sun (double primed)

δg - the perturbations in g caused by the moon (primed)/sun (double primed)

$\delta \Omega$ - the perturbations in Ω caused by the moon (primed)/sun (double primed)

$\delta \omega$ - the perturbations in ω caused by the moon (primed)/sun (double primed)

ϵ - the obliquity of the ecliptic

J_2, J_4 - zonal harmonic coefficients in the earth's gravitational potential ($J_2 = 1.08219 \times 10^{-3}$ and $J_4 = -2.123 \times 10^{-6}$)

t_0 - Julian date of January 0.5, 1900 (2415020.0 days)

$t - t_0$ - number of days from January 0.5, 1900

T - number of Julian Centuries (36525 days) from January 0.5, 1900

$\ell_{☾}$ - geocentric mean longitude of the moon

$\Omega_{☾}$ - geocentric mean longitude of the lunar node

$\tilde{\omega}_{☾}$ - geocentric mean longitude of the lunar perigee

$\ell_{\odot}$ - geocentric mean longitude of the sun

$\Gamma \equiv \tilde{\omega}_{\odot}$ - geocentric mean longitude of the solar perigee

$\ell \equiv \ell_{☾} - \tilde{\omega}_{☾}$ - argument of the principal elliptic term

$\ell' \equiv \ell_{\odot} - \tilde{\omega}_{\odot}$ - argument of the annual equation

$F \equiv \ell_{☾} - \Omega_{☾}$ - argument of the principal term in latitude

$D \equiv \ell_{☾} - \ell_{\odot}$ - half argument of the variation

ON THE ANALYTIC LUNAR AND SOLAR PERTURBATIONS OF A NEAR EARTH SATELLITE

INTRODUCTION

The usual analytical treatment of the secular and long period effects of the sun and moon upon a close earth satellite is based on a trigonometric expansion of the disturbing function with the angular equatorial elements of the moon as arguments (Kozai, 1959), (Musen, et. al., 1961), (Kaula, 1962), (Murphy and Felsentreger, 1966). While simple in form, the analytic integration of the perturbation equations resulting from this type of expansion is difficult because of nonlinearity in the elements of the series. In particular, the equatorial node of the lunar orbit oscillates between two limits.

The expansion presented in this paper, following Musen and Estes (1971), represents the disturbing function as a sum of products of two harmonic functions, one depending on the position of the satellite and the other on the position of the moon (sun) as given by the Hill-Brown lunar theory (Newcomb's solar theory) instead of osculating elements which are not known precisely (Musen and Felsentreger, 1972). The harmonic functions depending on the lunar and solar positions are represented as trigonometric series with the ecliptic elements ℓ, ℓ', F, D, and Γ of the lunar theory which are very nearly linear with respect to time. The expansion of perturbations is then in the form of trigonometric series with the ecliptic lunar elements and the equatorial elements ω and Ω of the satellite so that the analytic integration is simpler and will be valid for a longer time span.

As with other analytic theories of lunar and solar perturbations based on the development of the disturbing function into a trigonometric series, oblateness produces resonances at some critical inclination angles such as 46.4°, 56.1°, 63.4°, 69.0°, and 73.1°. Such resonances must be treated as special cases.

THE LUNAR AND SOLAR DISTURBING FUNCTION

The expansion of the lunar disturbing function is

$$R' = G m' \left\{ \frac{1}{r'} \left[\frac{r}{r'} P_1 (\cos \gamma') + \frac{r^2}{r'^2} P_2 (\cos \gamma') + \frac{r^3}{r'^3} P_3 (\cos \gamma') + \cdots \right] - \frac{r}{r'^2} \cos \gamma' \right\}$$

$$= G m' \left\{ \frac{r^2}{r'^3} P_2 (\cos \gamma') + \frac{r^3}{r'^4} P_3 (\cos \gamma') + \cdots \right\} \tag{1}$$

where P_i $(\cos\gamma')$ are Legendre polynomials and

$$\bar{r}\cdot\bar{r}' = r\,r'\cos\gamma'.$$

Then

$$\begin{aligned}
P_2(\cos\gamma') &= \frac{1}{2}(3\cos^2\gamma' - 1) \\
&= \frac{3}{2}\,[\lambda^2\lambda'^2 + 2\lambda\mu\lambda'\mu' + \mu^2\mu'^2 \\
&\quad + 2\lambda\nu\lambda'\nu' + 2\mu\nu\mu'\nu' + \nu^2\nu'^2] - \frac{1}{2} \\
&= \frac{1}{4}(1 - 3\nu^2)\,[1 - 3\nu'^2] + \frac{3}{4}(\lambda^2 - \mu^2)\,[\lambda'^2 - \mu'^2] \\
&\quad + 3\lambda\mu\,[\lambda'\mu'] + 3\lambda\nu\,[\lambda'\nu'] + 3\mu\nu\,[\mu'\nu']
\end{aligned}$$

$$\begin{aligned}
P_3(\cos\gamma') &= \frac{1}{2}\cos\gamma'\,(5\cos^2\gamma' - 3) \\
&= \frac{3}{8}\lambda\,(1 - 5\nu^2)\,[\lambda'\,(1 - 5\nu'^2)] \\
&\quad + \frac{3}{8}\mu\,(1 - 5\nu^2)\,[\mu'\,(1 - 5\nu'^2)] \\
&\quad + \frac{1}{4}\nu\,(3 - 5\nu^2)\,[\nu'\,(3 - 5\nu'^2)] \\
&\quad + \frac{5}{8}\lambda\,(\lambda^2 - 3\mu^2)\,[\lambda'\,(\lambda'^2 - 3\mu'^2)] + \frac{5}{8}\mu\,(3\lambda^2 - \mu^2)\,[\mu'\,(3\lambda'^2 - \mu'^2)] \\
&\quad + \frac{15}{2}\lambda\mu\nu\,[2\lambda'\mu'\nu'] \\
&\quad + \frac{15}{4}\nu\,(\lambda^2 - \mu^2)\,[\nu'\,(\lambda'^2 - \mu'^2)]
\end{aligned}$$

and we have, in notation similar to that of Musen and Estes (1971)

$$
\begin{aligned}
R' = & \frac{G\,m'\,a^2}{a'^3}\{a_{20}\,C'_{20} + a_{21}\,C'_{21} + b_{21}\,S'_{21} + a_{22}\,C'_{22} + b_{22}\,S'_{22}\} \\
& + \frac{G\,m'\,a^3}{a'^4}\{a_{31}\,C'_{31} + b_{31}\,S'_{31} + b_{32}\,S'_{32} + a_{33}\,C'_{33} + b_{33}\,S'_{33} \\
& + a_{34}\,C'_{34} + b_{34}\,S'_{34}\} \\
& + \cdots
\end{aligned}
\tag{2}
$$

where

$$a_{20} = \frac{1}{4}\left(\frac{r}{a}\right)^2 (1 - 3\,\nu^2)$$

$$a_{21} = \frac{3}{4}\left(\frac{r}{a}\right)^2 (\lambda^2 - \mu^2) \qquad b_{21} = \frac{3}{2}\left(\frac{r}{a}\right)^2 \lambda\,\mu$$

$$a_{22} = 3\left(\frac{r}{a}\right)^2 \mu\,\nu \qquad b_{22} = 3\left(\frac{r}{a}\right)^2 \lambda\,\nu$$

$$a_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \lambda\,(1 - 5\,\nu^2) \qquad b_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \mu\,(1 - 5\,\nu^2)$$

$$b_{32} = \frac{1}{4}\left(\frac{r}{a}\right)^3 \nu\,(3 - 5\,\nu^2)$$

$$a_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \lambda\,(\lambda^2 - 3\,\mu^2) \qquad b_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \mu\,(3\,\lambda^2 - \mu^2)$$

$$a_{34} = \frac{15}{2}\left(\frac{r}{a}\right)^3 \lambda\,\mu\,\nu \qquad b_{34} = \frac{15}{4}\left(\frac{r}{a}\right)^3 \nu\,(\lambda^2 - \mu^2)$$

$$C'_{20} = \left(\frac{a'}{r'}\right)^3 (1 - 3\,\nu'^2)$$

$$C'_{21} = \left(\frac{a'}{r'}\right)^3 (\lambda'^2 - \mu'^2) \qquad S'_{21} = 2\left(\frac{a'}{r'}\right)^3 \lambda'\,\mu'$$

$$C'_{22} = \left(\frac{a'}{r'}\right)^3 \mu' \nu' \qquad S'_{22} = \left(\frac{a'}{r'}\right)^3 \lambda' \nu'$$

$$C'_{31} = \left(\frac{a'}{r'}\right)^4 \lambda' (1 - 5 \nu'^2) \qquad S'_{31} = \left(\frac{a'}{r'}\right)^4 \mu' (1 - 5 \nu'^2)$$

$$S'_{32} = \left(\frac{a'}{r'}\right)^4 \nu' (3 - 5 \nu'^2)$$

$$C'_{33} = \left(\frac{a'}{r'}\right)^4 \lambda' (\lambda'^2 - 3 \mu'^2) \qquad S'_{33} = \left(\frac{a'}{r'}\right)^4 \mu' (3 \lambda'^2 - \mu'^2)$$

$$C'_{34} = 2 \left(\frac{a'}{r'}\right)^4 \lambda' \mu' \nu' \qquad S'_{34} = \left(\frac{a'}{r'}\right)^4 \nu' (\lambda'^2 - \mu'^2)$$

The solar disturbing function is of the same form with all primes replaced by double primes.

In the equatorial elements of the satellite,

$$\lambda = \cos (f + \omega) \cos \Omega - \sin (f + \omega) \sin \Omega \cos i$$

$$\mu = \cos (f + \omega) \sin \Omega + \sin (f + \omega) \cos \Omega \cos i$$

$$\nu = \sin (f + \omega) \sin i.$$

Then

$$a_{20} = \frac{1}{4}\left(\frac{r}{a}\right)^2 \left\{1 - \frac{3}{2} \sin^2 i + \frac{3}{2} \sin^2 i \cos (2 f + 2 \omega)\right\}$$

$$a_{21} = \frac{1}{4}\left(\frac{r}{a}\right)^2 \left\{\frac{1}{2} \sin^2 i \cos 2 \Omega + \frac{1}{2} \cos 2 \Omega \cos (2 f + 2 \omega)\right.$$

$$\left. - \cos i \sin 2 \Omega \sin (2 f + 2 \omega) + \frac{1}{2} \cos^2 i \cos 2 \Omega \cos (2 f + 2 \omega)\right\}$$

$$b_{21} = \frac{3}{2}\left(\frac{r}{a}\right)^2 \left\{\frac{1}{2}\sin^2 i \sin\Omega\cos\Omega + \frac{1}{2}(1+\cos^2 i)\sin\Omega\cos\Omega\cos(2f+2\omega)\right.$$

$$\left. + \frac{1}{2}\cos i\,(\cos^2\Omega - \sin^2\Omega)\sin(2f+2\omega)\right\}$$

$$a_{22} = 3\left(\frac{r}{a}\right)^2 \left\{\frac{1}{2}\sin i \sin\Omega\sin(2f+2\omega) + \frac{1}{2}\sin i\cos i\cos\Omega\right.$$

$$\left. - \frac{1}{2}\sin i\cos i\cos\Omega\cos(2f+2\omega)\right\}$$

$$b_{22} = 3\left(\frac{r}{a}\right)^2 \left\{-\frac{1}{2}\sin i\cos i\sin\Omega + \frac{1}{2}\sin i\cos\Omega\sin(2f+2\omega)\right.$$

$$\left. + \frac{1}{2}\sin\cos i\sin\Omega\cos(2f+2\omega)\right\}$$

$$a_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \left\{\cos\Omega\left(1-\frac{3}{4}\sin^2 i\right)\cos(f+\omega) + \frac{3}{4}\sin^2 i\cos\Omega\cos(3f+3\omega)\right.$$

$$\left. + \cos i\sin\Omega\left(\frac{9}{4}\sin^2 i - 1\right)\sin(f+\omega) - \frac{3}{4}\sin^2 i\cos i\sin\Omega\sin(3f+3\omega)\right\}$$

$$b_{31} = \frac{3}{8}\left(\frac{r}{a}\right)^3 \left\{\sin\Omega\left(1-\frac{3}{4}\sin^2 i\right)\cos(f+\omega) + \frac{3}{4}\sin^2 i\cos\Omega\cos(3f+3\omega)\right.$$

$$+ \cos i\cos\Omega\left(1-\frac{9}{4}\sin^2 i\right)\sin(f+\omega)$$

$$\left. + \frac{3}{4}\sin^2\cos i\cos\Omega\sin(3f+3\omega)\right\}$$

$$b_{32} = \frac{1}{4}\left(\frac{r}{a}\right)^3 \left\{\sin i\left(3-\frac{15}{4}\sin^2 i\right)\sin(f+\omega) + \frac{5}{4}\sin^3\sin(3f+3\omega)\right\}$$

$$a_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \left\{\frac{3}{4}\sin^2 i \cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\cos(f+\omega)\right.$$

$$+\frac{3}{4}\cos i \sin^2 i \sin\Omega(\sin^2\Omega - 3\cos^2\Omega)\sin(f+\omega)$$

$$+\frac{1}{4}(1+3\cos^2 i)\cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\cos(3f+3\omega)$$

$$\left.+\frac{1}{4}\cos i\,(3+\cos^2 i)\sin\Omega(\sin^2\Omega - 3\cos^2\Omega)\sin(3f+3\omega)\right\}$$

$$b_{33} = \frac{5}{8}\left(\frac{r}{a}\right)^3 \left\{\frac{3}{4}\sin^2 i \sin\Omega(3\cos^2\Omega - \sin^2\Omega)\cos(f+\omega)\right.$$

$$+\frac{3}{4}\sin^2 i \cos i \cos\Omega(\cos^2\Omega - 3\sin^2\Omega)\sin(f+\omega)$$

$$+\frac{1}{4}\sin\Omega(1+3\cos^2 i)(3\cos^2\Omega - \sin^2\Omega)\cos(3f+3\omega)$$

$$\left.+\frac{1}{4}\cos\Omega\cos i\,(3+\cos^2 i)(\cos^2\Omega - 3\sin^2\Omega)\sin(3f+3\omega)\right\}$$

$$a_{34} = \frac{15}{2}\left(\frac{r}{a}\right)^3 \left\{\frac{1}{4}\cos i \sin i \cos 2\Omega\cos(f+\omega)\right.$$

$$+\frac{1}{4}\sin i\,(1-3\cos^2 i)\cos\Omega\sin\Omega\sin(f+\omega)$$

$$-\frac{1}{4}\cos i \sin i \cos 2\Omega\cos(3f+3\omega)$$

$$\left.+\frac{1}{4}\sin i\,(1+\cos^2 i)\sin\Omega\cos\Omega\sin(3f+3\omega)\right\}$$

$$b_{34} = \frac{15}{4}\left(\frac{r}{a}\right)^3 \left\{-\frac{1}{4}\sin 2\,i \sin 2\,\Omega \cos(f+\omega)\right.$$

$$+\frac{1}{4}\sin i\,(1-3\cos^2 i)\cos 2\,\Omega \sin(f+\omega)$$

$$+\frac{1}{4}\sin 2\,i \sin 2\,\Omega \cos(3\,f+3\,\omega)$$

$$\left.+\frac{1}{4}\sin i\,(1+\cos^2 i)\cos 2\,\Omega \sin(3\,f+3\,\omega)\right\}.$$

Using the well-known formulas of elliptic motion

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 d\,g = 1+\frac{3}{2}e^2,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^2 \cos 2\,f\,d\,g = \frac{5}{2}e^2,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^3 \cos f\,d\,g = -\frac{5}{2}e-\frac{15}{8}e^3,$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r}{a}\right)^3 \cos 3\,f\,d\,g = -\frac{35}{8}e^3,$$

we find for the averaged potential due to the P_2 term

$$[R_2'] = \frac{G\,m'\,a^2}{a'^3}\left\{\frac{1}{4}\left(1+\frac{3}{2}e^2\right)\left(1-\frac{3}{2}\sin^2 i\right)C_{20}'\right. \tag{3}$$

$$+\frac{15}{16}e^2\sin^2 i\,(\cos 2\,\omega\,C_{20}')$$

$$+\frac{3}{8}\sin^2 i\left(1+\frac{3}{2}e^2\right)C_{210}'$$

$$+\frac{15}{16}e^2\,(1+\cos^2)\,(\cos 2\,\omega\,C_{210}')$$

$$+ \frac{15}{8} e^2 \cos i \, (\sin 2\omega S'_{210}) + \frac{15}{4} e^2 \sin i \, (\sin 2\omega S'_{220})$$

$$- \frac{15}{4} e^2 \sin i \cos i \, (\cos 2\omega C'_{220})$$

$$+ \frac{3}{2} \sin i \cos i \left(1 + \frac{3}{2} e^2\right) C'_{220}$$

where

$$C'_{210} = \cos 2\Omega C'_{21} + \sin 2\Omega S'_{21}$$

$$C'_{220} = \cos \Omega C'_{22} - \sin \Omega S'_{22}$$

$$S'_{210} = \cos 2\Omega S'_{21} - \sin 2\Omega C'_{21}$$

$$S'_{220} = \sin \Omega C'_{22} + \cos \Omega S'_{22}$$

and that due to the P_3 term

$$[R'_3] = \frac{G m' a^3}{a'^4} \left\{ + \frac{15}{16} e \left(1 + \frac{3}{4} e^2\right) \left(\frac{3}{4} \sin^2 i - 1\right) [\cos \omega C'_{310}] \right. \tag{4}$$

$$+ \frac{15}{16} e \left(1 + \frac{3}{4} e^2\right) \cos i \left(1 - \frac{9}{4} \sin^2 i\right) [\sin \omega S'_{310}]$$

$$- \frac{315}{256} e^3 \sin^2 i \, [\cos 3\omega C'_{310}]$$

$$+ \frac{315}{256} e^3 \sin^2 \cos i \, [\sin 3\omega S'_{310}]$$

$$+ \frac{15}{8} e \left(1 + \frac{3}{4} e^2\right) \sin i \left(\frac{5}{4} \sin^2 i - 1\right) [\sin \omega S'_{32}]$$

$$- \frac{175}{128} e^3 \sin^3 i \, [\sin 3\omega S'_{32}]$$

$$- \frac{75}{64} e \left(1 + \frac{3}{4} e^2\right) \sin^2 i \, [\cos \omega \, C'_{330}]$$

$$+ \frac{75}{64} e \left(1 + \frac{3}{4} e^2\right) \cos i \sin^2 i \, [\sin \omega \, S'_{330}]$$

$$- \frac{175}{256} e^3 (1 + 3 \cos^2 i) \, [\cos 3 \Omega \, C'_{330}]$$

$$+ \frac{175}{256} e^3 (3 + \cos^2 i) \cos i \, [\sin 3 \omega \, S'_{330}]$$

$$+ \frac{75}{32} e \left(1 + \frac{3}{4} e^2\right) \sin i \, (3 \cos^2 i - 1) \, [\sin \omega \, S'_{340}]$$

$$- \frac{75}{16} e \left(1 + \frac{3}{4} e^2\right) \sin i \cos i \, [\cos \omega \, C'_{340}]$$

$$- \frac{525}{128} e^3 \sin i \, (1 + \cos^2 i) \, [\sin 3 \omega \, S'_{340}]$$

$$+ \frac{525}{64} e^3 \cos i \sin i \, [\cos 3 \omega \, C'_{340}] \Big\}$$

where

$$C'_{310} = \cos \Omega \, C'_{31} + \sin \Omega \, S'_{31}$$

$$S'_{310} = \sin \Omega \, C'_{31} - \cos \Omega \, S'_{31}$$

$$C'_{330} = \cos 3 \Omega \, C'_{33} + \sin 3 \Omega \, S'_{33}$$

$$S'_{330} = \sin 3 \Omega \, C'_{33} - \cos 3 \Omega \, S'_{33}$$

$$C'_{340} = \cos 2 \Omega \, C'_{34} - \sin 2 \Omega \, S'_{34}$$

$$S'_{340} = \sin 2 \Omega \, C'_{34} + \cos 2 \Omega \, S'_{34}$$

The satellite elements appearing in these equations and those which follow now designate mean instead of osculating elements.

Primed (double primed) quantities appearing in the disturbing function depend only upon the position of the moon (sun). The coordinates of the moon are obtained analytically from E. W. Brown's theory where if $\lambda_{☾}$ is the true longitude of the moon measured in the plane of the ecliptic and β is the latitude above the plane of the ecliptic, Brown's tables express a'/r', β and $\delta\lambda \equiv \lambda_{☾} - \ell_{☾}$ by sums of periodic terms whose arguments are algebraic sums of multiples of ℓ, ℓ', F, D and Γ. Then

$$\lambda' = \cos\lambda_{☾} \cos\beta$$

$$\mu' = \sin\lambda_{☾} \cos\beta \cos\epsilon - \sin\beta \sin\epsilon$$

$$\nu' = \sin\beta \cos\epsilon + \sin\lambda_{☾} \cos\beta \sin\epsilon$$

where

$$\cos\lambda_{☾} = \cos(\ell' + D + \Gamma + \delta\lambda)$$

$$= \cos(\ell' + D + \Gamma) \cos\delta\lambda - \sin(\ell' + D + \Gamma) \sin\delta\lambda$$

$$\sin\lambda_{☾} = \sin(\ell' + D + \Gamma) \cos\delta\lambda + \cos(\ell' + D + \Gamma) \sin\delta\lambda$$

$$\sin\delta\lambda = \delta\lambda - \frac{1}{6}(\delta\lambda)^3 + \cdots$$

$$\cos\delta\lambda = 1 - \frac{1}{2}(\delta\lambda)^2 + \frac{1}{24}(\delta\lambda)^4 - \cdots$$

$$\sin\beta = \beta - \frac{1}{6}\beta^3 + \cdots$$

$$\cos\beta = 1 - \frac{1}{2}\beta^2 + \cdots$$

The coordinates of the sun are obtained in similar fashion from Newcomb's theory which expresses the ecliptic longitude and latitude and the common logarithm of r''/a'' by sums of periodic terms with the same ecliptic elements as the lunar theory. From the relation $r^n = e^{n \ln r}$ we have

$$\left(\frac{a''}{r''}\right)^n = \exp\left\{-\frac{n}{M_0}\log\left(\frac{r''}{a''}\right)\right\}$$

$$= 1 - \frac{n}{M_0}\left(\log\frac{r''}{a''}\right) + \frac{n^2}{2\,M_0^2}\left(\log\frac{r''}{a''}\right)^2 - \cdots$$

where M_0 is the logarithmic modulus,

$$\frac{1}{M_0} = 2.3025851.$$

The series manipulations involved in the above calculations were performed by electronic computer with programs developed by Estes (1971) for expanding the earth's tidal potential.

THE MAIN PROBLEM

The secular and long period perturbations in the orbital elements are given by the variation equations

$$\begin{aligned}
\frac{d\,\delta a}{d\,t} &= \frac{2}{n\,a}\,\frac{\partial\,[R]}{\partial g} \\
\frac{d\,\delta e}{d\,t} &= \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\,[R]}{\partial g} - \frac{\sqrt{1-e^2}}{n\,a^2\,e}\,\frac{\partial\,[R]}{\partial \omega} \\
\frac{d\,\delta i}{d\,t} &= \frac{\cos i}{n\,a^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,[R]}{\partial \omega} - \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,[R]}{\partial \Omega} \\
\frac{d\,\delta \Omega}{d\,t} &= \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,[R]}{\partial i} + \frac{d\,\dot{\Omega}}{d\,e}\,\delta e + \frac{d\,\dot{\Omega}}{d\,i}\,\delta i \\
\frac{d\,\delta \omega}{d\,t} &= \frac{\sqrt{1-e^2}}{n\,a^2\,e}\,\frac{\partial\,[R]}{\partial e} - \frac{\cos i}{n\,a^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,[R]}{\partial i} + \frac{d\,\dot{\omega}}{d\,e}\,\delta e + \frac{d\,\dot{\omega}}{d\,i}\,\delta i \\
\frac{d\,\delta g}{d\,t} &= -\frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\,[R]}{\partial e} - \frac{2}{n\,a}\,\frac{\partial\,[R]}{\partial a} + \frac{d\,\dot{g}}{d\,e}\,\delta e + \frac{d\,\dot{g}}{d\,i}\,\delta i
\end{aligned} \tag{5}$$

where (Brouwer, 1959)

$$\dot{g} = n \left\{ 1 - \frac{3 J_2 R_e^2}{4 a^2 (1 - e^2)^{3/2}} (1 - 3 \cos^2 i) + \frac{3 J_2^2 R_e^2}{128 a^4 (1 - e^2)^{7/2}} \right.$$

$$\times [10 - 25 e^2 + 16 \sqrt{1 - e^2} - 6 (10 - 15 e^2 + 16 \sqrt{1 - e^2}) \cos^2 i$$

$$\left. + (130 - 25 e^2 + 144 \sqrt{1 - e^2}) \cos^4 i] - \frac{45 J_4 R_e^4 e^2}{128 a^4 (1 - e^2)^{7/2}} (3 - 30 \cos^2 i + 35 \cos^4 i) \right\}$$

$$\dot{\omega} = n \left\{ - \frac{3 J_2 R_e^2}{4 a^2 (1 - e^2)^2} (1 - 5 \cos^2 i) + \frac{3 J_2^2 R_e^4}{128 a^4 (1 - e^2)^4} [- 10 - 25 e^2 \right.$$

$$+ 24 \sqrt{1 - e^2} - 6 (6 - 21 e^2 + 32 \sqrt{1 - e^2}) \cos^2 i$$

$$+ 5 (86 - 9 e^2 + 72 \sqrt{1 - e^2}) \cos^4 i]$$

$$- \frac{15 J_4 R_e^4}{128 a^4 (1 - e^2)^4} [3 (4 + 3 e^2) - 18 (8 + 7 e^2) \cos^2 i$$

$$\left. + 7 (28 + 27 e^2) \cos^4 i] \right\},$$

$$\dot{\Omega} = n \left\{ - \frac{3 J_2 R_e^2 \cos i}{2 a^2 (1 - e^2)^2} + \frac{3 J_2^2 R_e^4 \cos i}{32 a^4 (1 - e^2)^4} [4 - 9 e^2 + 12 \sqrt{1 - e^2} \right.$$

$$- (40 - 5 e^2 + 36 \sqrt{1 - e^2}) \cos^2 i]$$

$$\left. - \frac{15 J_4 R_e^4 (2 + 3 e^2) \cos i}{32 a^4 (1 - e^2)^4} (3 - 7 \cos^2 i) \right\}.$$

The main problem denotes the secular and long period effects resulting from setting $[R]$ to be $[R_2'] + [R_2'']$. Then

$$\frac{d \delta a}{d t} = 0$$

$$\left(\frac{d\,\delta\,e}{d\,t}\right)' = -\,n\,e\sqrt{1-e^2}\,\frac{M'}{M}\,\frac{a^3}{a'^3}\left\{-\frac{15}{8}\,(1+\cos^2 i)\,[\sin 2\,\omega\,C'_{210}]\right.$$

$$+\frac{15}{4}\,\cos i\,[\cos 2\,\omega\,S'_{210}]$$

$$+\frac{15}{2}\,\sin i\,[\cos 2\,\omega\,S'_{220}]$$

$$+\frac{15}{2}\,\sin i\cos i\,[\sin 2\,\omega\,C'_{220}]$$

$$\left.-\frac{15}{8}\,\sin^2 i\,[\sin 2\,\omega\,C'_{20}]\right\}$$

$$\left(\frac{d\,\delta\,i}{d\,t}\right)' = \frac{n\,\frac{M'}{M}}{\sqrt{1-e^2}}\,\frac{a^3}{a'^3}\left\{\frac{15}{8}\,e^2\sin i\cos i\,[\sin 2\,\omega\,C'_{210}]\right.$$

$$-\frac{15}{8}\,e^2\sin i\,[\cos 2\,\omega\,S'_{210}]$$

$$-\frac{15}{4}\,e^2\sin^2 i\,[\sin 2\,\omega\,C'_{220}]$$

$$+\frac{15}{4}\,e^2\cos i\,[\cos 2\,\omega\,S'_{220}]$$

$$-\frac{3}{4}\sin i\left(1+\frac{3}{2}e^2\right)S'_{210}$$

$$+\frac{3}{2}\cos i\left(1+\frac{3}{2}e^2\right)S'_{220}$$

$$\left.-\frac{15}{8}\,e^2\sin i\cos i\,[\sin 2\,\omega\,C'_{20}]\right\}$$

$$\left(\frac{d\,\delta\,\Omega}{d\,t}\right)' = \frac{n\,\frac{M'}{M}}{\sqrt{1-e^2}}\;\frac{a^3}{a'^3}\left\{-\frac{3}{4}\left(1+\frac{3}{2}e^2\right)\cos i\;C'_{20} + \frac{3}{4}\cos i\left(1+\frac{3}{2}e^2\right)C'_{210}\right.$$

$$+\frac{3}{2}\;\frac{\cos 2\,i}{\sin i}\left(1+\frac{3}{2}e^2\right)C'_{220}$$

$$+\frac{30}{16}\,e^2\cos i\,[\cos 2\,\omega\,C'_{20}] - \frac{30}{16}\,e^2\cos i\,[\cos 2\,\omega\,C'_{210}]$$

$$-\frac{15}{8}\,e^2\,[\sin 2\,\omega\,S'_{210}] + \frac{15}{4}\,e^2\,\mathrm{ctn}\,i\,[\sin 2\,\omega\,S'_{220}]$$

$$\left.-\frac{15}{4}\,e^2\,\frac{\cos 2\,i}{\sin i}\,[\cos 2\,\omega\,C'_{220}]\right\}$$

$$+\frac{d\,\dot{\Omega}}{d\,e}(\delta\,e)' + \frac{d\,\dot{\Omega}}{d\,i}(\delta\,i)'$$

$$\left(\frac{d\,\delta\,\omega}{d\,t}\right)' = \frac{n\,\frac{M'}{M}}{\sqrt{1-e^2}}\;\frac{a^3}{a'^3}\left\{\frac{3}{8}(4-5\sin^2 i+e^2)\,C'_{20} + \frac{3}{8}(5\sin^2 i-2-3\,e^2)\,C'_{210}\right.$$

$$+\frac{3}{2}\,\mathrm{ctn}\,i\left(5\sin^2 i-1-\frac{3}{2}e^2\right)C'_{220}$$

$$+\frac{30}{16}(\sin^2 i-e^2)\,[\cos 2\,\omega\,C'_{20}]$$

$$+\frac{30}{16}(1+\cos^2 i-e^2)\,[\cos 2\,\omega\,C'_{210}]$$

$$+\frac{15}{8}\,(2-e^2)\cos i\,[\sin 2\,\omega\,S'_{210}]$$

$$+\frac{15}{4}\,\mathrm{ctn}\,i\,(e^2-2\sin^2 i)\,[\cos 2\,\omega\,C'_{220}]$$

$$\left.+\frac{15}{4}\left(2\sin i - e^2\,\frac{(1+\sin^2 i)}{\sin i}\right)[\sin 2\,\omega\,S'_{220}]\right\}$$

$$+\frac{d\,\dot{\omega}}{d\,e}(\delta\,e)' + \frac{d\,\dot{\omega}}{d\,i}(\delta\,i)'$$

$$
\begin{aligned}
\left(\frac{d\,\delta g}{d\,t}\right)' = -\frac{n\left(\frac{M'}{M}\right)a^3}{a'^3} \Bigg\{ & \frac{1}{4}(7+3e^2)\left(1-\frac{3}{2}\sin^2 i\right) C'_{20} \\
& + \frac{3}{8}(7+3e^2)\sin^2 i\, C'_{210} \\
& + \frac{3}{2}(7+3e^2)\sin i \cos i\, C'_{220} \\
& + \frac{30}{16}(1+e^2)\sin^2 i\, [\cos 2\omega\, C'_{20}] \\
& + \frac{30}{16}(1+e^2)(1+\cos^2 i)\, [\cos 2\omega\, C'_{210}] \\
& + \frac{30}{8}(1+e^2)\cos i\, [\sin 2\omega\, S'_{210}] \\
& - \frac{15}{2}(1+e^2)\sin i \cos i\, [\cos 2\omega\, C'_{220}] \\
& + \frac{30}{4}(1+e^2)\sin i\, [\sin 2\omega\, S'_{220}]\Bigg\} + \frac{d\,\dot{g}}{d\,e}(\delta\, e)' + \frac{d\,\dot{g}}{d\,i}(\delta\, i)'
\end{aligned}
\tag{6}
$$

Similar expressions are obtained for the solar perturbations where all primes are replaced by double primes.

Tables I-III display the principal terms of the lunar longitude, latitude and (a'/r') as given by Brown's theory. For the purpose of this paper, the lunar and solar terms in longitude, latitude and parallax whose coefficients are less than 5×10^{-6} radians in magnitude are omitted. In addition all planetary terms and cosine terms in $\delta\lambda$ and β are omitted. The resulting accuracy of the functions depending on the position of the moon and sun is 1×10^{-4}, with 231 terms in C'_{200}, 235 terms in C'_{210}, 255 terms in C'_{220}, 11 terms in C''_{200}, 17 terms in C''_{210} and 16 terms in C''_{220}. Tables V-VIII list the terms of these functions whose coefficients exceed 10^{-5} radians.

Figures 1 and 2 compare perturbations of orbital elements obtained using the method of this paper with those obtained from the Geodyne program (numerical integration of the disturbing function, Eq. 1) and an analytic theory of long period and secular effects (Murphy and Felsentreger, 1966) using the equatorial elements of the moon.

Table I. Longitude of Moon $\delta\lambda$

Coef × 10^5 of sine	Multiples of ℓ	ℓ'	F	D	Γ
-61	0	0	0	1	0
7	0	0	0	4	0
-200	0	0	2	0	0
-60	0	1	0	-2	0
9	0	1	0	1	0
-4	0	2	0	-2	0
1	1	-2	0	-2	0
14	1	-1	0	-2	0
7	1	-1	0	2	0
19	1	0	-2	0	0
-19	1	0	0	-4	0
-2224	1	0	0	-2	0
10576	1	0	0	0	0
53	1	0	0	2	0
-2	1	1	0	-4	0
-53	1	1	0	0	0
-4	1	2	0	-2	0
5	2	-1	0	0	0
-103	2	0	0	-2	0
7	2	0	0	2	0
-1	2	1	0	-4	0
-4	2	1	0	0	0
17	3	0	0	0	0

Coef × 10^5 of sine	Multiples of ℓ	ℓ'	F	D	Γ
1145	0	0	0	2	0
-27	0	0	2	-2	0
-3	0	0	2	2	0
-324	0	1	0	0	0
-12	0	1	0	2	0
-4	0	2	0	0	0
1	1	-2	0	0	0
72	1	-1	0	0	0
4	1	0	-2	-2	0
-3	1	0	-2	2	0
2	1	0	0	-3	0
5	1	0	0	-1	0
-4	1	0	0	1	0
-22	1	0	2	0	0
-100	1	1	0	-2	0
-1	1	1	0	2	0
-1	2	-1	0	-2	0
-15	2	0	0	-4	0
373	2	0	0	0	0
-2	2	0	2	0	0
-4	2	1	0	-2	0
-6	3	0	0	-2	0

Table II. Latitude of Moon

Coef × 10^5 of sine	Multiples of ℓ	ℓ'	F	D	Γ
-2	0	0	1	-4	0
2	0	0	1	-1	0
-3	0	0	1	1	0
-1	0	0	3	-2	0
-4	0	1	-1	-2	0
-6	0	1	-1	2	0
-3	0	1	1	0	0
3	1	-1	1	0	0
-1	1	0	-1	-4	0
485	1	0	-1	0	0
-3	1	0	1	-4	0
490	1	0	1	0	0
-4	1	1	-1	-2	0
-4	1	1	1	-2	0
-1	2	0	-1	-4	0
-7	2	0	1	-2	0
2	3	0	1	0	0

Coef × 10^5 of sine	Multiples of ℓ	ℓ'	F	D	Γ
-302	0	0	1	-2	0
8950	0	0	1	0	0
57	0	0	1	2	0
-3	0	0	3	0	0
-2	0	1	-1	0	0
-14	0	1	1	-2	0
3	1	-1	-1	0	0
1	1	0	-3	0	0
-97	1	0	-1	-2	0
16	1	0	-1	2	0
-81	1	0	1	-2	0
7	1	0	1	2	0
-2	1	1	-1	0	0
-3	1	1	1	0	0
15	2	0	-1	0	0
30	2	0	1	0	0

Table III. $\frac{a'}{r'}$

Coef × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ
100000	0	0	0	0	0
825	0	0	0	2	0
-3	0	0	2	-2	0
-12	0	1	0	0	0
-9	0	1	0	2	0
-7	1	-1	0	-2	0
7	1	-1	0	2	0
-1	1	0	-2	2	0
-1	1	0	0	-3	0
5450	1	0	0	0	0
90	1	0	0	2	0
-2	1	0	2	-2	0
42	1	1	0	-2	0
-1	1	1	0	2	0
4	2	-1	0	0	0
-9	2	0	0	-2	0
8	2	0	0	2	0
-3	3	0	0	-2	0
1	4	0	0	0	0

Coef × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ
-29	0	0	0	1	0
8	0	0	0	4	0
56	0	1	0	-2	0
4	0	1	0	1	0
3	0	2	0	-2	0
34	1	-1	0	0	0
-21	1	0	-2	0	0
18	1	0	0	-4	0
1002	1	0	0	-2	0
-3	1	0	0	1	0
1	1	0	0	4	0
2	1	1	0	-4	0
-28	1	1	0	0	0
1	1	2	0	-2	0
11	2	0	0	-4	0
297	2	0	0	0	0
-3	2	1	0	0	0
18	3	0	0	0	0

Table IV. Development of sin λ'' and cos λ''

Coef. $\times 10^5$ of cosine in cos λ'' of sine in sin λ''	Multiples of ℓ	ℓ'	F	D	Γ
99972	0	1	0	0	1
1674-4.2T	0	2	0	0	1
32	0	3	0	0	1
1	0	4	0	0	1
2	0	1	0	1	1
- 1675 +4.2T	0	0	0	0	1
- 4	0	-1	0	0	1
- 2	0	1	0	-1	1
4	0	0	1	-1	0
- 4	0	2	-1	1	2

Development of $(a''/r'')^3$

Coef. of cosine $\times 10^5$	Multiples of ℓ	ℓ'	F	D	Γ
100042-.2T	0	0	0	0	0
- 1	0	0	0	1	0
5027-12.5T	0	1	0	0	0
126-.63T	0	2	0	0	0
3	0	3	0	0	0

Table V. Development of C'_{200}

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
75695	0	0	0	0	0	0
-60	0	0	2	-2	0	0
909	0	0	2	0	0	0
2369	1	0	0	-2	0	0
326	1	0	0	2	0	0
147	0	1	-1	3	1	0
-9826	0	1	-1	1	1	0
6	1	-1	1	-5	-1	0
-86	1	1	1	-1	1	0
-170	1	-1	1	-3	-1	0
5	2	1	-1	3	1	0
-5	0	2	-1	3	1	0
3	0	0	1	-3	-1	0
1	0	1	1	0	1	0
-3	0	1	2	-2	0	0
175	1	0	2	0	0	0
-3	2	0	2	-2	0	0
297	0	1	1	3	1	0
-4	0	1	3	-1	1	0
19	0	0	1	3	1	0
354	1	-1	-1	-3	-1	0
72	1	1	1	3	1	0
-13	2	-1	-1	-1	-1	0
-21	0	2	1	1	1	0
-5	2	-1	-1	-3	-1	0
-11	0	0	1	-1	-1	0
-4	1	2	1	-1	1	0
28	3	1	1	1	1	0
15	1	0	1	1	1	0
-5	3	1	1	-1	1	0
1	1	0	-1	-1	-1	0
717	0	2	0	4	2	0
18	0	2	0	6	2	0
-10	0	2	2	0	2	0
96	0	2	-2	2	2	0
-10	0	3	0	4	2	0
-17	1	-2	-2	-2	-2	0
853	1	-2	0	-4	-2	0
-663	1	-2	0	-2	-2	0
2	1	-2	0	0	-2	0
29	2	2	0	4	2	0
-13	2	-2	0	-4	-2	0
-71	2	2	0	0	2	0

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
6	0	1	-1	5	1	0
2055	0	0	0	2	0	0
-73	1	0	-2	0	0	0
-16	1	0	2	-2	0	0
-272	0	1	1	-1	1	0
9720	0	1	1	1	1	0
-813	1	1	-1	1	1	0
-269	1	-1	-1	-1	-1	0
33	1	1	-1	3	1	0
-27	2	1	1	-1	1	0
-12	0	2	1	-1	1	0
5	0	0	1	-1	1	0
-54	2	1	-1	1	1	0
28	0	0	2	2	0	0
32	1	0	-2	-2	0	0
1016	2	0	0	0	0	0
23	2	0	2	0	0	0
14	1	-1	-1	-5	-1	0
-19	0	1	-3	1	1	0
-7	1	-1	-3	-1	-1	0
1861	1	1	1	1	1	0
247	2	1	1	1	1	0
11	2	1	1	3	1	0
15	0	2	-1	1	1	0
17	0	0	1	1	1	0
-10	0	1	1	2	1	0
16	1	0	-1	-3	-1	0
9	2	-1	-1	-5	-1	0
-13	1	2	1	1	1	0
-1	1	-2	-1	-1	-1	0
23457	0	2	0	2	2	0
19	0	2	0	0	2	0
34	1	-2	0	-6	-2	0
4	0	2	-2	4	2	0
48	0	1	0	4	2	0
10	1	2	-2	2	2	0
-173	1	2	0	0	2	0
4491	1	2	0	2	2	0
174	1	2	0	4	2	0
594	2	2	0	2	2	0
-81	0	3	0	2	2	0
72	0	1	0	2	2	0
-24	0	2	0	3	2	0

Table V. Development of C'_{200} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
4	0	2	0	1	2	0
-8	1	-3	0	-4	-2	0
13	1	1	0	4	2	0
67	3	2	0	2	2	0
42	1	1	0	2	2	0
-14	3	2	0	0	2	0
-6	1	2	0	3	2	0
-3	2	3	0	0	2	0
-66	0	0	0	1	0	0
4	0	1	-1	0	1	0
31	0	0	0	4	0	0
12386	1	0	0	0	0	0
6	1	0	0	4	0	0
-130	0	1	-1	-1	1	0
-25	1	-1	-1	1	-1	0
-804	1	-1	1	-1	-1	0
-7	2	1	-1	-1	1	0
5	0	1	1	5	1	0
-2	2	-1	-1	1	-1	0
-8	0	2	-1	-1	1	0
1	3	1	1	3	1	0
-1	1	-2	-1	1	-1	0
-2	0	2	2	2	2	0
1	0	1	0	6	2	0
-1	1	-2	-2	0	-2	0
4	1	2	0	6	2	0
2	1	-1	0	-6	-2	0
-1	1	1	0	0	2	0
-2	0	1	-1	-3	1	0
1	1	-2	2	-4	-2	0
4	0	1	0	-4	0	0
2	2	1	0	4	2	0
-24	0	1	0	0	0	0
2	0	3	0	3	2	0
-22	0	1	0	2	0	0
-1	1	-2	0	-2	0	0
1	1	-2	0	2	0	0
76	1	-1	0	0	0	0
-4	1	-2	1	-1	-1	0
23	1	-1	0	2	0	0
1	1	1	-3	1	1	0
-3	1	0	0	-3	0	0
-2	2	-1	1	-5	-1	0
-6	1	2	-1	-1	1	0

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-9	1	3	0	0	2	0
40	1	-1	0	-4	-2	0
3	1	2	0	1	2	0
22	2	-2	0	-6	-2	0
-38	1	3	0	2	2	0
-3	1	-3	0	-2	-2	0
9	2	1	0	2	2	0
-8	2	3	0	2	2	0
4	0	1	-1	2	1	0
-2	1	1	1	2	1	0
-9	1	0	-2	2	0	0
62	1	0	0	-4	0	0
-4	0	1	1	-3	1	0
-154	1	1	-1	-1	1	0
-5	1	1	1	-3	1	0
-1	2	1	1	-3	1	0
5	1	0	2	2	0	0
1	1	1	1	5	1	0
-2	0	2	1	3	1	0
1	0	0	1	1	-1	0
3	1	0	1	3	1	0
-1	1	-2	-1	-3	-1	0
1	0	2	-2	0	2	0
2	1	2	-2	0	2	0
-1	1	2	0	-2	2	0
-2	0	1	0	0	2	0
3	3	2	0	4	2	0
-2	1	3	0	4	2	0
-2	1	2	2	0	2	0
8	1	-2	2	-2	-2	0
133	0	1	0	-2	0	0
1	0	0	0	4	2	0
10	0	1	0	1	0	0
2	0	1	0	1	2	0
6	0	2	0	-2	0	0
1	1	-2	0	0	0	0
-16	1	-1	0	-2	0	0
-6	1	0	-1	1	1	0
2	2	0	1	1	1	0
-2	1	-2	1	1	-1	0
-4	1	1	-1	-3	1	0
39	2	0	0	-4	0	0
5	1	2	-1	1	1	0
-6	1	0	1	-3	-1	0

Table V. Development of C'_{200} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
1	2	-1	0	-6	-2	0
-4	1	0	0	-1	0	0
110	2	0	0	-2	0	0
-9	2	-1	1	-3	-1	0
81	3	0	0	0	0	0
-2	1	-1	3	-1	-1	0
3	1	0	1	-1	-1	0
-2	2	2	1	1	1	0
-11	1	0	0	1	0	0
-20	1	-1	1	1	-1	0
7	1	1	0	-4	0	0
-64	1	1	0	0	0	0
-5	1	1	0	2	0	0
13	2	-1	0	0	0	0
-3	2	1	-1	-3	1	0
1	2	0	0	4	0	0
4	2	1	0	-2	0	0
1	3	-1	0	0	0	0
-5	3	-1	1	-1	-1	0
-1	3	1	0	0	0	0

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-1	4	2	0	0	2	0
-5	2	0	-2	0	0	0
38	2	0	0	2	0	0
-4	3	1	-1	1	1	0
2	3	0	2	0	0	0
-1	3	-1	-1	-1	-1	0
2	4	1	1	1	1	0
5	4	2	0	2	2	0
-1	2	0	-2	2	0	0
-66	2	-1	1	-1	-1	0
102	1	1	0	-2	0	0
2	1	1	0	1	0	0
3	1	2	0	-2	0	0
3	2	-1	0	2	0	0
-2	2	-1	1	1	-1	0
3	2	1	0	-4	0	0
-10	2	1	0	0	0	0
2	3	0	0	-4	0	0
4	3	0	0	2	0	0
6	4	0	0	0	0	0

Table VI. Development of C'_{210}

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
168	0	2	0	2	2	2
-94	0	2	0	3	2	-2
5	0	2	0	4	2	2
73	0	2	0	0	2	-2
69	0	2	0	6	2	-2
17	0	2	-2	4	2	-2
371	0	2	-2	2	2	-2
3	0	3	0	0	2	-2
-313	0	3	0	2	2	-2
14	0	3	0	3	2	-2
-39	0	3	0	4	2	-2
8	0	0	0	4	2	-2
3	0	0	0	2	2	-2
3	1	0	0	2	2	-2
-32	1	-3	0	-4	-2	2
-10	1	-3	0	-2	-2	2
2	1	1	0	6	2	-2
-3	1	-2	-2	-4	-2	2
-67	1	-2	-2	-2	-2	2
130	1	-2	0	-6	-2	2
-1	1	2	0	0	2	2
3303	1	-2	0	-4	-2	2
14	1	2	0	1	2	-2
32	1	2	0	2	2	2
-2567	1	-2	0	-2	-2	2
-25	1	2	0	3	2	-2
671	1	2	0	4	2	-2
18	1	2	0	6	2	-2
7	1	-2	2	-4	-2	2
-32	1	3	0	0	2	-2
-147	1	3	0	2	2	-2
4	1	3	0	3	2	-2
5	1	0	0	-4	-2	2
9	2	1	0	4	2	-2
85	2	-2	0	-6	-2	2
-52	2	-2	0	-4	-2	2
4	2	2	0	2	2	2
111	2	2	0	4	2	-2
-3	2	2	2	0	2	-2
-11	2	3	0	0	2	-2
-31	2	3	0	2	2	-2
6	3	1	0	2	2	-2
260	3	2	0	2	2	-2

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
90813	0	2	0	2	2	-2
16	0	2	0	1	2	-2
2777	0	2	0	4	2	-2
-3	0	2	0	5	2	-2
-39	0	2	2	0	2	-2
-14	0	2	2	2	2	-2
9	0	1	0	6	2	-2
188	0	1	0	4	2	-2
275	0	1	0	2	2	-2
-2	0	1	0	1	2	-2
-2	0	3	2	0	2	-2
-3	0	4	0	2	2	-2
-2	1	-4	0	-4	-2	2
2	1	0	0	4	2	-2
163	1	1	0	2	2	-2
51	1	1	0	4	2	-2
6	1	2	-2	0	2	-2
40	1	2	-2	2	2	-2
2	1	-2	-2	0	-2	2
-6	1	-2	0	-5	-2	2
-669	1	2	0	0	2	-2
6	1	-2	0	-4	-2	-2
-3	1	-2	0	-3	-2	2
17388	1	2	0	2	2	-2
-5	1	-2	0	-2	-2	-2
1	1	2	0	4	2	2
6	1	-2	0	0	-2	2
-15	1	2	2	0	2	-2
15	1	-1	0	-6	-2	2
153	1	-1	0	-4	-2	2
3	1	-1	0	-2	-2	2
-13	1	3	0	4	2	-2
36	2	1	0	2	2	-2
-3	2	-2	-2	-4	-2	2
-273	2	2	0	0	2	-2
4	2	2	0	1	2	-2
2301	2	2	0	2	2	-2
3	2	2	0	6	2	-2
8	2	-1	0	-6	-2	2
-2	2	-1	0	-4	-2	2
-2	2	3	0	4	2	-2
-56	3	2	0	0	2	-2
15	3	2	0	4	2	-2

Table VI. Development of C'_{210} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-5	3	3	0	2	2	-2
27	4	2	0	2	2	-2
5	0	2	-2	0	2	-2
2	1	-2	0	-8	-2	2
30	1	-2	2	-2	-2	2
3	2	-2	-2	-2	-2	2
2	5	2	0	2	2	-2
-3	0	0	0	1	0	2
106	0	0	0	2	0	2
2	0	0	0	4	0	2
-3	0	0	2	-2	0	2
47	0	0	2	0	0	2
7	0	1	0	-2	0	2
-1	0	1	0	0	0	2
-1	0	1	0	2	0	2
4	1	-1	0	0	0	2
1	1	-1	0	2	0	2
2	1	0	-2	-2	0	2
-4	1	0	-2	0	0	2
3	1	0	0	-4	0	2
123	1	0	0	-2	0	2
643	1	0	0	0	0	2
17	1	0	0	2	0	2
5	1	1	0	-2	0	2
-3	1	1	0	0	0	2
2	2	0	0	-4	0	2
6	2	0	0	-2	0	2
53	2	0	0	0	0	2
2	2	0	0	2	0	2
4	3	0	0	0	0	2
-2	2	1	0	0	2	-2
1	0	0	2	2	0	-2
9	1	0	2	0	0	-2
1	2	0	2	0	0	-2
-2	3	3	0	0	2	-2
-2	1	-1	1	-5	-1	2
95	0	1	1	-1	1	-2
-3386	0	1	1	1	1	-2
94	1	-1	-1	-1	-1	2
-1	1	1	1	-1	1	2
8	1	-1	-1	1	-1	2
-648	1	1	1	1	1	-2
9	2	1	1	-1	1	-2
2	0	1	-1	3	1	2

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-9	4	2	0	0	2	-2
-4	2	2	0	3	2	-2
-2	1	-3	0	-6	-2	2
-2	1	2	2	2	2	-2
4	2	2	-2	2	2	-2
3	2	-2	0	-8	-2	2
7855	0	0	0	0	0	2
-3	0	0	0	1	0	-2
106	0	0	0	2	0	-2
2	0	0	0	4	0	-2
-3	0	0	2	-2	0	-2
47	0	0	2	0	0	-2
7	0	1	0	-2	0	-2
-1	0	1	0	0	0	-2
-1	0	1	0	2	0	-2
4	1	-1	0	0	0	-2
1	1	-1	0	2	0	-2
2	1	0	-2	-2	0	-2
-4	1	0	-2	0	0	-2
3	1	0	0	-4	0	-2
123	1	0	0	-2	0	-2
643	1	0	0	0	0	-2
17	1	0	0	2	0	-2
5	1	1	0	-2	0	-2
-3	1	1	0	0	0	-2
2	2	0	0	-4	0	-2
6	2	0	0	-2	0	-2
53	2	0	0	0	0	-2
2	2	0	0	2	0	-2
4	3	0	0	0	0	-2
1	0	0	2	2	0	2
9	1	0	2	0	0	2
1	2	0	2	0	0	2
2	2	-2	2	-2	-2	2
-2	0	1	-1	5	1	-2
-4	0	1	1	-1	1	2
146	0	1	1	1	1	2
-6	0	0	1	1	1	-2
-4	1	-1	-1	-1	-1	-2
30	1	1	1	-1	1	-2
28	1	1	1	1	1	2
5	2	-1	-1	-1	-1	2
2	3	1	1	-1	1	-2
-51	0	1	-1	3	1	-2

Table VI. Development of C'_{210} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-147	0	1	-1	1	1	2
-5	0	2	-1	1	1	-2
283	1	1	-1	1	1	-2
59	1	-1	1	-3	-1	2
280	1	-1	1	-1	-1	2
-2	1	2	-1	1	1	-2
-2	2	1	-1	3	1	-2
3	0	1	1	2	1	-2
-104	0	1	1	3	1	-2
4	0	2	1	-1	1	-2
7	0	2	1	1	1	-2
-5	1	0	1	1	1	-2
-5	1	-1	-1	-5	-1	2
5	1	-1	-1	-3	-1	-2
-25	1	1	1	3	1	-2
5	1	2	1	1	1	-2
2	2	-1	-1	-3	-1	2
-36	2	1	1	1	1	-2
-10	3	1	1	1	1	-2
46	0	1	-1	-1	1	-2
3	0	2	-1	-1	1	-2
2	1	0	-1	1	1	-2
54	1	1	-1	-1	1	-2
2	1	2	-1	-1	1	-2
2	2	1	-1	-1	1	-2
2	3	-1	1	-1	-1	2

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
3423	0	1	-1	1	1	-2
-12	1	1	-1	1	1	2
-12	1	1	-1	3	1	-2
-3	1	-1	1	-3	-1	-2
-12	1	-1	1	-1	-1	-2
19	2	1	-1	1	1	-2
3	2	-1	1	-3	-1	2
4	0	1	1	3	1	2
-2	0	1	1	5	1	-2
-7	0	0	1	3	1	-2
-2	0	0	1	-1	1	-2
2	1	-1	-3	-1	-1	2
-123	1	-1	-1	-3	-1	2
1	1	1	1	3	1	2
-6	1	0	-1	-3	-1	2
-3	2	-1	-1	-5	-1	2
4	2	1	1	1	1	2
-4	2	1	1	3	1	-2
-2	0	1	-1	-1	1	2
7	0	1	-3	1	1	-2
4	0	0	1	-1	-1	2
-2	1	1	-1	-1	1	2
7	1	-1	1	1	-1	2
3	1	0	1	-3	-1	2
23	2	-1	1	-1	-1	2

Table VII. Development of C'_{220}

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-2	0	1	-1	5	1	-1
-15	0	1	1	-1	1	1
519	0	1	1	1	1	1
-6	0	0	1	1	1	-1
-14	1	-1	-1	-1	-1	-1
31	1	1	1	-1	1	-1
-1	1	-1	-1	1	-1	-1
-680	1	1	1	1	1	-1
-1	2	1	1	-1	1	1
2	3	1	1	-1	1	-1
-54	0	1	-1	3	1	-1
3592	0	1	-1	1	1	-1
-43	1	1	-1	1	1	1
2	1	1	-1	3	1	1
62	1	-1	1	-3	-1	1
294	1	-1	1	-1	-1	1
-3	2	1	-1	1	1	1
4	2	-1	1	-3	-1	1
16	0	1	1	3	1	1
-3	0	1	1	5	1	-1
1	0	0	1	3	1	1
-1	0	2	1	1	1	1
-5	1	0	1	1	1	-1
3	1	-1	-3	-1	-1	1
-129	1	-1	-1	-3	-1	1
4	1	1	1	3	1	1
-6	1	0	-1	-3	-1	1
-3	2	-1	-1	-5	-1	1
13	2	1	1	1	1	1
-4	2	1	1	3	1	-1
-10	3	1	1	1	1	-1
48	0	1	-1	-1	1	-1
7	0	1	-3	1	1	-1
4	0	0	1	-1	-1	1
-8	1	1	-1	-1	1	1
8	1	-1	1	1	-1	1
2	1	2	-1	-1	1	-1
3	2	1	-1	-1	1	-1
-3	2	-1	1	-1	-1	-1
811	0	2	0	2	2	1
20	0	2	0	3	2	-1
25	0	2	0	4	2	1
-15	0	2	0	0	2	-1
8	0	2	2	0	2	-1
3	0	2	2	2	2	-1

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-2	1	-1	1	-5	-1	1
99	0	1	1	-1	1	-1
-3553	0	1	1	1	1	-1
99	1	-1	-1	-1	-1	1
-5	1	1	1	-1	1	1
8	1	-1	-1	1	-1	1
99	1	1	1	1	1	1
5	2	-1	-1	-1	-1	1
10	2	1	1	-1	1	-1
8	0	1	-1	3	1	1
-525	0	1	-1	1	1	1
-6	0	2	-1	1	1	-1
297	1	1	-1	1	1	-1
-12	1	1	-1	3	1	-1
-9	1	-1	1	-3	-1	-1
-43	1	-1	1	-1	-1	-1
20	2	1	-1	1	1	-1
4	0	1	1	2	1	-1
-109	0	1	1	3	1	-1
4	0	2	1	-1	1	-1
-7	0	0	1	3	1	-1
8	0	2	1	1	1	-1
-2	1	0	1	3	1	-1
-5	1	-1	-1	-5	-1	1
19	1	-1	-1	-3	-1	-1
-26	1	1	1	3	1	-1
5	1	2	1	1	1	-1
2	2	-1	-1	-3	-1	1
-90	2	1	1	1	1	-1
1	3	1	1	1	1	1
-7	0	1	-1	-1	1	1
-1	0	1	-3	1	1	1
3	0	2	-1	-1	1	-1
2	1	0	-1	1	1	-1
56	1	1	-1	-1	1	-1
-1	1	-1	1	1	-1	-1
3	1	0	1	-3	-1	1
24	2	-1	1	-1	-1	1
2	3	-1	1	-1	-1	1
-18842	0	2	0	2	2	-1
-3	0	2	0	1	2	-1
-576	0	2	0	4	2	-1
-14	0	2	0	6	2	-1
-4	0	2	-2	4	2	-1
3	0	2	-2	2	2	1

Table VII. Development of C'_{220} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-77	0	2	-2	2	2	-1
2	0	1	0	4	2	1
-3	0	3	0	2	2	1
2	0	1	0	2	2	1
-3	0	3	0	3	2	-1
-2	0	0	0	4	2	-1
1	1	1	0	2	2	1
2	1	-3	0	-2	-2	1
-8	1	2	-2	2	2	-1
-27	1	-2	0	-6	-2	1
-6	1	2	0	0	2	1
-685	1	-2	0	-4	-2	1
-3	1	2	0	1	2	-1
-3608	1	2	0	2	2	-1
-23	1	-2	0	-2	-2	-1
6	1	2	0	4	2	1
-4	1	2	0	6	2	-1
-3	1	-1	0	-6	-2	1
-32	1	-1	0	-4	-2	1
-1	1	3	0	2	2	1
2	1	3	0	4	2	-1
-18	2	-2	0	-6	-2	1
57	2	2	0	0	2	-1
21	2	2	0	2	2	1
-23	2	2	0	4	2	-1
6	2	3	0	2	2	-1
-54	3	2	0	2	2	-1
2	4	2	0	0	2	-1
18115	0	0	0	0	0	1
-8	0	0	0	1	0	-1
246	0	0	0	2	0	-1
4	0	0	0	4	0	-1
-7	0	0	2	-2	0	-1
109	0	0	2	0	0	-1
16	0	1	0	-2	0	-1
-3	0	1	0	0	0	-1
1	0	1	0	1	0	-1
-3	0	1	0	2	0	-1
-2	1	-1	0	-2	0	-1
9	1	-1	0	0	0	-1
3	1	-1	0	2	0	-1
4	1	0	-2	-2	0	-1
-9	1	0	-2	0	0	-1

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
-2	0	1	0	6	2	-1
-39	0	1	0	4	2	-1
65	0	3	0	2	2	-1
-57	0	1	0	2	2	-1
8	0	3	0	4	2	-1
7	1	-3	0	-4	-2	1
-34	1	1	0	2	2	-1
-10	1	1	0	4	2	-1
14	1	-2	-2	-2	-2	1
1	1	-2	0	-6	-2	-1
139	1	2	0	0	2	-1
30	1	-2	0	-4	-2	-1
155	1	2	0	2	2	1
533	1	-2	0	-2	-2	1
5	1	2	0	3	2	-1
-139	1	2	0	4	2	-1
3	1	2	2	0	2	-1
7	1	3	0	0	2	-1
1	1	-1	0	-4	-2	-1
30	1	3	0	2	2	-1
-7	2	1	0	2	2	-1
-2	2	2	0	0	2	1
11	2	-2	0	-4	-2	1
-478	2	2	0	2	2	-1
2	2	3	0	0	2	-1
2	3	2	0	2	2	1
-3	3	2	0	4	2	-1
-6	4	2	0	2	2	-1
-8	0	0	0	1	0	1
246	0	0	0	2	0	1
4	0	0	0	4	0	1
-7	0	0	2	-2	0	1
109	0	0	2	0	0	1
16	0	1	0	-2	0	1
-3	0	1	0	0	0	1
1	0	1	0	1	0	1
-3	0	1	0	2	0	1
-2	1	-1	0	-2	0	1
9	1	-1	0	0	0	1
3	1	-1	0	2	0	1
4	1	0	-2	-2	0	1
-9	1	0	-2	0	0	1
8	1	0	0	-4	0	1

Table VII. Development of C'_{220} (continued)

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
8	1	0	0	−4	0	−1
283	1	0	0	−2	0	−1
1482	1	0	0	0	0	−1
−1	1	0	0	1	0	−1
39	1	0	0	2	0	−1
−2	1	0	2	−2	0	−1
12	1	1	0	−2	0	−1
−8	1	1	0	0	0	−1
2	2	−1	0	0	0	−1
5	2	0	0	−4	0	−1
13	2	0	0	−2	0	−1
121	2	0	0	0	0	−1
5	2	0	0	2	0	−1
−1	2	1	0	0	0	−1
10	3	0	0	0	0	−1
3	0	0	2	2	0	1
21	1	0	2	0	0	1
3	2	0	2	0	0	1
−6	1	−2	2	−2	−2	1
−1	1	2	−1	1	1	−1
−1	2	1	−1	3	1	−1
−1	0	0	1	−1	1	−1

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
283	1	0	0	−2	0	1
1482	1	0	0	0	0	1
−1	1	0	0	1	0	1
39	1	0	0	2	0	1
−2	1	0	2	−2	0	1
12	1	1	0	−2	0	1
−8	1	1	0	0	0	1
2	2	−1	0	0	0	1
5	2	0	0	−4	0	1
13	2	0	0	−2	0	1
121	2	0	0	0	0	1
5	2	0	0	2	0	1
−1	2	1	0	0	0	1
10	3	0	0	0	0	1
12	3	2	0	0	2	−1
3	0	0	2	2	0	−1
21	1	0	2	0	0	−1
3	2	0	2	0	0	−1
1	1	2	−1	1	1	1
1	2	1	−1	3	1	1
1	0	0	1	−1	1	1

Table VIII. Development of C''_{200}

Coef. × 10^5 of cosine	Multiples of ℓ	ℓ'	F	D	Γ	Ω
76291	0	0	0	0	0	0
23724	0	2	0	0	2	0
-2	0	3	-1	1	3	0
56	0	4	0	0	2	0
1	0	5	0	0	2	0
2	0	3	0	0	0	0
-198	0	1	0	0	2	0
2	0	1	1	-1	1	0
1389	0	3	0	0	2	0
3827	0	1	0	0	0	0
96	0	2	0	0	0	0

Development of C''_{210}

Coef. × 10^5 of cosine	ℓ	ℓ'	F	D	Γ	Ω
-1	0	1	0	0	2	2
7917	0	0	0	0	0	2
199	0	1	0	0	0	-2
5	0	2	0	0	0	-2
91851	0	2	0	0	2	-2
5376	0	3	0	0	2	-2
-3	0	2	0	-1	2	-2
8	0	5	0	0	2	-2
-8	0	3	-1	1	3	-2
-768	0	1	0	0	2	-2
199	0	1	0	0	0	2
5	0	2	0	0	0	2
170	0	2	0	0	2	2
10	0	3	0	0	2	2
3	0	2	0	1	2	-2
219	0	4	0	0	2	-2
8	0	1	1	-1	1	-2

Development of C''_{220}

Coef. × 10^5 of cosine	ℓ	ℓ'	F	D	Γ	Ω
-7	0	1	0	0	2	1
18257	0	0	0	0	0	1
458	0	1	0	0	0	-1
11	0	2	0	0	0	-1
-19057	0	2	0	0	2	-1
-1115	0	3	0	0	2	-1
-45	0	4	0	0	2	-1
-2	0	1	1	-1	1	-1
159	0	1	0	0	2	-1
458	0	1	0	0	0	1
11	0	2	0	0	0	1
820	0	2	0	0	2	1
48	0	3	0	0	2	1
2	0	4	0	0	2	1
-2	0	5	0	0	2	-1
2	0	3	-1	1	3	-1

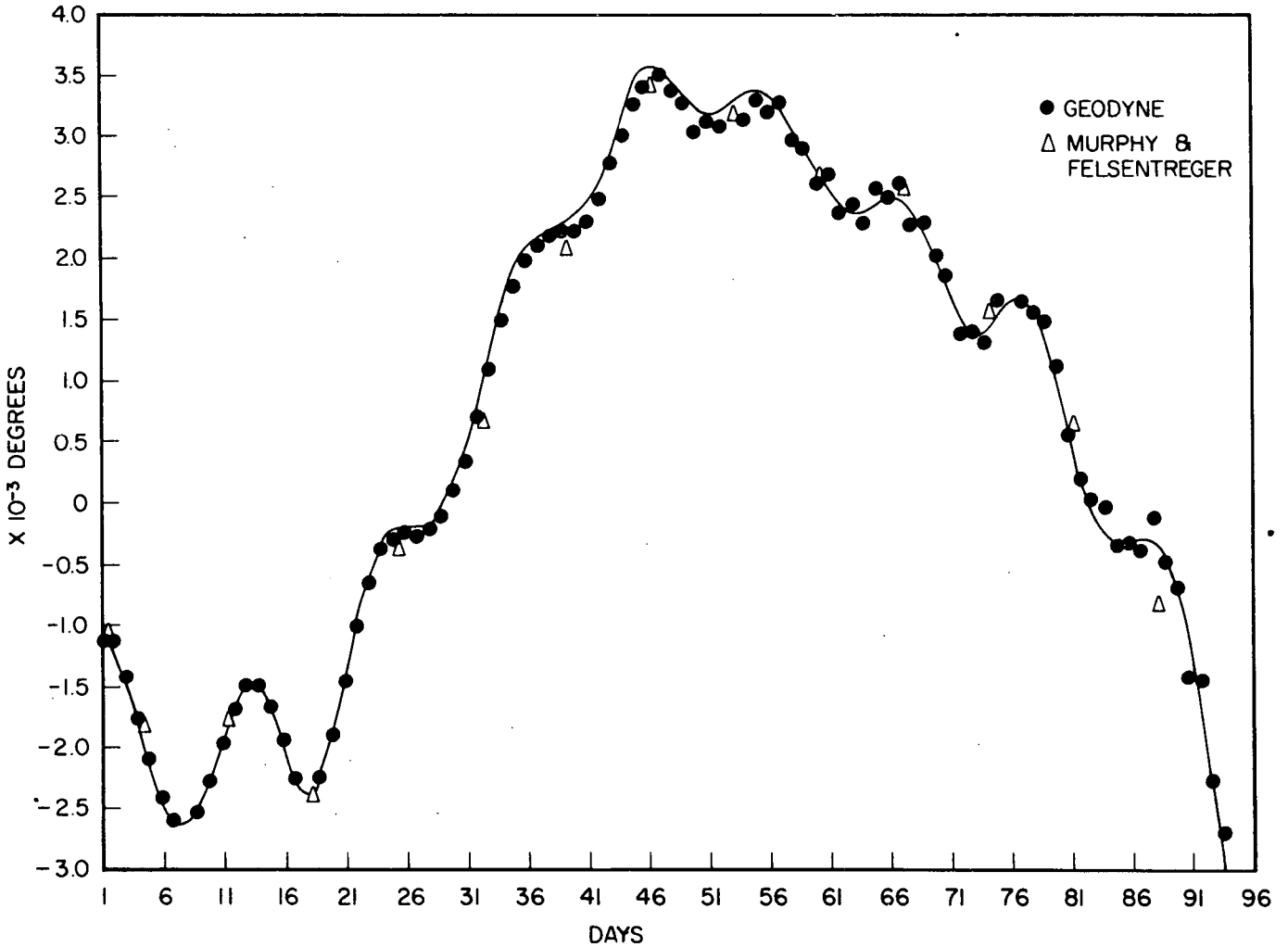

Figure 1. δi at the epoch March 17.5075, 1958 for the Vanguard I satellite.

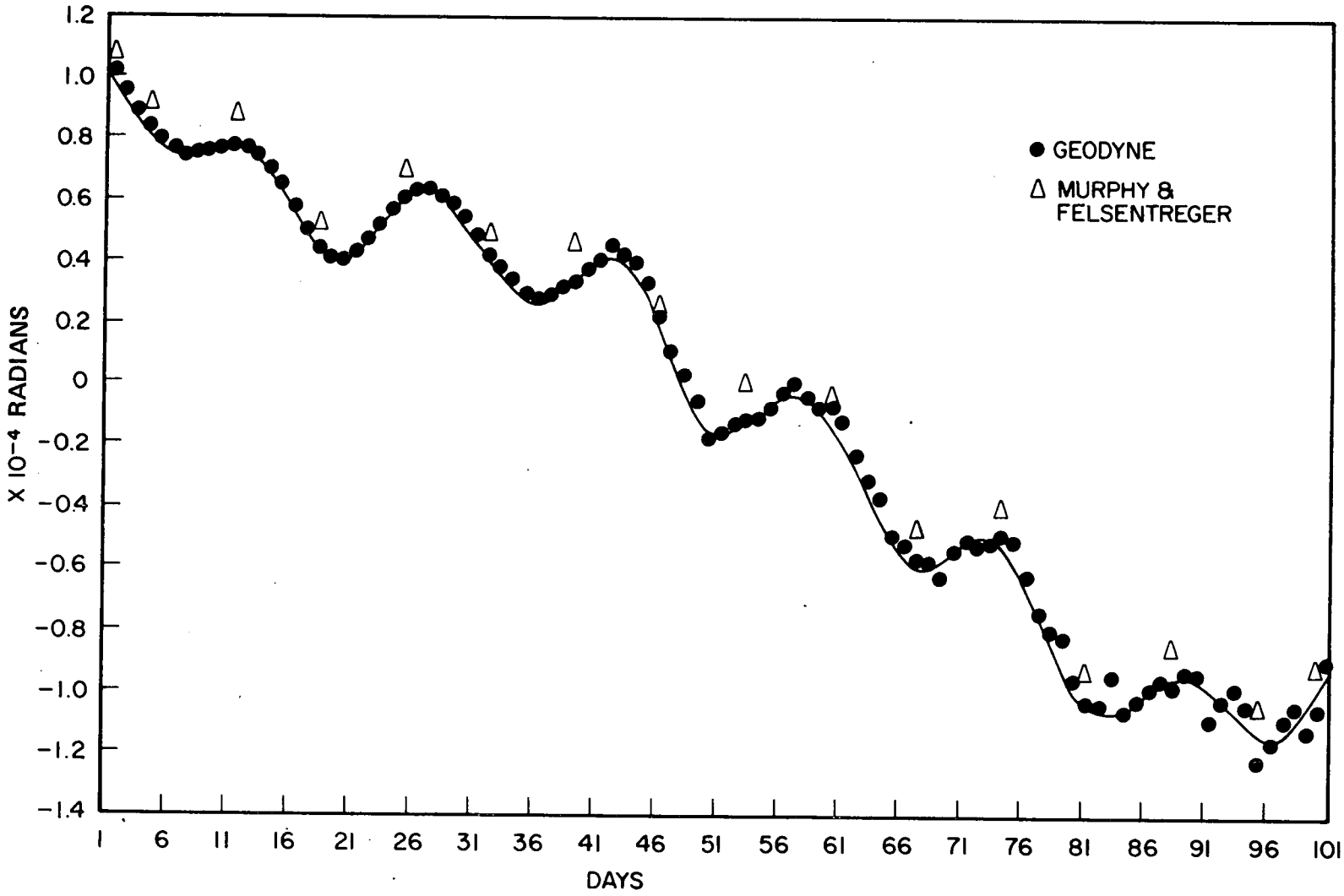

Figure 2. δe at the epoch March 17.5075, 1958 for the Vanguard I satellite.

The lunar elements used in the calculation of the perturbed orbital elements (assumed linear in the formal integration of the variation equations [6]) are given by

$$\ell = 296.^\circ 104608 + 13.^\circ 0649924465 (t - t_0)$$

$$+ 0.^\circ 0006889 ((t - t_0) \times 10^{-4})^2$$

$$\ell' = 358.^\circ 475845 + 0.^\circ 9856002670 (t - t_0)$$

$$- 0.^\circ 0000112 ((t - t_0) \times 10^{-4})^2$$

$$F = 11.^\circ 250889 + 13.^\circ 229350449 (t - t_0)$$

$$- 0.^\circ 0002407 ((t - t_0) \times 10^{-4})^2$$

$$D = 350.^\circ 737486 + 12.^\circ 1907491914 (t - t_0)$$

$$- 0.^\circ 0001076 ((t - t_0) \times 10^{-4})^2$$

$$\Gamma = 281.^\circ 220833 + 0.^\circ 0000470684 (t - t_0)$$

$$+ 0.^\circ 0000339 ((t - t_0) \times 10^{-4})^2$$

and the obliquity is given by

$$\epsilon = 23.^\circ 452294 - 0.^\circ 0035626 (t - t_0) \times 10^{-4}$$

$$- 0.^\circ 000000123 ((t - t_0) \times 10^{-4})^2.$$

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