CURRENT DISTRIBUTION ON A CYLINDRICAL ANTENNA
WITH PARALLEL ORIENTATION IN A LOSSY MAGNETOPLASMA

C. A. KLEIN, P. W. KLOCK, AND G. A. DESCHAMPS

SCIENTIFIC REPORT NO. 19
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ENGINEERING EXPERIMENT STATION
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS 61801
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ABSTRACT

The current distribution and impedance of a thin cylindrical antenna with parallel orientation to the static magnetic field of a lossy magnetoplasma is calculated with the method of moments. The electric field produced by an infinitesimal current source is first derived. Results are presented for a wide range of plasma parameters. Reasonable answers are obtained for all cases except for the overdense hyperbolic case. A discussion of the numerical stability is included which not only applies to this problem but other applications of the method of moments.
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CHAPTER I. INTRODUCTION

Since man has launched space vehicles into the ionosphere, there has been strong interest in the effect of plasmas on antennas. A primary interest is understanding the effects of the plasma on the current distribution and the antenna impedance. Since cylindrical antennas have a simple geometry, are well-understood in free space, and have wide usage, they are most commonly studied in plasma problems. Because one cannot analytically solve Maxwell's equations for a lossy magnetoplasma with cylindrical boundary conditions, a current distribution usually is assumed. This assumed current distribution, though, represents only the limiting case of an infinitesimal antenna. Approximations made by other workers give results valid for infinite length antennas. These approximations simplify the analytical work but give results of questionable validity for the finite antenna.

The approach used in this report eliminates these approximations. The method of moments is used to solve for current distribution on the antenna. Basically, this method converts an integral equation into a matrix equation which can be easily solved by computer methods. From the current distribution one can trivially find the impedance at the feed point. Thus, the current distribution and impedance for the finite antenna over a wide range of plasma densities and applied magnetic fields can be calculated.

The geometry used in this problem is shown in Figure 1. The orientation of the antenna is restricted to the z direction parallel to the applied magnetic field. The antenna is described by the half-length H and radius a. For a thin-wire antenna, the condition a << H.
Odd \( N \) number of segments

Figure 1. Geometry of Problem
must hold. For the method of moments, the antenna is divided into an odd number of segments $N$.

The first step in applying the method of moments is solving for the electric field due to an infinitesimal source. Exact details of how the method of moments is applied will be discussed. Results are given for a wide range of plasma parameters and are compared to the limiting cases of free space and quasi-static models. Comparisons are also made to experimental data. A sensitivity theory is discussed which applies not only to this problem but to many other applications of the moment method.
CHAPTER II. FIELDS DUE TO INFINITESIMAL SOURCE

2.0 An Overview

To determine the current distribution along an antenna it is first necessary to find the fields produced by an infinitesimal current source. The derivation given here used the notation and technique developed by Mittra and Deschamps [1]. In their paper they outline the derivation of the field components for any orientation of an infinitesimal source. The parallel orientation case worked out in detail here, however, is not the same as the specific case completed in their paper.

In this report the plasma will be characterized by normalized plasma parameters $X$, $Y$, and $Z$ where

$$X = \left(\frac{\omega_p}{\omega}\right)^2, \quad Y^2 = \left(\frac{\omega_c}{\omega}\right)^2, \quad Z = \frac{\nu}{\omega}$$

$$\omega_p = \left[\frac{N_e}{\omega_o m e_o}\right]^{1/2} = \text{plasma frequency}, \quad \omega_c = \frac{eB_o}{m} = \text{cyclotron frequency}$$

$$\omega = \text{operating frequency}$$

$$\nu = \text{collision frequency}$$

$N_o$ = average electron density

$e$ = magnitude of electronic charge

$m$ = mass of electron

$B_o$ = applied magnetic field.

The use of $X$, $Y$, and $Z$ parameters allows one to indicate results on commonly used plasma diagrams.
Before giving the lengthy details, we wish to give an overview of the derivation. Maxwell's equations can be combined to yield a relation for the electric field caused by an infinitesimal source in the plasma. Taking the Fourier transform allows one to easily solve for the transform of the z component of the electric field. The main feature of this technique is that in performing the inverse Fourier transform, the integral can be broken into a sum of functions such that one can analytically integrate terms that are singular when observation and source points coincide. These terms represent the near field and are dominant in the determination of the antenna current and impedance. Terms that are finite when observation and source points coincide need to be numerically integrated, but these represent far-field terms and need not be as accurate because of their small relative size. The rest of this chapter gives the details; the final expressions for the field due to the infinitesimal source are given by Equations (2.80) through (2.88).

2.1 Maxwell's Equations

Using \( e^{j\omega t} \) time convention, Maxwell's equations for the plasma are:

\[
\text{curl } \overline{E} = -j\omega \mu \overline{H} \tag{2.2a}
\]
\[
\text{curl } \overline{H} = j\omega \varepsilon \overline{\varepsilon} \cdot \overline{E} + \overline{J} \tag{2.2b}
\]
\[
\text{div } \overline{H} = 0 \tag{2.2c}
\]
\[
\text{div } \overline{\varepsilon} \cdot \overline{E} = \rho / \varepsilon_0 \tag{2.2d}
\]

where
When (2.2a) and (2.2b) are combined, the following equation is obtained:

\[ \text{curl curl } \bar{E} = k_o^2 \bar{\varepsilon} \cdot \bar{E} - j \omega \mu \bar{J}, \quad k_o^2 = \omega^2 \mu \varepsilon_o. \quad (2.4) \]

At this point it becomes convenient to transform the problem into \( k \)-space by applying the Fourier transform. Let \( \bar{E}' \) and \( \bar{J}' \) represent the transforms of \( \bar{E} \) and \( \bar{J} \). The transform \( E'_x \) of \( E_x \) is defined by

\[ E'_x(k_x, k_y, k_z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_x(x, y, z) e^{-j(k_x x + k_y y + k_z z)} \, dx \, dy \, dz \quad (2.5) \]

with the transforms of other components defined similarly. Note that the del operator can be replaced by \( jk \). Then, using a vector identity, the transform of (2.4) can be written as

\[ \bar{k}(\bar{k} \cdot \bar{E}') - \bar{k} \cdot \bar{k} \bar{E}' + k_o^2 \bar{\varepsilon} \bar{E}' = j \omega \mu \bar{J}'. \quad (2.6) \]

Next one can expand this equation by components and write it in matrix
Because of the radial symmetry about the z axis, $\mathbf{k}$ can be represented by spherical coordinates $(\Gamma, \Psi, \alpha)$,

$$
k_x = \Gamma \sin \Psi \cos \alpha, \quad k_y = \Gamma \sin \Psi \sin \alpha, \quad k_z = \Gamma \cos \Psi. \quad (2.8)
$$

Making these changes and collecting terms, we obtain

$$
\overline{\mathbf{M}} \overline{\mathbf{E}}' = j \frac{n}{\kappa_0} \overline{\mathbf{J}}',
$$

where

$$
\overline{\mathbf{M}} = \begin{bmatrix}
\epsilon - \Gamma_1^2 (1 - \sin^2 \Psi \cos^2 \alpha) & -j \epsilon' + \Gamma_1^2 \sin^2 \Psi \sin \alpha \cos \alpha & \Gamma_1^2 \sin \Psi \cos \Psi \cos \alpha \\
-j \epsilon' + \Gamma_1^2 \sin^2 \Psi \sin \alpha \cos \alpha & \epsilon - \Gamma_1^2 (1 - \sin^2 \Psi \sin^2 \alpha) & \Gamma_1^2 \sin \Psi \cos \Psi \sin \alpha \\
\Gamma_1^2 \sin \Psi \cos \Psi \cos \alpha & \Gamma_1^2 \sin \Psi \cos \Psi \sin \alpha & \epsilon_z - \Gamma_1^2 \sin^2 \Psi
\end{bmatrix}
$$

and $\Gamma_1 = -\Gamma/k_0$.

To solve for $\overline{\mathbf{E}}'$ we can write

$$
\overline{\mathbf{E}}' = \overline{\mathbf{M}}^{-1} j \frac{n}{\kappa_0} \overline{\mathbf{J}}'.
$$

Finding $\overline{\mathbf{M}}^{-1}$ is a straightforward but lengthy algebraic manipulation. One finally obtains
\[ M^{-1} = \frac{[a_{ij}]}{\Delta} \quad (2.11) \]

\[ \Delta = \det(M) = \Gamma_1^4 (\varepsilon \sin^2 \psi + \varepsilon_z \cos^2 \psi) - \Gamma_1^2 [\varepsilon \varepsilon_z (1 + \cos^2 \psi) + (\varepsilon^2 - \varepsilon_z^2) \sin^2 \psi] + (\varepsilon^2 - \varepsilon_z^2) \varepsilon_z \]

\[ a_{11} = \Gamma_1^4 \sin^2 \psi \cos^2 \alpha - \Gamma_1^2 [\varepsilon \sin^2 \psi + \varepsilon_z (1 - \sin^2 \psi \sin^2 \alpha)] + \varepsilon \varepsilon_z \]

\[ a_{21} = \Gamma_1^4 \sin^2 \psi \sin \alpha \cos \alpha - \Gamma_1^2 \sin^2 \psi (\varepsilon_z \sin \alpha \cos \alpha - j\varepsilon') - j\varepsilon' \varepsilon_z \]

\[ a_{31} = \Gamma_1^2 \sin \psi \cos \psi [\Gamma_1^2 \cos \alpha - (\varepsilon \cos \alpha - j\varepsilon' \sin \alpha)] \]

\[ a_{12} = \Gamma_1^4 \sin^2 \psi \sin \alpha \cos \alpha - \Gamma_1^2 \sin^2 \psi (\varepsilon_z \sin \alpha \cos \alpha + j\varepsilon') + j\varepsilon' \varepsilon_z \]

\[ a_{22} = \Gamma_1^4 \sin^2 \psi \sin^2 \alpha - \Gamma_1^2 [\varepsilon \sin^2 \psi + \varepsilon_z (1 - \sin^2 \psi \cos^2 \alpha)] + \varepsilon \varepsilon_z \]

\[ a_{32} = \Gamma_1^2 \sin \psi \cos \psi [\Gamma_1^2 \sin \alpha - (\varepsilon \sin \alpha + j\varepsilon' \cos \alpha)] \]

\[ a_{13} = \Gamma_1^2 \sin \psi \cos \psi [\Gamma_1^2 \cos \alpha - (\varepsilon \cos \alpha + j\varepsilon' \sin \alpha)] \]

\[ a_{23} = \Gamma_1^2 \sin \psi \cos \psi [\Gamma_1^2 \sin \alpha - (\varepsilon \sin \alpha - j\varepsilon' \cos \alpha)] \]

\[ a_{33} = \Gamma_1^4 \cos^2 \psi - \Gamma_1^2 \varepsilon (1 + \cos^2 \psi) + \varepsilon^2 - \varepsilon_z^2 \]

where \( \Gamma_1 = \Gamma/k_0 \).

It should be noted that \( \det(M) = 0 \) is the dispersion relation for the plasma. Since, for a fixed polar angle \( \psi \), this is a second-degree polynomial in \( \Gamma_1^2 \), there will be exactly two roots. These two roots are the squares of the indices of refraction for two characteristic plane waves traveling at an angle \( \psi \) to the magnetic field. The proper square root must be taken to satisfy the radiation condition. One can appreciate
the conceptual difficulties inherent in understanding plasma problems since unlike free space, there is not even a unique wavelength in a given direction. The exact form of the indices of refraction will be given shortly.

To solve our problem, only one of the nine terms of (2.11) will be needed. For parallel orientation of the antenna, there is only source current in the z direction. Only $E'_z$ is constrained by the boundary condition. Therefore, the rest of this chapter is directed to finding the inverse transform of

$$E'_z = (a_{33}/\Delta) \frac{i\hbar}{k_0} J'_z. \quad (2.12)$$

### 2.2 Inverse Transform

Conceptually all that remains is to perform the inverse transform

$$E_z(x, y, z) = \frac{i\hbar}{k_0} \left( \frac{1}{2\pi} \right)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{a_{33}}{\Delta} \right) e^{-jk \cdot r} \, dk. \quad (2.13)$$

Recall that $J'_z$ is unity since our problem is formulated for a Dirac $\delta$ source. If spherical coordinates are also used for the observation point, (2.13) becomes

$$E_z(R, \theta, \phi) = \frac{i\hbar}{k_0} \left( \frac{1}{2\pi} \right)^3 \int_0^{2\pi} \int_0^\pi \int_0^\pi \left( \frac{a_{33}}{\Delta} \right) e^{-j\Gamma R [\sin \theta \sin \psi \cos(\alpha - \phi) + \cos \theta \cos \psi]}$$

$$\cdot \Gamma^2 \sin \psi \, d\alpha \, d\psi \, d\Gamma$$

where $x = R \sin \theta \cos \psi$, $y = R \sin \theta \sin \phi$, $z = R \cos \theta$.

First we can relatively easily perform the $\alpha$ integration using the following standard integration formula [2].
\[ J_0(\Gamma R \sin \theta \sin \psi) = \frac{1}{2\pi} \int_0^{2\pi} e^{-j\Gamma R \sin \theta \sin \psi \cos(\alpha-\phi)} d\alpha. \]  

(2.15)

Applying (2.15) to (2.14) produces

\[ E_z = \frac{1}{k_0} \left( \frac{1}{2\pi} \right)^2 \int_0^{\pi} \int_0^{\pi} (\frac{a_{33}}{\Delta}) J_0(\Gamma R \sin \theta \sin \psi) e^{-j\Gamma \cos \theta \cos \psi} \Gamma^2 \sin \psi \, d\psi \, d\Gamma \]  

\[ \frac{a_{33}}{\Delta} = \frac{\Gamma_1^4 \cos^2 \psi - \Gamma_1^2 \epsilon (1 + \cos^2 \psi) + \epsilon^2 - \epsilon'^2}{\Gamma_1^4 (\epsilon \sin^2 \psi + \epsilon_z \cos^2 \psi) - \Gamma_1^2 [\epsilon \epsilon_z (1 + \cos^2 \psi) + (\epsilon^2 - \epsilon'^2 \sin^2 \psi)] + (\epsilon^2 - \epsilon'^2) \epsilon_z} \]

(2.16)

\[ \Gamma_1 = \Gamma/k_0. \]

At this point it is convenient to write \( \Delta \) in factored form.

\[ \Delta(\Gamma, \psi) = (\epsilon \sin^2 \psi + \epsilon_z \cos^2 \psi)(\Gamma_1^2 - n_1^2)(\Gamma_1^2 - n_2^2) \]  

(2.17)

where

\[ n_1^2, n_2^2 = \frac{B(\psi) \pm \sqrt{B^2(\psi) - 4A(\psi) C(\psi)}}{2A(\psi)} \]

\[ A(\psi) = \epsilon \sin^2 \psi + \epsilon_z \cos^2 \psi \]

\[ B(\psi) = \epsilon \epsilon_z (1 + \cos^2 \psi) + (\epsilon^2 - \epsilon'^2 \sin^2 \psi) \]

\[ C(\psi) = (\epsilon^2 - \epsilon'^2) \epsilon_z. \]

Recall \( n_1 \) and \( n_2 \) represent indices of refraction in the plasma mentioned earlier.

The ability to do the integration in (2.16) depends upon it being possible to break \( \frac{a_{33}}{\Delta} \) into a sum of functions with different dominant orders of \( \Gamma \). As mentioned in the overview, it will be shown that those terms which are singular when observation and source point coincide can
be integrated in closed form. These represent near-field terms and will be dominant in this problem. Those terms that are finite when observation and source points coincide will need to be integrated numerically but will be small compared to singular terms.

By successive synthetic division one can show

$$\frac{a_{33}}{\Delta} \cos^2 \psi, \frac{1}{A(\psi)} + \frac{k_o^2(\varepsilon^2 + \varepsilon'^2 \cos^2 \psi)}{A^2(\psi)} \sin^2 \psi \quad (2.18)$$

$$-k_o^2 \sin^2 \psi \frac{\{k_o^2[\varepsilon(\varepsilon^2 - \varepsilon'^2) - (n_1^2 + n_2^2)(\varepsilon^2 + \varepsilon'^2 \cos^2 \psi)] + n_2^2(\varepsilon^2 + \varepsilon'^2 \cos^2 \psi)k_o^4\}}{(\Gamma^2 - n_1^2k_o^2)(\Gamma^2 - n_2^2k_o^2)A^2(\psi)}$$

where $A(\psi) = \varepsilon \sin^2 \psi + \varepsilon_z \cos^2 \psi$.

Let the various parts of the $z$ component of the field be defined as follows

$$E_{zs} = \frac{jn}{k_o} \cos^2 \psi, \frac{1}{2\pi} \int_0^\pi \int_0^{2\pi} \cos^2 \psi \frac{1}{\varepsilon \sin^2 \psi + \varepsilon_z \cos^2 \psi} J_o(\Gamma R \sin \theta \sin \psi) d\psi d\Gamma$$

$$E_{zfs} = -jn \frac{k_o}{k_o} \left( \frac{1}{2\pi} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{\cos^2 \psi \sin^2 \psi}{A^2(\psi)} e^{-j\Gamma R \cos \theta \cos \psi} \sin \psi d\psi d\Gamma$$

$$E_{zf} = jn \frac{k_o}{k_o} \left( \frac{1}{2\pi} \right)^2 \int_0^\pi \int_0^{2\pi} \frac{\sin^2 \psi \{k_o^2[\varepsilon(\varepsilon^2 - \varepsilon'^2) - (n_1^2 + n_2^2)(\varepsilon^2 + \varepsilon'^2 \cos^2 \psi)] + n_2^2(\varepsilon^2 + \varepsilon'^2 \cos^2 \psi)k_o^4\}}{(\Gamma^2 - n_1^2k_o^2)(\Gamma^2 - n_2^2k_o^2)A^2(\psi)} e^{-j\Gamma R \cos \theta \cos \psi} \sin \psi d\psi d\Gamma$$

The total field is given by

$$E_z = E_{zs} + E_{zfs} + E_{zf} \quad (2.22)$$
When the total field is decomposed in this way, \( E_{zs} \) represents the most singular term in \( R \), \( E_{zs} \) the next most singular term, and \( E_{zf} \) represents a finite term. What remains now is the evaluation of each integral separately.

2.3 Integrating \( E_{zs} \)

The convergence of (2.19), the equation which defines \( E_{zs} \), at first appears to be questionable. One can note that for a three-dimensional Fourier transform, \( \nabla^2 \) with respect to the observation coordinates \( R \) and \( \theta \), can be replaced by \(-\Gamma^2\). Therefore, (2.19) can be written as

\[
E_{zs} = \frac{-\text{in}}{k_0} \left( \frac{1}{2\pi} \right)^2 \nabla^2 I_s(R, \theta) \tag{2.23}
\]

\[
\nabla^2 I_s(R, \theta) = -\int_0^\infty \int_0^\pi \frac{\Gamma^2 \cos^2 \psi \sin \psi}{\varepsilon \sin^2 \psi + \varepsilon z \cos^2 \psi} e^{-j\Gamma R \cos \psi \cos \theta} J_0(\Gamma R \sin \psi \sin \theta) d\theta d\psi \tag{2.24}
\]

where the order of integration and differentiation has been formally interchanged.

Suppose we make the following change of variables to cylindrical coordinates.

\[
z = R \cos \theta, \quad \rho = R \sin \theta
\]

\[
\gamma = \Gamma \cos \psi, \quad u = \Gamma \sin \psi.
\]

Then we can write (2.24)

\[
\nabla^2 I_s(R, \theta) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \int_0^\infty \int_0^\pi \frac{\cos^2 \psi / \sin \psi}{\varepsilon \sin^2 \psi + \varepsilon z \cos^2 \psi} e^{-j\gamma z \frac{J_0(\rho \psi)}{\psi}} d\psi d\Gamma \tag{2.26}
\]

where Bessel's equation has been used to explicitly perform differentiation with respect to \( \rho \) outside the integral.
Completing the change of variable gives

$$\nabla^2 I_s(\rho, z) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \int_0^\infty \int_{\gamma=-\infty}^{\infty} \frac{\gamma^2 e^{-j\gamma z}}{u(ue^2 + \varepsilon z^2)} J_0(u\rho) \, du \, d\gamma. \tag{2.27}$$

First, to perform the $\gamma$ integration, we will perform the integral

$$J = \int_{-\infty}^{\infty} \frac{\gamma^2 e^{-j\gamma z}}{ue^2 + \varepsilon z^2} \, d\gamma. \tag{2.28}$$

At this point we can explicitly change the order of differentiation with respect to $z$ and the integration, thus completing the formal interchange indicated earlier.

$$J = -\frac{3}{z^2} \int_{-\infty}^{\infty} \frac{e^{-j\gamma z}}{u e^2 + \varepsilon z^2} \, d\gamma. \tag{2.29}$$

At last we have an integration that can be performed with the aid of the residue theorem of complex variables. The integrand in Equation (2.29) is of exponential order; and therefore, for positive $z$ we may close the integration path over the lower-half complex $\gamma$ plane. There is only one simple pole at

$$\gamma = -j \sqrt{\varepsilon/\varepsilon_z} u.$$

One obtains

$$J = -\frac{3}{z^2} \frac{\pi}{\sqrt{\varepsilon z}} \frac{e^{-j\sqrt{\varepsilon/\varepsilon_z} z u}}{\sqrt{\varepsilon u}}. \tag{2.30}$$

We substitute into (2.27) and perform differentiation with respect to $z$

$$\nabla^2 I_s = \frac{-\pi \sqrt{\varepsilon}}{\sqrt{\varepsilon z}} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \int_0^\infty e^{-j\sqrt{\varepsilon/\varepsilon_z} z u} J_0(u\rho) \, du. \tag{2.31}$$

This final integration for $E_{zs}$ can be performed to obtain
where the following definite integral has been used [3].

\[
\int_0^\infty e^{-at} J_0(bt) \, dt = \frac{1}{\sqrt{a^2 + b^2}}.
\] (2.33)

Finally

\[
E_{zs} = \frac{\ln \sqrt{\varepsilon}}{4\pi k_0 \varepsilon_z} \frac{1}{\rho} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \rho} \frac{1}{\sqrt{\varepsilon_z^2 + \varepsilon_z^\prime 2}}.
\] (2.34)

or

\[
E_{zs} = \frac{-\ln \sqrt{\varepsilon}}{4\pi k_0 \varepsilon_z} \frac{1}{(\varepsilon_z^2 + \varepsilon_z^\prime 2)^{3/2}} \left[ \frac{2\varepsilon z^2 - \varepsilon z^\prime 2}{\varepsilon z^2 + \varepsilon z^\prime 2} \right].
\] (2.35)

The dependence on distance can be seen more clearly by noting that \(R' = \sqrt{\varepsilon \varepsilon z^2 + \varepsilon_z^\prime 2} \rho^2\) is used as a plasma-scaled distance in some statics problems. Clearly

\[
E_{zs} = \frac{-\ln \varepsilon}{4\pi k_0 \varepsilon_z} \cdot \frac{1}{R'^3} \cdot \left[ \frac{2\varepsilon z^2 - \varepsilon z^\prime 2}{\varepsilon z^2 + \varepsilon z^\prime 2} \right]. \quad (2.35a)
\]

**2.4 Integrating \(E_{zfs}\)**

We repeat (2.20), which defines \(E_{zfs}\),

\[
E_{zfs} = -jnk_0 \left( \frac{1}{2\pi} \right)^2 \int_0^\infty \int_0^\pi \frac{(\varepsilon^2 + \varepsilon z^2 \cos^2 \psi) \sin^3 \psi}{A^2(\psi)} \, e^{-jR' \cos \theta \cos \psi} \, J_0(R' \sin \theta \sin \psi) \, d\Gamma d\psi
\]

\[
A(\psi) = \varepsilon \sin^2 \psi + \varepsilon_z \cos^2 \psi.
\]
This term will be artificially divided into two terms \( E_{zs1} \) and \( E_{zs2} \) for ease of integration.

\[
E_{zs1} = -jnk_0 \left( \frac{1}{2\pi} \right)^2 \varepsilon^2 I_{fs1} \tag{2.36}
\]

where

\[
I_{fs1} = \int_0^\infty \int_0^\pi \frac{\sin^3 \psi}{A^2(\psi)} e^{-jR \cos \theta \cos \psi} J_0(\Gamma R \sin \theta \sin \psi) d\psi d\Gamma. \tag{2.37}
\]

\[
E_{zs2} = -jnk_0 \left( \frac{1}{2\pi} \right)^2 \varepsilon^2 I_{fs2} \tag{2.38}
\]

where

\[
I_{fs2} = \int_0^\infty \int_0^\pi \frac{\cos^2 \psi \sin^3 \psi}{A^2(\psi)} e^{-jR \cos \theta \cos \psi} J_0(\Gamma R \sin \theta \sin \psi) d\psi d\Gamma. \tag{2.39}
\]

Again it will be useful to apply the same change of variables used for \( E_{zs1} \).

\[
z = R \cos \theta \quad \rho = R \sin \theta \quad \gamma = \Gamma \cos \psi \quad u = \Gamma \sin \psi. \tag{2.40}
\]

Applying these changes, (2.37) becomes

\[
I_{fs1} = \int_{u=0}^\infty \int_{\gamma=-\infty}^{\gamma=\infty} \frac{u^3}{(\varepsilon u^2 + \varepsilon z \gamma^2)^2} e^{-j\gamma z} J_0(\rho u) d\gamma du. \tag{2.41}
\]

To do the \( \gamma \) integration we need to evaluate

\[
J = \int_{-\infty}^{\infty} \frac{e^{-j\gamma z}}{(\varepsilon u^2 + \varepsilon z \gamma^2)^2} d\gamma. \tag{2.42}
\]

This integration can be done with the aid of the residue theorem. One can close the integration path in the lower-half complex \( \gamma \) plane and evaluate the residue at the second-order pole, \( \gamma = -j\sqrt{\varepsilon/\varepsilon_z} u \), obtaining
Equation (2.41) becomes

\[ I_{\text{fsl}} = \frac{\pi}{2\epsilon \varepsilon_z} \int_0^\infty \left( uz + \frac{1}{\sqrt{\epsilon/\varepsilon_z}} \right) e^{-\sqrt{\epsilon/\varepsilon_z} zu} J_0(\rho u) \, du \]  

(2.44)

or

\[ I_{\text{fsl}} = \frac{\pi}{2\epsilon \varepsilon_z} \left[ \frac{-z}{\sqrt{\epsilon/\varepsilon_z}} \frac{\partial}{\partial z} + \frac{1}{\sqrt{\epsilon/\varepsilon_z}} \right] \int_0^\infty e^{-\sqrt{\epsilon/\varepsilon_z} zu} J_0(\rho u) \, du \]  

(2.45)

by recognizing the integral as the derivative of a different function.

Applying the integration formula

\[ \int_0^\infty e^{-au} J_0(\rho u) \, du = \frac{1}{\sqrt{\rho^2 + a^2}} \]  

(2.46)

obtained from Dwight [4], (2.45) may be written

\[ I_{\text{fsl}} = \frac{\pi}{2\epsilon \sqrt{\varepsilon_z}} \left[ \frac{z}{\sqrt{\varepsilon_z}} \frac{\partial}{\partial z} + \frac{1}{\sqrt{\varepsilon_z}} \right] \frac{1}{\sqrt{\varepsilon_z} \rho^2 + a^2}. \]  

(2.47)

Performing the indicated differentiation, \( E_{z\text{fsl}} \) may be finally written as

\[ E_{z\text{fsl}} = \frac{-jnk_0}{8\pi\sqrt{\varepsilon_z}} \frac{1}{(\varepsilon_z \rho^2 + \varepsilon_z z^2)^{1/2}} \left[ 1 + \frac{\varepsilon_z^2}{\varepsilon_z \rho^2 + \varepsilon_z z^2} \right]. \]  

(2.48)

or

\[ E_{z\text{fsl}} = \frac{-jnk_0}{8\pi} \cdot \frac{1}{R' \cdot \varepsilon_z} \left[ 1 + \frac{\varepsilon_z^2}{\varepsilon_z \rho^2 + \varepsilon_z z^2} \right]. \]  

(2.48a)

The integration for \( E_{z\text{fsl}} \) will be done similarly. Using the same change of variables,
$I_{fs2} = \int_{u=0}^{\infty} \int_{\gamma=-\infty}^{\infty} \frac{\gamma^2 u^3}{(\epsilon u^2 + \epsilon z^2)^2 (u^2 + \gamma^2)} e^{-j\gamma z} J_0(\rho u) \, d\gamma \, du. \quad (2.49)$

Again the $\gamma$ integration is done first

$$J = \int_{-\infty}^{\infty} \frac{2 e^{-j\gamma z} \, d\gamma}{(\epsilon u^2 + \epsilon z^2)^2 (u^2 + \gamma^2)}. \quad (2.50)$$

This integration can be performed with the aid of the residue theorem.

Note in this case that there will be a pole at $\gamma = -u$ that did not occur for $E_{zfs1}$. One obtains the following equation

$$J = \frac{-\pi}{\epsilon z (\epsilon z - \epsilon) u^3} \left\{ \frac{\epsilon e^{-u z}}{\epsilon z - \epsilon} + e^{\epsilon z} \frac{u z}{2 \sqrt{\epsilon z} (\epsilon z - \epsilon)} \right\}. \quad (2.51)$$

The final integration of $E_{zfs2}$ uses the exact methods as $E_{zfs1}$ did. Differentiation with respect to $z$ is moved outside the integral, the integration formula (2.46) is applied, and differentiation with respect to $z$ is performed. After simplification one obtains

$$E_{zfs2} = \frac{j n k_0}{4\pi} \frac{\epsilon^2}{(\epsilon z - \epsilon)} \left\{ \frac{1}{(\epsilon z - \epsilon)^2} - \frac{1}{2\sqrt{\epsilon z} \rho^2 + \epsilon z^2} - \frac{\epsilon z + \epsilon}{\epsilon z \rho^2 + \epsilon z^2} \right\} \quad (2.52)$$

or

$$E_{zfs2} = \frac{j n k_0}{4\pi} \frac{\epsilon^2}{(\epsilon z - \epsilon)} \left\{ \frac{1}{(\epsilon z - \epsilon)^2} - \frac{1}{2\sqrt{\epsilon z} \rho^2 + \epsilon z^2} - \frac{\epsilon z + \epsilon}{\epsilon z \rho^2 + \epsilon z^2} \right\} \quad (2.52a)$$
Equation (2.21), which defines $E_{zf}$, can be written

$$E_{zf} = j n k_o^3 \left( \frac{1}{2 \pi} \right)^2 P(R, \theta) \tag{2.53}$$

$$P(R, \theta) = \int_0^{\pi} \int_0^\infty N(\Gamma, \psi)e^{-j\Gamma R \cos \theta \cos \psi} \frac{J_0(\Gamma R \sin \theta \sin \psi)}{(\Gamma^2 - n_1^2 k_o^2)(\Gamma^2 - n_2^2 k_o^2)} \cdot \sin^3 \psi \, d\Gamma d\psi \tag{2.54}$$

$$N(\Gamma, \psi) = \Gamma^2[\epsilon(\epsilon^2 - \epsilon'^2) - (n_1^2 + n_2^2)(\epsilon^2 + \epsilon'^2 \cos^2 \psi)]$$

$$+ n_1^2 n_2^2 (\epsilon^2 + \epsilon'^2 \cos^2 \psi) k_o^2 \tag{2.55}$$

Note that the integrand of (2.54) is even in $\psi$. Therefore, the limits of integration can be changed

$$P(R, \theta) = \int_0^{\pi/2} \int_0^\infty N(\Gamma, \psi) \sin^3 \psi e^{-j\Gamma R p} J_0(\Gamma R q) \frac{d\Gamma d\psi}{A^2(\psi)(\Gamma^2 - n_1^2 k_o^2)(\Gamma^2 - n_2^2 k_o^2)} \tag{2.56}$$

where we have also substituted

$$p = \cos \theta \cos \psi \tag{2.57}$$

$$q = \sin \theta \sin \psi$$

The Bessel function can be changed to a more tractable form by using the integral definition

$$J_0(\Gamma R q) = \frac{1}{\pi} \int_0^{\pi/2} \left[ e^{-j\Gamma R q \cos \alpha} + e^{j\Gamma R q \cos \alpha} \right] d\alpha. \tag{2.58}$$

Making this substitution and changing the order of integrations, we can organize $P(R, \theta)$ as
\[ P(R, \theta) = \frac{1}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \left( I_1 + I_2 \right) \frac{\sin^3 \psi}{A^2(\psi)} \, d\psi \, d\theta \]  
(2.59)

where

\[ I_1 = \int_{-\infty}^{\infty} \frac{N(\Gamma, \psi) e^{-jR(p + q \cos \alpha)}}{(R^2 - n_1^2k_o^2)(R^2 - n_2^2k_o^2)} \, d\Gamma \]  
(2.60)

\[ I_2 = \int_{-\infty}^{\infty} \frac{N(\Gamma, \psi) e^{-jR(p - q \cos \alpha)}}{(R^2 - n_1^2k_o^2)(R^2 - n_2^2k_o^2)} \, d\Gamma . \]  
(2.61)

The next step is obviously integration by the residue method. However, one must take care to consider the signs of \((p + q \cos \alpha)\) and \((p - q \cos \alpha)\) for various ranges of \(\psi\) and \(\theta\). Also one must remember that the square roots of \(n_1^2\) and \(n_2^2\) must be taken so that \(n_1\) and \(n_2\) have negative imaginary parts to satisfy the radiation condition.

For the range of the angles permitted by (2.59) for \(I_1\), it can easily be seen \(p + q \cos \alpha > 0\). Therefore, the integration path can be closed in the lower-half \(\Gamma\) plane to obtain

\[ I_1 = -2\pi j \frac{k_o^3}{k_o} \left[ \frac{N(n_1k_o, \psi) e^{-jn_1k_o R(p + q \cos \alpha)}}{2n_1(n_1^2 - n_2^2)} + \frac{N(n_2k_o, \psi) e^{-jn_2k_o R(p + q \cos \alpha)}}{-2n_2(n_1^2 - n_2^2)} \right]. \]  
(2.62)

For \(I_2\), the sign of \((p - q \cos \alpha)\) depends on the exact range of \(\theta\), \(\psi\), and \(\alpha\). Recalling (2.57)

\[ p = \cos \theta \cos \psi \]  
(2.57)

\[ q = \sin \theta \sin \psi, \]

one can see
\[ p - q \cos \alpha > 0 \] whenever either \( 0 < \psi < \pi/2 - \theta \) and \( 0 < \alpha < \pi/2 \) or whenever \( \pi/2 - \theta < \psi < \pi/2 \), and \( \cos^{-1} \frac{p}{q} < \alpha < \pi/2 \).

(2.63)

In this case the integration path can also be closed in the lower-half \( \Gamma \) plane to obtain

\[
I_2 = -\frac{2\pi i}{k_o} \left[ \frac{N(n_{1k}, \psi) e^{-jn_{1k} R (p - q \cos \alpha)}}{2n_1(n_1^2 - n_2^2)} + \frac{N(n_{2k}, \psi) e^{-jn_{2k} R (p - q \cos \alpha)}}{-2n_2(n_1^2 - n_2^2)} \right].
\]

(2.64)

However,

\[ p - q \cos \alpha < 0 \] for \( \frac{\pi}{2} - \theta < \psi < \frac{\pi}{2} \) and \( 0 < \alpha < \cos^{-1} \frac{p}{q} \).

(2.65)

Then, the integration path must be taken in the upper half of the complex-\( \Gamma \) plane to obtain

\[
I_2 = \frac{2\pi i}{k_o} \left[ \frac{N(-n_{1k}, \psi) e^{jn_{1k} R (p - q \cos \alpha)}}{-2n_1(n_1^2 - n_2^2)} + \frac{N(-n_{2k}, \psi) e^{jn_{2k} R (p - q \cos \alpha)}}{2n_2(n_1^2 - n_2^2)} \right].
\]

(2.66)

Gathering the results of (2.62), (2.64), and (2.66) and carefully noting over which range each is valid, we can rewrite (2.59). Note that \( N \) is an even function of its first argument, and that the second term of \( I_1 \) and \( I_2 \) is the first term with all \( n_1 \)'s and \( n_2 \)'s interchanged.

\[
P(R, \theta) = R_{n_1}(R, \theta) + P_{n_2}(R, \theta)
\]

(2.67)

\[
P_{n_2}(R, \theta) = P_{n_1}(R, \theta) \quad \text{with } n_1 \text{ and } n_2 \text{ interchanged}
\]

\[
P_{n_1}(R, \theta) = -\frac{i}{k_o^3} \int_0^\pi/2 \int_0^\pi/2 \frac{N(n_{1k}, \psi) e^{-jn_{1k} R (p + q \cos \alpha)}}{n_1(n_1^2 - n_2^2)} \frac{\sin^3 \psi}{A^2(\psi)} d\psi.
\]
Note that the first integral's range can be broken into the ranges of the last three integrals. One can then distribute the integrand of the first integral to the last three. Recognizing the exponential form of the cosine function one finds

\[ P_{nl}(R, \theta) = \int_{0}^{\pi/2-\theta} \int_{0}^{\pi/2} H(n_1, n_2, \psi) e^{-in_1k_0Rp} \cos(n_1k_0Rq \cos \alpha) d\theta d\psi \]

\[ + \int_{\pi/2-\theta}^{\pi/2} \int_{\cos^{-1}p/q}^{\pi/2} H(n_1, n_2, \psi) e^{-in_1k_0Rp} \cos(n_1k_0Rq \cos \alpha) d\theta d\psi \]

\[ + \int_{\pi/2-\theta}^{\pi/2} \int_{\cos^{-1}p/q}^{\pi/2} H(n_1, n_2, \psi) e^{-in_1k_0Rq \cos \alpha} \cos(n_1k_0Rp) d\theta d\psi \]

(2.69)

where we have let
\[ H(n_1, n_2, \psi) = \frac{-24}{k_o^3} \frac{N(n_1k_o, \psi) \sin^3 \psi}{A^2(\psi) n_1(n_1^2 - n_2^2)}. \] (2.70)

A little algebra simplifies (2.70) to

\[ H(n_1, n_2, \psi) = \frac{-24}{k_o^3} n_1 [\epsilon(\epsilon^2 - \epsilon'{}^2) - n_1^2(\epsilon^2 + \epsilon'{}^2 \cos^2 \psi)] \sin^3 \psi \]
\[ \frac{1}{A^2(\psi)(n_1^2 - n_2^2)}. \] (2.70a)

Now if we formally add and subtract the term

\[ \int_{\pi/2}^{\pi/2} \int_0^{\pi/2} \cos^{-1} p/q \]
\[ \pi/2 - \theta \]

\[ H(n_1, n_2, \psi) e^{-jn_1k_oR_p} \]
\[ \cos(n_1k_oR_q \cos \psi) \, d\psi \]

from \[ P_{nl}(R, \theta) \], it completes the range of one integral so that we can use the identity [5]

\[ J_0(n_1k_oR_q) = \frac{2}{\pi} \int_0^{\pi/2} \cos(n_1k_oR_q \cos \psi) \, d\psi. \] (2.71)

Using these two manipulations, \[ P_{nl}(R, \theta) \] further simplifies to

\[ P_{nl}(R, \theta) = \frac{\pi}{2} \int_0^{\pi/2} H(n_1, n_2, \psi) e^{-jn_1k_oR_p} J_0(n_1k_oR_q) \, d\psi \]
\[ + \int_{\pi/2}^{\pi/2} \int_0^{\pi/2} \cos^{-1} p/q \]
\[ \int_{\pi/2-\theta}^{\pi/2} \int_0^{\pi/2} H(n_1, n_2, \psi) [e^{-jn_1k_oR_p} \cos(n_1k_oR_q \cos \alpha) - e^{-jn_1k_oR_p} \cos(n_1k_oR_q \cos \alpha)] \, d\psi. \] (2.72)

Simply expanding exponentials

\[ P_{nl}(R, \theta) = \frac{\pi}{2} \int_0^{\pi/2} H(n_1, n_2, \psi) e^{-jn_1k_oR_p} J_0(n_1k_oR_q) \, d\psi \]
\[ + \int_{\pi/2}^{\pi/2} \int_0^{\pi/2} H(n_1, n_2, \psi) \sin n_1k_oR(p - q \cos \alpha) \, d\psi. \] (2.73)
At this point it appears that no more exact simplifications are possible. It will be necessary then to make judicious approximations to put $P_n^1(R, \theta)$ into a suitable form for calculation. The approximations will be easier to justify in cylindrical coordinates where

$$z = R \cos \theta$$

$$\rho = R \sin \theta.$$  

(2.74)

Recall that $z$ is to be interpreted as the distance along the antenna from the dipole source and $\rho$ will be $a$, the radius of the antenna, which is assumed to be small compared to the length.

First, note that the sine term in the second integral of (2.73) is

$$\sin n_1 k_0 [z \cos \psi - a \sin \psi \cos \alpha] = \sin (n_1 k_0 z \cos \psi)$$

$$\cos (n_1 k_0 a \sin \psi \cos \alpha) - \cos (n_1 k_0 z \cos \psi) \sin (n_1 k_0 a \sin \psi \cos \alpha).$$

(2.75)

For small $a$ we make a first-order approximation of (2.75) by

$$\sin (n_1 k_0 z \cos \psi) - n_1 k_0 a \sin \psi \cos \alpha \cos (n_1 k_0 z \cos \psi).$$

(2.76)

Now we can perform the indicated $\alpha$ integration in (2.73) on this simplified term (2.76) to get

$$\sin (n_1 k_0 z \cos \psi) \cos^{-1} \frac{z \cos \psi}{a \sin \psi} - n_1 k_0 a \sin \psi \cos (n_1 k_0 z \cos \psi) \sin \alpha$$

where

$$\sin \alpha = \sqrt{1 - \left[\frac{z \cos \psi}{a \sin \psi}\right]^2}.$$  

(2.77)

We shall also make an approximation for the Bessel function in (2.73):

$$J_0(n_1 k_0 Rq) = J_0(n_1 k_0 a \sin \psi) \approx 1$$

(2.78)

for small $a$. Finally, we can write
\[ P_{\text{n}1}(R, \theta) = \frac{\pi}{2} \int_{0}^{\pi/2} H(n_1, n_2, \psi) e^{-j n \frac{k}{o} z \cos \psi} d\psi \]

\[ + j \int_{\pi/2-\theta}^{\pi/2} H(n_1, n_2, \psi) \left[ \sin(n \frac{k}{o} z \cos \psi \cos \frac{\pi}{2} - \frac{z \cos \psi}{a \sin \psi} \right] d\psi \]

\[ - n \frac{k}{o} a \sin \psi \cos(n \frac{k}{o} z \cos \psi) \sqrt{1 - \left( \frac{z \cos \psi}{a \sin \psi} \right)^2} d\psi \] (2.79)

where

\[ \theta = \tan^{-1} \frac{a}{z}. \]

2.6 Final Field Expressions for Infinitesimal Source

At last all the fields from an infinitesimal source have been found. They are repeated here for references purposes.

\[ E_z = E_{zs} + E_{zfs1} + E_{zfs2} + E_{zf} \] (2.80)

\[ E_{zs} = -\frac{j\eta e^2}{4\pi k o e z} \cdot \frac{1}{R^3} \cdot \left[ \frac{-2\epsilon z^2 - \epsilon_\rho^2}{\epsilon z^2 + \epsilon_\rho^2} \right] \] (2.81)

\[ E_{zfs1} = \frac{-j\eta k o}{8\pi} \cdot \frac{1}{R^1} \cdot \left[ 1 + \frac{\epsilon z^2}{\epsilon z^2 + \epsilon_\rho^2} \right] \] (2.82)

\[ E_{zfs2} = \frac{j\eta k o}{4\pi} \cdot \frac{\epsilon^2}{\epsilon - \epsilon} \left\{ \frac{1}{\epsilon z - \epsilon} - \frac{1}{\sqrt{\epsilon z^2 + \rho^2}} \right\} - \frac{1}{2R^1} \left[ \frac{\epsilon z - \epsilon}{\epsilon z^2 - \epsilon} - \frac{\epsilon z^2}{\epsilon z^2 + \epsilon_\rho^2} \right] \] (2.83)

\[ R' = \frac{1}{\sqrt{\epsilon}} \sqrt{\epsilon z^2 + \epsilon_\rho^2} \] (2.83a)
\[ E_{zf} = jnk_0 \left( \frac{k_0}{2\pi} \right)^2 [P_{n_1}(z, \rho) + P_{n_2}(z, \rho)] \]  
(2.84)

\[ P_{n_1}(z, \rho) = \frac{n}{2} \int_0^{\pi/2} H(n_1, n_2, \psi) e^{-jn k_0 z \cos \psi} \, d\psi \]

\[ + j \int_{\pi/2}^{\pi/2-\theta} H(n_1, n_2, \psi) \left[ \sin(n k_0 z \cos \psi) \cos^{-1} \frac{z \cos \psi}{a \sin \psi} \right] \, d\psi \]

\[ - n_1 k_0 a \sin \psi \cos(n k_0 z \cos \psi) \sqrt{1 - \frac{z \cos \psi}{a \sin \psi}} \right] \, d\psi \]  
(2.85)

where

\[ H(n_1, n_2, \psi) = -\frac{24}{k_0^3} n_1 \left[ \frac{\tan^{-1} a}{z} \right] \frac{[\epsilon(\epsilon^2 - \epsilon'^2) - n_1^2(\epsilon^2 + \epsilon'^2 \cos^2 \psi)] \sin^3 \psi}{A^2(\psi)(n_1^2 - n_2^2)} \]  
(2.85a)

\[ n_1^2, n_2^2 = \frac{B(\psi) \pm \sqrt{B^2(\psi) - 4A(\psi)C(\psi)}}{2A(\psi)} \quad ; \quad \text{Im}(n_1) < 0, \text{Im}(n_2) < 0 \]  
(2.86)

\[ A(\psi) = \epsilon \sin^2 \psi + \epsilon' \cos^2 \psi \]

\[ B(\psi) = \epsilon \epsilon' \left( 1 + \cos^2 \psi \right) + (\epsilon^2 - \epsilon'^2) \sin^2 \psi \]  
(2.87)

\[ C(\psi) = (\epsilon^2 - \epsilon'^2) \epsilon \]  
(2.88)

\[ P_{n_2} = P_{n_1} \text{ with all } n_1' \text{'s and } n_2' \text{'s interchanged.} \]
CHAPTER III. APPLICATION OF METHOD OF MOMENTS

In the last section we derived an expression for the electric field produced by an infinitesimal source. It can easily be seen that the current distribution on the antenna, $i(z)$, must satisfy the integral equation

$$\int_0^l i(z') K(z - z') \, dz' = e(z)$$

(3.1)

where $e(z)$ is

$\begin{cases} E & \text{applied at antenna feed,} \\ 0 & \text{everywhere else} \end{cases}$

and where $l$ is the length of the antenna, and $K(z - z') = E_z(z - z')$ is the Green's function evaluated on the surface of the antenna. Thus (3.1) basically specifies that the tangential $E$ field on the surface of the antenna must be zero.

The general theory of the method of moments will be presented, following the formulation by Harrington [6]. Then it will be shown how this method can be applied to this problem. First define the inner product of two functions on some interval $I$ as

$$\langle f(z), g(z) \rangle = \int_I f(z) g(z) \, dz.$$  

(3.2)

Now consider solving the inhomogeneous equation

$$L(f) = g$$

(3.3)

where $f$ and $g$ are functions defined on the common interval $I$, $L$ is a linear operator, $g$ is known, and $f$ is unknown.

Suppose $f$ is approximated as a sum of $n$ known functions weighted by $n$ unknown coefficients $a_n$. 
The $f_j$ functions are called expansion functions. Now by substituting (3.4) into (3.3) and using the linearity of $L$, the equation becomes

$$a_L(f_j) = g.$$  

(3.5)

Now define a set of $n$ weighting functions or testing functions, $w_i, i = 1, n$. By taking the inner product of both sides of (3.5) with respect to each testing function, the equation becomes

$$\sum_{j=1}^{n} a_j < w_i, Lf_j > = < w_i, g >, i = 1, 2, \ldots, n.$$  

(3.6)

This can be written in matrix form as

$$[\lambda_{ij}][a_j] = [g_i]$$  

(3.7)

where

$$[\lambda_{ij}] = \begin{bmatrix} < w_1, Lf_1 > & < w_1, Lf_2 > & \cdots \\ < w_2, Lf_1 > & < w_2, Lf_2 > & \cdots \\ \vdots & \vdots & \ddots \\ < w_n, Lf_1 > & < w_n, Lf_2 > & \cdots \end{bmatrix}$$  

(3.8)

$$[a_j] = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad [g_i] = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$  

(3.9)

Equation (3.7) can be solved for $[a_j]$ simply by inversion, provided $[\lambda_{ij}]$ is nonsingular.
Thus, \( f \) has been solved as a weighted sum of the expansion functions.

The method of moments, as it has thus been described, is very general. The particular choice of expansion and testing functions used for this problem will simplify the form of the solution considerably. Also, this choice affects the accuracy attainable for a given number of functions. Now we will show how to apply the method to this antenna problem by first identifying the functions and operators in (3.3). The unknown function \( f \) is \( i(z) \) defined on the interval \([0, \ell]\) representing the antenna. Likewise, the known function \( g \) is the tangential electric field \( e(z) \). The linear operator \( L \) is the integral operator

\[
L(h) = \int_0^\ell h(z') E_z(z - z') \, dz'
\]

or described in words, convolution with the Green's function along the antenna.

Next expansion functions must be devised for \( i(z) \). We will follow what is called the method of subsections, that is, expansion functions are chosen that are defined on only a subsection of the antenna interval. First, the antenna is divided into odd \( n \) number of segments, where each segment has length \( \frac{2H}{n} \). Then, the expansion functions are defined as the pulse functions

\[
p_i(z) = \begin{cases} 
1 & \text{if } (i - 1) \frac{2H}{n} \leq z < i \frac{2H}{n} \\
0 & \text{otherwise}
\end{cases}
\]

Now we can express \( i(z) \) as

\[
i(z) = \sum_{i=1}^{n} i_p p_i(z).
\]
Figure 2 shows how the use of pulse expansion functions replaces the actual current with a step approximation.

Now a set of testing functions must be chosen. The simplest choice utilizes the point-matching method. For this method, we choose \( w_i \) to be \( \delta(z - m_i) \), where \( \delta(z) \) is the Dirac delta function and \( m_i \) is the midpoint of the \( i \)th antenna segment. Thus, by the sampling property of the \( \delta \) function, the right-hand side of (3.6) is

\[
< w_i, g > = g(m_i).
\] (3.14)

Therefore, for this problem \( [g_i] = [e_i] \), where \( [e_i] \) is the vector of tangential electric fields evaluated at the midpoint of each segment.

Similarly, we can see that element \( e_{ij} \) of (3.8) is the electric field at the midpoint of the \( i \)th segment caused by the uniform current over the \( j \)th segment.

Finally, the original integral equation (3.1) has been transformed into the following matrix equation

\[
Z I = E
\] (3.15)

where \( Z = [z_{ij}] \); \( z_{ij} \) is the electric field produced by the uniform current over the \( j \)th segment evaluated at the midpoint of the \( i \)th segment.

\[
I = \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix}, \quad \text{\( i_k \) is the current over \( k \)th segment}
\]

\[
E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ E \text{ applied tangential electric field at midpoint of } k \text{th segment} \end{bmatrix}
\]

The solution is obviously

\[
I = Z^{-1} E = YE.
\] (3.16)
Figure 2. Use of Pulse Expansion Functions
Suppose the feed is at the $k^{th}$ segment. Then the impedance of the antenna is simply $\frac{v_k}{i_k}$ where $v_k$ is the voltage over the gap. In this report the feed is always at the center of the antenna.

Note some very important properties of the matrix $Z$. First, there are only $n$ unique elements. Segments the same distance apart have the same effect on each other. Thus $z_{ij} = z_{i+k,j+k}$. Also, by symmetry $z_{ij} = z_{ji}$. Therefore we can write $Z$ as

$$
Z = \begin{bmatrix}
    z_1 & z_2 & z_3 & \cdots & z_n \\
    z_2 & z_1 & z_2 & & \\
    z_3 & z_2 & z_1 & & \\
    & & & \ddots & \\
    z_n & & & z_2 & z_1 \\
\end{bmatrix} \quad (3.17)
$$

The matrix $Z$ is a complex symmetric matrix. Unfortunately, most of the useful properties of real symmetric matrices or complex Hermitian matrices do not apply to complex symmetric matrices. The element $z_1$ is called the self-element since it is the field at the midpoint of a segment carrying a uniform current. Since the fields decrease strongly with distance, it will be shown that $Z$ is strongly diagonal.

The $Z$ matrix has been discussed somewhat generally. Next, we will discuss the method for calculating the elements from the Green's function derived in the last section. There are three different sets of
expressions based on three different models for the distribution of current over one segment. The basic model used assumes that the current over one segment is uniformly distributed over the axis line; this current distribution will be called the current filament model. Since the field is evaluated at the radius, it remains finite.

Recalling that the length of one segment is $2\Delta l$, we can write the field caused by the current filament as the superposition of fields caused by infinitesimal sources distributed along the filament. Thus,

$$E_{z, \text{filament}}(z, a) = \frac{1}{2\Delta l} \int_{-\Delta l}^{\Delta l} E_z(z', a)\, dz'. \quad (3.18)$$

The factor of $1/2\Delta l$ is included to make the effective current over the segment equal for the two models. This integration was done on each of the four component fields of $E_z$. For $E_{zs}$, $E_{zfs1}$, and $E_{zfs2}$, one must perform integrals of the form

$$I = \int_{-\Delta l}^{\Delta l} \frac{(z - z')^n}{[(z - z')^2 + a^2]^m} \, dz'. \quad (3.19)$$

where $n$ is 0 or 2 and $m$ is $1/2$, $3/2$, or $5/2$. These integrals can be found in common integral tables. Therefore, the integrated forms of $E_{zs}$, $E_{zfs1}$, and $E_{zfs2}$ will be simply presented later.

Integrating $E_{zf}$, the component of the Green's function $E_z$, which cannot be written in closed form, requires more explanation. Recall that $E_{zf}$ is given by (2.84)

$$E_{zf} = jnk_o \left( \frac{k_o}{2\pi} \right)^2 \left[ P_{n1}(z, \rho) + P_{n2}(z, \rho) \right]. \quad (2.84)$$

The first term can be written as
where

\[ P_{nl} = I_{zfa} + I_{zfb} \quad (3.20) \]

\[ I_{zfa} = \frac{\pi}{2} \int_{0}^{\pi/2} H(n_1, n_2, \psi) e^{-jn_1k_0z \cos \psi} d\psi \quad (3.21) \]

\[ I_{zfb} = j \int_{\pi/2-\theta}^{\pi/2} H(n_1, n_2, \psi) \left[ \sin(n_1k_0z \cos \psi) \cos^{-1}\left(\frac{z \cos \psi}{a \sin \psi}\right) \right. \]

\[ \left. - n_1k_0a \sin \psi \cos(n_1k_0z \cos \psi) \sqrt{1 - \left(\frac{z \cos \psi}{a \sin \psi}\right)^2} \right] d\psi \quad (3.22) \]

where

\[ \theta = \tan^{-1}\frac{a}{z}. \]

Remember \( n_1, n_2 \) and \( H \) are functions of \( \psi \).

We will make the approximation that \( I_{zfb} \) can be neglected. To justify this, first note that the integration width for the self-element where \( z = 0 \) is \( \theta = \frac{\pi}{2} \). For the first segment away from the source element, \( z = 2\Delta \), and since only thin wires are being treated, \( \frac{a}{z} \) is small. Therefore for all elements, except the self-element, \( \theta = \frac{a}{z} \).

We can now make small angle approximations in (3.22) and perform the integration to get

\[ I_{zfb} = -j H(n_1, n_2, \frac{\pi}{2}) \frac{3\pi}{8} n_1k_0a^2 \quad \frac{\Delta^2}{z}, \quad z \geq 2\Delta. \quad (3.23) \]

Because of the \( \frac{a^2}{z} \) term, \( I_{zfb} \) should be small compared to \( I_{zfa} \). For \( z = 0 \), Equation (3.22) simplifies to

\[ I_{zfb} = j \int_{0}^{\pi/2} H(n_1, n_2, \psi) n_1k_0a \sin \psi d\psi. \quad (3.24) \]

Direct numerical integration of (3.21), (3.22), and (3.24) indicates that \( I_{zfb} \) may be neglected for all terms including the self-element.
Moreover, one should note that the singular field terms are orders of magnitude larger than \( E_{zf} \) for all but the farthest segments; therefore, \( E_{zf} \) need not have an extremely small relative error.

Now, when \( I_{zfb} \) is neglected, \( E_{zf} \) can be integrated over one segment, according to (3.18). Note that \( z \) occurs only in the exponential \( e^{-j n_1 k_0 z \cos \psi} \). Therefore, one obtains the integrated form of \( E_{zf} \) by integrating the exponential. One can easily verify that

\[
\frac{1}{2\Delta z} \int_{-\Delta z}^{\Delta z} e^{-j n_1 k_0 (z - z') \cos \psi} \, dz' = e^{-j n_1 k_0 z \cos \psi} \frac{\sin(n_1 k_0 \cos \psi \Delta z)}{n_1 k_0 \cos \psi \Delta z}.
\]

(3.25)

One does the exact same integration for the exponential containing \( n_2 \). The integrated form of \( E_{zf} \) is then only \( E_{zf} \) with the exponentials replaced by the right-hand side of (3.25).

We will summarize the fields produced by a current filament mode. To avoid proliferation of variable names, the integrated form has the same name as the corresponding term of the Green's function. We have substituted the antenna radius \( a \) for \( \rho \).

\[
E_z \text{ filament model} = E_{zs} + E_{zfs1} + E_{zfs2} + E_{zf}
\]

(3.26)

\[
E_{zs} = \frac{\ln}{8\pi k_0 e \Delta z} \left[ \frac{z + \Delta z}{[(z + \Delta z)^2 + a'^2]^{3/2}} - \frac{z - \Delta z}{[(z - \Delta z)^2 + a'^2]^{3/2}} \right]
\]

(3.27)

where \( a'^2 = \frac{c}{e} \frac{z}{a} \)

\[
E_{zfs1} = \frac{-jk_0 \eta}{8\pi (2\Delta z)} \left\{ 2 \ln \frac{z + \Delta z + \sqrt{(z + \Delta z)^2 + a'^2}}{z - \Delta z + \sqrt{(z - \Delta z)^2 + a'^2}} + \frac{z + \Delta z}{\sqrt{(z + \Delta z)^2 + a'^2}} + \frac{z - \Delta z}{\sqrt{(z - \Delta z)^2 + a'^2}} \right\}
\]

(3.28)
\[ E_{zfs2} = \frac{j k_o n}{4 \pi} \frac{\varepsilon_r^2}{(\varepsilon - \varepsilon) 2 \Delta l} \left\{ \frac{1}{\varepsilon - \varepsilon} \ln \frac{z + \Delta l + \sqrt{(z + \Delta l)^2 + a^2}}{z - \Delta l + \sqrt{(z - \Delta l)^2 + a^2}} \right\} \]

\[ - \frac{1}{\varepsilon - \varepsilon} \ln \frac{z + \Delta l + \sqrt{(z + \Delta l)^2 + a^2}}{z - \Delta l + \sqrt{(z - \Delta l)^2 + a^2}} \]

\[ - \frac{1}{2 \varepsilon} \left( \frac{z + \Delta l}{\sqrt{(z + \Delta l)^2 + a^2}} - \frac{z - \Delta l}{\sqrt{(z - \Delta l)^2 + a^2}} \right) \] (3.29)

\[ E_{zf} = j n k_o (\frac{k_o^2}{2}) (p_{n1} + p_{n2}) \] (3.30)

\[ p_{n1} = \frac{\pi}{2} \int_0^{\pi/2} H(n_1, n_2, \psi) e^{-j n_1 k_o \varepsilon_r \cos \psi} \frac{\sin(n_1 k_o \cos \psi \Delta l)}{n_1 k_o \cos \psi \Delta l} d\psi \]

\[ (3.31) \]

\[ H(n_1, n_2, \psi) = -\frac{2 j}{k_o n_1} \frac{[\varepsilon (\varepsilon_r^2 - \varepsilon_r^2) - n_1^2 (\varepsilon_r^2 + \varepsilon_r^2 \cos^2 \psi)] \sin^3 \psi}{A^2(\psi) (n_1^2 - n_2^2)} \]

\[ (3.32) \]

\[ n_1^2, n_2^2 = \frac{B(\psi) \pm \sqrt{B^2(\psi) - 4A(\psi) C(\psi)}}{2A(\psi)} \]; \( \text{Im}(n_1) < 0 \), \( \text{Im}(n_2) < 0 \) (3.33)

\[ A(\psi) = \varepsilon \sin^2 \psi + \varepsilon_r \cos^2 \psi \]

\[ (3.34) \]

\[ B(\psi) = \varepsilon \varepsilon_r (1 + \cos^2 \psi) + (\varepsilon_r^2 - \varepsilon_r^2) \sin^2 \psi \]

\[ C(\psi) = (\varepsilon_r^2 - \varepsilon_r^2) \varepsilon_r \]

\[ p_{n2} = p_{n1} \text{ with all } n_1 \text{'s and } n_2 \text{'s interchanged.} \] (3.35)
Equations (3.32) to (3.35) are the same as Equations (2.80) to (2.89) but are reproduced for reference.

It has been shown [7] that one obtains better answers if $\sqrt{2} \alpha$ is used in the calculations as an effective radius in place of $\alpha$. This correction compensates for the assumption that the current is on the axis instead of distributed on the surface.

One can note that many simplifications in the preceding formulas are possible and in many cases necessary for limiting cases of the general anisotropic plasma. For free space, the $E_{\text{zfs2}}$ term has zeros in both the numerator and the denominator. One can clearly see that for plasma parameters $X = 0$, corresponding physically to free space with an applied magnetic field, and that for $Y^2 = 0$, corresponding physically to an isotropic plasma, there is no off-diagonal dielectric element $\varepsilon'$, and, therefore, $E_{\text{zfs2}} = 0$. In the limiting case of free space, $E_{\text{zfs2}}$ must be set equal to zero. One must also analytically treat removable singularities in the expressions for $E_{\text{zf}}$. For the isotropic case, the two indices of refraction $n_1$ and $n_2$ are equal. The expression for $H$ can be rewritten to avoid an indeterminate form.

Next one can compare the fields produced by the current filament model to the fields produced by $\delta$-source model. The $\delta$-source model is physically unrealistic; however, it should give the same values as the filament model for observation points far from the segment with the current. Far from the source, numerical calculations show that for all field components one may use the $\delta$-source model to avoid subtracting nearly equal terms present in the filament model expression. In general, no more than 21 segments were used and, therefore, the $\delta$-source model was not needed for computation. One can note that the
The $E_z$ term is nearly independent of distance. Therefore, one would not expect the fields due to the two models to be significantly different for any distance. Calculations verify this.

A third current distribution model was considered. One might object to the filament model as being unrealistic, especially for elements very close to the current segment. Therefore, we have examined a model for which the current is uniform on the surface of the segment. We call this model the cylindrical current model. This model can be obtained from the filament model by summing up the contribution of distributed current filaments placed along the surface of the current segment. When one integrates around the segment, one needs to replace the radius $a$ with the radial distance $2a \sin \frac{\phi}{2}$.

(See Figure 3.) Recall that $E_z$ has the form $E_z(z, a)$. Therefore, we write

$$E_z \text{ cylindrical} = \frac{1}{2\pi} \int_0^{2\pi} E_z \text{ filament} (z, 2a \sin \frac{\phi}{2}) \, d\phi.$$  

(3.36)

Since the cylindrical model is needed only in the very near region, and since in this region the $E_z \text{ filament}$ term strongly dominates the other terms, this cylindrical integration is applied only to the $E_z \text{ filament}$ term. Then substituting $E_z \text{ filament}$ into Equation (3.36) one obtains

$$E_z \text{ cylindrical} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\ln}{8\pi k \epsilon \Delta \ell} \left[ \frac{z + \Delta \ell}{[(z + \Delta \ell)^2 + 4a^2 \sin^2 \frac{\phi}{2}]}^{3/2} - \frac{z - \Delta \ell}{[(z - \Delta \ell)^2 + 4a^2 \sin^2 \frac{\phi}{2}]}^{3/2} \right] \, d\phi.$$  

(3.37)
Figure 3. Cylindrical Model as Distributed Current Filaments on Surface of Segment
One can see that (3.37) essentially is an integral of the form

\[ I = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{[4a'^2 \sin^2 \frac{\phi}{2} + b^2]^{3/2}} \]  

(3.38)

Making the change of variable

\[ \theta = \phi + \pi \]  

(3.39)

and simplifying yields

\[ I = \frac{1}{4\pi a'^3} k^3 \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{3/2}} \]  

(3.40)

where

\[ k = \frac{2a'}{\sqrt{b^2 + 4a'^2}} \]  

and \[ a'^2 = \frac{\varepsilon}{\varepsilon - a^2} \].  

(3.41)

Note how similar (3.40) is to the definition of an elliptic integral. Applying a clever transform from Gradshteyn and Ryzhik [8] allows us to write

\[ I = \frac{1}{4\pi a'^3} k^3 \frac{k^3}{1 - k^2} E(k) \]  

(3.42)

where \( E(k) \) is an elliptic integral of the second kind. Equation (3.42) can be applied to (3.37) to obtain

\[ E_{\text{cylindrical}} = \frac{1}{8\pi k_0 \varepsilon \Delta k} \left[ \frac{z + \Delta \ell}{4\pi a'^3} \frac{k_+^3}{1 - k_+^2} E(k_+) \right. 
- \left. \frac{(z - \Delta \ell)}{4\pi a'^3} \frac{k_-^3}{1 - k_-^2} E(k_-) \right] 
\]

(3.43)

\[ k_+ = \frac{2a'}{[(z + \Delta \ell)^2 + 4a'^2]^{1/2}} , \quad k_- = \frac{2a'}{[(z - \Delta \ell)^2 + 4a'^2]^{1/2}} \cdot \]

Now one can compare the fields due to a current filament to the fields due to the equivalent current flowing on the cylindrical surface. Although one might expect much different values for the most singular
terms; the results were extremely close. Except near \( z = \Delta l \), the fields generally agree to better than 1 per cent. See Table 1 for a comparison of the two models for various distances and plasma parameters.

One can note that the field for the cylindrical model is singular for \( z = \Delta l \), and that it changes sign crossing this point. This can easily be understood from a quasi-static model. From this viewpoint, the current causes rings of charge to accumulate on the edges of the cylinder, and the electric field is the electrostatic field due to these two rings. Clearly for \( z = \Delta l \), one is evaluating the field at the ring and it is singular there. The filament model remains finite at \( z = \Delta l \) because for this model the charge is on the axis, an effective distance of \( a' \) from the point of observation.

After comparing the three current distribution models, we decided that the filament model was entirely adequate for \( E_{zfs1} \), \( E_{zfs2} \), and \( E_{zf} \). For \( E_{zs} \) we use the cylindrical model although it is only slightly different from the filament model. Therefore, each element \( z_i \) of \( Z \), as shown in (3.17), is computed as

\[
z_i = E_z [(i - 1) 2\Delta l, a]
\]

where

\[
E_z = E_z(z, a);
\]

the cylindrical model of \( E_z \) is used for \( E_{zs} \) and the filament model is used for \( E_{zfs1} \), \( E_{zfs2} \), and \( E_{zf} \).

There are still a few small matters to discuss before calculating \( I \), the current vector. First, one may consider applying one volt to the feed obtaining the resulting electric field.
### TABLE 1.

**COMPARISON OF ELECTRIC FIELDS DUE TO FILAMENT AND CYLINDRICAL CURRENT DISTRIBUTION MODELS**

\[ H = 1, \quad a = 1/144., \quad \Delta z = \frac{1}{23} \quad k_o = .1, \quad z = m\Delta z \]

<table>
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<th>Y</th>
<th>Z</th>
<th>segments</th>
<th>( E_{zs} ) cylindrical ((z, a))</th>
<th>( E_{zs} ) filament ((z, \sqrt{2} a))</th>
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<td>30.41</td>
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since the gap distance $d$ is $2A\xi$. Second, note that $E_z$ filament gives the field due to a distributed source with the same effective current as the $\delta$-source model. This $\delta$ source was one ampere at one point; when it is distributed, the current at any point is only $\frac{1}{2A\xi}$ ampere. Therefore, one should multiply all elements of $Z$ by $2A\xi$ to get true fields. Equivalently, this last factor of $2A\xi$ is factored from the $Z$ matrix into the $E$ vector. Therefore, considering both comments about the $2A\xi$ factor, the vector $E$ is computed as

\[
E = [e_i] = \begin{cases} 
-\frac{1}{(2A\xi)^2}, & i = \text{center element} \\
0, & i \neq \text{center element}
\end{cases}
\]  

(3.46)

One must also make slight corrections between effective and actual dimensions of the antenna. The use of pulse expansion functions assumes a finite current at the end of the antenna when the true current must be zero. To compensate for this, dummy segments have been added to each end. These segments are not used in calculations, but only make the length of the antenna for which the nonzero currents can exist shorter than the actual physical length $2H$. Therefore, for segments one uses

\[
2A\xi = \frac{2H}{n + 2} \quad \text{instead of} \quad 2A\xi = \frac{2H}{n}.
\]

The exact number of segments to add was partially determined empirically. Using one extra segment on each end results in a good, but not perfect, linear current distribution for a very short
antenna in free space. The other dimensional correction, which has already been noted, is that one uses an effective radius of \( \sqrt{2} a \) instead of the actual radius \( a \) in all calculations using the filament model.

In most cases, 21 active antenna segments were used. Trials with 33 and 45 segments show only slight changes. It is not at all clear what is an optimum number of segments to use because one cannot always assume that using more terms gives a better approximation.
CHAPTER IV. NUMERICAL DIFFICULTIES

The field expressions for the filament current model appear deceptively simple. Even though these expressions have required a lengthy derivation, most of the effort expended on this problem has been in performing numerical computations. For the terms in closed form, one must guard against loss in accuracy due to subtracting nearly equal terms, and for the terms indicated as integrals, one must perform difficult numerical integrations. The purpose of this section is to describe the numerical analysis required in writing an efficient computer program. Particular attention is paid to the numerical integrations needed for $E_{zf}$. The general question of stability of the method is discussed in Chapter V.

The computer program that is used can be found in the Appendix. For computational purposes, it was convenient to formulate the field expressions with dielectric constants normalized to $\varepsilon$. Thus, in the program $\varepsilon'$ is actually $\varepsilon'/\varepsilon$, $\varepsilon_z = \varepsilon_z / \varepsilon$, and the field expressions have been correspondingly modified. One result of this renormalization is that one uses $n_1^{\sqrt{\varepsilon}}$ in place of $n_1$. In this case, one must be careful to compute $n_1^{\sqrt{\varepsilon}}$ as the square root of $n_1^2\varepsilon$ with the negative imaginary part rather than as the product of the square roots of $n_1^2$ and $\varepsilon$ taken individually.

For the singular terms in closed form, $E_{zs}$, $E_{zfsl}$, and $E_{zfsl'}$, one must be careful to avoid subtracting nearly equal quantities. This problem increases with increasing distance $z$. As has been already indicated, if fields are needed more than 30 segments away, the $\delta$-source model gives entirely adequate values. A series expansion of the singular terms has
been made, but the range of validity for the filament and the $6$-source models overlap and, therefore, the series expansion is not needed.

The largest source of numerical difficulty is performing the numerical integration for the finite term $E_{zf}$. The shape of the graph of the integrand varies with the three plasma parameters, the free-space wavenumber $k_o$, and the distance $z$. Figure 4 shows real and imaginary parts of the integrand for various parameters scaled so that the difference between maximum and minimum function values is unity. For some plasma parameters the integrand is well-behaved, but for many values, the integrand is nearly singular at some point. The integrand starts at zero, goes through a strong positive peak and through a strong negative peak.

Essentially, one has to subtract the effects of the two large and nearly equal peaks. As $Y^2$ or $X$ gets closer to 1, and as the loss factor $Z$ decreases, the peaks get narrower and steeper until the integrand resembles the doublet function.

Now one can appreciate the numerical difficulty. The standard technique of singularity subtraction is not applicable because of the complicated form of the integrand. One cannot even find analytically the zero crossing of the integrand. An integration scheme with moderate-sized uniform steps might only randomly sample the nearly-singular region or overstep it completely. If the steps were small enough to accurately sample the peaks, there would be an excessive number over regions where the function is relatively flat. Therefore, we have chosen a variable-step Romberg integration scheme [9].

The Romberg quadrature scheme is based on Richardson extrapolation of successive trapezoidal-rule estimates of the area under the integrand. Suppose one needs the integral of some function $f(x)$ over the interval
Figure 4. Normalized Integrand in the Numerical Integration of $E_Z$ for Various Plasma Parameters and Distances
($k_o = .576, H = \pi/2, n = 21$)
[a, b]. Let $h = a - b$. The trapezoidal-rule estimate of the area for $2^k$ equally spaced segments is

$$T_{ok} = \frac{h}{2^k} \left[ \frac{1}{2} f(x_0) + f(x_1) + \cdots + f(x_{2^k-1}) + \frac{1}{2} f(x_{2^k}) \right]$$

where

$$x_n = a + (n - 1) \frac{h}{2^k}.$$ 

Extrapolations $T_{nk}$ can be generated by the formula

$$T_{nk} = \frac{1}{4^n - 1} (4^n T_{n-1,k+1} - T_{n-1,k}).$$

These results can be tabulated as a triangular array

$$
\begin{array}{cccc}
T_{00} & & & \\
T_{01} & T_{10} & & \\
T_{02} & T_{11} & T_{20} & \\
T_{03} & T_{12} & T_{21} & T_{30} \\
& & & \\
\end{array}
$$

The elements of the first column are the trapezoidal-rule estimates; each other element is generated from the element to the left and the element above the element to the left. The accuracy of the estimates becomes better as one moves down and to the right. The procedure stops when the last two elements of a row are sufficiently close. Recall that the trapezoidal rule has an error term on the order of $h^3$. The application of (4.2) to $T_{00}$ and $T_{01}$ subtracts the $h^3$ error term leaving only an error on the order of $h^5$ for $T_{10}$. Substituting (4.1) into (4.2) shows that $T_{10}$ is exactly the Simpson rule estimate; similarly, $T_{20}$ is the third Newton-Cotes formula. One might suspect that $T_{k0}$ is the $k^{th}$ Newton-Cotes formula, but fortunately that is not true. High-
order Newton-Cotes formulas suffer severely due to round-off error while Romberg's method converges very well.

Due to the particular integrand in this problem, we have partially followed modifications suggested by Miller and Burke [10]. This modification is applicable to any numerical integration where the function is well-behaved except in a small subinterval. The original interval is divided into m subintervals and Romberg's method is applied to each. Convergence is checked by means of a test ratio $T_r$. If

$$\frac{|T_{01} - T_{10}|}{|T_{10}|} < T_r \quad (4.4)$$

only the midpoint of the subinterval needs to be computed and $T_{10}$ can be used as the area of the subinterval. If (4.4) does not hold, two additional points in the subinterval are computed and the program checks if

$$\frac{|T_{11} - T_{20}|}{|T_{20}|} < T_r .$$

If (4.5) is true, $T_{20}$ is used as the estimate of the area. However, if (4.5) is not satisfied, the program subdivides the interval and applies Romberg's method on that new subinterval rather than subdividing all intervals as the unmodified version would require. Thus, no estimate higher than $T_{20}$ is computed for an interval. This scheme is efficient since the midpoint of the new subinterval has already been calculated. If one had tried using an adaptive Gaussian quadrature scheme, one could not have used any previous calculations. A distinctive feature of our
method is that our program saves all calculations made. Thus when a subinterval converges, one already has the function values for the end points and midpoint of the next subinterval to be integrated. To be practical, one must limit the maximum number of subdivisions allowed and the minimum size of a subinterval.

The exact error of this method is hard to estimate precisely. One may calculate an error estimate as the sum of the differences between the two best area estimates for each subinterval. This is a very pessimistic guess of the error. Even so, in most cases the relative error calculated in this way will be far less than $T_r$ although there is no assurance that this is always true. The usual value used for $T_r$ is $10^{-4}$.

A very interesting pathological case exists for this modified integration scheme. The $T_{20}$ estimate term of (4.3) is exact for a polynomial of degree up to 5. When the program tried integrating $y = x^8$ over $[0, 1]$ as a test problem, it kept subdividing the first interval until it had reached the maximum number of subdivisions allowed. The problem is analogous to integrating a parabola with the trapezoidal rule. Consider approximating the function $y = x^2$ with a straight line over $[0, \varepsilon]$. The true value of the integral over the interval will be $\frac{1}{3} \varepsilon^3$ but the integral of the approximation will be $\frac{1}{2} \varepsilon^3$. Clearly the absolute error of the approximation decreases as $\varepsilon$ decreases but the relative error does not. When a maximum number of subdivisions is specified, there is no practical problem. Absolute error on the first interval is small enough, and for any interval not containing the 0, the relative error converges. In practice, this pathology occurs when the function behaves as a high-order polynomial.
The Romberg method has been described for the case of a real function. For a complex function, such as in this problem, we have required that the test ratio be satisfied for both the real and imaginary parts of the estimates before going to the next interval.

The integration of $E_{zf}$ is done slightly differently than formally indicated. The integrands of $P_{n1}$ and $P_{n2}$ are added together before integrating. This not only eliminates the need for two separate applications of Romberg's method, but also simplifies the handling of terms common to $P_{n1}$ and $P_{n2}$.
CHAPTER V. STABILITY OF METHOD OF MOMENTS

After many trial runs, we wished to test the stability of our method. Even if the models used were perfect and electric fields could be precisely analytically formulated, there would still be computational errors that would be transmitted to the final answers. Knowing the sensitivity of the final answers to both matrix manipulation errors and errors in computing the fields allows one to estimate how accurately fields must be calculated to get results that are reliable. Early results obtained from field calculations with deliberate perturbations showed a noticeable sensitivity. The stability theory developed here applies not only to this particular plasma problem, but also to any problem using the method of moments.

This sensitivity analysis will use norms of complex vectors and matrices to estimate error. Because of the wide variety in definitions of matrix norms, we wish to develop our analysis in detail and carefully define our matrix and vector notation for complex matrices and vectors. Our definition will follow Forsythe and Moler [11].

All vectors \( \mathbf{x} \) will be assumed to be column vectors. \( \mathbf{A}^T \) and \( \mathbf{x}^T \) will indicate the transpose of matrix \( \mathbf{A} \) and vector \( \mathbf{x} \), that is, the original matrix or vector with rows and columns interchanged. \( \mathbf{A}^H \) and \( \mathbf{x}^H \) will indicate the matrix or vector that is the complex conjugate of the transpose of the original matrix or vector. A matrix that is equal to its own transpose conjugate is said to be Hermitian.

Next we define vector norms. The Euclidean norm of a complex vector \( \mathbf{x} \) is defined as

\[
|\mathbf{x}|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{\mathbf{x}^H \mathbf{x}}
\]  

(5.1)
where \( x_1, x_2, \ldots, x_n \) are components of \( x \). For real vectors this norm is the usual concept of the length of a vector. We can also define vector norms

\[
\| x \|_1 = \sum_{i=1}^{n} |x_i| \quad (5.2)
\]

\[
\| x \|_\infty = \max_{1 \leq i \leq n} |x_i|. \quad (5.3)
\]

These norms are not quite as geometrically obvious as the Euclidean norm. Figure 5 shows all possible unit-length vectors in two dimensions for each of the three norms. Mathematically, one can define an infinite number of norms, but these three are the most useful. All vector norms must satisfy the following three properties.

\[
\| cx \| = |c| \cdot \| x \| \quad \text{for any real scalar } c, \quad (5.4)
\]

\[
\| x \| \geq 0 \text{ and } \| x \| = 0 \text{ if and only if } x = \text{null vector}, \quad (5.5)
\]

\[
\| x + y \| \leq \| x \| + \| y \| \quad \text{(triangle inequality).} \quad (5.6)
\]

Next we define a matrix norm for square matrices \( A \) as

\[
\| A \| = \max_{x \neq 0} \frac{\| Ax \|}{\| x \|} \quad (5.7)
\]

or equivalently

\[
\| A \| = \max_{\| x \| = 1} \| Ax \|. \quad (5.8)
\]

Each of the three vector norms thus produces a matrix norm that is said to be compatible with the generating vector norm. Because of this compatibility, these matrix norms satisfy the three vector norm
Figure 5. Unit Vectors under Different Norms
properties (5.4), (5.5), and (5.6), when $x$ and $y$ are replaced by matrices. From the definition (5.7) one can see

$$||Ax|| \leq ||A|| \cdot ||x|| \quad \text{for all } A \text{ and } x. \quad (5.9)$$

The main advantage of the two non-Euclidean norms is their ease of computation. It can be shown [12] that

$$||A||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \quad (5.10)$$

$$||A||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|. \quad (5.11)$$

Finally we can introduce a scalar function of a matrix that measures the degree of ill-conditioning for certain operations with this matrix.

Define the condition number of square matrix $A$ as

$$\text{cond} (A) = ||A|| \cdot ||A^{-1}||. \quad (5.12)$$

For real matrices, the condition number with respect to the Euclidean norm can be calculated

$$\text{cond}_2(A) = \sqrt{\frac{\lambda_1}{\lambda_n}} \quad (5.13)$$

where $\lambda_1$ and $\lambda_n$ are the largest and smallest eigenvalues in absolute magnitude of the symmetric matrix $AA^T$. If matrix $A$ is symmetric

$$\text{cond}_2(A) = \frac{|\lambda_1|}{|\lambda_n|} \quad (5.14)$$

where $\lambda_1$ and $\lambda_n$ are the largest and smallest eigenvalues of $A$. 
If matrix A is complex, (5.13) can be applied where one uses the largest and smallest eigenvalues of $AA^H$ which will be Hermitian and thus have real eigenvalues.

For the case of real symmetric matrices and the Euclidean norm, one can most easily understand the application of the condition number. One can think of premultiplication by a matrix as a vector function in which components of the vector in the direction of the eigenvectors of the matrix are stretched or shrunk by the respective eigenvalues. Consider $y = Ax$ as a mapping of vectors $x$ onto vectors $y$. It is geometrically clear that $y = Ax$ maps the $n$-dimensional sphere of all unit length $x$'s onto an $n$-dimensional ellipsoid with semi-major axis $\lambda_1$ and semi-minor axis $\lambda_n$. One can see that $|\lambda_1|/|\lambda_n|$ is the ratio of possible distortions a vector $x$ can undergo. Also note that the condition number shows the loss in accuracy for computation of components of $x$ that are not in the direction of the eigenvector associated with the largest eigenvalue. The geometrical interpretation with other norms is not as clear.

It is to be stressed that cond(A) is a far better measure of ill-conditioning than the smallness of the determinant $A$. Even if $A$ is normalized, the smallness of det($A$) has a weak relation to the degree of ill-conditioning. Supporting examples are given by Forsythe and Moler.

Now we will indicate how the condition number can be used to measure sensitivity of a solution to a linear system. Consider the problem

$$Ax = y$$  \hspace{1cm} (5.15)

with unknown vector $x$. Suppose now that $y$ is precisely known but that
A and, therefore, $x$ are subject to uncertainty. Then

$$(A + \delta A)(x + \delta x) = y$$  \hspace{1cm} (5.16)$$

where $\delta A$ and $\delta x$ are uncertainties in $A$ and $x$. Forsythe and Moler show from Equations (5.4) to (5.9) that

$$\frac{||\delta x||}{||x + \delta x||} \leq \text{cond}(A) \frac{||\delta A||}{||A||}. \hspace{1cm} (5.17)$$

Thus we can relate relative uncertainty in $A$ to relative uncertainty in $x$ by a function of $A$ itself. It should be noted that (5.17) is the sharpest inequality possible; i.e., the equality will occur for some $\delta A$. Recall that any of the three norms can be used in (5.17). Occasionally it is useful to write (5.17) as

$$\frac{||\delta x||}{||x + \delta x||} \leq ||A^{-1}|| \cdot ||\delta A||. \hspace{1cm} (5.18)$$

To make practical calculations of the condition number, the condition number based on the infinity norm has been used. Equation (5.10) shows that the norm of $A$ is the sum of the magnitudes of the elements of the row that gives the maximum sum. For the special form of the impedance matrix $Z$, this is always the center row. For the inverse matrix $Z^{-1}$, in general, one must check all the rows above and including the center row. However, if $k_o$ is small and the current distribution is essentially triangular, then the center row will have the maximum sum of the magnitude of the elements. Multiplying the norm of $Z$ by the norm of $Z^{-1}$ gives the condition number which determines maximum error for the current on any segment. It should be noted
that since $Z$ and $Z^{-1}$ are symmetric, $\text{cond}_\infty(Z) = \text{cond}_1(Z)$. Thus, if one estimates error in $Z$ as $||\delta Z||_1$, one can use the same condition number to measure relative error in current $I$ as $||\delta I||_1/||I + \delta I||_1$.

Although the use of the Euclidean norm is geometrically clearer, it requires far more computations. First, one must form the Hermitian matrix $ZZ^H$ before finding eigenvalues. This multiplication, in itself, takes on the order of $n^3$ multiplications. Rather than finding all the eigenvalues, one needs to find only the largest and smallest. A computer program has been written to find the largest eigenvalue by iteration. The iteration scheme operates on the principle that successive premultiplications of a vector by a matrix orients the resulting vector in the direction of the eigenvector associated with the largest eigenvalue in magnitude. Each iteration takes on the order of $n^2$ operations and the convergence of this scheme is so poor that the convergence can be hastened by a linear extrapolation of two successive vectors. Finding the minimum eigenvalue is even harder. All of the eigenvalues of a matrix can be shifted by subtracting the maximum eigenvalue from the diagonal, thus making the former minimum eigenvalue the new maximum eigenvalue in magnitude. This, however, does not appear to be an efficient numerical procedure because small eigenvalues tend to be close together causing slow convergence. Even worse, more accuracy is required so that when the largest eigenvalue is added to the final answer to obtain the true minimum eigenvalue, all significant figures are not lost. One can invert the matrix and find the maximum eigenvalue of that matrix. This eigenvalue will be the reciprocal of the minimum eigenvalue of the original matrix, but the inversion itself will take on the order of $n^3$ operations. All considered, using the
infinity norm, which is easy to calculate and gives the error in a Chebyshev sense, appears to be more practical.
CHAPTER VI. RESULTS

Current distributions and antenna impedances have been calculated for various plasma parameters. Tables 2 through 9 and Figures 7 through 14 give the results for eight different plasma conditions. A graph with axes $X$ and $Y^2$ is very useful for mapping plasma conditions. (See Figure 6.) The lines plotted on this graph correspond to various resonances in a lossless plasma. When dispersion surfaces are drawn over this plot, this display is known as a CMA diagram [13]. From this display one can see that the nature of the characteristic waves in the plasma changes abruptly as one crosses any one of these lines; hence, these lines form regions with similar properties. Each region has been sampled in the results presented. For our formulation, it is necessary to have some collision loss; therefore, for all cases, except for free space which corresponds to $X = 0$ and $Y^2 = 0$, $Z$ has been arbitrarily set to .05. In each case $H = 1$, $a = 1/250$, and results are presented for three values of $k_o$. Since there is symmetry around the center feed, the graphs of current distribution show the current from the end point to the feed. The condition number based on the infinite norm, as described in the last chapter, is presented with each calculation.

Balmain has developed an analytical expression for the input impedance using a quasi-static approximation which assumes a linear current distribution [14]. The impedance predicted by Balmain's formulation is presented for comparison. Where our method has determined a linear current distribution, the two values should agree.

The first case presented is the degenerate plasma condition of free space. (See Table 2 and Figure 7.) One can see that the data
Figure 6. Plasma Parameter Diagram Showing Cases Run
TABLE 2.

FREE-SPACE RESULTS (X = 0, Y^2 = 0, Z = 0)

(See Figure 7)

\( \varepsilon = 1.0 \quad \varepsilon' = 0 \quad \varepsilon_z = 1.0 \)

<table>
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<tr>
<th>( k_0 (m^{-1}) )</th>
<th>IMPEDANCE (OHM) – COMPUTED</th>
<th>ADMITTANCE (MILLIMHO) – COMPUTED</th>
<th>( \text{COND}_\infty )</th>
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</tr>
<tr>
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<td>35.8</td>
</tr>
<tr>
<td>1.5</td>
<td>68.86 + j 16.52</td>
<td>13.73 – j 3.295</td>
<td>139</td>
</tr>
</tbody>
</table>
Figure 7. Current Distribution for Free Space ($X = 0, Y^2 = 0, Z = 0$)

H = 1.000
X = 0.000
Y^2 = 0.000
Z = 0.000
KO = 0.100
KO = 0.750
KO = 1.500

MID POINTS OF SEGMENTS
FEED

MAG OF CURRENT (X10^-3)

0.0
1.5
3.0

PHASE OF CURRENT

0.0
1.5
3.0

MID POINTS OF SEGMENTS
FEED
TABLE 3.

UNDERDENSE ELLIPTIC PLASMA RESULTS (X = .1, Y^2 = .1, Z = .05)

(See Figure 8)

\[
\varepsilon = 0.8893 - j \ 6.760 \times 10^{-3} \quad \varepsilon' = 0.0348 + j \ 3.878 \times 10^{-3} \quad \varepsilon_z = 0.9002 - j \ 4.988 \times 10^{-3}
\]

<table>
<thead>
<tr>
<th>( k_0 (m^{-1}) )</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>Cond(_\infty)</th>
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</thead>
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<td>Computed</td>
<td>Balmain</td>
<td>Computed</td>
</tr>
<tr>
<td>.1</td>
<td>38.57 - j 5051</td>
<td>44.92 - j 6092</td>
<td>1.512 \times 10^{-3} + j .1980</td>
</tr>
<tr>
<td>.75</td>
<td>14.84 - j 547.2</td>
<td>5.989 - j 812.3</td>
<td>4.953 \times 10^{-2} + j 1.826</td>
</tr>
<tr>
<td>1.5</td>
<td>63.02 - j 31.88</td>
<td>2.994 - j 406.2</td>
<td>12.64 + j 6.392</td>
</tr>
</tbody>
</table>
Figure 8. Current Distribution for Underdense Elliptic Plasma ($X = .1, Y^2 = .1, Z = .05$)
Table 4.

<table>
<thead>
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<th>Impedance (OM)</th>
<th>Computed</th>
<th>Balmain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_0 ) (m(^{-1}))</td>
<td>( \epsilon' = 5.464 + j \cdot 1.098 )</td>
<td>( \epsilon' = 5.464 + j \cdot 1.098 )</td>
</tr>
<tr>
<td>( \epsilon_z = 0.6610 - j \cdot 1.995 \times 10^{-2} )</td>
<td>( \epsilon_z = 0.6610 - j \cdot 1.995 \times 10^{-2} )</td>
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</tr>
<tr>
<td>0.1</td>
<td>8216. - j 1581.</td>
<td>8277. - j 17910.</td>
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<td>0.75</td>
<td>1096. - j 1988.</td>
<td>1104. - j 2388.</td>
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<tr>
<td>1.5</td>
<td>572.7 - j 806.4</td>
<td>551.8 - j 1194.</td>
</tr>
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</table>
Figure 9. Current Distribution for Underdense Elliptic Plasma ($X = 0.4$, $Y^2 = 0.5$, $Z = 0.05$)
TABLE 5.
UNDERDENSE HYPERBOLIC PLASMA RESULTS (X = .9, Y^2 = .9, Z = .05)
(See Figure 10)

\[ \epsilon = -3.729 - j 4.389 \]
\[ \epsilon' = -1.059 + j 7.258 \times 10^{-2} \]
\[ \epsilon_z = -5.555 \times 10^{-2} + j 1.858 \times 10^{-2} \]

<table>
<thead>
<tr>
<th>( k_0 (m^{-1}) )</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>Cond ( \infty )</th>
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</thead>
<tbody>
<tr>
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<td>Balmain</td>
<td>Computed</td>
</tr>
<tr>
<td>( .1 )</td>
<td>612.9 + j 541.0</td>
<td>1157. + j 728.2</td>
<td>.9171 - j .8095</td>
</tr>
<tr>
<td>( .75 )</td>
<td>138.3 + j 182.9</td>
<td>154.2 + j 97.09</td>
<td>2.630 - j 3.478</td>
</tr>
<tr>
<td>( 1.5 )</td>
<td>119.5 + j 164.9</td>
<td>77.10 + j 48.55</td>
<td>2.881 - j 3.976</td>
</tr>
</tbody>
</table>
Figure 10. Current Distribution for Underdense Hyperbolic Plasma ($X = .9$, $Y^2 = .9$, $Z = .05$)
TABLE 6.

UNDERDENSE ELLIPTIC PLASMA RESULTS (X = .5, Y^2 = 1.5, Z = .05)

(See Figure 11)

\[ \varepsilon = 1.948 - j 0.2383 \quad \varepsilon' = -1.172 + j 0.2333 \quad \varepsilon_z = 0.5103 - j 0.02494 \]

<table>
<thead>
<tr>
<th>( k_0 ) (m^{-1})</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>( \text{Cond}_\infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>\begin{align*} \text{Computed} \ \text{Balmain} \end{align*}</td>
<td>\begin{align*} \text{Computed} \ \text{Balmain} \end{align*}</td>
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<td></td>
</tr>
<tr>
<td>.1</td>
<td>282.2 - j 2286.</td>
<td>364.7 - j 3161.</td>
<td>5.321\times10^{-2} + j .4309</td>
</tr>
<tr>
<td>.75</td>
<td>64.56 - j 145.8</td>
<td>48.62 - j 421.5</td>
<td>2.540 - j 5.735</td>
</tr>
<tr>
<td>1.5</td>
<td>387.4 + j 246.1</td>
<td>24.31 - j 210.8</td>
<td>1.839 - j 1.168</td>
</tr>
</tbody>
</table>
Figure 11. Current Distribution for Underdense Elliptic Plasma ($X = .5$, $Y^2 = 1.5$, $Z = .05$)
TABLE 7.

OVERDENSE ELLIPTIC PLASMA RESULTS (X = 1.5, Y^2 = .5, Z = .05)

(See Figure 12)

\[ \varepsilon = -1.927 - j \cdot 0.4376 \quad \varepsilon' = 2.045 - j \cdot 0.4119 \quad \varepsilon_z = 0.4963 - j \cdot 7.481 \times 10^{-2} \]

<table>
<thead>
<tr>
<th>( k_0 ) (m(^{-1}))</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>( \text{Cond}_\infty )</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Balmain</td>
<td>Computed</td>
</tr>
<tr>
<td>( 0.1 )</td>
<td>510.2 + j 2268.</td>
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<td>9.435 \times 10^{-2} - j \cdot 0.4196</td>
</tr>
<tr>
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<td>71.67 + j 422.4</td>
<td>90.44 + j 411.8</td>
<td>0.3904 - j \cdot 2.301</td>
</tr>
<tr>
<td>( 1.5 )</td>
<td>45.74 + j 328.3</td>
<td>45.22 + j 205.9</td>
<td>0.4163 - j \cdot 2.988</td>
</tr>
</tbody>
</table>
Figure 12. Current Distribution for Overdense Elliptic Plasma \((X = 1.5, Y^2 = .5, Z = .05)\)
TABLE 8.

OVERDENSE ELLIPTIC PLASMA RESULTS (X = 2.5, Y^2 = .5, Z = .05)

(See Figure 13)

\[
\epsilon = -3.876 - j \cdot 0.7294 \quad \epsilon' = 3.415 + j \cdot 0.6865 \quad \epsilon_z = -1.494 - j \cdot 1.1247
\]

<table>
<thead>
<tr>
<th>k_o (m^-1)</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>Cond_{\infty}</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Computed</td>
<td>Balmain</td>
<td>Computed</td>
</tr>
<tr>
<td>.1</td>
<td>212.4 + j 1151.</td>
<td>266.0 + j 1497.</td>
<td>.1544 - j .8394</td>
</tr>
<tr>
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<td>31.29 + j 252.1</td>
<td>35.46 + j 194.8</td>
<td>.4847 - j 3.906</td>
</tr>
<tr>
<td>1.5</td>
<td>22.27 + j 208.1</td>
<td>17.73 + j 99.71</td>
<td>.5086 - j 4.752</td>
</tr>
</tbody>
</table>
Figure 13. Current Distribution for Overdense Elliptic Plasma \(X = 2.5, Y^2 = .5, Z = .05\)
TABLE 9.

OVERDENSE HYPERBOLIC PLASMA RESULTS \(X = 2.0, Y^2 = 2.0, Z = 0.05\)

(See Figure 14)

\[
\begin{align*}
\varepsilon &= 2.9655 - j 0.2958 \\
\varepsilon' &= -0.9420 + j 0.0 \\
\varepsilon_z &= -0.3289 - j 0.6644
\end{align*}
\]

<table>
<thead>
<tr>
<th>(k_0 (m^{-1}))</th>
<th>IMPEDANCE (OHM)</th>
<th>ADMITTANCE (MILLIMHO)</th>
<th>Cond(_\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Computed</td>
<td>Balmain</td>
<td>Computed</td>
</tr>
<tr>
<td>(0.1)</td>
<td>147.1 - j 1516</td>
<td>792.0 - j 1972</td>
<td>6.340 (10^{-2}) + j 0.6534</td>
</tr>
<tr>
<td>(0.75)</td>
<td>-23.27 - j 21.73</td>
<td>105.6 - j 262.9</td>
<td>22.96 + j 22.43</td>
</tr>
<tr>
<td>(1.5)</td>
<td>-3180. - j 1872</td>
<td>52.79 - j 131.4</td>
<td>-0.2335 + j 0.1375</td>
</tr>
</tbody>
</table>
Figure 14. Current Distribution for Overdense Hyperbolic Plasma (X = 2.,\nY^2 = 2.,Z = .05)
agree generally with well-known results. The condition number for the impedance matrix $Z$ is alarmingly high. For $k_0 = 1.5$, which corresponds to a dipole slightly longer than a half wavelength, the condition number is 140. This means that an error of .1 per cent in formulating the self term could cause a .14 per cent error in the input impedance. One should be aware that for free space the Green's function is much simpler and allows one to use better expansion functions in the method of moments. Mautz [15] and Klock [16] have done computer work for more arbitrarily shaped, thin-wire structures that agrees better with measurements than our results for free space. However, condition numbers for some of these results are still high.

Results for anisotropic plasmas are given in Tables 3 through 9 and Figures 8 through 14. The validity of results varies with the region. Although there is not a unique wavelength in an anisotropic plasma, in several cases the current distribution looks like a frequency-scaled, free-space current distribution. Results appear best in regions which make an antenna look shorter, and worse in regions which make an antenna appear longer, just as in free space the stability was best for short antennas. For example, when $H = 1$ and $k_0 = 1.5$, an antenna in free space is approximately half-wave resonant, but an antenna in a plasma with $X = 2$ and $Y^2 = 2$ is approximately full-wave resonant. When the current distribution is no longer linear, Balmain's approximation is no longer valid and there is no standard analytical approximation.

Several regions require special comment. In the overdense hyperbolic region, our results predict a negative radiation resistance for $k_0 = 7.5$ and 1.5, which is clearly incorrect. The overdense elliptic
region \((X^2 > 1, Y^2 < 1)\) is more believable but has some unusual properties. The current decreases rapidly from the feed, making the current distribution curve concave upward, the reverse of the usual curvature of a sinusoidal distribution. The condition numbers indicate a much better conditioned problem than free space. Unlike other regions, the condition number decreases and the impedance agrees better with Balmain's approximation as \(k_o\) increases, although the current distribution is moving away from the assumed linear distribution. In general, regions separated only by a hybrid frequency resonance show no drastic changes.

There is little experimental work on current distribution which can be compared to our theoretical results. Some work was done by Snyder and Mittra [17], but they were unable to accurately determine the plasma parameters. An attempt was made to compare our results to work done by Ishizone et al. [18]. Their results did not indicate their antenna radius or a plasma loss parameter \(Z\) and so our computations were made with an arbitrary \(H/a\) ratio of 250 and a \(Z\) parameter of .05. Their measurements are all for overdense plasmas \((X\) from 4. to 50) but in both elliptic and hyperbolic regions \((Y^2\) from 0 to 1.04). In all cases their current distributions have standing-wave characteristics, often with the current increasing at the feed. However, for all of their cases for which we have made computations, our current distribution simply decreases rapidly from the feed similar to our computed results for \(X = 2.5, Y^2 = .5\). Thus, there is rather poor qualitative agreement between our results and their measurements.

In trying to understand why blatantly incorrect answers are sometimes obtained, one can note certain patterns between the condition number, the validity of the impedance, and the relative importance of the four terms of the Green's function \(E_z\). (See Tables 10, 11, and 12.)
### TABLE 10

**ELECTRIC FIELD CALCULATIONS FOR H = 1, k = .1, a = \(\frac{1}{250}\), X = .1, \(\chi^2 = .1\), Z = .05**

<table>
<thead>
<tr>
<th>FIELD ELEMENTS</th>
<th>No.</th>
<th>(E_1)</th>
<th>(E_2)</th>
<th>(E_F = E_1 + E_2)</th>
<th>(E_T)</th>
<th>(\text{SUM} = \text{E}_1 + \text{E}_2 + \text{E}_T)</th>
</tr>
</thead>
<tbody>
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<td>-0.4002E-07</td>
<td>-0.5690E-06</td>
<td>-0.1542E-03</td>
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<td>-0.6918E-03</td>
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<tr>
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<td>0.1341E-05</td>
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<td>-0.5300E-06</td>
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<td>0.1273E-02</td>
</tr>
<tr>
<td>3</td>
<td>0.1114E-04</td>
<td>-0.1154E-06</td>
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<td>0.1853E-02</td>
</tr>
<tr>
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<td>0.355E-03</td>
<td>-0.4014E-05</td>
<td>-0.1192E-04</td>
<td>-0.1161E-02</td>
<td>-0.1887E-00</td>
<td>0.2434E-02</td>
</tr>
<tr>
<td>5</td>
<td>0.1256E-03</td>
<td>-0.1653E-05</td>
<td>-0.4976E-05</td>
<td>-0.1867E-00</td>
<td>0.3015E-02</td>
<td>0.1254376E-03</td>
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<tr>
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<td>0.6362E-02</td>
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<td>-0.1991E-00</td>
<td>0.3599E-01</td>
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<tr>
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<td>-0.1885E-00</td>
<td>0.417E-02</td>
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<tr>
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<td>-C.6295 E 00</td>
<td>-C.522700 F 05</td>
<td>-C.546721 E 06</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 11.**

**ELECTRIC FIELD CALCULATIONS FOR H = 1, k = 1, a = {\frac{1}{2}}, \lambda = 2,\ y^2 = 2, z = 0.05, X = 2.**
<table>
<thead>
<tr>
<th>FILM ELEMENTS</th>
<th>E / EFS1 / EFS2</th>
<th>E ZF</th>
<th>E / EFS1 / EFS2</th>
<th>E / EFS1 / EFS2</th>
<th>E / EFS1 / EFS2</th>
<th>E / EFS1 / EFS2</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.7377E-01</td>
<td>0.665E-01</td>
<td>0.5747E-00</td>
<td>0.1687E-06</td>
<td>0.9006E-06</td>
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</tr>
<tr>
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</tr>
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<td>0.6065E-06</td>
<td>0.3225E-06</td>
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</tr>
<tr>
<td>4</td>
<td>0.118E-01</td>
<td>0.382E-02</td>
<td>0.5529E-00</td>
<td>0.1673E-06</td>
<td>0.8887E-06</td>
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<tr>
<td>5</td>
<td>0.115E-01</td>
<td>0.222E-02</td>
<td>0.5529E-00</td>
<td>0.6886E-06</td>
<td>0.3654E-06</td>
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<tr>
<td>6</td>
<td>0.117E-01</td>
<td>-0.665E-03</td>
<td>-0.5486E-00</td>
<td>0.1486E-06</td>
<td>0.1841E-06</td>
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<tr>
<td>7</td>
<td>0.110E-01</td>
<td>-0.216E-03</td>
<td>-0.5473E-00</td>
<td>0.2005E-06</td>
<td>0.1061E-06</td>
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<tr>
<td>8</td>
<td>0.117E-01</td>
<td>-0.220E-03</td>
<td>-0.5324E-00</td>
<td>0.1257E-06</td>
<td>0.6642E-06</td>
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<td>9</td>
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<td>-0.391E-04</td>
<td>-0.5324E-00</td>
<td>0.8405E-06</td>
<td>0.4435E-06</td>
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<td>10</td>
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<td>-0.566E-04</td>
<td>-0.5254E-00</td>
<td>0.5915E-06</td>
<td>0.3103E-06</td>
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<tr>
<td>11</td>
<td>0.229E-06</td>
<td>0.124E-04</td>
<td>0.5244E-00</td>
<td>0.2254E-06</td>
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<tr>
<td>12</td>
<td>0.114E-04</td>
<td>0.160E-04</td>
<td>0.5200E-00</td>
<td>0.3218E-06</td>
<td>0.1624E-06</td>
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<tr>
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<td>0.103E-06</td>
<td>0.124E-04</td>
<td>0.5154E-00</td>
<td>0.2476E-06</td>
<td>0.1295E-06</td>
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<tr>
<td>14</td>
<td>0.135E-06</td>
<td>0.275E-01</td>
<td>0.5104E-00</td>
<td>0.1945E-06</td>
<td>0.1015E-06</td>
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<tr>
<td>15</td>
<td>0.149E-04</td>
<td>0.304E-04</td>
<td>0.5074E-00</td>
<td>0.1550E-06</td>
<td>0.8106E-06</td>
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<tr>
<td>16</td>
<td>0.116E-05</td>
<td>0.345E-04</td>
<td>0.5095E-00</td>
<td>0.1263E-06</td>
<td>0.6568E-06</td>
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<tr>
<td>17</td>
<td>0.111E-05</td>
<td>0.391E-04</td>
<td>0.4972E-00</td>
<td>0.1040E-06</td>
<td>0.5792E-06</td>
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<tr>
<td>18</td>
<td>0.116E-06</td>
<td>0.449E-04</td>
<td>0.4932E-00</td>
<td>0.8634E-06</td>
<td>0.4479E-06</td>
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<tr>
<td>19</td>
<td>0.147E-06</td>
<td>0.488E-04</td>
<td>0.7299E-00</td>
<td>0.3755E-06</td>
<td>0.2719E-06</td>
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<tr>
<td>20</td>
<td>0.118E-06</td>
<td>0.484E-04</td>
<td>0.6191E-00</td>
<td>0.3184E-06</td>
<td>0.2719E-06</td>
<td>0.3184E-06</td>
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</table>
As expected, the impedance matrix $Z$ is best conditioned when it is most nearly diagonal, or when the singular terms which decrease rapidly with distance are dominant over the finite term. For the overdense elliptic plasma which gives good results as compared to Balmain's approximation, the field contribution on a segment distance $2H$ from the source due to the finite term is two orders of magnitude less than the total field, while for the overdense hyperbolic case $X = 2, Y^2 = 2$, which gives negative radiation resistances, the finite term is an order of magnitude larger than the total field. The hypothesis that the difficulties encountered in some regions are due to an incorrect evaluation of the $E_{zf}$ term, either through invalid approximations or poor numerical quadrature, is strengthened by the tendency of results to get worse as $k_o$ increases. In the singular terms, $k_o$ appears only in the coefficient, while in the finite term, $k_o$ is a factor in the neglected terms of a Taylor series. A method is currently being investigated to evaluate fields for which the finite term is significant by an asymptotic expansion of the transformed Green's function.

Calculations were made to test the sensitivity theory discussed in the last chapter. Consistent with the high condition numbers, it was found that a 1 per cent perturbation of the diagonal term in the $Z$ matrix can cause a 30 per cent change of the impedance both for free space and the hyperbolic overdense plasma. One should note that we are unable to obtain the condition number of an error-free matrix. If errors have occurred which make the matrix more diagonal, a deceptively low condition will be calculated. In cases where large error must be present, such as for the overdense hyperbolic plasma, one
cannot claim that the results approximate the true results within definite limits when only the condition number of the matrix with errors is known.
CHAPTER VII. CONCLUSIONS

Using the method of moments we have been able to calculate the current distribution on an antenna oriented parallel to the static magnetic field in a plasma. For most plasma regions, reasonable impedances are obtained that agree with the Balmain approximation when $k \rho$ is small. Calculations on the overdense hyperbolic plasma, however, give unrealizable results. The exact cause of this discrepancy is not well understood; work is continuing on a different method of calculating the fields that will be more accurate and faster for segments far from the feed.

A method of quantitatively analyzing the degree of ill-conditioning has been discussed. This method applies not only to this problem but to any application of the method of moments. In particular, it is suggested that the stability of free-space, thin-wire problems be evaluated. This method can also be applied to inversion problems arising from remote sensing studies.
APPENDIX (LISTING OF PROGRAM USED)

C PREPARES E'S TO BE INVERTED
C LIMITS CHANGES
       COMPLEX E,E1,EZ,EZS,EZFS1,T(55),EZFS2,EZFN1,T1(30,25),EZ,AD,ZIMP,
       1 EZFN2,U,ER(21),Y1(21),THOL,TEZ(21)
       DIMENSION PM(30),PP(30),ZZ(30),SM(2),PM1(3,30),PP1(3,30)
       REAL SC1(12)/1.0,7.5,6.5,5.4,4.3,3.2,2.6,2.1,1.5,1.0,1.0,1.0,
       1 REAL K0/576/,PER(1)/0.,0.01,1./
       REAL SS(4)/4.53,5.35,13.47,50./
       REAL SP(2)/-3.,1.50/,K1(3)/1.0,7.5,1.5/
       PI=3.161593
       H=1.
       A=1./25C.
       CALL CCP1PL(5,1,,-3)
       C M IS NUMBER OF SEGMENTS
       B=SORT(2,1)*A
       M=21
       XM=M
       DO 100 KJOB=1,1
       READ (5,25) XC,YC,ZC
       25 FORMAT (3F10.6)
       XC=SQR(T(YC)
       SS=0.
       SS1=0.
       NPLOT=3
       C XC,YC, AND ZC ARE 3 PLASMA PARAMETERS (DON'T USE X,Y,Z)
       DL=H/(XM+2.)
       U=(1.,0.)-(0.,1.)*ZC
       E=(1.,0.)-U*XC/(U**2-YC**2)
       E1=E1/E
       EZ=(1.,0.)-XC/U
       EZ=EZ/E
       DO 11 IXX=1,NPLOT
       KO=K1(IXX)
       PRINT 24,KO,U,XC,YC2,ZC,E,E1,EZ,M
       24 FORMAT(1I,1,KO=E,F6.3,F6.3,1X,'E1=',2G16.7,5X,'XC=',2G16.7,5X,'YC=',
       1 G16.7,5X,'ZC=',I1,
       PRINT 4
       4 FORMAT(1H1,FIELD ELEMENTS'/ NO. ',T17,'E1S',T43,'EZFS1','EZFS2',
       1 T75,'EZFN1','T109','SUM')
       C ECRYPT FOR EFS, E2DL1 FOR EZFS1
       KK=MNO(30,M)
       DO 2 I=1,KK
       X=I-1
       Z=X*2.*DL
       CALL E2DL1(KG,E,E1,EZ,B,DL,Z,EZFS1,EZFS2)
       CALL ECRYPT(KO,E,E1,EZ,AD,DL,Z,EZS)
       CALL EZFN(KO,E,E1,EZ,DL,Z,EZFS1,EZFN1,ER(I),Y1(I))
       T(I)=EZS*EZFS1+EZFN1+EZFS2
       2 WRITE (6,51)I,EZS,EZFS1,EZFN1,T(I),EZFS2
       51 FORMAT (1H1,I3,6E15.4,2E18.7/1H,33X,2E15.4)
       IF(M .LT. 31) GO TO 12
E2DL1

KK=M1NO(50,M)
DO 5 I=31,KK
X=I-1
Z=X*2.*DL
CALL E2DL1(KO,E,E1,EZ,B,DL,Z,EZS,EZFS1,EZFS2)
CALL EZFI1(KO,E,E1,EZ,DL,Z,EZFN1,ER(I),Y1(I))
T(I)=EZS+EZFS1+EZFN1+EZFS2
5 WRITE (6,51) I,EZS,EZFS1,EZFN1,ER(I),Y1(I)

CALL EZFN(KO,E,E1,EZ,DL,Z,EZFN1,ER(I),Y1(I))
T(I)=EZS+EZFS1+EZFN1+EZFS2
5 WRITE (6,51) I,EZS,EZFS1,EZFN1,ER(I),Y1(I)
12 WRITE (6,104)
104 FORMAT (1HO,'INTEGRATION AND ERROR'//"NO.",T15,'UNNORMALIZED EZFN1'
1*105,156,'ERROR*')
COND=0.
DO 71 I=1,M
WRITE (6,3) I,Y1(I),ER(I)
3 FORMAT (1H ,13,5X,2E15.7,5X,2E15.7)
DO 6 J=1,M
K=M *(I-1)+J
KK=IABS(J-1)+1
6 T1(K)=T(KK)
M1=IABS(M/2+I-1)+1
COND=COND+CABS(T(M1))
71 CONTINUE
WRITE (6,72) COND
72 FORMAT (1H ,5X,'NORM BEFORE INVERSION =',E16.7)
CHOLD=COND
CALL LINEQ(M,T1)
I=0
CONDIT=0.
DO 73 I=1,M
COND=0.
DO 74 J=1,M
KK=M *(I-1)+J
74 COND=COND+CABS(T1(KK))
IF (COND .LE. CONDIT) GO TO 73
I=1
CONDIT=COND
73 CONTINUE
WRITE (6,75) CONDIT,I
75 FORMAT (1H ,5X,'NORM AFTER INVERSION =',E16.7)
COND=COND*CHOLD
WRITE (6,76) COND
76 FORMAT (1H ,5X,'ESTIMATE OF CONDITION =',E16.7)
PRINT 7
7 FORMAT('1')
M=M**2-2
9 FORMAT('6',I4,3(5X,2E16.7))
PRINT 15
15 FORMAT('6',I4,3(5X,2E16.7))
PRINT 14
14 PRINT 9,I,T1(I),T1(I+1),T1(I+2)
C
M=M**2
\[ F = -1 / (2 \cdot DL) \]

DO 13 I = 1, M2

13 \[ T1(I) = F \cdot T1(I) \]

MM = ((M2-1)/2+1)

AD = T1(MM)*1000.

ZIMP = 1./T1(MM)

M1 = ((M-1)/2) * M

M3 = (M1+1)/2

M4 = M4+1

M5 = M1+M3

DO 17 I = M4, M5

XX = CARDS(T1(I))

XY = ATAN2(AIMAG(T1(I)), REAL(T1(I)))

17 \[ T1(I) = CMPLX(XX, XY) \]

PRINT 21

21 FORMAT('0///', 10X, 'CURRENTS (MAG AND PHASE)' )

DO 20 I = 1, M3

20 PRINT 22, T1(I+M1)

22 FORMAT(' ', 10X, 'CURRENTS (MAG AND PHASE)' )

PRINT 23, AD, ZIMP

23 FORMAT('0///', 10X, 'ADMITTANCE = 2G16.7, MILLIMHOS'/

1 10X, 'IMPEDEANCE = 2G16.7, OHMS')

C START PLOTTING

100 M4 = M3+1

DO 30 I = 1, M3

30 PM(I+1) = REAL(T1(I+M1))

IF (PM(I+1) > SS) SS = PM(I+1)

PM(IXX, I) = -PM(I+1)

PP(I+1) = AIMAG(T1(I+M1))

PP(I+1) = 0.

PM(IXX, I) = 0.

PP(I) = 0.

PP(IXX, I) = 0.

11 CONTINUE

SS = SS/4.

DO 53 K = 1, 7

53 IF (SS.GE.1.) GO TO 54

54 SS = SS/10.

55 K = K-1

DO 55 I = 2, 12

56 SM(2) = SC1(I-1)/10.*K+1.E-6

SM(I) = 0.

C START PLOTTING

CALL CCPIRA

CALL CCP1PL(4.5, 2, -3)

CALL CCP1PL(0, 0.3)

CALL CCP1PL(0.6, 2)

CALL FACTOR (0.5)

CALL CP2SY(-7.9, 5.3, -90, 0.3)

CALL CP3NR(0, 0, -3, 90, 3)
CALL CCP2SY(-7.4,.5,.3,'X= ',90.,3)
CALL CCP3NR(0.,0.,-3,XC,90.,3)
CALL CCP2SY(-7.0,.5,.3,'Y= ',90.,3)
CALL CCP3NR(0.,0.,-.3,YC,90.,3)
CALL CCP2SY(-6.6,.5,.3,'Z= ',90.,3)
CALL CCP3NR(0.,0.,-.3,ZC,90.,3)
YP=.5
XP=-6.2
DO 52 K=1,NPLOT
L=K
IF (K.EQ.3) L=0
CALL CCP2SY(XP,YP,.3,'K0=',90.,3)
CALL CCP3NR(0.,0.,-.3,K1(K),90.,3)
CALL CCP2SY(0.,0.,-.15,L,90.,-1)
52 XP=XP+.4
CALL FACTOR(1.1)
CALL CCP5AX(0.,0.,'MAG OF CURRENT',14.4,000,180.,SM)
CALL CCP2SY(.3,1.370,.15,'MID POINTS OF SEGMENTS',90.,22)
CALL CCP2SY(.3,5.800,.15,'FEED',90.,4)
DO 46 I=1,M4
XXI=I-1
ZZ(I)=12./(XXI+1.)*XXI
CALL CCP2SY(0.,ZZ(I),07,13,90.,-1)
DO 46 K=1,NPLOT
L=K
IF (K.EQ.3) L=0
PM1(K,1)=(PM1(K,1)+SM(11))/SM(2)
46 CALL CCP1PL(PM1(K,1),ZZ(I),90.,-1)
DO 49 K=1,NPLOT
CALL CCP1PL(PM1(K,1),ZZ(I),90.,3)
DO 49 I=2,M4
49 CALL CCP1PL(PM1(K,1),ZZ(I),2)
CALL CCP1PL(2.5,0.,-3)
CALL FACTOR(0.1)
CALL CCP5AX(2.0,0.,'PHASE OF CURRENT',6,4.000,180.,SP)
CALL CCP2SY(.3,1.370,.15,'MID POINTS OF SEGMENTS',90.,22)
CALL CCP2SY(.3,5.800,.15,'FEED',90.,4)
DO 47 I=2,M4
47 CALL CCP2SY(PPI(K,1),ZZ(I),007,13,90.,-1)
DO 47 K=1,NPLOT
L=K
IF (K.EQ.3) L=0
PPI(K,1)=PPI(K,1)/SP(2)
47 CALL CCP2SY(PPI(K,1),ZZ(I),007,13,90.,-1)
DO 50 K=1,NPLOT
CALL CCP1PL(PPI(K,2),ZZ(2),3)
DO 50 I=3,M4
50 CALL CCP1PL(PPI(K,1),ZZ(I),2)
CALL CCP1PL(4.,-2.,-3)
1000 CONTINUE
CALL EXIT
END
SUBROUTINE LINEQ(LL,C)
C 2/5/71 C KLEIN CHANGED ARITH IFs TO LOGICAL IFs
COMPLEX C(I),STOR,STO,ST,S
DIMENSION LR(58)
DO 20 I=1,LL
LR(I)=I
20 CONTINUE
M1=0
DO 18 M=1,LL
K=M
DO 2 I=M,LL
K1=M1+I
K2=M1+K
IF (CABS(C(K1))-CABS(C(K2))) .LE. 0. GO TO 2
6 K=I
2 CONTINUE
LS=LR(M)
LR(M)=LR(K)
LR(K)=LS
K2=M1+K
STOR=C(K2)
J1=0
DO 7 J=1,LL
K1=J1+K
K2=J1+M
ST=STOR
C(K1)=STOR
C(K2)=STO/STOR
J1=J1+LL
7 CONTINUE
K1=M1+M
C(K1)=1./STOR
DO 11 I=1,LL
11 K1=M1+I
ST=C(K1)
C(K1)=0.
J1=0
DO 10 J=1,LL
K1=J1+I
K2=J1+M
C(K1)=C(K1)-C(K2)*ST
J1=J1+LL
10 CONTINUE
11 CONTINUE
M1=M1+LL
18 CONTINUE
J1=0
DO 9 J=1,LL
IF(J-LR(J) .EQ. 0) GO TO 8
14 LR(J)=LR(J)
J2=(LRJ-1)*LL
21 DO 13 I=1,LL
K2 = J2 + 1
K1 = J1 + 1
S = C(K2)
C(K2) = C(K1)
C(K1) = S

13 CONTINUE
LR(J) = LR(LRJ)
LR(LRJ) = LRJ
IF(J - LR(J) .NE. 0) GO TO 14

8 J1 = J1 + LL
9 CONTINUE
RETURN
END
SUBROUTINE EZFN(KO, E, E1, EZ, DL, Z, EZFN1, ERROR, Y1)

REAL KO
REAL*8 XY/100/

COMPLEX E, E1, EZ, X, EZFN1, ERROR, Y1
REAL*8 P12/1.  5707963268/
COMPLEX *16 Y, DAPHI1, DAPHIO, DPHI2, ERR
X=DPHI0(DBLE(KO), DCMPLX(DBLE(REAL(E)), DBLE(AIMAG(E))))
1 DCMPLX(DBLE(REAL(E1)), DBLE(AIMAG(E1))))
2 DCMPLX(DBLE(REAL(EZ)), DBLE(AIMAG(EZ))))
3 DBLE(DL), DBLE(Z)
CALL ROMBREG(0, P12, DAPHI1, A, Y, 10-4, 10-3, ERR)
ERROR=ERR
Y1=Y
K=REALZ(DPHI2(XY))
X=Y
EZFN1=(0., 15.)*KG**2*X
RETURN
END
SUBROUTINE E2DL1 (KO, E, E1, E2, A, DL, Z, E2S, E2FS1, E2FS2)
C THIS SUBROUTINE CALCULATES FIRST THREE E TERMS USING 2 DL MODEL
C COMPLEX E, E1, E2, E2S, E2FS1, E2FS2, A1, ZAP, ZAM, ZAPH, ZAMH, ZF2, ZF3
C RETURNS COMPLEX NUMBERS
C THIS REPLACES SUBROUTINE E2DL
REAL KO
A12=EZ*A**2
ZPDL=Z+DL
ZMDL=Z-DL
ZPDL2=ZPDL**2
ZMDL2=ZMDL**2
ZAP = ZPDL**2 + A12
ZAM=ZMDL**2 + A12
ZAPH=CSQRT(ZAP)
ZAMH=CSQRT(ZAM)
ZF2=CLLOG((ZPDL+ZAPH)/(ZMDL+ZAMH))
ZF3=ZPDL/ZAPH - ZMDL/ZAMH
C EZS SIMPLIFIED 2/7/70
EZS=((0.,15.)/(KO*DL*E)*((ZPDL/((ZAP+ZAPH)-ZMDL/((ZAM+ZAMH))))
EZFS1=(0.,-7.5)*((K0/0.1)*((2.,0.)*ZF2-ZF3)
IF ( REAL(EZ) .NE. 1. .OR. AIMAG(EZ) .NE. 0. .OR. REAL (E1)
1 *NE. 0. .OR. AIMAG(E1) .NE. 0. ) GO TO 1
EZFS2 = (0.,0.)
GO TO 2
1 T1=ALOG (((ZPDL/SQRT(ZPDL)*D)**2)/(ZMDL/SQRT(ZMDL)^2))
EZFS2 = (0.,15.)*E1**2/(T-(1.,0.)*E1**2*(KO/0.1)**2*ZF2*.5,0.)
1 (T-(1.,0.)*ZF3*.5,0.)*((1.,0.)-EZ))
C 1
2 RETURN
END
SUBROUTINE ECYL(KG, E, El, EZ, A, DL, Z, EIZ)

C THIS EVALUATES EIZ USING ELLIPTICAL INTEGRALS
COMPLEX E, El, EZ, EIZ, X, A12, KP2, KM2
REAL KO

C THIS FUNCTION STATEMENT EVALUATES ELLIPTICAL INTEGRAL
EL(X)=(((A4*X+A3)*X +A2)*X+A1)*X+1.
1 =((((B4*X+B3)*X +B2)*X +B1)*X)*CLOG(X)
DATA A1, A2, A3, A4/.4432514, 6.260601E-?, 4.757384E-2, 1.736506E-7/
DATA B1, B2, B3, B4/.2A99837, 9.220180E-2, 4.69698E-2, 5.264496E-3/
ZPDL=Z+DL
ZMDL=Z-DL
A12=EIZ*A**2
KP2=4.*A12/(ZPDL**2+4.*A12)
KM2=4.*A12/(ZMDL**2+4.*A12)
EIZ=(0., 15.)/((KO*3.141593*DL*A)*CLOG(EIZ))/
1 (CLOG(KP2)*EL((1., 0.)-KP2)/ZPDL-
2 CLOG(KM2)*EL((1., 0.)-KM2)/ZMDL)
RETURN
END
SUBROUTINE RDMBR G(A,B,T,TTT,NMIN,Y,TR,HMIN,ERR)

IMPLICIT COMPLEX*16(A-G,O-W,Y-Z)
IMPLICIT REAL*8(H,X)
REAL *8 TR,A,B,DREALZ,DIMAGZ

TNT MUST BE DECLARED EXTERNAL IN CALLING PROGRAM

ADAPTIVE INTERVAL ROMBERG INTEGRATION ROUTINE

DIMENSION FF2(10),FF4(10),HF(10)
NSUB=10
KODE=0
HMAX=(B-A)/NMIN
ERR=(.0,.0,.0,.0)
Y=(.0,.0,.0,.0)
X4=A
F4=TNT(A)

DO 1 JJ=1,NMIN

INITIALIZE LARGEST INTERVAL POSSIBLE

X0=X4

F0=F4
X4=X0+HMAX
F4=TNT(X4)
X2=X0+.5*HMAX
LEVEL=0
F2=TNT(X2)
H=HMAX

START LOOP

4 W=F0+F4

H FACTORED OUT OF T(N,M) TERMS

T00=W/2DO
WW=W+2DO*F2
T01=WW/4DO
T10=(4DO*T01-T001/3DO)
IF (DREALZ(T10).EQ.0.,0DO) GO TO 20
IF (DMAX1(DABS(DREALZ(T10))-DREALZ(TT1))/DREALZ(TT1)).GT.
1 DABS((DIMAGZ(T10)-DIMAGZ(T01))/DIMAGZ(T10)).LF.
2 TR) GO TO 8

GO TO 21

20 IF (DABS((DIMAGZ(T10)-DIMAGZ(T01))/DIMAGZ(T10)).LE.TR) GO TO 8

1 CONTINUE

F1=TNT(X0+.25*H)
F3=TNT(X4-.25*H)
T2=1(WW+2DO*(F1+F3))/8DO
T11=(4DO*T02-T011/3DO
T20=(16DO*T11-T101)/15DO
IF (DREALZ(T10).EQ.0.,0DO) GO TO 22
IF (DMAX1(DABS(DREALZ(T10)-DREALZ(T11))/DREALZ(T11)).GT.
2 .LF., TR) GO TO 2

GO TO 23

22 IF (DABS((DIMAGZ(T20)-DIMAGZ(T10))/DIMAGZ(T10)).LE.TR) GO TO 8

23 CONTINUE

IF (LEVEL .LT. NSUB .AND. H .GT.HMIN) GO TO 3
KODE = 1
GO TO 2
C
SUBDIVIDE

3  \( H = H/2 \).
LEVEL = LEVEL + 1
FF2(LEVEL) = F3
HF(LEVEL) = H
FF4(LEVEL) = F4
F4 = F2
F2 = F1
X4 = X3 + H
GO TO 4
8  \( Y = Y + T10 \times H \)
ERR = ERR + (T10 - T01) \times H
GO TO 9
C
UNSUBDIVIDE

2  \( Y = Y + T20 \times H \)
ERR = ERR + (T20 - T11) \times H
9  IF(LEVEL .EQ. 0) GO TO 1
F0 = F4
F2 = FF2(LEVEL)
F4 = FF4(LEVEL)
H = HF(LEVEL)
X0 = X4
X4 = X0 + H
LEVEL = LEVEL - 1
GO TO 4
1 CONTINUE
IF(KODE .EQ. 1) PRINT 5
5 FORMAT('ACCURACY SPECIFIED CANNOT BE OBTAINED')
RETURN
END
COMPLEX FUNCTION DHPHIO*16(KO,E,E1,EZ,DL,Z)

IMPLICIT REAL*8(A-H,0-Z)

C EXTENSIVE CHANGES 7-28-71 TWO INTEGRAND HALVES SUMMED AND COUNTER
C FOR TIMES CALLED ADDED
C CHANGES MADE 5/1/71 HC ELIMINATED AND INCLUDED IN HPHI
C CHANGES MADE 4/20/71 ABOUT SQRT OF N1
C DON'T DO UNIAXIAL CASE
C CHANGES MADE 4/15/71 HPHI NO LONGER NEEDS HCFO

COMPLEX*16 N1,U1,U2,COEF,H,E1,EZ,E,N12,ST,DSCP,N22,DUM,E12,A,B,C,
H1,D1,TNT,DHPHI2
REAL*8 KC
I=0
DHPHI2=(C00,000)

RETURN
ENTRY TNT(PHI)
I=I+1
SP=DSIN(PHI)
SP2=SP**2
CP=DCOS(PHI)

IF(CDABS(E1).EQ.CD0) GO TO 1

H FOR E1/=0
E12=E1**2
CP2=CP**2
A=SP2+EZ*CP2
B=EZ*(100*CP2)+(100-E12)*SP2
C=(100-E12)*EZ
DSCP=CDSQRT(B**2-400*A*C)
N12=(B-DSCP)/(200*A)
N22=(B+DSCP)/(200*A)
DUM=(000,-200)*SP*SP2/(A**2*(N12-N22))
D1=100*E12*CP2
H=DUM*(100-E12-N12*D1)
H1=-DUM*(100-E12-N22*D1)
GO TO 2

H FOR E1=0
1 N12=(100,CD0)
N22=N12
H=(000,200)*SP*SP2
H1=(000,000)

C CALCULATING SIN( ) / ( ) TERM
2 ST=CDSQRT(N12*E)
IF(DIMAGZ(ST).GT.000) ST=-ST
U1=(ST*KO*CP)

U2=U1*DL
IF(CDABS(U2).LT.1D-3) GO TO 3
COEF=CDRIZ(U2)/U2
GO TO 4

3 COEF=(100,000)

4 TNT =H**CDEXP(-(100,100)*U1*CDABS(Z))/COEF*ST
ST=CDSQRT(N22*E)
IF(DIMAGZ(ST).GT.000) ST=-ST
U1=(ST*KO*CP)
U2=U1*DL
IF ( COABS(U2) .LT. 1D-3 ) GO TO 6
   COEF=COSIN(U2)/U2
   GO TO 8
6  COEF=(1D0,0D0)
8  TNT=H1* CDEXP(-1D0,1D0)*U1*DABS(Z)*COEF*ST*TNT
9  RETURN
ENTRY DPHI2(X)
   DPHI2=DCMPLX(DBLE(FLOAT(I)),0D0)
RETURN
END
REFERENCES


The current distribution and impedance of a thin cylindrical antenna with parallel orientation to the static magnetic field of a lossy magnetoplasma is calculated with the method of moments. The electric field produced by an infinitesimal current source is first derived. Results are presented for a wide range of plasma parameters. Reasonable answers are obtained for all cases except for the overdense hyperbolic case. A discussion of the numerical stability is included which not only applies to this problem but to other applications of the method of moments.
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