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TWO DIFFERENT APPROACHES FOR  
A CONTROL LAW OF SINGLE GIMBAL  
CONTROL MOMENT GYROS

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16. ABSTRACT  In the field of momentum exchange attitude control systems, single gimbal control moment gyros (SGCMG's) are of increasing interest. This report presents a gimbal angle approach and a gimbal rate approach for the SGCMG control law including the singularity avoidance. Both approaches are compared and some illustrative examples are given.					
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## DEFINITION OF SYMBOLS

Symbol	Definition
$a_{ik}$	coordinate transformation matrix
A	Jacobian matrix of $f(\delta)$
B	redistribution matrix
C	Jacobian matrix of $g(\delta)$
D	redistributability matrix
E	unit matrix
$E(\delta)$	redistribution number
$f(\delta)$	angular momentum vector function
F	Jacobian matrix of $h(H)$
$g(\delta)$	regularity
G	regularity matrix
$h(H)$	gimbal angle vector function
$h_j$	angular momentum of the $j$ th SGCMG
H	total angular momentum vector
$\vec{H}_j$	angular momentum vector of the $j$ th SGCMG
t	time
T	torque
u	redistribution vector

## DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
$X_1, X_2, X_3$	spacecraft fixed axes
$Y_{1j}, Y_{2j}, Y_{3j}$	axes correlated to the jth SGCMG
$z$	arbitrary vector
$Z_{1j}, Z_{2j}, Z_{3j}$	axes fixed in the jth SGCMG
$\beta$	gimbal axis angle within the cluster
$\delta$	gimbal angle vector
$\delta_j$	gimbal angle of the jth SGCMG
$\Delta_i$	subdeterminant of matrix A
$\theta$	particular gimbal angle
$\lambda$	Lagrange multiplier vector
$\lambda_j$	eigenvalue of matrix G
$\omega$	angular velocity vector

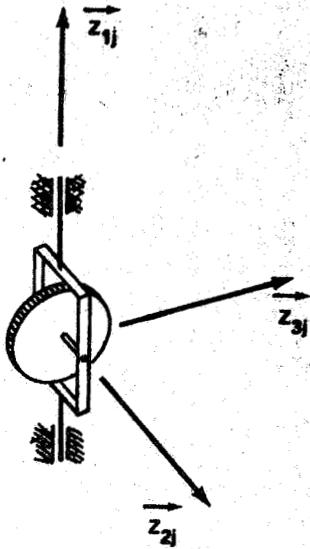
## TWO DIFFERENT APPROACHES FOR A CONTROL LAW OF SINGLE GIMBAL CONTROL MOMENT GYROS

### INTRODUCTION

Double gimbal control moment gyros (DGCMG's) are well suited for space applications as shown by W. B. Chubb and S. M. Seltzer (Reference 1) in their investigation of the Skylab Attitude and Pointing Control System. However, DGCMG's have hardware problems since there are two gimbals and four bearings with each DGCMG. Thus, single gimbal control moment gyros (SGCMG's) are now of increasing interest in the field of momentum exchange attitude control systems. The SGCMG's are presently being considered for the High Energy Astronomy Observatory (HEAO) and Large Space Telescope (LST) projects. Bendix, Sperry Rand, and TRW Systems have published reports relating to the use of SGCMG's. (See References 2, 3, and 4.) These reports, as well as simulations done by MSFC, show that problems occur from the singularities which appear in the SGCMG cluster. A singularity is present if the cluster cannot produce a torque in a given direction in the three-dimensional space; then, at least in one direction, the spacecraft will not be controllable. The saturation singularity surface is given by the outer angular momentum envelope and can be avoided only by use of a desaturation system employing external torques. However, there are singularities within the angular momentum envelope which, theoretically, can be handled with desaturation by external torques. Thus, a specific control law for the SGCMG's is necessary to avoid the singularities. Some control laws have been investigated in the aforementioned reports with little success. In this report, two different approaches for the most general, nonlinear control law will be discussed. The numerical computation of these control laws will not be given. A comparison between the two approaches will be made on a qualitative basis, since quantitative measures are a function of a particular CMG configuration. Thus, a comparison on a quantitative basis can only be made if a specified CMG cluster is given; however, some illustrations are included.

## MATHEMATICAL DESCRIPTION

Let  $N$  be the number of single gimbal control moment gyros within the spacecraft. Let  $X = \{\vec{X}_1, \vec{X}_2, \vec{X}_3\}$  be an orthonormal frame fixed in the spacecraft and  $Z_j = \{\vec{Z}_{1j}, \vec{Z}_{2j}, \vec{Z}_{3j}\}$  an orthonormal frame fixed in the  $j$ th SGCMG,  $j = 1(1)N$ . The frame  $Z_j$  is defined as follows (Fig. 1):



- $Z_{1j}$  — axis is along the gimbal axis,
- $Z_{2j}$  — axis is along the wheel axis,
- $Z_{3j}$  — axis completes the right-handed frame.

The angular momentum of the  $j$ th SGCMG can be described in tensor notation using the summation convention

Figure 1. SGCMG fixed frame.

$$\vec{H}_j = X^H_{ij} \vec{X}_i = Z_j^H_{ij} \vec{Z}_{ij}, \quad i = 1(1)3, \quad (1)$$

where  $X^H_{ij}$  and  $Z_j^H_{ij}$  are the coordinates of the angular momentum corresponding to frame  $X$  and  $Z_j$ , respectively. From the definition of frame  $Z_j$  for a completely balanced wheel,

$$Z_j^H_{ij} = [0 \ h_j \ 0]^T, \quad (2)$$

where  $h_j$  is the  $j$ th SGCMG angular momentum.

It is practical to introduce a second spacecraft fixed frame  $Y_j = \{\vec{Y}_1, \vec{Y}_2, \vec{Y}_3\}$  which is correlated to the  $j$ th SGCMG in the following way:

$Y_{1j}$  — axis is along the gimbal axis,

$Y_{2j}$  — axis is along the zero gimbal angle direction,

$Y_{3j}$  — axis completes the right-handed frame.

Then the transformations between frame  $X$ ,  $Y_j$ , and  $Z_j$  are characterized by

$$\vec{X}_i = {}^{XY_j}a_{ik} \vec{Y}_{kj} = {}^{XY_j}a_{ik} {}^{YZ_j}a_{kl} \vec{Z}_{lj}, k = 1(1)3, l = 1(1)3 \quad (3)$$

Here the matrix  ${}^{XY_j}a_{ik}$  depends on the geometrical configuration of the  $j$ th SGCMG within the cluster. Therefore,  ${}^{XY_j}a_{ik}$  is time-invariant.

The matrix  ${}^{YZ_j}a_{ik}$  is an explicit function of the gimbal angle  $\delta_j$  of the  $j$ th SGCMG:

$${}^{YZ_j}a_{ik} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\delta_j & -\sin\delta_j \\ 0 & \sin\delta_j & \cos\delta_j \end{bmatrix} \quad (4)$$

Applying equation (4) to equation (2) one gets

$$Y_j H_{ij} = {}^{YZ_j}a_{ik} Z_j H_{kj} = [0 \ h_j \cos\delta_j \ h_j \sin\delta_j]^T \quad (5)$$

Here the theorem was used that vector coordinates are transformed just as the base vectors of the corresponding frames.

The angular momentum of the total CMG cluster will be evaluated in frame  $X$  which is the only frame used further,

$${}^X H_i = \sum_{j=1}^N {}^{XY_j} a_{ik} Y_j H_{kj} \quad (6)$$

Using equation (5), equation (6) can be rewritten using the convenient matrix notation,

$$H = f(\delta), \quad H = [{}^X H_1, {}^X H_2, {}^X H_3]^T, \quad (7)$$

$$\delta = [\delta_1, \delta_2, \dots, \delta_N]^T.$$

It results that the angular momentum of the CMG cluster is a nonlinear  $3 \times 1$  vector function  $f$  of the  $N \times 1$  gimbal angle vector  $\delta$ . From equation (7) the outer angular momentum envelope can be determined by variation of the gimbal angles. It has to be pointed out that the angular momentum envelope may have windows. Therefore, numerical investigations are recommended in each particular case.

The torque that can be produced by the CMG cluster follows from the derivative of the angular momentum with respect to the inertial space. It yields

$$T = \dot{H} + \tilde{\omega} H, \quad (8)$$

where the  $3 \times 1$  torque vector  $T$  and the  $3 \times 3$  skew symmetric matrix  $\tilde{\omega}$  appear. The matrix  $\tilde{\omega}$  corresponds to the angular velocity  $\omega$  of the frame  $X$  relative to the inertial frame. Usually the angular velocity is small and the second term in equation (8) is neglected. Thus, it remains

$$T = \dot{H} = \frac{\partial f}{\partial \delta} \dot{\delta} = A \dot{\delta}, \quad (9)$$

where the  $3 \times N$  Jacobian matrix  $A$  of the nonlinear vector function appears. The CMG cluster is termed singular if the rank of matrix  $A$  is smaller than three; otherwise it may be termed regular:

$$\left. \begin{array}{l} \text{Rank } A = 3 \text{ Regularity} \\ \text{Rank } A < 3 \text{ Singularity} \end{array} \right\} \quad (10)$$

Usually singular surfaces can be found within the outer momentum envelope. If the angular momentum vector reaches a point on a singular surface, then the CMG cluster cannot produce a torque in an arbitrary direction. The resulting torque is constrained to a plane (Rank  $A = 2$ ) or a line (Rank  $A = 1$ ). This means that the controllability of the spacecraft is lost and arbitrary attitude errors may occur. Therefore, it is necessary to use at least three SGCMG's,  $N \geq 3$ , and it is very important to find a control law for the SGCMG's which guarantees the avoidance of singularities.

For the formulation of a feasible control law some measure of the CMG cluster's regularity is needed. Such a measure can be found that is analogous to the degree of controllability introduced by P. C. Müller and H. I. Weber (Reference 5) using the quadratic symmetric matrix

$$G = AA^T = G^T \quad . \quad (11)$$

One degree of regularity  $g$  is defined by the determinant of matrix  $G$ :

$$g = \det G = \det AA^T \quad . \quad (12)$$

Another degree of regularity is found from the eigenvalues  $\lambda_j$  of matrix  $G$ :

$$g^* = \text{Min} \left| \lambda_j \right| , \quad j = 1, 2, \dots , N \geq 3 \quad . \quad (13)$$

Both degrees of regularity are positive for regular CMG clusters but they vanish in the case of singularities and depend on the gimbal angle vector  $\delta$ .

The numerical computation of equation (12) can be simplified using the Binet-Cauchy formula

$$g = \sum_{i=1}^{\binom{N}{3}} \Delta_i^2 , \quad \binom{N}{3} = \frac{N(N-1)(N-2)}{3!} , \quad N \geq 3 \quad , \quad (14)$$

where  $\Delta_i$  denotes the  $\binom{N}{3}$  distinct subdeterminants of order 3 extractable from the  $3 \times N$  matrix A. Thus, even in the case of numerous SGCMG's there will be no essential difficulties in computation of the determinant equation (12).

In general a control law can be found by inversion of equation (7) or equation (9). In the first approach, the gimbal angle vector  $\delta$  has to be controlled and the angular momentum vector H will be commanded. Using the second approach, the gimbal rate vector  $\dot{\delta}$  is controlled and the torque vector T is commanded.

## GIMBAL ANGLE APPROACH FOR A CONTROL LAW

The spacecraft will be controllable by the CMG cluster's angular momentum if the Jacobian matrix A has full rank 3 for each angular momentum vector within the outer envelope. Optimal controllability will be given if the matrix A has an optimal regularity or, using equation (12), if

$$g(\delta) \stackrel{!}{=} \text{Max} \quad . \quad (15)$$

Thus, for each point within the angular momentum envelope, a gimbal angle vector  $\delta$  has to be found satisfying equation (15). Since the gimbal angle vector is subject to equation (7) too, which can be interpreted as a constraint equation, necessary conditions for a local extremum can be found by the method of Lagrange multipliers. It yields (See Reference 6)

$$\left. \begin{array}{l} H - f(\delta) = 0 \\ C - \lambda^T A = 0 \end{array} \right\} , \quad (16)$$

with the  $1 \times N$  Jacobian matrix  $C = \partial g / \partial \delta$ , the  $3 \times N$  Jacobian matrix  $A = \partial f / \partial \delta$ , the  $3 \times 1$  vector  $\lambda$  of the Lagrange multipliers, and  $N > 3$ . By equation (16) there are  $(3 + N)$  equations to solve for  $N$  gimbal angles and 3 Lagrange multipliers. The solution of equation (16) will be a nonlinear  $N \times 1$  vector function,

$$\delta = h(H) \quad , \quad (17)$$

which can be named the inverse of equation (7) or the gimbal angle control law.

The following cases now have to be discerned:

1.  $h(H)$  and  $\partial h/\partial H$  exist everywhere within the  $H$ -space of the envelope,
2.  $h(H)$  exists and  $\partial h/\partial H$  does not exist everywhere within the  $H$ -space of the envelope,
3.  $h(H)$  does not exist everywhere within the  $H$ -space of the envelope.

Case 1. The commanded trajectory of the angular momentum vector can be achieved within the envelope by a smooth control of the gimbal angle vector. Specifically, by differentiation of equation (17) one obtains

$$\dot{\delta} = \frac{\partial h}{\partial H} \dot{H} = F(H)T \quad , \quad (18)$$

where  $F$  is the  $N \times 3$  Jacobian matrix corresponding to equation (17). Introducing equation (7) into equation (18) one obtains

$$\dot{\delta} = F(\delta)T \quad , \quad (19)$$

which defines an SGCMG control law without singularities within the  $\delta$  space of the cube:  $-\pi \leq \delta_j \leq +\pi$ ,  $j = 1, 2, \dots, N > 3$ .

Case 2. The commanded angular momentum trajectory can be achieved within the envelope by a discontinuous control of the gimbal angle vector. This means that jumps occur in the gimbal angles at some surfaces within the envelope. During the time intervals needed for the jumps, the spacecraft will show some attitude errors. If the jump time interval approximates zero, the spacecraft errors vanish also; however, infinite gimbal rates are necessary in the limiting case. Such a jump phenomenon can be called internal desaturation because the CMG cluster is turned into a better regularity without any external torque.

Case 3. The commanded trajectory of the angular momentum vector cannot always reach the outer envelope. There are holes within the envelope and, if the trajectory of the angular momentum vector is commanded to such a hole, external desaturation has to be provided by external torques.

Programs for constraint parameter optimization problems can be used for the solution of equation (16). Starting points may be the origin of the envelope and/or some known points on the envelope corresponding to the axis of the frame.

## GIMBAL RATE APPROACH FOR A CONTROL LAW

The spacecraft can be controlled by the CMG cluster's torque if the Jacobian matrix  $A$  has the full rank 3 and especially if optimal regularity is required:

$$g(\delta) \stackrel{!}{=} \text{Max} \quad . \quad (20)$$

However, the torque equation, equation (9), with the gimbal rate vector  $\dot{\delta}$  as a controlled variable cannot be combined immediately with equation (20) for inversion. Thus, the general solution of linear equations first has to be treated, as by C. R. Rao and S. K. Mitra (Reference 7).

A general solution of the consistent equation  $T = A\dot{\delta}$  is

$$\dot{\delta} = A^- T + (E - A^- A)z \quad , \quad (21)$$

where  $A^-$  is any  $N \times 3$  generalized inverse matrix of  $A$ ,  $E$  is the  $N \times N$  unit matrix, and  $z$  is an arbitrary  $N \times 1$  vector.

Assuming the full rank 3 of matrix  $A$  the right inverse can be used as a generalized inverse matrix,

$$A^- = A^T (AA^T)^{-1} \quad . \quad (22)$$

Then, due to the properties of symmetric matrices, it yields

$$E - A^{-1}A = E - A^T(AA^T)^{-1}A = BB^T, \quad (23)$$

where B is an  $N \times (N - 3)$  matrix. Further,  $\text{Rank } BB^T = N - 3$  and  $AB \equiv 0$ . Since z is an arbitrary vector, it follows that

$$u = B^T z, \quad (24)$$

where u is an arbitrary  $(N - 3) \times 1$  vector, equally. From equation (21) to equation (24), it follows that

$$\dot{\delta} = A^T(AA^T)^{-1}T + Bu, \quad (25)$$

which is the inverse of equation (9) or the gimbal rate control law. The gimbal rate vector depends not only on the commanded torque T but also on an arbitrary vector u which produces no physical torque on the spacecraft. Thus, the vector u can be used for a redistribution of the gimbal angle vector. It is obvious from equation (25) that the gimbal angles are not completely redistributable by u since T is an unknown, commanded function of time. However, if at least the degree of regularity is redistributable by u, then singularity avoidance seems to be possible. For the synthesis of a redistribution law, equation (20) has to be combined with equation (25) which can be done in two different ways: (1) linear redistribution and (2) nonlinear redistribution.

Case 1. From equation (20) it follows, examining the time history of the CMG cluster between zero angular momentum and outer angular momentum envelope, that

$$\frac{d}{dt} g(\delta) \stackrel{!}{=} \text{Max} \quad (26)$$

Since the degree of regularity g is a nonlinear function of the gimbal angle vector, it yields

$$\dot{g} = \frac{\partial g}{\partial \delta} \dot{\delta} = C \dot{\delta} \quad , \quad (27)$$

with the  $1 \times N$  Jacobian matrix  $C$  known from equation (16). Using equation (25), one obtains

$$\dot{g} = CA^T (AA^T)^{-1} T + CBu \quad . \quad (28)$$

Since the first term of equation (28) is commanded, only the second term of equation (28) can be used to redistribute the regularity and to satisfy equation (26). There are two subcases possible: (1)  $D = CB \equiv 0$ , locally unredistributable and (2)  $D = CB \neq 0$ , redistributable. In case (1) the regularity  $g$  of the CMG cluster is locally unredistributable and therefore no redistribution law can be defined. In case (2) a suitable redistribution law is

$$u = u_o \operatorname{sgn} d_i(\delta) \quad , \quad (29)$$

where  $d_i$  are the elements of the  $1 \times (N - 3)$  matrix  $D = CB$  depending on the gimbal angle vector  $\delta$ . The redistribution gain  $u_o$  is arbitrary. However, by examining equation (28) it results that for an efficient redistribution, even in the neighborhood of a singularity  $g \approx 0$ , it is necessary that

$$\|u_o\| > g^{-1/2} \|T\| \quad , \quad (30)$$

where  $\|u_o\|$  denotes the norm of the vector  $u_o$ .

Case 2. For an optimal satisfaction of equation (20), the absolute reachable regularity for each gimbal angle vector  $\delta$  within the gimbal angle cube has to be found. This can be done only in the limit case of zero torque  $T \equiv 0$  or infinite redistribution control  $\|u\| \rightarrow \infty$ , because the commanded torque is an unknown function of time. Assuming zero torque  $T \equiv 0$ , it remains from equation (25):

$$\dot{\delta} = Bu \quad . \quad (31)$$

This nonlinear differential equation has to be solved for arbitrary redistribution vectors  $u$  with all possible gimbal angle vectors  $\delta$  as initial conditions under simultaneous computation of the degree of regularity  $g$ . It is obvious that this can be accomplished, in general, only with considerable complexity. However, in the case of four SGCMG's, equation (31) changes to two determined equations

$$\dot{\delta} = +Bu \quad , \quad \dot{\delta} = -Bu \quad , \quad (32)$$

where  $u$  is a scalar. By solution of equation (32) there will be found, for each gimbal angle vector  $\delta$  within the admissible cube, the optimal sign of the redistribution control characterized by some number  $E$ . Thus, a redistribution law will be

$$u = u_0 \operatorname{sgn} E(\delta) \quad , \quad u_0 \rightarrow \infty \quad . \quad (33)$$

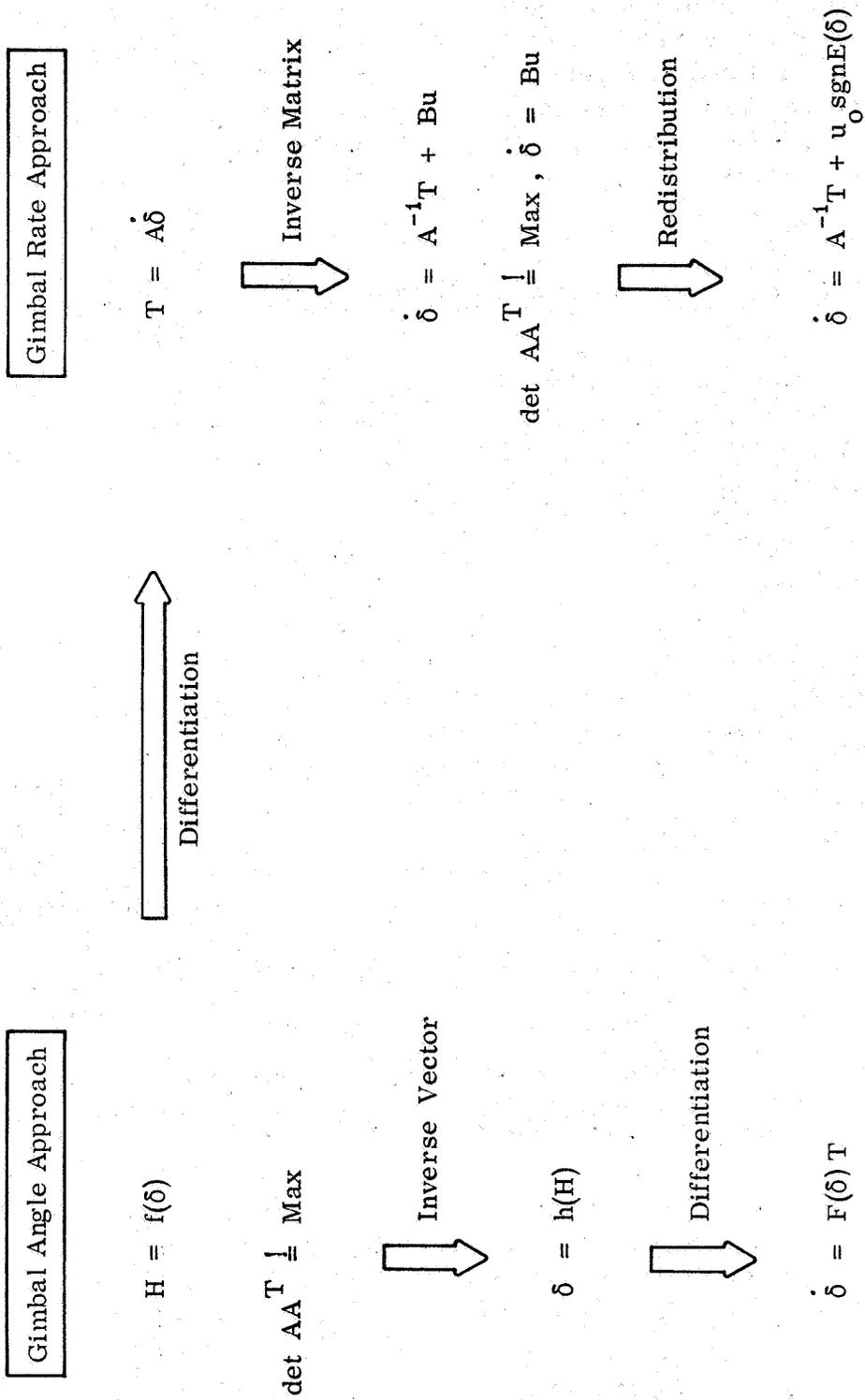
However, in practice, infinite gimbal rates assumed for the synthesis of the redistribution law equation (33) are not feasible. Therefore, the optimal redistribution may be approximated only, since the additional influence of the commanded torque will change things continuously.

The numerical computation of the linear control law equation (25) and the linear redistribution law equation (29) is simple. For the solution of the nonlinear differential equation (32), resulting in the nonlinear redistribution law equation (33), presently available programs can be used.

## COMPARISON OF GIMBAL ANGLE AND GIMBAL RATE APPROACH

A comparison of the gimbal angle and the gimbal rate approach is made in Table 1 assuming some simplifications. It is quite clear that the gimbal angle approach is much more suitable for the control law of the SGCMG's than the gimbal rate approach. The reason is that the fundamental property of the CMG cluster, the regularity  $g = \det AA^T$ , depends on the gimbal angles and not on the gimbal rates. Therefore, in the nonlinear

TABLE 1. COMPARISON OF GIMBAL ANGLE AND GIMBAL RATE APPROACH



gimbal rate approach, an integration is necessary to find a redistribution law. The linear gimbal rate approach, on the other hand, may be restricted due to the local information which results from the differentiation process characterizing the gimbal rate approach. In addition, a large torque command can overcome the provided redistribution and drive the CMG cluster to a small regularity.

From the standpoint of technical realization, the linear gimbal rate approach is preferable because all computations can be done on line at the spacecraft. On the other side, the gimbal angle approach and the nonlinear gimbal rate approach need an intensive off-line computation and the resulting control laws have to be stored in the onboard computer in discrete form. However, the gimbal angle approach is defined over the three-dimensional angular momentum space, while the nonlinear gimbal rate approach is defined over the N-dimensional gimbal rate space.

## EXAMPLES

For a regular four-CMG cluster, as shown in Figure 2, the mathematical description will be given. For the illustration of the gimbal angle approach, an optimal gimbal angle vector will be determined for zero angular momentum and the linear gimbal rate approach will be treated in the neighborhood of a singularity.

Let the four SGCMG's have equal angular momentum in the corresponding frames  $Z_j$ . Thus, from equation (2) one obtains

$$Z_j^H_{ij} = [0 \ h \ 0]^T, \quad j = 1(1)4 \quad . \quad (34)$$

Further, equation (5) results in

$$Y_j^H_{ij} = h[0 \ \cos\delta_j \ \sin\delta_j]^T, \quad j = 1(1)4 \quad . \quad (35)$$

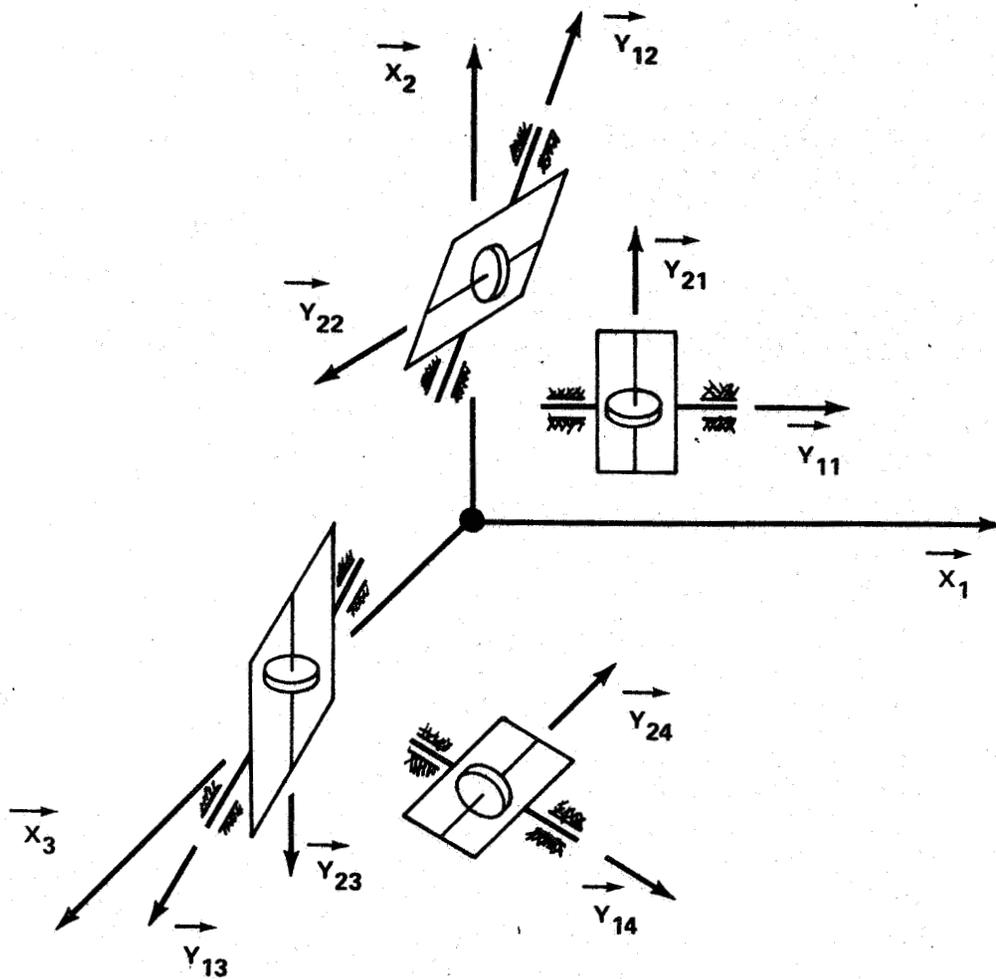


Figure 2. CMG cluster configuration.

The transformation matrices are

$${}^{XY_1}a_{ik} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}, \quad (36)$$

$${}^{XY_2}a_{ik} = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ \sin \beta & 0 & -\cos \beta \\ 0 & 1 & 0 \end{bmatrix}, \quad (37)$$

$$XY_{3a_{ik}} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & -1 & 0 \\ \sin\beta & 0 & -\cos\beta \end{bmatrix}, \quad (38)$$

and

$$XY_{4a_{ik}} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ -\sin\beta & 0 & \cos\beta \\ 0 & -1 & 0 \end{bmatrix}, \quad (39)$$

where  $\beta$  is a constant angle. From equation (6) and (7) one obtains

$$H = \begin{bmatrix} \sin\beta \sin\delta_1 + \sin\beta \sin\delta_2 + \sin\beta \sin\delta_3 + \sin\beta \sin\delta_4 \\ \cos\delta_1 - \cos\beta \sin\delta_2 - \cos\delta_3 + \cos\beta \sin\delta_4 \\ \cos\beta \sin\delta_1 + \cos\delta_2 - \cos\beta \sin\delta_3 - \cos\delta_4 \end{bmatrix} \quad (40)$$

and the Jacobian matrix, equation (9), is

$$A = \begin{bmatrix} \sin\beta \cos\delta_1 & \sin\beta \cos\delta_2 & \sin\beta \cos\delta_3 & \sin\beta \cos\delta_4 \\ -\sin\delta_1 & -\cos\beta \cos\delta_2 & \sin\delta_3 & \cos\beta \cos\delta_4 \\ \cos\beta \cos\delta_1 & -\sin\delta_2 & -\cos\beta \cos\delta_3 & \sin\delta_4 \end{bmatrix}. \quad (41)$$

The minors  $\Delta_i$  of the matrix equation (41) are

$$\begin{aligned} \Delta_i &= \sin\beta \sin\delta_{i+2} \sin(\delta_{i+1} + \delta_{i+3}) \\ &\quad - \sin\beta \cos\beta \cos\delta_{i+2} \sin(\delta_{i+1} - \delta_{i+3}) \\ &\quad + 2\sin\beta \cos^2\beta \cos\delta_{i+1} \cos\delta_{i+2} \cos\delta_{i+3}, \\ i &= 1(1)4 \text{ and } \delta_5 \equiv \delta_1, \delta_6 \equiv \delta_2, \delta_7 \equiv \delta_3. \end{aligned} \quad (42)$$

For illustration of the gimbal angle approach, an optimal gimbal angle vector will be determined. In the case  $H \equiv 0$ , there exists the particular solution

$$\delta_1 = -\delta_2 = \delta_3 = -\delta_4 = \theta \quad . \quad (43)$$

In regard to equation (43) one obtains from equation (42)

$$\Delta_i = \Delta = -\sin\beta \sin\theta \sin 2\theta + 2 \sin\beta \cos^2\beta \cos^3\theta, \quad (44)$$

$$i = 1(1)4 \quad .$$

Thus, with equation (14), the regularity of the CMG cluster for  $H \equiv 0$  is

$$g = \det AA^T = \sum_{i=1}^4 \Delta_i^2 = 4 \Delta^2(\theta) \quad . \quad (45)$$

The regularity, equation (45), is plotted for  $\beta = 45 \text{ deg}$  in Figure 3. It is obvious that there are six singularities and two optimal gimbal angles. Thus, in this simple case one obtains

$$\delta = h(H=0) = 0 \quad . \quad (46)$$

Such an optimal gimbal angle vector has to be found for each angular momentum vector within the envelope.

For illustration of the gimbal rate approach, a linearization in the neighborhood of the singularity  $\delta_1 = \delta_3 = 0$ ,  $\delta_2 = -\delta_4 = -\pi/2$  is performed. At time  $t = t_0$  let

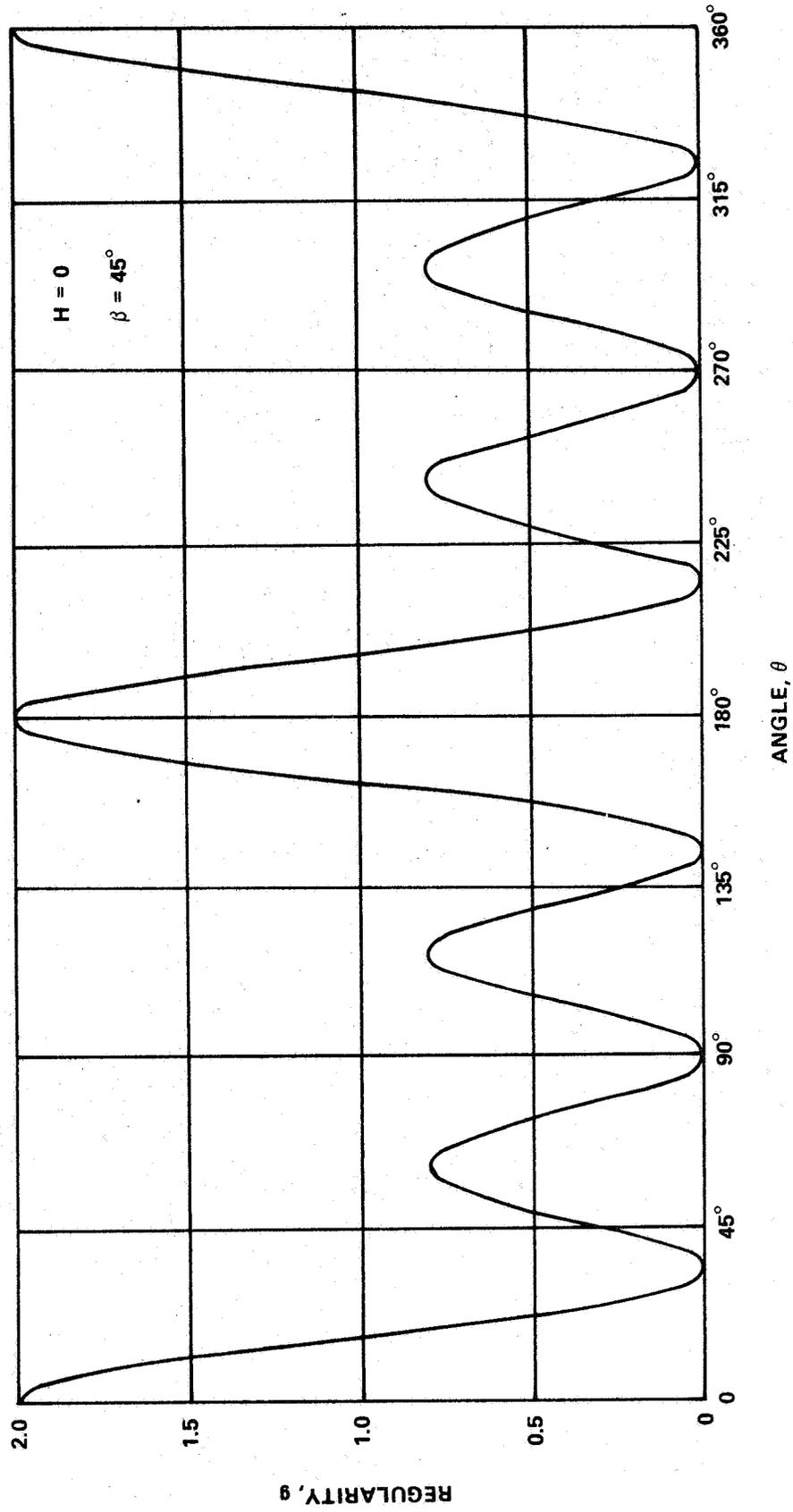


Figure 3. CMG cluster regularity for zero angular momentum.

$$\delta = \begin{bmatrix} 0 \\ -\frac{\pi}{2} + \theta_0 \\ 0 \\ \frac{\pi}{2} - \theta_0 \end{bmatrix}, \quad \dot{\delta} = \begin{bmatrix} \dot{\theta}_{10} \\ \dot{\theta}_{20} \\ \dot{\theta}_{30} \\ \dot{\theta}_{40} \end{bmatrix}, \quad (47)$$

where  $\theta_0 \ll 1$  and  $\dot{\theta}_{j0}$ ,  $j = 1(1)4$ , is arbitrary. Then, one obtains from equation (41) the following Jacobian matrix neglecting all terms of third and higher order:

$$A = \begin{bmatrix} \sin\beta & \sin\beta\theta_0 & \sin\beta & \sin\beta\theta_0 \\ 0 & -\cos\beta\theta_0 & 0 & \cos\beta\theta_0 \\ \cos\beta & 1 + \frac{1}{2}\theta_0^2 & -\cos\beta & 1 - \frac{1}{2}\theta_0^2 \end{bmatrix}, \quad (48)$$

with the regularity

$$g = \det AA^T = 8 \sin^2\beta \cos^2\beta (1 + \cos^2\beta) \theta_0^2. \quad (49)$$

Further, it yields

$$A^T(AA^T)^{-1} = \frac{4}{g} \begin{bmatrix} \sin\beta \cos^2\beta (1 + \cos^2\beta) \theta_0^2 & 0 & \sin^2\beta \cos^3\beta \theta_0^2 \\ 0 & -\sin^2\beta \cos\beta (1 + \cos^2\beta) \theta_0^2 & \sin^2\beta \cos^2\beta \theta_0^2 \\ \sin\beta \cos^2\beta (1 + \cos^2\beta) \theta_0^2 & 0 & -\sin^2\beta \cos^3\beta \theta_0^2 \\ 0 & \sin^2\beta \cos\beta (1 + \cos^2\beta) \theta_0^2 & \sin^2\beta \cos^2\beta \theta_0^2 \end{bmatrix} \quad (50)$$

and

$$B = \frac{2}{g^{1/2}} \begin{bmatrix} -\sin\beta \cos\beta \theta_0 - \sin\beta \cos^2\beta \theta_0^2 \\ \sin\beta \cos^2\beta \theta_0 \\ \sin\beta \cos\beta \theta_0 - \sin\beta \cos^2\beta \theta_0^2 \\ \sin\beta \cos^2\beta \theta_0 \end{bmatrix} \quad (51)$$

The matrix C according to equation (27) is

$$C = 8 \sin^2\beta \cos^2\beta (1 + \cos^2\beta) \theta_0 [0, 1, 0, -1] \quad (52)$$

and the matrix D, in this example a scalar, is

$$D = CB \equiv 0 \quad (53)$$

Thus, it results that the CMG cluster is locally unredistributable even though the regularity, equation (49), does not vanish.

## CONCLUSION

Two different approaches for a control law of single gimbal control moment gyros have been presented but the gimbal angle approach seems to be better suited for a control law than the gimbal rate approach because of the singularity problem. The gimbal angle approach requires off-line computation of suitable gimbal angles for each point within the three-dimensional angular momentum envelope and the commanded variable is the angular momentum itself. The linear gimbal rate approach can be implemented by online computation and the commanded variable will be the torque. The nonlinear gimbal rate approach also requires off-line computation of suitable redistribution controls for each point within the N-dimensional gimbal angle cube. However, a general statement on the controllability of spacecrafts with an SGCMG cluster has not been found. Only detailed numerical computations will answer this question.

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The information in this report has been reviewed for security classification. Review of any information concerning Department of Defense or Atomic Energy Commission programs has been made by the MSFC Security Classification Officer. This report, in its entirety, has been determined to be unclassified.

This document has also been reviewed and approved for technical accuracy.

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