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BOUNDARY LAYER TRANSITION:  
A REVIEW OF  
THEORY, EXPERIMENT  
AND RELATED PHENOMENA

By

E. L. Kistler  
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LEC Document No. 675-44-00450

February 1971

Prepared by:

Lockheed Electronics Company  
Houston Aerospace Systems Division  
Houston, Texas

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For

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National Aeronautics and Space Administration  
Manned Spacecraft Center  
Houston, Texas

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## SUMMARY

The overall problem of boundary layer flow transition is reviewed from the point of view that evidence seems to indicate a need for basic new physical hypotheses to be injected into classical fluid mechanics math models based on the Navier-Stokes equations. The Navier-Stokes equations are challenged as inadequate for the investigation of fluid transition since they are based on several assumptions which should be expected to alter significantly the stability characteristics of the resulting math model. This point is not proved, but the document collects strong prima facie evidence to this effect.

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## LIST OF SYMBOLS

<u>Symbol</u>	<u>Definition</u>
$A^1, A^2, A^3$	Constant; eqn. IV-14.
$b$	Impact parameter, taken as radius of a cylindrical shell about the trajectory of molecule $i$
$\tilde{B}, \tilde{\tilde{B}}$	Coefficients of linear viscoelasticity; eqn. III-5.
$\vec{B}_k$	Force on molecule $k$ due to external field; eqn. III-51
$c_i$	Imaginary part (amplification factor) of complex disturbance wave; eqn. II-9.
$c_r$	Real part (or of phase velocity) of complex disturbance wave; eqn. II-9
$C_{12}^{(2)}$	Pair correlation function, $C_{12}^{(2)} = 0$ indicates no correlation between molecules 1 and 2; eqn. III-54.
$C, \tilde{C}$	Coefficients of linear viscoelasticity, eqn. III-5
$D_v$	Function depending on vibrational frequency; eqn. III-26
$\mathcal{D}$	Tensor operator defined by eqn. III-75.
$e$	Specific internal energy; eqn. III-31. $e = E/mN$
$e$	Random thermal energy.
$e^*$	Local macroscopic rotational energy.
$E$	Total internal energy of the gas; eqn. III-28.
$E$	Modulus of elasticity; eqn. III-19
$f^{(N)}$	Distribution function in $\gamma$ -space of $6N$ dimensions; eqn. III-50.

<u>Symbol</u>	<u>Definition</u>
$f_1^{(1)}, f_2^{(1)}$	First-order singlet distribution functions; eqn. III-53, 54.
$f_{12}^{(1)}$	First-order pair distribution function; eqn. III-53.
$\vec{F}_{12}$	Pair interaction force; eqn. III-53.
$\vec{F}_k$	Force on molecule $k$ due to all other molecules in the system; eqn. III-51
$g_{ij}$	Degeneracy of quantized rotation; eqn. III-32
$h$	Planck's constant; eqn. III-26.
$H$	Hamiltonian function; eqn. III-43.
$I$	Moment of inertia of diatomic molecule; eqn. III-26.
$J$	Rotation quantum number; eqn. III-26.
$J$	Collision integral; net rate at which molecules are gained into a particular state due to collisions; eqn. III-75.
$J$	Jacobian
$k$	Boltzman's constant; eqn. III-28
$\kappa$	Coefficient of bulk viscosity, $\kappa = 3\lambda + 2\mu$ ; eqn. III-25
$K$	Kinetic energy function; eqn. III-40.
$\vec{\ell}$	Internal angular momentum (spin angular momentum) of fluid substructure; eqn. IV-30.
$\vec{L}$	Total angular momentum; eqn. IV-30. $\vec{L} = \vec{\ell} + \vec{R} \times \vec{u}$ .

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<u>Symbol</u>	<u>Definition</u>
L	Lagrangian function; eqn. III-40.
m	Mass of a single molecule.
n	Number density of molecules; eqn. III-74.
N	Number of molecules in the gas; eqn. III-28.
$N_A$	Amplitude parameter of free-stream unsteadiness, $\Delta u_0 / u_0$
p	Thermodynamic pressure; eqn. II-1
P	Static stress eqn. IV-22.
$P_K$	Generalized momentum; eqn. III-42.
P	Poisson's ratio; eqn. III-19.
q	Arbitrary flow parameter, such as density or velocity; eqn. II-2
q'	Perturbation or fluctuation value of q ; eqn. II-2.
$q_N$	Generalized position coordinates.
$\dot{q}_N$	Generalized velocity coordinates.
Q	Mean-flow value of the parameter q ; eqn. II-2.
Q	Energy partition function; eqn. II-29, III-30.
$r_i$	Position coordinate of an individual molecule; eqn. III-41
$r_e$	Equilibrium nuclear separation; eqn. III-39.
$r_J$	Nuclear separation due to rotation at different quantum numbers.
R	Universal gas constant; eqn. III-31.
$Re_x$	Length Reynolds number, $Re_x = \frac{u_1 x}{\nu}$ .

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<u>Symbol</u>	<u>Definition</u>
$Re_{cr}$	Critical Reynolds number, $Re_{cr} = \frac{u_1 x_{cr}}{\nu}$
$Re_{tr}$	Transition Reynolds number, $Re_{tr} = \frac{u_1 x_{tr}}{\nu}$
$Re_r$	Energy-thickness Reynolds number; Morisetti (1968)
$R_L$	Minimum Reynolds number at which a particular wave becomes unstable.
$R_U$	Maximum unstable Reynolds number for a particular wave.
S	Effective range of intermolecular force; eqn. III-39.
S	Arbitrary physical property of the system, depends on dynamical state; eqn. III-47.
St	Stanton number
t	Time, sec.
T	Temperature of the gas; eqn. III-28.
$T_m$	Constant in the stress-strain relation; represents time required by the fluid to build up the equilibrium stress; eqn. III-10.
$u_1, u_l$	Freestream velocity (longitudinal component).
$\vec{u}$	Total velocity vector; (used also as x-axis velocity component).
u,v,w	Total velocity components; eqn. II-1
v	Translational velocity of an individual molecule. $v = (v_1, v_2, v_3)$ .
U,V,W	Velocity components of meanflow; eqn. II-3.

<u>Symbol</u>	<u>Definition</u>
$U_0$	Mean value of longitudinal velocity component in freestream.
$x$	Distance downstream from the leading edge.
$x, y, z$	Position coordinates; eqn. II-1.
$x^*, y^*, z^*$	Dimensional Parameters under change of notation in eqn. II-4 and remainder of Section II.
$X_w$	Frequency parameter, $xw/U_0$ .

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<u>Greek Symbols</u>	<u>Definition</u>
$\alpha$	Eulerian angles defining the orientation of single molecule, $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ .
$\alpha$	Wavenumber; $\alpha = 2\pi/\lambda$ .
$\alpha_\delta, \beta_\delta$	Dimensionless wavenumber; eqn. II-6, II-10.
$\gamma$	Strain.
$\dot{\gamma}$	Rate of strain; eqn. III-1
$\delta$	Boundary layer displacement thickness.
$\epsilon_{\text{rot Q.M.}}$	Quantized rotational energy; eqn. III-26.
$d\epsilon$	Sector around the axis of the collision trajectory;
$\theta$	Phase angle used for incorporating stress lead or lag for Maxwellian fluid; eqn. III-10.
$\Theta$	Characteristic rotational temperature; eqn. III-34.
$\kappa$	Coefficient of bulk viscosity; eqn. III-25, IV-24.
$\lambda$	Wavelength.
$\mu$	Coefficient of shear viscosity; eqn. III-1.
$\nu$	Frequency of vibration of molecule; eqn. III-39.
$\nu$	Kinematic viscosity.
$\nu_1, \nu_2$	Bulk and shear coefficients of spin viscosity; eqn. IV-34.
$\xi$	Coefficient of vortex viscosity; eqn. IV-34.
$\xi$	Wave propagation angle, measured from the free-stream direction; eqn. II-7.
$\rho$	Freestream density (mass per unit volume).

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Greek  
Symbols

Definition

$\rho(q_i, p_i, t)$	Continuous distribution function in phase space.
$\tau$	Shear stress; eqn. III-1.
$\tau_o$	Small time interval over which the distribution functions are averaged.
$\phi(y)$	Periodic disturbance; eqn. II-12
$\Phi$	Potential energy function; eqn. III-40.
$\psi^{(i)}$	Stream function; eqn. II-11, II-12
$\psi$	Arbitrary summation invariant; eqn. III-77, 78.
$\omega, \omega_\delta$	Disturbance wave frequency; eqn. II-10
$\omega$	Angular velocity in space-fixed coordinate system, $\omega = [\omega_1, \omega_2, \omega_3]$ .
$\omega_o$	Continuum spin field; eqn. IV-34.
$\Omega$	Phase of freestream oscillations, $\Omega = \omega * t^*$

Mathematical  
Symbols

Definition

$\approx$	Approximately equal to
$\equiv$	Defined as
$\equiv$	Identical to
$\Rightarrow$	Implies
$\langle\langle \ \ \rangle\rangle$	Ensemble average
$\bar{\quad}$	Overbar; time averaged
$\vec{\quad}$	Overbar Arrow; indicates vector or tensor quantity
$\delta_{ij}$	Kronecker Delta.
$\epsilon_{ijk}$	Levi-Civita tensor, or permutation symbol
$\frac{D(\ )}{dt}, \frac{d(\ )}{dt}$	Material derivative $\frac{D(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \vec{u} \cdot \nabla(\ )$
$\delta(\ )$	Small increment
$\nabla$	Nabla operator; $\nabla = \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z}$
$\nabla^2$	Laplacian operator; $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$(\ )^*$	As necessary, or for emphasis, dimensional quantities may be denoted by asterisk.
$(\ )^*$	Used to denote conjugate tensor

## I. INTRODUCTION

This document presents a review of the general problem of boundary layer transition from a new point of view. It is felt that despite many advances in observing, understanding, and even predicting boundary layer transition, there still has been too little success in describing the actual physical mechanism (or mechanisms) by which a flow changes from laminar to turbulent.

No work to date adequately describes and predicts the transition process. Therefore, it is necessary to consider at least the following two possible sources of the difficulty. One possibility is that an adequate set of equations is available, but that they are so difficult to analyze that the problem has not yet been completely solved. A second possibility not frequently discussed is that the conventional equations might fail to represent some important physical processes occurring in a real flow as it begins transition. In other words, the present equations cannot reveal the importance of some parameter or physical mechanism not imbedded in them. The mathematics can never create missing physics.

The first possibility has been considered exhaustively for almost a century, and some new physical characteristics of the problem indeed have been "discovered". However, these characteristics actually have been in the classical mathematics all along. From the standpoint of basic hypotheses, no really new physics has been injected, and the basic turbulence mechanism still is unknown.

This review primarily considers the second possibility and seeks to identify missing physics. It also describes qualitatively why some of the missing physics could be important and how it might alter the stability of a laminar flow. This background gives quite a new overall perspective and should provide the basis for a new approach to the problem of describing flow transition and turbulence.

## II. STABILITY THEORY

### II.1 Background

One of the earliest approaches to predicting transition was to assume that turbulence has its origin in an instability of the laminar flow. As Mack (1969) points out, such an approach tells nothing about turbulence, but it does explain why the original laminar flow can no longer exist. This basic premise has been the foundation for all rational approaches to boundary layer transition.

Although it is recognized today that instability of one laminar state may only result in shift to a new and different laminar state, the above premise is even stronger today than when put forth. One views transition of the laminar boundary layer as merely the response of a very complicated physical system for some one or many forcing functions.

One might say that advances in boundary layer stability research can be categorized as follows:

- (a) Advances in understanding how the specified oscillator (physical system) responds to various disturbances; in other words, how the amplifications grow thru wave interactions.
- (b) Advances in understanding the type of disturbances which excite the real physical system, and which therefore must be modeled.
- (c) Advances in the realism of the math model thru incorporation of more physically descriptive terms.

- (d) Advances in the analytical/numerical techniques used to study system response.
- (e) Advances in the basic concepts of what constitutes stability of physical systems.
- (f) Advances in quality and controllability of experiments.

To date, stability theory actually can predict neither the details of the nonlinear process by which the flow changes from laminar to turbulent, nor the "location" of transition. What it can do is define approximately which boundary-layer profiles are unstable, and by how much. It also can identify by an approximate analysis those frequencies for which the system is most responsive, and how system parameters governing the flow will delay or enhance transition.

Since the Schubauer-Skramstad (1943) experiments, it has been accepted generally that in the majority of cases turbulence does indeed arise from an instability of the laminar flow. However, because of the complexity of the problem, there still is much unknown and much confusion about the connection between stability and transition.

Classical stability theory considers individual periodic disturbances whose amplitudes are small enough so that a linear theory can be used (Figure II-1). The wave number in the freestream direction is  $\alpha_x = 2\pi/\lambda_x$ , where  $\lambda_x$  is the wavelength. Early work assumed the disturbance was two-dimensional, with propagation parallel to the freestream. Later studies have considered oblique disturbances, with propagation at an angle  $\psi$  to the freestream direction. The disturbance propagates in a downstream direction with phase velocity  $c_r$ . The phase velocity is less than

freestream velocity  $u_1$ , so there is some point in the boundary layer where the mean velocity is equal to  $c_r$ . This point is called the critical point, and it is now known to be very important to boundary layer stability.

Numerical results from stability theory can be presented in the form of neutral-stability diagrams such as shown in Figure II-2. They show regions of stability and instability, separated by a line of neutral stability. One can think of these diagrams as indicating whether or not a given wave will be unstable in propagating through the system at any given system Reynolds number  $Re_x = \frac{u_1 x}{\nu}$ . They can also be thought of as indicating whether or not the system, at a given  $Re_x$  will be unstable when excited by a disturbance of some given wave number.

Two kinds of diagrams are found in Figure II-2. The neutral-stability curve of type (a) shows all dimensionless wave numbers ( $\alpha_\delta$ ) are damped at sufficiently high Reynolds numbers, where  $\delta$  is thickness of the boundary layer. The mean flow is said to have inviscid stability. Since decreasing Reynolds number by increasing viscosity can cause instability, it is clear that viscosity can have a destabilizing influence in addition to its more intuitively possible role of damping out disturbances! The flat plate, or Blasius, boundary layer is an example of a flow which is unstable only through the action of viscosity.

This dual role of viscosity is a key point. Early investigations of fluid instability omitted viscous effects on disturbances because air was thought of as an "inviscid" fluid and the influence of viscosity on the growth of disturbances

would therefore be negligible. Hence, early investigations treated only the Rayleigh equation. It was not until the viscous terms were retained that a critical Reynolds number was obtained. Later sections of this report will show that still other terms have been neglected based on similar assumptions, and the implication of these omissions will be discussed.

With a neutral-stability curve of type-(b), there is a neutral wave number at infinite Reynolds number. Hence, wave numbers smaller than  $(\alpha\delta)_s$  are unstable no matter how great the Reynolds number, and the mean flow is said to have inviscid instability. A boundary layer in an adverse pressure gradient is an example of flow of this kind.

In both cases (a) and (b), all disturbances less than the maximum  $\alpha\delta$  value on the neutral-stability curve are unstable for some range of Reynolds number. However, there is a minimum critical Reynolds number,  $Re_{cr}$ , below which no amplification is possible. Often the objective of stability theory is to compute  $Re_{cr}$ , but it turns out that this has limited significance and cannot be relied upon to indicate the relative instability of various mean flows. The computed  $Re_{cr}$  is always less than  $Re_{tr}$ , and it definitely is not proper to identify  $Re_{cr}$  with the transition point (Mack; 1969).

The frequency which is proportional to  $\alpha c_r$  is very useful in practice. A disturbance introduced into the boundary layer with a particular frequency will remain essentially at that frequency as it passes downstream, but the wave number will change. A disturbance of frequency  $f$  and a low wave number

will pass through the unstable region of wave numbers (Fig. II-2a). It will damp as it moves from the leading edge to the downstream position where its wave number corresponds to  $R_L$ , the first neutral point. Then, between  $R_L$  and  $R_U$  (the second neutral point) it will amplify. Downstream of  $R_U$  it will again damp. If the magnitude of the disturbance becomes sufficiently large before  $R_U$  is reached, then nonlinear processes take over and eventually lead to transition because the disturbance will grow even though linear theory says it should damp. That is, linear theory would become invalid for the description.

The neutral-stability curve only identifies the range of unstable frequencies. It is important also to calculate amplification rates which tell how fast each frequency is growing, and which frequency is growing fastest. Furthermore, it is important to calculate overall growth of a disturbance at constant frequency as it travels through the unstable region. With this information, and given some initial disturbance spectrum, it is possible to identify the frequency which has the largest amplitude at each Reynolds number. See Figure II-3 for an illustration of typical results, and Figure II-4 and II-5 for a comparison with experiment.

One might suspect that there is some sort of critical amplitude, and when the largest wave reaches that critical condition, nonlinear effects begin to dominate and there is a "triggering" of the transition process. Note however that the physical mechanism of the transition process is still not identified.

## II.2 Formulation of Stability Theory

Generally the equations used to represent the physical system merely define conservation of mass and balance of linear momentum. These two continuum principles are expressed thru the continuity and Navier-Stokes equations. For a viscous incompressible fluid, the equations are:

$$\begin{aligned}
 & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\
 & \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \\
 & \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) \\
 & \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
 \end{aligned}
 \tag{II-1}$$

In Section IV, these equations will be derived and it will be shown that they are the result of approximations which certainly must influence the stability of the fluid system. Specifically, these equations omit terms representing fluid elasticity and internal angular momentum by assumption that such effects are negligible for air. This is reminiscent of the early assumption regarding importance of viscosity on disturbances in air, and is a clue as to what effects need to be examined in more detail.

Beginning with Eqns. II-1, all flow quantities are divided into a meanflow term and a fluctuation term: i.e., for some parameter  $q(x,y,z,t)$  it is assumed that

$$q(x,y,z,t) = Q(x,y,z) + q'(x,y,z,t) . \quad \text{II-2}$$

The mean-flow quantity  $Q$  is assumed to satisfy conventional boundary layer approximations, and negligible mean flow terms are dropped. The equations sometimes are reduced even further by the assumption of parallel flow. Reshotko (1969) has noted that this assumption is not always valid, and mean growth of the boundary layer may at times be an important factor in the stability of the flow. Donaldson (1969) and others have investigated a growing boundary layer and found that it can influence stability (as might be expected). Hence, each individual investigator must determine if the nonparallel conditions are significant for the particular flow he is considering. Parallel flow equations are achieved by the following conditions placed on the mean flow:

$$U = U(y) , \quad W = W(y) , \quad V = 0 . \quad \text{II-3}$$

The parallel flow equations next are put in dimensionless form using freestream conditions and boundary layer thickness as characteristic parameters. For convenience, notation usually is changed at this point so that dimensional quantities are identified by an asterisk (\*) superscript. For boundary conditions it is assumed that the condition of no-slip at the wall also applies to disturbance velocities at the wall, and that disturbances go to zero as  $y \rightarrow \infty$ .

The boundary conditions are homogeneous, and solutions to the set of parallel flow equations with these conditions will exist only for certain combinations of flow and disturbance parameters. Combinations for which the boundary conditions

are satisfied are eigenvalues of the problem, and the objective of stability analyses is to find the boundary between stable and unstable eigenvalues.

The disturbance quantities are taken to be periodic, having the general form:

$$q'(x,y,z,t) = q(y) \exp i[\alpha_\delta x + \beta_\delta z - \omega_\delta t] \quad \text{II-4}$$

where under a change of notation it is customary at this point to let nondimensional parameters be

$$x = \frac{x^*}{\delta}, \quad z = \frac{z^*}{\delta}, \quad t = \frac{t^*}{\delta} U, \quad \text{II-5}$$

and asterisk quantities become dimensional.

The term  $q(y)$  is the complex amplitude function of a typical flow variable  $q'$ ; the  $\alpha_\delta$  and  $\beta_\delta$  are dimensionless wave numbers  $2\pi\delta/\lambda_x^*$  and  $2\pi\delta/\lambda_z^*$ ; and  $\lambda_x^*$  and  $\lambda_z^*$  are wavelengths in the  $x$  and  $z$  directions, respectively. The dimensionless frequency is  $\omega_\delta = \omega^* \delta / U^*$ . At time  $t$ , the amplitude of  $q'$  is constant along a line

$$\left( \alpha_\delta x + \beta_\delta z \right) = \text{constant}, \quad \text{II-6}$$

and this line therefore is a line of constant phase in the  $xz$ -plane. If  $\beta_\delta = 0$ , then the wave propagates in the freestream direction. If  $\beta_\delta \neq 0$ , then the wave is inclined at an angle

$$\psi = \tan^{-1} \frac{\beta_\delta}{\alpha_\delta} \quad \text{II-7}$$

with respect to the freestream. Note that even if the boundary layer is two-dimensional the oblique waves or disturbances are three-dimensional.

If  $\alpha_\delta$ ,  $\beta_\delta$ , and  $\omega_\delta$  are all real, then the disturbance does not grow, and so it propagates through the parallel flow with constant amplitude  $|q'(y)|$ . If the frequency  $\omega_\delta$  is complex, then the amplitude will change with time, and this is referred to as temporal growth. If  $\alpha_\delta$  and  $\beta_\delta$  are complex, the amplitude will change with  $x$  and  $z$ , and this is referred to as spatial growth. If all three quantities are complex, then the disturbance wave grows both in time and space.

For temporal theory the disturbances are written in the form

$$q'(x,y,z,t) = q(y) \exp i[\alpha_\delta x + \beta_\delta z - \alpha_\delta ct] \quad \text{II-8}$$

where the complex frequency  $\omega_\delta$  has been replaced by  $\alpha_\delta c$ , with  $c$  being complex:

$$c = c_r + ic_i . \quad \text{II-9}$$

The real part ( $c_r$ ) of the complex wave velocity  $c$  is equal to the phase velocity in the  $x$  direction, and the imaginary part ( $c_i$ ) is the amplification factor.

In spatial theory the complex  $c$  is not introduced. Instead,  $\alpha_\delta$ ,  $\beta_\delta$ , and  $\omega_\delta$  are written as complex: i.e.,

$$\omega_{\delta} = (\omega_{\delta})_r + i(\omega_{\delta})_i$$

$$\alpha_{\delta} = (\alpha_{\delta})_r + i(\alpha_{\delta})_i$$

II-10

$$\beta_{\delta} = (\beta_{\delta})_r + i(\beta_{\delta})_i .$$

Then, in order to have no temporal growth, the frequency must be real, so  $(\omega_{\delta})_i = 0$  .

Depending on which type of disturbance growth a particular author wishes to investigate, these disturbance expressions are substituted into the equations of motion and various additional levels of assumptions about orders of magnitude are made. The equations generally are analyzed in two-dimensional form for simplicity, but several investigations have gone so far as to consider three-dimensional disturbances in a two-dimensional mean flow.

There are many special forms of the stability equations. For parallel flow, these equations are discussed extensively by Betchov and Criminale (1967). One of the most important special forms is the Orr-Sommerfeld equation. The mean laminar flow in the x-direction is assumed to be influenced by a disturbance which is composed of a number of discrete partial fluctuations, each of which consists of a wave propagated in the x-direction.

When the disturbance also is two-dimensional the resulting form of the equation is less complicated, and it is possible to introduce a stream function  $(x,y,t)$  to represent a single

oscillation of the disturbance. The disturbance is assumed to be of the form:

$$\psi(x,y,t) = \phi(y)e^{i(\alpha x - \omega_d t)}, \quad \text{II-11}$$

where  $\phi$  is the amplitude function of the fluctuation and is assumed to depend only on  $y$ .

Then the velocity perturbation components  $u'$  and  $v'$  are of the form

$$\begin{aligned} u' &= \frac{\partial \psi}{\partial y} = \phi'(y)e^{i(\alpha x - \omega_d t)} \\ v' &= -\frac{\partial \psi}{\partial x} = -i\alpha\phi(y)e^{i(\alpha x - \omega_d t)}. \end{aligned} \quad \text{II-12}$$

where prime on  $\phi$  denotes not perturbation but differentiation with respect to  $y$ .

These are introduced into the continuity and momentum equations, the pressure term is eliminated, and the remaining terms are nondimensionalized. The resulting equation is

$$\boxed{(U - c)(\phi'' - \alpha^2\phi) - U''\phi = -\frac{i}{\alpha R}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi)}, \quad \text{II-13}$$

which is the fundamental stability equation for the disturbance. It is known as the Orr-Sommerfeld equation (1907, 1908), and in its several forms (depending on the nature of the disturbance assumed), it is at the heart of almost all basic work on stability of incompressible steady parallel flows.

If viscosity is considered to act only in the establishment of the mean flow and is assumed to be so weak that it has a negligible effect on the disturbances, then the Orr-Sommerfeld equation reduces to the following much simpler inviscid equation

$$\boxed{(U - c)(\phi'' - \alpha^2 \phi) - U''\phi = 0}. \quad \text{II-14}$$

This is known as the inviscid Orr-Sommerfeld equation, but is also known more extensively as the Rayleigh equation (1880, 1887). The Rayleigh form was developed before the viscous form, and the majority of early papers on stability of fluids used this frictionless form as their point of departure. As a result, an extensive body of work on inviscid stability has built up over the past 100 years. However, from this equation no critical Reynolds number was realized. It was only much later (1907, 1908) that the more complete viscous equation was developed and finally solved (Tollmien, 1929) for critical Reynolds number.

### II.3 Special Problems

Mack (1969) describes several special forms of the stability equation, and discusses analytical and numerical techniques used over the years to obtain solutions. Even with modern computers there are great difficulties. For one thing, every numerical integration scheme has errors, and growth of these errors will be determined by the most rapidly growing solution. It is well known that after the integration has proceeded a certain distance, the error may actually overwhelm the solution.

Gortler and Velte (1969) discuss the phenomenon of "numerical instability" in steady solution of the Navier-Stokes equations. In some cases this numerical instability definitely may yield misleading results. The problem normally is overcome in computational fluid dynamics by stability requirements involving computational mesh size, incorporation of "artificial viscosity", and other suppression techniques. However, successful use of these techniques in the computation of mean flows does not make it clear how they can be applied to problems where the task is to investigate some physical instability. If numerical techniques are to be applicable to stability problems for physical systems, the numerical instability must be considerably smaller than the physical instability in order to distinguish between them, and in order for the two to be uncoupled. It is believed that this problem is not sufficiently understood at this time, and that further work must be done before numerical integration can be used effectively and confidently in studying flow transition via analysis of the dynamics of the system.

Taylor (1915) was the first to point out the possible destabilizing influence of viscosity, but Prandtl (1921) was the first to demonstrate clearly that a stable inviscid flow can be made unstable by viscosity. Prandtl's discovery was preceded by experimental observations where he detected occasional wave forms with slowly increasing amplitude which contradicted the accepted stability of laminar motion with respect to small disturbances. Following these observations, Prandtl and Tietjens performed calculations taking into account the influence of viscosity on disturbances, but only over a very small region of the velocity profile in the immediate neighborhood of the wall.

Their calculations yielded the unexpected result that introduction of a small value of viscosity into the equation did not produce damping but instead amplification for all Reynolds numbers and all wavelengths of disturbances for velocity profiles which had been shown to be stable when viscosity was neglected! The velocity profiles they investigated were composed of straight segments. Heisenberg extended their work by studying the stability of curved velocity profiles, taking into account the effects of viscosity. However, he still did not succeed in obtaining a critical Reynolds number (Schlichting; 1955).

The problem was finally resolved when Tollmien (1929) demonstrated that the influence of viscosity on disturbances must be taken into account not only in the immediate vicinity of the wall, as supposed by Prandtl and Tietjens, but must also be accounted for in the neighborhood of the critical layer where the velocity of wave propagation of disturbances becomes equal to the velocity of the main flow. Tollmien also showed that this influence of viscosity becomes evident only if the curvature of the velocity profile is accounted for. As a result of this analysis, he was able to find a limit of stability, the critical Reynolds number, for flow in the boundary layer of a flat plate at zero incidence. This history is important and is emphasized because it clearly illustrates how key features of the problem were for years discarded as negligible effects. The point is emphasized in relation to the suggestion of this report that molecular effects heretofore neglected are also a key part of the problem.

The first attempt to develop a compressible stability theory was made by Küchemann (1938), but he too neglected the influences of viscosity, mean temperature gradient, and the curvature of the velocity profile. The earliest fruitful theoretical investigation of the stability of a compressible boundary layer was made by Lees and Lin (1946). They developed an asymptotic viscous theory and also conducted a detailed investigation of purely inviscid theory. As a consequence, they found that the flat-plate compressible boundary layer is unstable to purely inviscid disturbances, which is quite unlike the incompressible Blasius boundary layer where the instability is viscous in origin.

This inviscid instability increases with increasing Mach number, and leads to a major difference between the incompressible and compressible theories. A second major difference between incompressible and compressible theories results from the fact that the mean flow relative to the disturbance phase velocity can be supersonic. Mack (1965) found that whenever the relative flow is supersonic over some portion of the boundary layer profile there is an infinity of wave numbers for the single phase velocity! These additional inviscid neutral disturbances are called higher modes, and were discussed earlier.

Another feature of compressible flow, found by Dunn and Lin (1955), is that Squire's theorem does not hold. They showed that for compressible flow a three-dimensional disturbance is more unstable than a two-dimensional one, and Mack (1967) later showed that for all supersonic Mach numbers a three-dimensional disturbance is more unstable than a two-dimensional one.

#### II.4 Relationship Between Stability and Transition

The relationship between stability and transition at supersonic speeds is even more of an open question than at low speeds. Experiments [Laufer-Vrebalovich (1960), Demetriades (1960), Kendall (1967)] have been performed that firmly establish the existence of instability waves in supersonic and hypersonic laminar boundary layers, but they do not demonstrate how transition is caused by the instability waves. In essence, they deal with macroscopic amplification characteristics. From all this, however, it does appear that if the external disturbance level is sufficiently low then transition occurs as a direct result of laminar instability. The key problem is that the "physical mechanism" still is unknown.

A description by Tani (1967) of the transition process indicates that, in the absence of large disturbing influences, the sequence of transition processes actually involves four distinct stages in the following order: (1) amplification of weak disturbances, (2) further nonlinear development of the disturbances, (3) development of high-shear-layer disturbances, and finally (4) development of turbulent randomness.

Tani notes that the transition is preceded by the appearance of weak oscillations of the type predicted by linearized theory of laminar instability provided all sources of disturbance are sufficiently small. However, he points out that even in the most carefully controlled experiments an initially 2D wave develops into a 3D pattern with the rate of growth varying in the spanwise direction. He concludes that the wave grows locally at a rate determined by local

Reynolds number, which may vary in the spanwise direction due to minor irregularities in the free stream or the upstream boundary layer.

While the wave is sufficiently weak, it develops downstream in a manner predicted by the linearized theory. However, when the wave ceases to be weak, its development begins to deviate from that of a theoretical prediction, and nonlinear effects manifest before any actual breakdown. The most striking feature of the nonlinearity is a pronounced increase in three-dimensionality characterized by spanwise variation in wave amplitude, with peaks and valleys occupying "fixed" spanwise positions. As already noted, associated with this variation in wave amplitude is a spanwise variation in local mean velocity which is indicative of the existence of a system of streamwise vortices.

Benney and Lin (1960) found that the interaction between a 2D Tollmien-Schlichting wave and a 3D wave with periodic spanwise variation will exhibit a system of slowly growing secondary vorticity in the streamwise direction when the 2D component predominates. The result is a mean velocity defect at the peaks and an excess at the valleys. This is consistent with the observations in air made by Klebanoff, Tidstrom, and Sargent (1962), and as already noted it reverses findings of the early results of Klebanoff and Tidstrom (1959). The understanding of the effect of pre-existing three-dimensionality then becomes important in developing a unified view of transition.

The termination of the nonlinear development according to Tani is indicated by an abrupt increase in wave amplitude followed

by a subsequent concentration of vorticity in a thin layer called a "high-shear layer". This layer is related directly to the occurrence of an inflexional velocity profile for a considerable fraction of a cycle of the fluctuating velocity. (See Figures II-10 and II-11).

The hot-wire measurements of Klebanoff et al (1962) are also indicative of the shedding of vortices, which they call "hairpin eddies". These vortices are highly unstable and break down into smaller vortices which eventually develop into random fluctuations typical of turbulent flow.

The vortices generated by (or disintegrated from) the high shear layer travel downstream at a velocity greater than that of the primary wave. As they travel downstream, they break down into smaller vortices, which again break down into more smaller vortices. Hama, Long, and Hagarty (1957) using hydrogen bubbles in water observed this cascade process occurring several times. Tani indicates that it is during this cascade process of wave breakdown that the formerly periodic structure of the fluctuations is obliterated. He draws the conclusion that turbulence is initiated in small localized regions in the form of "turbulence spots." These spots grow as they move downstream until they merge to form the fully turbulent boundary layer (Figure II-12). Hence, there is the linear amplification followed by nonlinear development with associated 3D enhancement and high shear layers from which hairpin eddies are shed and cascade downstream. These hairpin eddies cascade until their periodic structure is obliterated into random fluctuations which appear as the turbulent spots.

The idea of turbulent spots is relatively new. It was originally put forth by Emmons in 1951, and involves the concept that each point of the boundary layer has a definite probability of being turbulent. But then how small is a point, and if significant processes begin at a "point", what is the smallest significant scale of the process? Clearly it is not the scale of the "hairpin eddies"!

The process of formation, growth, and coalescing of turbulent spots has been studied in detail by Emmons (1951), Schubauer and Klebanoff (1955), Elder (1960), and Spangenberg and Rowland (1960). Spangenberg and Rowland report from their optical studies that the turbulent spots grow so rapidly during the first few microseconds that they appear to explode from the smooth outline of the laminar layer.

According to Spangenberg and Rowland, the first manifestation of turbulent breakdown was the intermittent appearance of ripples on the outer surface of the boundary layer. One or more visible shock waves were usually seen at the crest of each ripple. As the ripples moved downstream, each divided into several segments. Each of the segments then became the source of a shock wave. In a matter of microseconds after the appearance of these disturbances, low density boundary layer air was belched from the disturbance area, and the erupted spot then grew as described by other authors.

At first the turbulent spots were thought to occur completely at random in both time and space, but later indications were that the process was not entirely random. Instead there

appeared to be a production frequency with a range which varied as the test conditions were varied. As for location, the bursts seemed to be predominantly generated near spanwise peaks in wave amplitude.

The terminology, "physical mechanism", frequently appears in the literature, and is generally accepted as the big unknown. Usually the amplification process is described as the "physical mechanism" of transition. However, a careful and important distinction needs to be made between the physical mechanism (of transition) and these multiple wave amplification processes.

Wave amplification processes really do not show how the complex wave patterns degenerate into the small but macroscopic randomness observed, measured, and described as turbulence. Certainly much more is known today about these amplification processes, and their description is an important part of identifying the progressive conditions which precede the development of turbulence, but that is about all that can be attributed to them.

For example, the description of wave amplification processes does not predict, demonstrate, or explain the formation of turbulent bursts. The math model of classical laminar boundary layer equations makes no provision for energy exchanges that could develop into and maintain randomness. This omission needs to be recognized, and the model needs to be modified to include physical coupling processes which would allow fluid randomness.

Regarding characterizations of turbulence, Kovaszny (1967) points out that the turbulent boundary layer is but a special subclass of a broad category of quasi-parallel turbulent shear flows. He discussed (1) unconstrained flows with no solid wall present, (2) fully constrained flows, and (3) half-constrained flows. The boundary layer, which is in this latter class, has all the complicating features of both constrained and unconstrained flows.

He states that upon close examination it is apparent that the turbulent boundary layer has four rather distinct regions. Very close to the wall there is a viscous sublayer (often called laminar sublayer) where viscosity dominates even though there still are large random fluctuations. Then there is a wall-dominated turbulent layer where the flow scales linearly with distance from the wall, and is known as the region of the "Law of the Wall". Next, far out from the wall there is a large outer region of nearly homogeneous turbulence which follows the "Law of the Wake". Finally, there is a superlayer which provides the turbulent-nonturbulent interface with the free stream.

Each of these regions apparently has its own characteristic length scale, and each has peculiarities which are not understood at this time. More recently, as Kovaszny points out, it has become evident that the random fluctuations in the sublayer are quite large, and it is inappropriate to refer to it as a "laminar sublayer". Movies by Kline and Reynolds (1967) are reported to show that this sublayer exhibited a very strong three-dimensionality.

An important feature of Kovasznay's 1967 paper was his calling attention to work at Stanford by H. K. Moffatt (1965). The existence of a very strong shear layer has already been discussed. Now it is found from Moffatt's work that any rapid shearing motion has two important effects on homogeneous turbulent flows. One effect is a change of energy level and an anisotropic redistribution of any disturbance wave number (i.e., changes in both orientation and magnitude), so that originally isotropically distributed wave number components are now preferentially distributed roughly perpendicular to the flow axis, and their total energy increases linearly with distance propagated.

The other important result is that the Reynolds stress increases linearly with time, and for short times after application of a strain, the stress established is proportional to the total strain experienced. This suggests an elastic rather than a viscous type of response. The elastic behavior implies that a readjustment of velocity profiles will proceed according to the wave equation and not according to a diffusion equation as normally thought. Moffatt notes that an analogy between turbulent flow and the flow of a non-Newtonian fluid was first pointed out by Rivlin (1957), and was further recommended by Liepmann (1961).

These results match with concepts developed independently by Kistler (1969), and with observations and analyses by other investigators as discussed in the next section. To summarize, the developments through recent years have shown that wave amplification progresses through a linear phase and on into

a nonlinear phase. From the nonlinear phase, regions of high shear develop, and finally Moffatt has shown that this leads to viscoelastic phenomena. The next section therefore will briefly discuss viscoelasticity.

### III. FLUID CLASSIFICATIONS AND CHARACTERISTICS

#### III.1 Constitutive Equations

##### Introduction

This section introduces some of the more general concepts of continuum fluids which are known in the field of rheology. A Newtonian fluid (the fluid of classical aerodynamics and hydrodynamics) is a very special and simplified concept, and is only a small part of a large hierarchy of fluids.

A Newtonian fluid is one where the shearing stress  $\tau$  is linearly related to rate of strain  $\dot{\gamma}$  by a proportionality factor  $\mu$ , called the shear viscosity. Such a relationship is called the constitutive equation for the fluid. Usually  $\mu$  is taken to be a constant, and dependence on temperature is neglected even in dynamical situations. Thus, for a Newtonian fluid,

$$\tau = \mu \dot{\gamma} = \mu \frac{\partial u}{\partial y} . \quad \text{III-1}$$

This relationship and its use in development of the classical equations of fluid mechanics will be discussed more in Section IV.

##### Non-Newtonian Fluids

There are many other fluids which are non-Newtonian, and roughly speaking they can be put into the following three categories, each of which has several sub-categories:

(1) Time-Independent Non-Newtonian

(a) Bingham Plastics

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- (b) Pseudoplastic Fluids
  - (c) Dilatant Fluids
- (2) Time-Dependent Non-Newtonian
- (a) Thixotropic Fluids
  - (b) Rheopectic Fluids
- (3) Viscoelastic Fluids
- (a) Maxwell Bodies
  - (b) Kelvin-Voigt Bodies

The relationship between Newtonian fluids and the time-independent non-Newtonian fluids is evident in Figure III-1. Time-dependent fluids develop a build-up in reaction to shearing. For example, a thixotropic fluid may behave essentially in a Newtonian manner when first sheared. However, after being sheared, if the fluid stands for a short period of time and is sheared again the stress build-up will be different than before. Thixotropic fluids also exhibit a hysteresis. These characteristics are shown in Figs. III-2 and III-3, respectively.

The Bingham Plastic exhibits a yield effect, and obeys an empirical relation of the form

$$\tau = \tau_0 + \eta_0 \dot{\gamma} . \quad \text{III-2}$$

Pseudoplastics obey an empirical relation similar to the Newtonian fluid. Ostwald suggested a power law which

encompasses pseudoplastics, Newtonian fluids, and dilutant fluids, as follows:

$$\tau = K\dot{\gamma}^n \quad \text{III-3}$$

- $n < 1$  ; pseudoplastic power-law fluids
- $n = 1$  ; Newtonian fluids
- $n > 1$  ; dilutant power-law fluids

There are many other empirical relations proposed for these fluids, but it is not important to go into the details of them here.

Another class of fluids, the viscoelastic fluids, are thought to be very important. Again, many empirical relations have been proposed to describe these fluids which exhibit combined characteristics of viscosity and elasticity. Most of the models proposed are a simple combination of Newtonian viscosity and Hookean elasticity. Obviously, more complex relationships could be developed (such as power law viscosity and Hookean elasticity, etc., etc.). The simple model mentioned would obey a relation of the form

$$\dot{\gamma} = \frac{\tau}{\mu_0} + \frac{\dot{\tau}}{\lambda}, \quad \text{III-4}$$

or

$$\left(\frac{\mu_0}{\lambda}\right)\dot{\tau} + \tau = \mu_0\dot{\gamma},$$

where

$$\frac{\mu_0}{\lambda} = \text{relaxation time; i.e., the time constant for exponential decay of stress at a constant strain.}$$

If the motion is stopped, then the stress relaxes as  $e^{-t\gamma/\mu_0}$ . For time varying processes, the elastic constants may actually be complex functions of frequency.

### General Linear Viscoelasticity

A general relationship for linear viscoelastic fluids could be written in the form

$$\left. \begin{aligned} \dots + \tilde{B}_{ijmn} \ddot{\gamma}_{mn} + \tilde{B}_{ijmn} \dot{\gamma}_{mn} + \tau_{ij} &= C_{ij} \\ + C_{ijmn} \gamma_{mn} + \tilde{C}_{ijmn} \dot{\gamma}_{mn} + \dots & \end{aligned} \right\} \text{III-5}$$

Then the more simple one-dimensional viscoelastic models and their mechanical analogies are:

- (1) Linear Elastic (generalized Hook's Law);

$$\tau_{ij} = C_{ijmn} \gamma_{mn} \quad \text{III-6}$$

- (2) Linear Viscous (Newtonian fluid);

$$\tau_{ij} = \tilde{C}_{ijmn} \dot{\gamma}_{mn} \quad \text{III-7}$$

- (3) Maxwell Body;

$$\tilde{B}_{ijmn} \dot{\tau}_{mn} + \tau_{ij} = \tilde{C}_{ijmn} \dot{\gamma}_{mn} \quad \text{III-8}$$

- (4) Kelvin-Voigt Body;

$$\tau_{ij} = C_{ijmn} \gamma_{mn} + \tilde{C}_{ijmn} \dot{\gamma}_{mn} \quad \text{III-9}$$

(5) etc., for many combinations and extensions.

Viscoelastic fluids exhibit many effects which are strange to aerodynamics and gas dynamics, but which are well known in rheology. When viscoelastic fluids are sheared rapidly they exhibit normal stress effects (or cross stress) which cannot exist in purely viscous fluids. Cross stress effects were first predicted by Reiner in 1945, but were observed for the first time by Weissenberg in 1947. Cross stresses cause a viscoelastic fluid to climb a rotating shaft or exhibit an axial load on a rotating disk, as shown in Fig. III-4, and these characteristics now generally are called the Weissenberg effect. Similar to the use of Reynolds number in viscous flows to express the ratio of inertial to viscous effects, in viscoelastic flows the Weissenberg number expresses the ratio of elastic to viscous effects!

There are numerous other viscoelastic effects, and three of them should be mentioned:

- (1) Viscoelastic effects are known to generate many types of secondary flows; (depending on factors such as geometry, motion of boundary surfaces, etc.).
- (2) Viscoelasticity is known to depress the turbulent friction factor when certain critical conditions are exceeded; (this is known as the Toms effect, and is illustrated in Fig. III-5).
- (3) Viscoelasticity is known to alter the critical Reynolds number obtained from a stability analysis! (For example, Wen (1963) and Betchov (1965)).

Betchov investigated the Orr-Sommerfeld equation for a Blasius boundary layer with a complex viscosity. He integrated the equation numerically and found that the mean flow is not influenced much by time-dependent effects, but that neutrally stable oscillations are destabilized by the stress lagging behind the rate of strain. Betchov's results are shown in Fig. III-6, where  $\delta$  is the boundary layer thickness and  $\theta$  is the phase angle defined by

$$\left. \begin{aligned} v_o &= \frac{v}{[1 + (\alpha c T_M)^2]^{1/2}} \\ \theta &= - \text{arc tan } (\alpha c T_M) . \end{aligned} \right\} \text{III-10}$$

Similar results were reported by Chun and Schwarz (1968) for Poiseuille flow of a Second-Order Fluid. They found a comparable strong shift in stability as the fluid becomes more non-Newtonian. Bogue and White (1970) discuss other studies of non-Newtonian effects on hydrodynamic stability.

### Linear Elasticity

Symmetry assumptions play a big role in determining the simplicity or complexity of the stress-strain relationships, and in the static and dynamic characteristics displayed by the model. Consider linear elasticity, defined by

$$\tau_{ij} = C_{ijmn} \gamma_{mn} \quad \text{III-11}$$

where  $C_{ijmn}$  are constants, with  $(i,j,m,n, = 1,2,3)$ . This relationship involves  $3^4 = 81$  independent constants.

The assumption that only symmetric strain is present is expressed by  $\gamma_{mn} = \gamma_{nm}$ , which requires that  $C_{ijmn} = C_{ijnm}$ , and the problem reduces to contain only  $9 \times 6 = 54$  independent constants. The alternative assumption that the stress is symmetric is expressed by  $\tau_{ij} = \tau_{ji}$ , which requires that  $C_{ijmn} = C_{jimn}$ , and again the number of independent constants is reduced to  $6 \times 9 = 54$ . The combined assumptions of symmetry in both stress and strain is expressed by the joint requirements

$$C_{ijmn} = C_{jimn} \quad \text{and} \quad C_{ijmn} = C_{ijnm}, \quad \text{III-12}$$

so the stress-strain relation may be given by

$$\tau_{\alpha} = C_{\alpha\beta} \gamma_{\beta} \quad \text{III-13}$$

where

$$(\alpha, \beta = 1, 2, 3, 4, 5, 6)$$

and the problem contains only  $6^2 = 36$  independent constants.

Considering the further assumption of isotropy, the most general isotropic tensor of rank four is represented by

$$C_{ijmn} = A^1 \delta_{ij} \delta_{mn} + A^2 \delta_{im} \delta_{jn} + A^3 \delta_{jn} \delta_{mj}. \quad \text{III-14}$$

which may be rewritten in terms of a symmetric and a skew-symmetric part. Dropping the skew-symmetric part and substituting into

$$\tau_{ij} = C_{ijmn} \gamma_{mn} \quad \text{III-15}$$

gives the linear isotropic law for stress-strain symmetry:

$$\tau_{ij} = \lambda \delta_{ij} \gamma_{mm} + 2\mu \gamma_{ij} \quad \text{III-16}$$

Here there are only two independent constants, given by

$$\left. \begin{aligned} \lambda &= A^1 \\ 2\mu &= A^2 + A^3 \end{aligned} \right\} \quad \text{III-17}$$

and the new coefficients ( $\lambda$  and  $2\mu$ ) are known as Lamé's constants.

Contraction of  $\tau_{ij}$  gives

$$\left. \begin{aligned} \tau_{ii} &= 3\tilde{\lambda} \gamma_{ii} + 2\tilde{\mu} \gamma_{ii} \\ &= (3\tilde{\lambda} + 2\tilde{\mu}) \gamma_{ii} \end{aligned} \right\} \quad \text{III-18}$$

and a common alternate arrangement of the coefficients is found to be

$$\left. \begin{aligned} E &= \frac{\mu(3\tilde{\lambda} + 2\tilde{\mu})}{(\tilde{\lambda} + \tilde{\mu})} = \text{modulus of elasticity} \\ P &= \frac{\tilde{\lambda}}{2(\tilde{\lambda} + \tilde{\mu})} = \text{Poisson's ratio} \end{aligned} \right\} \quad \text{III-19}$$

### Linear Viscosity

Results similar to linear elasticity are obtained for linear viscosity, except that stress is now related to rate of strain by  $\tilde{C}_{ijmn}$  independent coefficients:

$$\tau_{ij} = \tilde{C}_{ijmn} \dot{\gamma}_{mn} . \quad \text{III-20}$$

The general linear isotropic form for  $\tilde{C}_{ijmn}$  is

$$\tilde{C}_{ijmn} = A' \delta_{ij} \delta_{mn} + A'' \delta_{im} \delta_{jn} + A''' \delta_{in} \delta_{mj} , \quad \text{III-21}$$

which also may be written as

$$\tilde{C}_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + \nu (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) . \quad \text{III-22}$$

where the last term is the skew-symmetric part. The  $\lambda$ ,  $\mu$ , and  $\nu$  are not the same coefficients discussed in linear elasticity.

Now regardless of how the fluid is strained, if there is assumed to be symmetry of the resulting stress (i.e.,  $\tau_{ij} = \tau_{ji}$ ), then the skew-symmetric coefficients vanish since

$$(\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \equiv 0 . \quad \text{III-23}$$

This result is the same as would be obtained under the assumption of symmetry in the rate of strain. Substitution of the isotropic symmetric coefficient into the stress/rate-of-strain relation yields

$$\begin{aligned}
\tau_{ij} &= \left\{ \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \right\} \dot{\gamma}_{mn} \\
&= \lambda \delta_{ij} \delta_{mn} \dot{\gamma}_{mn} + \mu \delta_{im} \delta_{jn} \dot{\gamma}_{mn} + \mu \delta_{in} \delta_{jm} \dot{\gamma}_{mn} \\
&= \lambda \delta_{ij} \dot{\gamma}_{mm} + \mu \delta_{im} \dot{\gamma}_{jm} + \mu \delta_{in} \dot{\gamma}_{jn} \\
&= \lambda \delta_{ij} \dot{\gamma}_{mm} + 2\mu \dot{\gamma}_{ij}
\end{aligned}
\tag{III-24}$$

Contraction gives

$$\tau_{ii} = (3\lambda + 2\mu) \dot{\gamma}_{ii} = \kappa \dot{\gamma}_{ii}
\tag{III-25}$$

where

$$\kappa \equiv (3\lambda + 2\mu) = \text{coefficient of bulk viscosity} .$$

It is conventional in classical fluid mechanics to assume the coefficient of bulk viscosity is zero. This assumption, due to Stokes, is named after him and yields the familiar relation  $\lambda = -2/3\mu$  .

However, it is now well known that the bulk viscosity coefficient is not zero, and that it is indeed important in cases such as the relaxation of diatomic gases excited by ultrasonic waves. Frequently it is stated in the literature that the Stokes relation can be derived from the Kinetic Theory for monatomic gases, but Truesdell (1952b) points out that actually the Stokes relation is a basic assumption of that theory.

Constitutive equations are not exact descriptions of real materials! The best that can be said of any mathematical model of material behavior is that it provides a useful description of certain features of the behavior of some real material under specific (limited) conditions of operation. That same real material may under other circumstances exhibit quite different types of rheological behavior.

Constitutive equations of exceeding complexity and generality are available in the literature. Although they generally are too complicated for use, they do serve as a framework within which observed behavior under a wide range of conditions can be categorized. Also, they serve as sources for simpler equations which are valid under more restricted conditions of flow.

### III.2 Effects of Molecular Structure

Giesekus (1964) has shown that a suspension of soft-elastic dumbbells represents what is called an elastico-viscous liquid of type  $N = 1$ , and that if the hydrodynamic interaction of "dumbbell spheres" is accounted for, the normal stress in the direction of the flow gradient will have a finite value.

Oldroyd (1964) shows that in steady simple shearing flow with a finite velocity gradient, the elastico-viscous liquid of type  $N$  will exhibit a variable apparent viscosity depending on the rate of shear. It will also exhibit differences in the normal stresses along and perpendicular to the streamlines, and will exhibit phenomena such as the previously mentioned Weissenberg climbing effect.

Virk, et al, (1966) found from data on the Toms phenomenon in turbulent flow of dilute polymer solutions that the onset of drag reduction occurs only after the wall shear stress exceeds a "critical" value which is characteristic of the macromolecule in solution. This critical wall shear stress corresponds to a turbulence scale characteristic of the smallest eddies near the pipe wall becoming small enough with respect to some macromolecular scale, which he chooses as the "coil diameter", taken as the root mean square radius of gyration of the unperturbed macromolecular coil.

Even the largest macromolecular coils have coil diameters of only a few tenths of a micron. Therefore Virk's hypothesis that in these suspensions the individual macromolecule is the effective particle would imply that the turbulence structure extends to exceedingly small scale!

Townsend (1956) indicated that the scale of the smallest significant dissipative eddies near the pipe wall is roughly  $v/U$ , and that of the corresponding energy containing eddies is roughly ten times larger. On this basis, and using Virk's data, Fabula et al (1966) claim that the individual macromolecules are too small to interfere with the turbulence structure in a particle manner. However, it is quite important to realize that Townsend's indication of the smallest significant eddies is uncertain, and therefore open to question. Fabula's (1966) comment that Townsend's (1956) work represents "present" understanding is certainly not consistent with the results discussed in the previous section of this report concerning the "point" origin of turbulent spots.

Lieber (1964) has shown that the mechanical evolution of molecular clusters by binary elastic collision depends in a crucial manner on symmetry and uniformity in terms of molecular geometry, mass distribution, collision attitude, and orientation of relative and resultant velocity vectors.

He notes that most of what is factually known about turbulence is only phenomenological in character, and is expressed by describables and measurables that refer to the macro-scale. His contention, concurred with here, is that these phenomena necessarily have molecular counterparts, i.e., the observables are but macroscopic manifestations of molecular phenomena.

Binary encounters play a central role in the kinetic theory of gases, and it is reasonable to expect that the production and phenomena of turbulence is at least in part attributable to a succession of binary encounters whose spatial distributions and initial conditions are functions of time.

Lieber found that formation of a binary cluster is mechanically impossible when the colliding bodies are smooth rigid non-loaded spheres, but that clusters are mechanically admissible when the bodies can support and transfer angular momentum. Smooth rigid elastic spheres (with the point mass assumption) cannot transfer angular momentum by collision and consequently their initial angular momenta is conserved under all collisions. If these smooth rigid elastic bodies have less than spherical symmetry then kinetic energy can be exchanged between translation and rotation, and conversely. Under these conditions there are certain classes of initial conditions which do lead to clusters.

Lieber also found that nonuniformity in the inertial as well as geometrical properties of colliding bodies may further enhance clustering and modify dissipation. He reasoned that there must be a limit to the growth of such an aggregate, because when it becomes so large that it does not have sufficient speed to overtake other bodies it will no longer be able to direct their velocities and will cease to grow.

To test his ideas on binary clustering, Lieber conceived an experiment to establish whether or not aspects of molecular geometry not represented by the Navier-Stokes equations are relevant to the production and decay of turbulence. The experiments were conducted by a team under Bogdonov at Princeton in 1956 and, after initial difficulties with reproducibility, the experiments indeed did confirm the existence of differences in rate of decay of turbulence in excess of ten percent, attributable strictly to molecular differences (in that case between air and argon). With these positive experimental results, there clearly is reason to believe that important aspects of the mechanisms which account for either production or decay of turbulence are not embodied in the Navier-Stokes equations.

Since the experiments of Weissenberg, additional experiments have been conducted, and cross-stress effects have been reported for fluids such as air [Reiner (1957, 1958, 1960), Popper and Reiner (1958), Foux and Reiner (1964), and Bousso (1964)]! While normal stress effects now are widely accepted for that class of fluids referred to as Weissenberg fluids, the effects observed for air are still highly controversial [Taylor and Saffman (1957), and Oldroyd (1964)]. It seems, however, that the effects really should be expected

to be present even for simple diatomic gases, and that the only argument relates to how strong the effects will be, i.e., how observable. This point then emphasizes the question of how good are present experimental methods, and the answer must be able to stand in view of the situation before and after the experiments of Schubauer and Skramstad.

The dissipation mechanism discussed by Lieber (1949) suggests that in sufficiently strong shock waves the temperature (as a measure of molecular motion in translation that does not contribute to flow velocity) can indeed be anisotropic! Thus, the temperature field in flows with sufficiently large velocity gradients may in a sense be polarized. Lieber (1964) notes that the concept of a polarized temperature field is consistent with experimental results of Reiner (1960). Velocity gradients in Reiner's experiments were exceedingly large, and it would be expected that the kinetic energy in molecular rotation would be appreciable if the molecular geometry admitted transfer between translational and rotational energies. The concept of a polarized temperature is consistent with the dynamical exchange concepts of Kistler (1969), and specifically is suggested in work by Ried (1969).

Ericksen (1962a) explores orientation phenomena produced by flow in certain ideal incompressible viscoelastic fluids which tend to be unoriented at rest. He shows that the fluids under consideration continue to be unoriented and behave like Newtonian fluids when the velocity gradients are sufficiently small. However, when the velocity gradients are large and vorticity is not excessive, orientation may occur! When it does, the fluid becomes non-Newtonian!

Vorticity actually produces a stabilizing effect and resists particle orientation effects which could arise due to rapid compression or extension disturbances. Thus eddies tend to have a gyroscopic action giving stability, but on the other hand they also cause intense velocity gradients which increase in strength as the eddies cascade and disintegrate.

Clarke and McChesney (1964) develop a vorticity equation which is a generalization of Crocco's result for a chemically inert, nonrelaxing gas. They show that chemical reactions and internal state transitions can give rise to the production of vorticity in the gas. Hence, lack of chemical and internal equilibrium leads to nonzero vorticity. A general result for a reacting mixture has been given by Hayes and Wu (1958).

Clark and McChesney also show that in very weak fully dispersed shock waves, the effect of internal mode relaxation is exactly equivalent to the action of viscosity. Hence, internal mode relaxation should be expected to play some role in stability of disturbances, just as viscosity does. They conjecture that in near-equilibrium situations the effect of relaxation of an internal mode could be represented by an equivalent bulk viscosity. This is not adequate for treatment of stronger shocks. In nitrogen, for example, the lower rotational levels are excited almost instantaneously, but for  $J > 20$ , equilibration takes place at a different rate that is much slower.

### III.3 Diatomic Molecules

The simplest model of a rotating diatomic molecule is that of a dumbbell which rotates with moment of inertia  $I$  about each of two orthogonal axes, both of which are perpendicular

to the internuclear axis of the molecule. The molecule is considered to have angular momentum about each of two axes. This model assumes that the only angular momentum in the problem is that due to nuclear motion. There are other angular momenta which play a role in the complete problem. For example, there also are electronic orbital angular momenta, electron spin angular momenta, and nuclear spin angular momenta. However, these will not be considered here since they are much smaller effects.

There also could be an angular momentum component along the molecule axis of symmetry. This would give the molecule the properties of an axially symmetric spinning top, with a quantum-mechanical motion analogous to precession and nutation. However, the effective moment of inertia of the electrons about this axis is so small compared to that of the nuclei about the other two orthogonal axis that this spin component also is neglected.

If the molecule is assumed to be a nonrigid rotator where two atoms are connected by a massless spring, the internuclear distance (and therefore the moment of inertia  $I$ ) would increase with increasing rotational energy due to centrifugal stretching. In this case a quantum mechanical analysis of the rotational energy levels at a given vibrational level would give

$$\epsilon_{\text{rot Q.M.}} = J(J + 1) \frac{h^2}{8\pi^2 I} - D_v J^2 (J + 1)^2, \quad \text{III-26}$$

where

$D_v$  = function depending on vibrational frequency

$J$  = rotational quantum number of the nuclei

= 0,1,2,3,---

$h$  = Planck's constant.

Actually, even this expression is not complete because  $I = mr_J^2$ , and  $r_J$  stretches due to rotation at different quantum numbers.

If the rotator is rigid and allowed to rotate freely, then it may take on rotational energy levels given by

$$\epsilon_{\text{rot}_{Q.M.}} = J(J+1) \frac{h^2}{8\pi^2 I} ; J = 0,1,2,3,--- \quad \text{III-27}$$

Such energy levels are defined as consisting of all energy states having identical values of the energy  $\epsilon_J$ , and any level having more than one state is said to be degenerate. The number of states contained in a given energy level of common energy  $\epsilon_J$  is denoted by  $g_J$ , and is called the degeneracy of that level.

The internal energy is given by

$$E = NkT^2 \frac{\partial(\ln Q)}{\partial T} \quad \text{III-28}$$

where

- N = number of molecules
- k = Boltzman's constant
- T = temperature of the gas ,

and the partition function Q is given by

$$Q = \sum_J g_J \exp[-\epsilon_J/kT] \quad \text{III-29}$$

For the specific internal energy  $e = E/mN$  , since  $R = k/m$  and

$$Q = Q_{tr} Q_{rot} Q_{vib} Q_{el} , \quad \text{III-30}$$

one can write

$$\begin{aligned} e &= RT^2 \frac{\partial}{\partial T} \left[ (\ln Q) \right] \\ &= RT^2 \frac{\partial}{\partial T} \left[ (\ln Q_{tr}) \right] + RT^2 \frac{\partial}{\partial T} \left[ (\ln Q_{rot}) \right] + \dots \\ &= e_{tr} + e_{rot} + \dots \end{aligned} \quad \text{III-31}$$

Now the degeneracy for rotation is given by

$$g_J = 2J + 1 , \quad \text{III-32}$$

so

$$Q_{\text{rot Q.M.}} = \sum_{J=0}^{\infty} (2J + 1) \exp \left[ \frac{-J(J + 1)h^2}{8\pi^2 IkT} \right]. \quad \text{III-33}$$

The group  $(h^2/8\pi^2 Ik)$  has the dimensions of temperature, and is called the characteristic rotational temperature:

$$\theta_r = \left( \frac{h^2}{8\pi^2 Ik} \right). \quad \text{III-34}$$

Hence the partition function can be written as

$$Q_{\text{rot Q.M.}} = 1 + 3e^{-\frac{2\theta_r}{T}} + 5e^{-\frac{6\theta_r}{T}} + \dots, \quad \text{III-35}$$

and numerical values of  $\theta_r$  must be found by spectroscopic study.

In most cases  $\theta_r$  is very small. Clark and McChesney give:

Gas	H <sub>2</sub>	N <sub>2</sub>	NO	O <sub>2</sub>
$\theta_r$ , °K	85.4	2.86	2.42	2.07

Therefore, at ordinary temperatures, the ratio  $\theta_r/T$  is very small, and the sum of terms representing the partition function can be replaced by an integral

$$Q_{\text{rot}} = \int_0^{\infty} (2J + 1) e^{-J(J+1)\theta_r/T} dJ \quad \text{III-36}$$

which is the area under a curve of  $(2J+1) \exp [-J(J+1)\theta_r/T]$  versus  $J$  ; that is, an approximation to the sum of all energy levels  $J = 0,1,2,3,---$  . This then is the classical value for heteronuclear molecules. In general, for the above conditions, the partition function is

$$Q_{\text{rot}} = \frac{T}{\sigma\theta_r}, \quad \text{III-37}$$

where a symmetry factor has been added ( $\sigma = 1$  for heteronuclear molecules, and  $\sigma = 2$  for homonuclear).

With this value for the partition function, it is found that rotation of diatomic molecules accounts for a specific internal energy per unit mass of

$$\begin{aligned} e_{\text{rot}} &= RT^2 \frac{\partial}{\partial T} \left[ (\ln Q_{\text{rot}}) \right] && \text{III-38} \\ &= RT . \end{aligned}$$

Thus, at sufficiently high temperatures (which actually are quite low), the rotation contributes  $RT$  to the internal energy per unit mass. The rotation is then said to be "fully excited". This does not mean that no further energy can be taken up in rotation. What it really means is that the rotation energy increases linearly with  $T$  .

Spectrograms giving characteristic temperature of vibration indicate that this is strictly a high-temperature problem, and it will not be discussed further here.

In the case of the rotation mode in a diatomic molecule such as nitrogen, only a few collisions are required for the mode to reach a new equilibrium state. The time required for transfer of energy from the translational mode is very short. Hence, in a diatomic gas at room temperature, there must be a considerable number of inelastic collisions constantly occurring to preserve the rotational distribution.

Clarke and McChesney use Ehrenfest's Adiabatic Principle and develop the criteria

$$\left( \frac{S}{8\pi^2 r_e} \cdot \frac{h}{\mu v} \right) \gg 1 \quad \text{III-39}$$

for near adiabatic translation-rotation interaction. Here  $r_e$  is the equilibrium nuclear separation, and  $S$  is the effective range of intermolecular force between some atom  $A$  and a molecule  $BC$  involved in the collision. They point out that in all cases except for hydrogen and helium at cryogenic temperatures this relation is not satisfied, and the transfer of energy from translation to rotation is an efficient process and should occur readily on impact. The time of rotation is comparable with the duration of the collision so that the rotation cannot undergo a sufficient number of rotations to preserve the adiabaticity of the collision.

Clarke and McChesney also indicate that Landau and Teller (1936) concluded the efficiency of the rotational-translational collision interaction depended on the ratio

$$\left( \frac{\text{effective duration of a collision}}{\text{natural period of rotation}} \right),$$

and this ratio has a value around unity so that the energy transfer is efficient. The interaction is not a weak collision – rather it is a "violent" collision.

It is clear that if the excited rotational modes do not relax fast enough, then there could be a pumping of considerable energy into these modes. There is a limit however, and this limit depends on translational temperature, i.e., it depends on whether or not one atom of the dumbbell can be overtaken from the "rear" and thereby receive a collision that further increases its angular momentum (similar to Lieber's analysis). All other collisions will reduce the angular momentum as well as probably altering its orientation. The orientation is of considerable importance because this would indeed be a true physical mechanism whereby wave disturbances can interact as they would in a solid (i.e., longitudinal and transverse waves interacting).

It is clear from the above discussion that classical continuum theories of fluids as used in aerodynamics do not describe all the known properties of real fluids. The continuum replaces the physical fluid of molecules with a mathematical model which in certain cases may not accurately simulate characteristics of the real fluid. The degree of accuracy obviously depends on the "external" conditions and this usually is implied in the statement that there is only a limited range of validity of any mathematical model representing a physical system.

The above discussions clearly indicate that the structure of the molecular species can have profound effects upon a fluid's behavior. However, this statement is not limited

to the generally recognized fact that numerical values of transport coefficients differ from one fluid to another. It also applies to the fact that the set of dynamical equations and constitutive relations which are necessary for a description of the fluid's behavior may vary in form and number from one fluid to another. This applies not just to "suspension molecules", but also to basic "carrier molecules" such as nitrogen and oxygen (if the mean flow is air).

What really is needed is a generalization of the classical continuum theory to provide for greater detail in treating macroscopic manifestations of subcontinuum flow effects without sacrificing convenience of the field approach. The point of departure from traditional theory is incorporation of a general angular momentum principle for structured continua. This internal angular momentum is associated with configurational and kinematic aspects of a more general continuum "fluid particle" as discussed by Kistler (1969).

Two approaches to arriving at the desired equations will be mentioned. One approach is through a generalized Boltzmann equation, as discussed in papers by Dahler, Scriven, Curtis, McCoy, Condiff, Brenner, Parker, Kline, Allen, Green, and others. The second approach, not worked out yet, is through kinematical equations for a "continuum with discontinuities", using equations given by authors such as Truesdell, Jaunzemis, Eringen, etc. Though the necessary equations have been available for years, their use in developing a "Slip Theory" for boundary layers to account for diatomic molecular effects seems to be a new concept and first suggested here.

The above discussion has emphasized that a finite time is required for establishment of equilibrium in gases. For a gas in thermal equilibrium, individual molecules are constantly gaining and losing energy through collisions but the total change is zero (except for what is exchanged at the walls). The amount of energy either gained or lost per collision (measured by collisional efficiency) is of no importance in a condition of equilibrium. However, if conditions suddenly change so that the gas finds itself seeking a new state of equilibrium, then the rate of adjustment (measured by "relaxation time") is governed by collisional efficiency. If the efficiency is high, then molecules adjust rapidly and relaxation time is short. But, for a given collisional efficiency, if changes become too sudden (high shear rates, etc.), the relaxation effects become more and more important.

Parker (1959), using a molecular interaction potential consisting of an attractive component which acts between geometrical centers of molecules and a repulsive component assumed to originate from two centers of force in each molecule, obtained good results for the number of collisions necessary to establish rotational equilibrium. For nitrogen he obtained a rotational collision number of 4.01 for 300 °K, whereas experimental values obtained ultrasonically are 6, 3, 5, and 5.26. For oxygen he obtained 3.45 for 300 °K, whereas experimental values obtained ultrasonically are 3, 12, and 5. So, as noted earlier, the collision efficiency is good, and only a few collisions are necessary even at low temperatures.

A macroscopic transport process has the capability of sustaining preferred orientations among the constituent

molecules (or aggregate particles) whenever the geometrical configuration departs from smooth spherical and the mass configuration departs from symmetrical. This ability and tendency to polarize may be regarded as the natural consequence of biasing of molecular interactions inherent in the frictional interference which accompanies the transport.

The extent to which orientation occurs is governed in large measure by how small the interval of time between successive collisions in relation to the characteristic time for rotational disorientation! Even in a dilute gas state, molecules subjected to persuasive action of nearby boundaries may reorient appreciably and thereby revise their own influence upon a transport process.

Even though a precise determination of the modified transport coefficients for a diatomic gas is not the primary emphasis in this document, it is worth discussing the method by which they are obtained. Basically, in determining transport coefficients, the collisional equations are solved for a binary collision with some selected molecular mass, geometry, and force potential characteristics, and initial conditions on velocity vectors, etc. Therefore, at this point, the collisional equations themselves will be considered in order to gain insight into their completeness. Specifically, it is of interest to determine whether or not the equations used in calculating transport coefficients accurately account for molecular rotation effects in the collision dynamics.

### III.4 Collision Kinetics

Following Hirschfelder, Curtiss, and Byrd (1954)\*, the determination of appropriate collision kinematic equations begins with the laws of Newton expressed as Lagrangian equations of motion. The Lagrangian equations of motion are the Newtonian equations transformed to a generalized coordinate system. A Lagrangian function,  $L(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_{3N}; q_1, q_2, \dots, q_{3N})$ , with  $q_k(x_1, y_1, z_1, \dots, x_n, y_n, z_n)$  as generalized coordinates is defined by

$$L = K - \Phi , \quad \text{III-40}$$

where  $\Phi$  is the potential energy function for the entire system, and  $K$  is the kinetic energy of the system. Usually  $K$  is defined by

$$K = \frac{1}{2} \sum_i m \dot{r}_i^2 \quad \text{III-41}$$

which is the sum of the kinetic energies of the individual particles. One must be careful here, because for a diatomic particle this may not account for the kinetic energy of rotation. It is not clear that some of the formulations appearing in the literature have started with a sufficiently generalized Lagrangian.

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\*Hirschfelder, Curtiss, and Byrd (1954) hereafter will be referred to as HCB.

Generalized momentum of the system is written as

$$p_k = \frac{\partial L}{\partial \dot{q}_k}, \quad \text{III-42}$$

and if the generalized coordinates were Cartesian coordinates, this generalized momenta then would reduce to ordinary linear momenta.

Next, a Hamiltonian function of the system,  $H(q_1, \dots, q_{3N}; p_1, \dots, p_{3N})$ , is defined by

$$H(q_1, \dots, q_{3N}; p_1, \dots, p_{3N}) = \sum_i p_i \dot{q}_i - L. \quad \text{III-43}$$

Thus, the Lagrangian is a function of the coordinates and their time derivatives, whereas the Hamiltonian is a function of the coordinates and their conjugate momenta. In the above expression for the Hamiltonian, the  $\dot{q}_k$  appear explicitly, and also implicitly in the Lagrangian. Differentiation of  $H$  with respect to  $p_i$  and  $q_i$  gives

$$\left. \begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i + \sum_j p_j \frac{\partial \dot{q}_j}{\partial p_i} - \sum_j \frac{\partial \dot{q}_j}{\partial p_i} \frac{\partial L}{\partial \dot{q}_j} \\ \frac{\partial H}{\partial q_i} &= \sum_j p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial L}{\partial q_i} - \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{\partial \dot{q}_j}{\partial q_i} \end{aligned} \right\} \quad \text{III-44}$$

These may be simplified using Lagrange's equations of motion and the definition of conjugate momenta  $p_k$  to give

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i \end{aligned}$$

III-45

These are Hamilton's equations of motion, and are central to the development of a key equation in collision mechanics, the Liouville equation.

The Hamiltonian for a conservative system is numerically equal to the total energy of the system; i.e., it can be shown that

$$H = K + \Phi = E . \quad \text{III-46}$$

Consider now some property of the system,  $S = S(q_i, p_i, t)$ , which depends on the dynamical state of the system and in general depends explicitly on time. The change in  $S$  with time, following a point along a natural trajectory, is

$$\begin{aligned} \frac{DS}{Dt} &= \frac{\partial S}{\partial t} + \sum_i \left( \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial p_i} \dot{p}_i \right) \\ &= \frac{\partial S}{\partial t} + \sum_i \left( \frac{\partial S}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial S}{\partial p_i} \frac{\partial H}{\partial q_i} \right) \\ &= \frac{\partial S}{\partial t} + [S, H] . \end{aligned} \quad \text{III-47}$$

In the last equation, notation called the Poisson bracket has been introduced. It should be noted that an assumption of continuity along the trajectory has been introduced, and higher order terms have been dropped.

Proceeding traditionally, if one considers a large collection of noninteracting systems which differ from each other only in their initial conditions, the state of the entire collection of systems is described by a set of points. For a sufficiently large number of systems, the set of representative points can be specified by a continuous distribution function,  $\rho(q_i, p_i, t)$ , which is called the density in phase space. The choice of the density function is arbitrary at the initial time, but is fixed at any subsequent time by the equations of motion. Since there are no "sources" or "sinks" in phase space, the distribution function satisfies a generalized continuity equation:

$$\frac{\partial \rho}{\partial t} + \sum_i \left( \frac{\partial}{\partial q_i} (\dot{q}_i \rho) + \frac{\partial}{\partial p_i} (\dot{p}_i \rho) \right) = 0 . \quad \text{III-48}$$

Then using Hamilton's canonical equations and the Poisson bracket, the generalized continuity equation can be put in the form

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + [\rho H] = 0 ,} \quad \text{III-49}$$

which is called the Liouville equation.

A gas made up of  $N$  molecules may be represented by an ensemble described by the distribution function  $f^{(N)}(\vec{q}^N, \vec{p}^N, t)$  in  $\gamma$ -space of  $6N$  dimensions, and the variation of this distribution function with time must obey the Liouville equation. Thus, the time variation of the distribution function  $f^{(N)}$  is given by

$$\frac{Df^{(N)}}{Dt} = \frac{\partial f^{(N)}}{\partial t} + \left( \frac{\partial f^{(N)}}{\partial \vec{q}^N} \cdot \frac{\partial H}{\partial \vec{p}^N} - \frac{\partial f^{(N)}}{\partial \vec{p}^N} \cdot \frac{\partial H}{\partial \vec{q}^N} \right) = 0 .$$

III-50

With  $m_k$  as the mass of the molecule  $k$ ,  $\vec{F}_k$  as the force on molecule  $k$  due to all other molecules, and  $\vec{B}_k$  as the force on molecule  $k$  due to an external field, the above equation can be written as

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{k=1}^N \left\{ \frac{1}{m_k} \left( \vec{p}_k \cdot \frac{\partial f^{(N)}}{\partial \vec{q}_k} \right) + \left( (\vec{F}_k + \vec{B}_k) \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_k} \right) \right\} = 0$$

III-51

It usually is assumed that the macroscopic behavior of the gas is described with sufficient accuracy by a distribution function of low order. At sufficiently low densities, the macroscopic behavior is described by the first-order set of distribution functions  $f_i^{(1)}$ . These distribution functions are defined as the integral of  $f^{(N)}$  over the coordinates and momenta of all but one of the molecules. Thus, an equation for  $f_i^{(1)}$  may be obtained from the Liouville equation by integrating over the coordinates of  $(N-1)$  molecules. Performing this integration, and requiring  $f_i^{(1)}$

to vanish both at the walls of the container and also for  $|\vec{p}_i| \rightarrow \infty$ , the following equation is given by HCB:

$$\begin{aligned} \frac{\partial f_i^{(1)}}{\partial t} + \frac{1}{m_i} \left( \vec{p}_i \cdot \frac{\partial f_i^{(1)}}{\partial \vec{q}} \right) + \left( \vec{B}_i \cdot \frac{\partial f_i^{(1)}}{\partial \vec{p}_i} \right) \\ = - \frac{1}{(N-1)!} \iint \left( \vec{F}_i \cdot \frac{\partial f^{(N)}}{\partial \vec{p}_i} \right) d\vec{q}^{N-1} d\vec{p}^{N-1} \end{aligned}$$

III-52

In considering a system of identical molecules, the distribution function  $f^{(N)}$  is symmetric in the coordinates of all the molecules, because there is no physical differentiation among them. Therefore, in obtaining  $f^{(1)}$  it does not matter which molecule is considered. The  $f^{(1)}$  is adequate for description of all physical properties of gases which do not depend upon relative positions of two or more molecules, and so it provides a level of information sufficient for moderately dilute gases. For gases at higher density, or for dilute gases with local density concentrations, a knowledge of higher order distribution functions is required.

The equation for  $\frac{Df_i^{(1)}}{Dt}$  does not in itself define the behavior of  $f_i^{(1)}$ . That is, there is no unique integro-differential equation for  $f^{(1)}$ , and in order to remove this ambiguity it is necessary to invoke an additional condition which restricts the possible functions  $f^{(N)}$ . This is the condition of molecular chaos.

The measure of influence that the state of one particle has on the state of a different particle is called the correlation

between the particles. When the states of two particles are independent of one another, the particles are said to be uncorrelated, and this condition of vanishing correlations is what Liboff (1969) describes as the property or constraint of molecular chaos.

The incident velocities of two particles prior to collision are independent, but after collision they are not independent. After collision, the velocities are related thru the conservation equations. Hence, collisions create correlations, but they also destroy previous correlations. Molecular chaos in essence assumes that such encounters will distribute the molecular states at random, and without any correlation between velocity and position of two molecules prior to an encounter. Thus, there is to be no correlation remaining from a possible previous encounter between the molecules, and furthermore their force fields do not encounter. This clearly means that the molecular force fields are assumed to be short-ranged compared to the mean free path, so that the molecules travel a relatively long initial trajectory uninfluenced by other molecules until just the very last instant prior to collision.

Chapman and Cowling (1939) indicate that for gases at N.T.P. the mean free path is of the order of  $10^{-5}$  cm., which is several hundred times the diameter of the molecules, so at the beginning of the free paths which terminate in the collision of two molecules it indeed is unlikely that there is any new correlation between their velocities. However, they do point out that, say at 100 atm., the free path is comparable with the dimensions of a molecule and molecular chaos under these conditions may well be invalid.

Actually, the assumption of molecular chaos, involving random distribution of molecules in all states, is a condition corresponding to local equilibrium. Anytime there might be some phenomena in the fluid that destroys this equilibrium it will destroy the randomness, higher order distribution functions may become necessary, and molecular chaos would not hold.

The next step by HCB is to restrict the analysis to a single component gas and consider only two-body forces between the molecules. Under these restrictions,  $i = 1$ ,  $N = 2$ , and the equation for  $f_i^{(1)}$  becomes

$$\begin{aligned} \frac{\partial f_1^{(1)}}{\partial t} + \frac{1}{m} \left( \vec{p}_1 \cdot \frac{\partial f_1^{(1)}}{\partial \vec{q}} \right) + \left( \vec{B}_1 \cdot \frac{\partial f_1^{(1)}}{\partial \vec{p}_1} \right) \\ = - \iiint \left( \vec{F}_{12} \cdot \frac{\partial f_{12}^{(1)}}{\partial \vec{p}_1} \right) d\vec{q}_2 d\vec{p}_2 . \end{aligned}$$

III-53

If the intermolecular forces are short range, a collision diameter  $r_0$  can be defined such that  $\vec{F}_{12}$  is essentially zero when  $|\vec{r}_1 - \vec{r}_2| \geq r_0$ . Then all contributions to the collision integral above will come only from regions where  $|\vec{r}_1 - \vec{r}_2| < r_0$ .

Now the pair distribution function can be written as

$$f_{12}^{(2)} = f_1^{(1)} f_2^{(1)} + C_{12}^{(2)} , \quad \text{III-54}$$

where  $C_{12}^{(2)}$  is the correlation between particles. Thus, the case of vanishing correlation is expressed by

$$f_{12}^{(2)} = f_1^{(1)} f_2^{(1)}, \quad \text{III-55}$$

which was Liboff's definition of molecular chaos, and is assumed to hold outside the interaction sphere.

Inside the interaction sphere, the pair distribution function is now known explicitly. In order to proceed with developing a Boltzman equation from the Liouville equation, HCB then follow Kirkwood (1947), and use results where he showed that the Boltzman equation implies an assumption that the distribution functions  $f_i^{(1)}$  do not change appreciably during a collision. HCB then invoke the assumption that the pair distribution function does not change during a collision. For uncorrelated particles at a short time  $\delta t$  prior to a collision, they write

$$\begin{aligned} f_{12}^{(2)}(\vec{q}'_1, \vec{q}'_2; \vec{p}'_1, \vec{p}'_2; t - \delta t) \\ = f_1^{(1)}(\vec{q}'_1, \vec{p}'_1, t - \delta t) f_2^{(1)}(\vec{q}'_2, \vec{p}'_2, t - \delta t). \end{aligned} \quad \text{III-56}$$

Then, if the pair distribution function after the collision is the same as before the collision, it can also be written

$$\begin{aligned} f_{12}^{(2)}(\vec{q}_1, \vec{q}_2; \vec{p}_1, \vec{p}_2; t) \\ = f_1^{(1)}(\vec{q}'_1, \vec{p}'_1, t - \delta t) f_2^{(1)}(\vec{q}'_2, \vec{p}'_2, t - \delta t). \end{aligned} \quad \text{III-57}$$

Thus, for every  $(q_2, p_2)$  after a collision, the pair distribution function could be related to one-particle distribution functions at a time  $(t-\delta t)$  which is prior to the collision. For each point there is a different  $\delta t$ , and  $\delta t$  is of the order of magnitude of the duration of a collision, so it is small compared with macroscopic measurements.

The assumption of no change in distribution functions clearly limits the above approach to equilibrium situations where there are no sources or sinks in state space.

Regarding this short time interval, Kirkwood corrected for the various  $\delta t$  by time averaging the first-order equation for  $i$ -species over an interval somewhat longer than a collision. The time averaged distribution function then is denoted by  $\bar{F}_i^{(1)}$ , and the equation of change becomes:

$$\begin{aligned} \frac{\partial \bar{F}_i^{(1)}}{\partial t} + \frac{1}{m_i} \left( \vec{p}_i \cdot \frac{\partial \bar{F}_i^{(1)}}{\partial \vec{q}} \right) + \left( \vec{B}_i \cdot \frac{\partial \bar{F}_i^{(1)}}{\partial \vec{p}_i} \right) \\ = \sum_j \iiint \left[ \overline{f_i^{(1)} f_j^{(1)'}} - \overline{f_i^{(1)} f_j^{(1)}} \right] g_{ij} b \, db \, d\epsilon \, d\vec{p}_j \end{aligned}$$

III-58

If the  $\delta t$  represents an interval  $\frac{\tau_0}{2}$  to  $-\frac{\tau_0}{2}$ , where  $\tau_0$  is an interval of time comparable to the duration of a collision, and is the interval over which the distribution functions are time averaged, then HCB give

$$\begin{aligned}
\overline{f_i^{(1)} f_j^{(1)}} &= \frac{1}{\tau_o} \int_{-\tau_o/2}^{\tau_o/2} f_i(t + \tau) f_j(t + \tau) d\tau \\
&= f_i^{(1)}(t) f_j^{(1)}(t) + \frac{\tau_o^2}{24} \left[ f_i^{(1)} \frac{\partial^2 f_j^{(1)}}{\partial t^2} \right. \\
&\quad \left. + 2 \frac{\partial f_i^{(1)}}{\partial t} \frac{f_j^{(1)}}{\partial t} + f_j^{(1)} \frac{\partial^2 f_i^{(1)}}{\partial t^2} \right], \quad \text{III-59}
\end{aligned}$$

whereas

$$\begin{aligned}
\overline{f_i^{(1)} f_j^{(1)}} &= \left[ \frac{1}{\tau_o} \int_{-\tau_o/2}^{\tau_o/2} f_i^{(1)}(t + \tau) d\tau \right] \left[ \frac{1}{\tau_o} \int_{-\tau_o/2}^{\tau_o/2} f_j^{(1)}(t + \tau) d\tau \right] \\
&= f_i^{(1)}(t) f_j^{(1)}(t) + \frac{\tau_o^2}{24} \left[ f_i^{(1)} \frac{\partial^2 f_j^{(1)}}{\partial t^2} + f_j \frac{\partial^2 f_i}{\partial t^2} \right]. \quad \text{III-60}
\end{aligned}$$

Thus, if the distribution functions do not change appreciably in the time interval  $\tau_o$  over which they are averaged, the average of the product will be equal to the product of the averages. When this condition holds,

$$\overline{f_i^{(1)} f_j^{(1)}} = \overline{f_i^{(1)}} \overline{f_j^{(1)}}, \quad \text{III-61}$$

and Kirkwood's equation becomes the Boltzman equation obtained by physical arguments. Both Kirkwood's equation

and the Boltzman equation imply an assumption that the particles are uncorrelated before and after collision, because the collision integral must account for change in the pair distribution function, and yet there is no correlation term in the change equations given above.

Notation in Kirkwood's equation relates directly for that used in the physical derivation of the Boltzman equation. In the physical derivation, the probable number of molecules of type-i lost from the momentum range  $\vec{p}_i$  to  $(\vec{p}_i + d\vec{p}_i)$  in the position range  $\vec{r}$  to  $(\vec{r} + d\vec{r})$  because of collisions with molecules of type j during the time interval dt is given by

$$\Gamma_{ij}^{(-)} d\vec{r} d\vec{p} dt . \quad \text{III-62}$$

As before, consider some target molecule i, and let molecule j approach with relative momenta  $(\vec{p}_j - \vec{p}_i)$ . The initial relative velocity will be

$$\vec{g}_{ji} = \frac{\vec{p}_j}{m_j} - \frac{\vec{p}_i}{m_i} , \quad \text{III-63}$$

and is the same parameter as in Kirkwood's equation. An impact parameter b is taken as the inner radius of a cylindrical shell about the trajectory of molecule i. This shell will characterize the collision (i.e., the potential field of the molecule) because any molecule j located within the cylinder will be impacted and any j-molecule outside will pass uninfluenced. Clearly this does not characterize more general potentials and diatomic

geometry. The parameter  $d\epsilon$  identifies a sector around the axis of the cylinder. The probable number of  $j$ -molecules within any sector is

$$f_j^{(1)}(\vec{q}, \vec{p}_j, t) g_{ij} b \, db \, d\epsilon \, dt, \quad \text{III-64}$$

where

$$g_{ij} = g_{ji} = |\vec{g}_{ji}|. \quad \text{III-65}$$

The probable number of  $i$ -molecules in the volume element  $d\vec{r}$  about  $\vec{r}$  with momentum in the range  $d\vec{p}_i$  about  $\vec{p}_i$  is

$$f_i^{(1)}(\vec{r}, \vec{p}_i, t) \, d\vec{r} \, d\vec{p}_i \quad \text{III-66}$$

Then

$$\begin{aligned} \Gamma_{ij}^{(-)} \, d\vec{r} \, d\vec{p}_i \, dt &= d\vec{r} \, d\vec{p}_i \, dt \iiint f_j^{(1)}(\vec{r}, \vec{p}_j, t) \\ &\quad \cdot f_i^{(1)}(\vec{r}, \vec{p}_i, t) g_{ij} b \, db \, d\epsilon \, dt \end{aligned} \quad \text{III-67}$$

and hence

$$\Gamma_{ij}^{(-)} = \iiint f_i^{(1)} f_j^{(1)} g_{ij} b \, db \, d\epsilon \, d\vec{p}_j. \quad \text{III-68}$$

A similar expression is obtained for the probable number of molecules of the  $i$ th kind which in time  $dt$  join the group of points which started from  $(\vec{r}, \vec{p}_i)$  because of collision with molecules of type  $j$ . This is denoted by  $\Gamma_{ij}^{(+)}$   $d\vec{r} d\vec{p}_i dt$ , and it is found that

$$\Gamma_{ij}^{(+)} = \iiint f_i^{(1)'} f_j^{(1)'} g_{ij}' b' d\epsilon' d\vec{p}_j', \quad \text{III-69}$$

where primes indicate conditions after a collision. For smooth spheres with spherically symmetric potential functions, the collision is reversible and the following equalities hold:

$$\left. \begin{aligned} g_{ij} &= g_{ij}' \\ b &= b' \\ d\vec{p}_i d\vec{p}_j &= d\vec{p}_i' d\vec{p}_j' \end{aligned} \right\} \quad \text{III-70}$$

These relations are not valid for more general bodies and potentials.

Under conditions where they are valid, then the collision integral becomes

$$\sum_j \iiint [f_i^{(1)'} f_j^{(1)'} - f_i^{(1)} f_j^{(1)}] g_{ij} b db d\epsilon d\vec{p}_j, \quad \text{III-71}$$

and is analogous to the collision integral in Kirkwood's equation except that here the distribution functions are not time averaged.

The Boltzman equation also frequently is written in coordinate-velocity phase space,  $f_i(\vec{q}, \vec{v}_i, t)$ , instead of coordinate-momenta phase space, and in coordinate-velocity space it is conventional to drop the superscript.

### III.5 Kinetic Theory for Nonspherical Molecules

Curtis (1956), Dahler (1959), Dahler and Scriven (1963), Dahler (1965), and others have developed an extensive and more general theory for the kinematics of nonspherical molecules where molecular angular momentum and molecular orientation must be accounted for.

The dynamical state of a single molecule is now described by a set of 12 coordinates: coordinates of the center of mass,  $q = (q_1, q_2, q_3)$ ; the linear velocity,  $v = (v_1, v_2, v_3)$ ; the Eulerian angles,  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ ; and the angular velocity in the space-fixed coordinate system  $\omega = (\omega_1, \omega_2, \omega_3)$ . The state of a gas made up of such molecules is described by a distribution function  $f(\vec{q}, \vec{v}, \vec{\alpha}, \vec{\omega}, t)$  in the corresponding generalized space.

This distribution function is defined so that the number of molecules with  $\vec{q}$  between  $\vec{q}$  and  $(\vec{q} + d\vec{q})$ , with  $\vec{v}$  between  $\vec{v}$  and  $(\vec{v} + d\vec{v})$ , with  $\vec{\alpha}$  between  $\vec{\alpha}$  and  $(\vec{\alpha} + d\vec{\alpha})$ , and with  $\vec{\omega}$  between  $\vec{\omega}$  and  $(\vec{\omega} + d\vec{\omega})$  is

$$f(\vec{q}, \vec{v}, \vec{\alpha}, \vec{\omega}, t) d\vec{q} d\vec{v} d\vec{\alpha} d\vec{\omega} . \quad \text{III-72}$$

This distribution is normalized so that

$$n = \int f \, d\vec{v} \, d\vec{\alpha} \, d\vec{\omega} \quad , \quad \text{III-73}$$

where  $n$  is the number density of molecules.

Curtis (1956) follows this pattern and develops an equation

$$\mathcal{D}f(\vec{q}, \vec{v}, \vec{\alpha}, \vec{\omega}, t) = J \quad , \quad \text{III-74}$$

where  $J$  is the net rate at which molecules are gained into a particular state due to collisions, and the operator  $\mathcal{D}$  is defined by

$$\mathcal{D}f = \frac{\partial f}{\partial t} + \frac{\partial}{\partial \vec{q}} \cdot (\vec{v}f) + \frac{1}{m} \frac{\partial}{\partial \vec{v}} \cdot (\vec{B}f) + \sum_i \left[ \frac{\partial (\dot{\alpha}_i f)}{\partial \alpha_i} + \frac{\partial (\dot{\omega}_i f)}{\partial \omega_i} \right] \quad . \quad \text{III-75}$$

His equation then is a generalized Boltzman integro-differential equation describing the time variation of the distribution function. The parameter  $J$  is just the collision integral.

Curtis defines a summational invariant  $\psi^{(i)}$  as a function of  $\vec{v}$  and  $\vec{\omega}$ , such that the following relation holds:

$$\psi_1^{(i)} + \psi_2^{(i)} = \psi_1^{(i)'} + \psi_2^{(i)'} \quad , \quad \text{III-76}$$

The subscripts refer to the molecules in the collision, and primed values are before an encounter whereas unprimed are final (in contrast to earlier notation used herein).

Curtis then defines four independent summational invariants

$$\begin{aligned}
 \psi^{(1)} &= 1 \\
 \psi^{(2)} &= m\vec{v}, \quad \text{linear momentum} \\
 \psi^{(3)} &= \vec{I} \cdot \vec{\omega} + m[\vec{q} \times \vec{v}], \quad \text{angular momentum} \\
 \psi^{(4)} &= \frac{1}{2} mv^2 + \frac{1}{2} \vec{\omega} \cdot \vec{I} \cdot \vec{\omega}
 \end{aligned}
 \left. \vphantom{\begin{aligned} \psi^{(1)} \\ \psi^{(2)} \\ \psi^{(3)} \\ \psi^{(4)} \end{aligned}} \right\} \text{III-77}$$

where  $I$  is the moment-of-inertia tensor in the space fixed coordinate system.

Curtis demonstrates that

$$\int \psi^{(i)} J \, d\vec{v} \, d\vec{\omega} \, d\vec{\alpha} = 0, \quad \text{III-78}$$

so when his modified Boltzman equation is multiplied by a general summational invariant  $\psi^{(i)}$ , if the above condition is invoked, and the result integrated over  $\vec{v}$ ,  $\vec{\alpha}$ , and  $\vec{\omega}$ , the following general equation of change is obtained:

$$\begin{aligned}
 \frac{\partial}{\partial t} \left[ n \langle\langle \psi^{(i)} \rangle\rangle \right] + \frac{\partial}{\partial \vec{q}} \left[ n \langle\langle \vec{v} \psi^{(i)} \rangle\rangle \right] - \frac{n}{m} \langle\langle \vec{B} \cdot \frac{\partial}{\partial \vec{v}} \psi^{(i)} \rangle\rangle \\
 - \sum_k n \langle\langle \left( \dot{\alpha}_k \frac{\partial}{\partial \alpha_k} + \dot{\omega}_k \frac{\partial}{\partial \omega_k} \right) \psi^{(i)} \rangle\rangle = 0
 \end{aligned}
 \quad \text{III-79}$$

From this equation the four governing equations of change are found by inserting each of the four summational invariants in turn.

By the procedure of setting the collision terms to zero, Curtis may have discarded some important features of a diatomic collision. It seems in essence that he has invoked an assumption of reversibility of the collision, which is not valid for diatomic molecules. That is, for every "forward" collision there is not a unique inverse.

Lordi and Mates (1970) have investigated the rotational-translational energy transfer in collisions between homonuclear diatomic molecules, but assuming constant angular momentum and allowable inverse collisions. They used Parker's model for the intermolecular potential. The results for shear viscosity, thermal conductivity, and rotational relaxation times actually compared well with experimental values. Results were obtained for both a coplanar and a three-dimensional collision model, and for thermal conductivity and relaxation times the coplanar values more nearly agreed with experiment. The opposite result would be expected, and this discrepancy may have been a numerical problem with the 3D computer solution of the collision process.

Transport phenomena in monatomic gases have been described successfully in terms of atomic collision processes for quite some time, and the theory is described in well-known texts such as HCB, and Chapman and Cowling (1939). Gradually work has been done on idealized diatomic molecular models such as the rough or loaded spheres, but only recently has work been published for transfer processes in diatomic gases with more realistic models of the intermolecular potential.

Wang-Chang and Uhlenbeck (1951) did the first formal kinetic theory analysis of transport phenomena of polyatomic gases for a general potential. They used a semiclassical treatment of the molecular degrees of freedom, and obtained a Chapman-Enskog-type solution to the appropriate Boltzmann equation. Two cases were distinguished: easy transfer of energy from internal modes to translation, and difficult transfer. Their analysis is described in HCB.

Taxman (1958) did a purely classical study of the easy transfer case, not allowing inverse collisions. Sather and Dahler (1963), Condiff, Lu, and Dahler (1965), and Sandler and Dahler (1966) have evaluated loaded spheres, rough spheres, rigid spherocylinders, and other models. The dependence of the distribution function on the direction of the molecular angular momentum was included. Such an effect arises from the absence of inverse collisions in these idealized models, and it has a large effect on the transport properties!

With this background, Lordi and Mates obtained numerical solutions for the collision trajectories of two diatomic molecules using Parker's potential. They then performed a Monte Carlo evaluation of the transport properties. For the Parker model, each molecule had two repulsive force centers located on the internuclear axis, but not necessarily on the rigidly connected atomic mass centers. The attractive force center is located on the molecular center of mass. Both the attractive and repulsive parts of the intermolecular potential vary exponentially with distance between the respective centers.

They found, of course, that the problem was greatly simplified for molecules rotating only in the collision plane. The duration of the collision was found to be of the order of  $10^{-12}$  sec, and the molecules go through one or two rotations in that time. The scattering angle is close to that for a Lennard-Jones potential. Even for the simple coplanar collision there are many collision parameters which were found to influence significantly the rotational energy transfer. One of the most important was initial orientation of the molecules. The initial rotational energy and the potential-well depth also were quite significant.

For the out-of-plane rotations, they numerically solved the full three-dimensional equations of motion for a binary collision. Initial direction of rotation was as significant as initial orientation. In all, the deflection angles and rotational-energy change did not differ appreciably from the coplanar case, but the run times were significantly longer (30-50 sec. as compared to about 6 sec.).

The multidimensional integrals cannot be evaluated analytically for a general intermolecular potential, so a Monte Carlo evaluation was performed. Initial values of the collision parameters are obtained from suitable distributions of random variables. With these initial conditions, the numerical solution of the collision trajectory is performed, as discussed above. Then, with the solution for the trajectory, the integrands in the collision integrals can be obtained.

The results of the coplanar calculations showed that the values of the integrals after 500 trials were close to the values after 1,000 trials. Since even 500 trials is extremely time consuming, Lordi and Mates extended their work to obtain an approximate method of calculating transport properties. The basic technique is to seek a series solution to the Lagrangian form of the equations of motion for a coplanar collision. Using this technique they obtained rotational relaxation times which compare very favorably with the Monte Carlo solution, as shown below for nitrogen:

Temperature (°K)	Rotational Relaxation Time	
	Monte Carlo $p\tau_R$ (dyne-sec/cm <sup>2</sup> )	Approx. Soln. $p\tau_R$ (dyne-sec/cm <sup>2</sup> )
300	$7.73 \times 10^{-4}$	$8.17 \times 10^{-4}$
600	$1.63 \times 10^{-3}$	$1.85 \times 10^{-3}$
900	$2.84 \times 10^{-3}$	$2.78 \times 10^{-3}$

These data compare very well with experimental values, and therefore the method provides realistic diatomic transport data for use in appropriate generalized continuum equations for the dynamics of a diatomic gas.

#### IV. FLUID MATH MODELS

This section considers the classical macroscopic equations of motion for a continuum in order to indicate what factors in their derivation might be questionable if the equations are to be applied to stability analyses of a diatomic medium.

It is found that there are questionable areas, and these are discussed to clarify the nature of assumptions on which the classical equations rest. This review, by highlighting these assumptions, also provides a clue as to the type of modifications which might be made to the classical Navier-Stokes equations in order to model more accurately the momentum and energy exchange processes which are believed to occur in the dynamics of a diatomic fluid. These processes, of course, could very well influence a stability analysis, which is the primary concern of this work.

##### IV.1 Reynolds Transport Theorem

Reynolds Transport Theorem is a key part of the development of all the governing differential equations of a fluid. In essence, it provides the method for evaluating total rate of change with time of some physical parameter of the fluid. For some arbitrary physical parameter  $\mathcal{F}(\vec{x}, t)$ , which may be a scalar or vector quantity, the time rate of change of this parameter integrated over a changing volume  $v(t)$  may be expressed as the time rate of change of the same parameter integrated over a fixed volume  $v_0$  using the definition of the Jacobian  $J$  (see Aris, 1962):

$$\frac{d}{dt} \iiint_{v(t)} \mathcal{F}(\vec{x}, t) dv = \frac{d}{dt} \iiint_{v_0} \mathcal{F}(\vec{x}, t) J dv_0 . \quad \text{IV-1}$$

Since  $v_o$  is not a function of time, the derivative may be taken inside the integral to yield

$$\frac{d}{dt} \iiint_{v_o} \mathcal{F}(\vec{x}, t) J dv_o = \iiint_{v_o} \left[ \frac{d\mathcal{F}}{dt} J + \mathcal{F} \frac{dJ}{dt} \right] dv_o . \quad \text{IV-2}$$

But time derivative of the Jacobian may be written as

$$\frac{d(J)}{dt} = \frac{\partial(J)}{\partial t} + \vec{V} \cdot \nabla(J)$$

$$\vec{V} \cdot \nabla(J) = (\nabla \cdot \vec{V}) J ,$$

since the field is continuous and  $\frac{\partial J}{\partial t} = 0$  . Note: See eqn. IV-40 and comments in that section.

Thus

$$\begin{aligned} \frac{d}{dt} \iiint_{v(t)} \mathcal{F}(\vec{x}, t) dv &= \iiint_{v_o} \left[ \frac{d\mathcal{F}}{dt} J + \mathcal{F} \frac{dJ}{dt} \right] dv_o \\ &= \iiint_{v_o} \left[ \frac{d\mathcal{F}}{dt} J + \mathcal{F} (\nabla \cdot \vec{V}) J \right] dv_o \\ &= \iiint_{v_o} \left[ \frac{d\mathcal{F}}{dt} + \mathcal{F} (\nabla \cdot \vec{V}) \right] J dv_o \\ &= \iiint_{v(t)} \left[ \frac{d\mathcal{F}}{dt} + \mathcal{F} (\nabla \cdot \vec{V}) \right] dv \\ &= \iiint_{v(t)} \left[ \frac{\partial \mathcal{F}}{\partial t} + \nabla \cdot (\mathcal{F} \vec{V}) \right] dv . \quad \text{IV-4} \end{aligned}$$

Sometimes this expression is written as the sum of a volume integral and a surface integral. Note that the operation extracting the Jacobian from the bracket on the right hand side depended on treating the nabla operator as a conventional vector with the commutative property. This does not hold in general, and yields the same result only for continuous fields. But the assumption of continuous fields is a precept for the continuum method, so the question actually concerns the range of validity of a continuum theory. The fluid actually is not a continuum but instead is an ensemble of atoms, molecules and possibly collections of molecules which will be called particles.

Experience over at least the last century has established the usefulness and indeed validity of continuum equations for flow of diatomic molecules under not too rarefied or not too dense conditions. However, for some situations involving rapid shear and strong gradients in the flow, a medium of diatomic molecules might need to be analyzed as some type continuum with a finite number of discontinuities representing substructure boundaries. Such a flow might be modeled as horizontal lamina, with the conventional continuum approach given appropriate modifications to incorporate jump conditions at boundaries with container walls and at boundaries between lamina. Thickness of the lamina would represent a microscale characteristic length. It appears that the jump conditions (Jaunzemis (1967), Eringen (1967), Truesdell (1960, 1965)) would yield an additional equation of transport which would account for the same type of motion coupling effects that will be explored here for fluids with asymmetric stress (after the manner of Dahler, et al.).

## IV.2 Governing Differential Equations for Classical Model

As noted in the previous section, the Reynolds Transport Theorem is the basic relation used in developing all the governing conservation or balance equations of classical hydrodynamics.

A general equation of balance is obtained by equating time rate of change of the volume integral of some physical parameter to whatever external effects are assumed to be the cause of the change. The general balance equation is written as:

$$\frac{d}{dt} \iiint_{v(t)} \mathcal{F}(\vec{x}, t) dv = \text{External Effects} . \quad \text{IV-5}$$

Many of the questions raised here concerning the classical equations of hydrodynamics can be traced to inadequate use of this balance equation, because math models constructed from this balance equation incorporate only those effects which the investigator assumes are present. If he assumes that asymmetric stress and elasticity are negligible (or absent) then the model can never reveal any situations where these effects might not be negligible!

If external effects are viewed as the sum of body effects integrated over the volume of the fluid element plus surface effects integrated over the surface of the element, this can be written as

$$\text{External Effects} = \frac{\partial}{\partial t} \int_{v(t)} \mathcal{F} dv + \int_{s(t)} \mathcal{F} \vec{u} \cdot \vec{n} ds , \quad \text{IV-6}$$

where  $\vec{u}$  is the velocity of the center of mass of the element, and  $\vec{n}$  is the unit normal vector at any point on the surface. Then a tabulation of physical parameters and corresponding external effects is as follows.

Time Rate of Change Parameter↓	$\mathcal{F}$	External Effects
mass per unit vol.	$\rho$	0
linear momentum	$\rho\vec{u}$	$\vec{F}_S + \int_{v(t)} \vec{F}_B dv$
moment of momentum	$\vec{x} \times \rho\vec{u}$	$\int_{S(t)} (\vec{x} \times d\vec{F}_S) + \int_{v(t)} (\vec{x} \times \vec{F}_B) dv$
total angular momentum	$\vec{L}$	$\vec{M}_S + \int_{v(t)} \vec{M}_B dv$
energy	$E = \frac{\rho u^2}{2} + e + e^*$	$\frac{dQ}{dt} - \frac{dW}{dt}$

### Conservation of Mass

The balance equation for mass per unit volume then is written

$$\frac{d}{dt} \iiint_{v(t)} \rho dv = 0 \quad \text{IV-7}$$

By Reynolds Transport Theorem, the left hand side is expanded, and balance is written as

$$\iiint_{v(t)} \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) \right] dv = 0 \quad \text{IV-8}$$

Since  $v(t)$  is an arbitrary volume selected for investigation, this may be written

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad \text{IV-9}$$

or in index notation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0. \quad \text{IV-10}$$

Expanding the second term,

$$\boxed{\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} = 0} \quad \text{IV-11}$$

By definition,

$$\frac{\partial \rho}{\partial t} \equiv 0 \Rightarrow \text{steady flow}$$

and

$$\frac{\partial \rho}{\partial x_m} \equiv 0 \Rightarrow \text{incompressible flow}.$$

IV-12

Hence, for steady incompressible flow, conservation of mass yields

$$\frac{\partial u_j}{\partial x_j} = 0. \quad \text{IV-13}$$

### Conservation of Linear Momentum

Similarly, conservation of linear momentum yields

$$\boxed{\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = \rho f_i + \frac{\partial S_{ik}}{\partial x_k}} \quad \text{IV-14}$$

For details of the derivation see Aris (1962).

Now the stress tensor  $S_{ij}$  is frequently assumed to be separable into a static part and a dynamic part,

$$S_{ij} = -p\delta_{ij} + \tau_{ij}, \quad \text{IV-15}$$

where  $\delta_{ij}$  is the kronecker delta,  $p$  is the hydrostatic stress (not yet identified as pressure), and  $\tau_{ij}$  is the viscous stress tensor. The stress tensor  $\tau_{ij}$  need not be assumed to be purely viscous; any of the more general constitutive assumptions such as those discussed in Section III could be selected! Thus, the quality of a math model depends on judgement concerning what characteristics the fluid is assumed to process!

For the assumption that the fluid is linear, isotropic, and purely viscous, the constitutive equation is

$$\tau_{ij} = \alpha_{ijmn} \dot{\gamma}_{mn}, \quad \text{IV-16}$$

where

$$\dot{\gamma}_{mn} = \frac{\partial u_m}{\partial x_n}, \quad \text{IV-17}$$

and

$$\alpha_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) + \mu (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \quad \text{IV-18}$$

Hence,

$$\begin{aligned} \tau_{ij} &= \lambda \delta_{ij} \delta_{mn} \dot{\gamma}_{mn} + \mu (\delta_{im} \delta_{jn} \dot{\gamma}_{mn} + \delta_{in} \delta_{jm} \dot{\gamma}_{mn}) \\ &\quad + \mu (\delta_{im} \delta_{jn} \dot{\gamma}_{mn} - \delta_{in} \delta_{jm} \dot{\gamma}_{mn}) \\ &= \lambda \delta_{ij} \dot{\gamma}_{mn} + \mu (\dot{\gamma}_{ij} + \dot{\gamma}_{ji}) + \mu (\dot{\gamma}_{ij} - \dot{\gamma}_{ji}) \end{aligned} \quad \text{IV-19}$$

and

$$\begin{aligned} S_{ij} &= -p \delta_{ij} + \lambda \delta_{ij} \dot{\gamma}_{mn} + \mu (\dot{\gamma}_{ij} + \dot{\gamma}_{ji}) + \mu (\dot{\gamma}_{ij} - \dot{\gamma}_{ji}) \\ &= -p \delta_{ij} + \lambda \delta_{ij} \frac{\partial u_m}{\partial x_m} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \mu \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \end{aligned} \quad \text{IV-20}$$

Contraction gives

$$S_{ii} = -p \delta_{ii} + \lambda \delta_{ii} \dot{\gamma}_{mm} + 2\mu \dot{\gamma}_{ii}, \quad \text{IV-21}$$

so the total stress is

$$\begin{aligned} S &= \sum_{i=1}^3 S_{ii} = -3p + 3\lambda \dot{\gamma}_{ii} + 2\mu \dot{\gamma}_{ii} \\ &= -3p + (3\lambda + 2\mu) \dot{\gamma}_{ii} \end{aligned} \quad \text{IV-22}$$

Under static conditions,  $S = -3p$ , and  $S$  is identified as the hydrostatic pressure  $P$ . Hence, at equilibrium,

$$S = -3p \equiv P . \quad \text{IV-23}$$

This condition is assumed to hold also in dynamic situations not too far from equilibrium, which requires the assumption that bulk viscosity  $\kappa$  is identically zero; i.e.,

$$\kappa \equiv (3\lambda + 2\mu) \equiv 0 , \quad \text{IV-24}$$

since  $\dot{\gamma}_{ii} \neq 0$ . This is called Stoke's assumption, and it is now known that it does not hold for many important cases.

Furthermore, the parameter  $p$  classically is identified with thermodynamic pressure for the static condition, and it is assumed that this also holds for dynamic situations involving small deformations. However, experimental results with many fluids show that when density variations are large there does not seem to be any correlation between  $\lambda$  and  $\mu$  and consequently  $p$  cannot be the average normal stress (see Eskinazi, 1967). Furthermore, for most fluid motions with large dilitation rates the ratio  $\frac{\lambda}{\mu}$  is positive rather than negative, and may be as high as 200. Under dynamic situations,  $p$  should not be identified as the thermodynamic pressure.

Since the Orr-Sommerfeld stability equation incorporates these assumptions, it is clear that they alter results obtained with that equation, and it also is clear that when oscillations become large the assumptions involving small deformation are not valid!

Expanding the equation of motion gives

$$\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i u_j \frac{\partial \rho}{\partial x_j} \rho u_i \frac{\partial u_i}{\partial x_j} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho f_i + \frac{\partial S_{ik}}{\partial x_k}$$

IV-25

Again,

$$\frac{\partial \rho}{\partial t} \equiv 0 \Rightarrow \text{steady flow}$$

and

$$\frac{\partial \rho}{\partial x_j} \equiv 0 \Rightarrow \text{incompressible flow .}$$

Hence, for steady incompressible flow,

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho f_i + \frac{\partial S_{ik}}{\partial x_k} .$$

IV-26

This equation, with suitable assumptions on a constitutive expression for the stress tensor, is the basic equation used in most fluid stability analyses. Only limited work has been done with more general unsteady compressible forms.

### Moment of Momentum

If the stress tensor is written as the sum of a symmetric part and an anti-symmetric part, this gives

$$\begin{aligned}
S_{ij} &= S_{ij}^{(s)} + S_{ij}^{(a)} \\
&= S_{ij}^{(s)} + \frac{1}{2} \epsilon_{ijk} \epsilon_{kmn} S_{mn} \\
&= S_{ij}^{(s)} + \frac{1}{2} \epsilon_{ijk} A_k,
\end{aligned}
\tag{IV-27}$$

where  $A_k$  is a pseudo-vector.

Balance of moment of the linear momentum equation for the steady incompressible case gives

$$\rho \frac{d}{dt} (\epsilon_{kij} x_i u_j) = \rho \epsilon_{kij} x_i f_j + (\epsilon_{kij} x_i S_{lj})_{,l} - A_k$$

IV-28

The pseudo-vector  $A_k$  resulted from the anti-symmetric part of the stress tensor. By various arguments (Aris, 1962; Frederickson, 1964) it is generally assumed that the fluids of classical hydrodynamics and aerodynamics cannot support asymmetrical shear, so  $A_k$  is taken to be zero.

#### Total Angular Momentum

Balance of total angular momentum  $\vec{L}$  gives

$$\begin{aligned}
\frac{d}{dt} \iiint_{v(t)} \rho L_k \, dv &= \iiint_{v(t)} \rho (\epsilon_{kij} x_i f_j + G_k) \, dv \\
&+ \iint_{S(t)} n_l (\epsilon_{kij} x_i S_{lj} + C_{lk}) \, dS,
\end{aligned}$$

IV-29

or for steady incompressible flow

$$\rho \frac{dL_k}{dt} = \rho \epsilon_{kij} x_i f_j + \rho G_k + (\epsilon_{kij} x_i S_{lj})_{,l} + C_{lk,l}, \quad \text{IV-30}$$

where  $G_k$  is a body couple and  $C_{lk}$  is a surface couple, and  $L_k = l_k + \epsilon_{kij} x_i u_j$  represents the sum of any internal or spin angular momentum plus the moment of linear momentum.

### Spin Angular Momentum

Subtracting the equation for moment of momentum from the total angular momentum gives

$$\rho \frac{dl_k}{dt} = \rho G_k + C_{lk,l} + A_k, \quad \text{IV-31}$$

where

$$l_k = L_k - \epsilon_{kij} x_i u_j. \quad \text{IV-32}$$

From this equation it is clear that the antisymmetric part of the stress tensor, if it exists, will contribute to a rate of increase of internal or spin angular momentum! Thus, when the stress tensor is not symmetric, the moment of momentum is not conserved in the classical sense. Loss of moment of momentum will show up as increased spin angular momentum. This then represents an energy coupling between translational and rotational modes, and clearly would need to be considered in any stability analysis of the system!

### Fluid Mechanics With Antisymmetric Stress

Grad (1952), Dahler (1959), Dahler and Scriven (1963), Condiff and Dahler (1964), and Dahler (1965) discuss in detail the theory of structured continua. For the fluid mechanical effects of antisymmetric stress, Condiff and Dahler ignore submolecular forms of angular momentum and express the internal angular momentum  $\vec{l}$  in terms of a microscopic spin field  $\vec{\omega}_0$  and a tensor field  $\vec{\mathbb{I}}$  which simulates the average moment of inertia of the molecules.

In general,  $\vec{\mathbb{I}}$  is a function of position and time, but if the individual molecules are not too flexible or eccentric in shape then  $\vec{\mathbb{I}}$  can be regarded as a constant dyadic. If the fluid is isotropic, then

$$\vec{\mathbb{I}} = I \underline{\underline{U}}, \quad \text{IV-33}$$

where  $\underline{\underline{U}}$  is the unit dyadic and  $I$  is a constant scalar.

Condiff and Dahler consider a linear Stokesian fluid with the symmetric part of the stress tensor independent of internal strain of the spin field (hence dependent only on the symmetrized velocity gradient tensor). Furthermore, they assume the couple stress tensor is symmetric and depends only upon internal strain (i.e., only upon the symmetrized spin gradient tensor).

The constitutive equations then are

$$\begin{aligned}
S &= S^{(s)} + S^{(a)} \\
C &= C^{(s)} \\
S^{(s)} &= \underline{U} \left( -p + \frac{1}{3} \phi \nabla \cdot \vec{u} \right) + \eta \left( \nabla \vec{u} + (\nabla \vec{u})^* - \frac{2}{3} \underline{U} \nabla \cdot \vec{u} \right) \\
S^{(a)} &= \xi \text{ dual } (\nabla \times \vec{u} - 2\vec{\omega}_o) \\
C^{(s)} &= \underline{U} (v_1 \nabla \cdot \vec{\omega}_o) + v_2 \left( \nabla \vec{\omega}_o + (\nabla \vec{\omega}_o)^* - \frac{2}{3} \underline{U} \nabla \cdot \vec{\omega}_o \right) .
\end{aligned}$$

IV-34

The coefficients  $\phi$  and  $\eta$  here are respectively the bulk and shear coefficients of viscosity,  $p$  is assumed to be the thermodynamic pressure,  $v_1$  and  $v_2$  are bulk and shear coefficients of spin viscosity, and  $\xi$  is a phenomenological coefficient which they call the vortex viscosity (in earlier papers by Dahler this was called spin viscosity).

The spin field is a manifestation of noncentral intermolecular forces. Since the spin angular momentum equation explicitly contains the pseudo-vector  $\vec{A}$ , it demonstrates that anti-symmetric stress is the mechanism for transforming moment of linear momentum into internal (spin) angular momentum! This is just a coupling between internal spin and vorticity. For fluids with internal spin, there is not the usual conservation of vorticity. This will be considered further in the next section.

Condiff and Dahler insert the above constitutive equations into the transport equations for conservation of mass, balance of linear momentum, and balance of spin angular

momentum, and obtain the following polar fluid analogues of the Navier-Stokes equations of motion:

$$\begin{aligned}
 \frac{d\rho}{dt} &= -\rho \nabla \cdot \vec{u} \\
 \rho \frac{d\vec{u}}{dt} &= -\nabla p + \rho \vec{f} + 2\xi \nabla \times \vec{\omega}_o + \left( \phi + \frac{1}{3} \eta - \xi \right) \nabla \nabla \cdot \vec{u} \\
 &\quad + (\xi + \eta) \nabla^2 \vec{u} \\
 \rho I \frac{d\vec{\omega}_o}{dt} &= \left( \nu_2 + \frac{1}{3} \nu_1 \right) \nabla \nabla \cdot \vec{\omega}_o + (\nu_1 + \nu_2) \nabla^2 \vec{\omega}_o \\
 &\quad + 2\xi (\nabla \times \vec{u} - 2\vec{\omega}_o) + \rho \vec{G}
 \end{aligned}
 \tag{IV-35}$$

These equations represent the dynamics of the system with internal structure, and stability analyses of a diatomic fluid should be performed with this type of coupled system!

#### Conservation of Vorticity

Consider now the classical proof of conservation of vorticity. By definition, the vorticity  $\vec{\zeta}$  is

$$\vec{\zeta} \equiv \nabla \times \vec{u} .
 \tag{IV-36}$$

Then divergence of the vorticity is

$$\nabla \cdot \vec{\zeta} = \nabla \cdot (\nabla \times \vec{u}) ,
 \tag{IV-37}$$

The claim frequently is made that

$$\nabla \cdot \vec{\zeta} \equiv 0 \quad \text{IV-38}$$

is always true; i.e., that the vorticity vector is always solenoidal. When the operator  $\text{div}$  is applied to a vector function  $\vec{\zeta}$  it gives at each point the rate per unit volume at which the physical entity is issuing from that point. If  $\text{div } \vec{\zeta}$  is positive at the point, then the point is a source and if  $\text{div } \vec{\zeta}$  is negative the point is a sink. When

$$\text{div } \vec{\zeta} \equiv 0 \quad \text{IV-39}$$

the net outward flow over any part of the region is zero, and so it is claimed that vorticity is conserved within a fluid and only created at solid boundaries or at slip interfaces between two fluid layers.

The procedure of cyclic permutation of the triple scalar product where one of the members is the nabla operator is not in general a valid procedure. The nabla operator is not a true vector, and is not in general commutative. That is, in general

$$\nabla \cdot \vec{\zeta} \neq \vec{\zeta} \cdot \nabla \quad \text{IV-40}$$

and

$$\nabla \times \vec{\zeta} \neq -\vec{\zeta} \times \nabla \quad \text{IV-41}$$

Thus, order of the product cannot be interchanged.

The correct approach is to perform the operations in the order indicated. First consider

$$\begin{aligned}
 \vec{\zeta} &= \nabla \times \vec{u} = \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \times (\vec{i}u + \vec{j}v + \vec{k}z) \\
 &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{u} & \vec{v} & \vec{w} \end{vmatrix} \\
 &= \vec{i} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - \vec{j} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \vec{k} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)
 \end{aligned}
 \tag{IV-42}$$

Then

$$\begin{aligned}
 \nabla \cdot \vec{\zeta} &= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \left[ \vec{i} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \right. \\
 &\quad \left. - \vec{j} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \vec{k} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \right] \\
 &= \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - \frac{\partial}{\partial y} \left( \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \\
 &= \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial z \partial x} - \frac{\partial^2 u}{\partial z \partial y}
 \end{aligned}
 \tag{IV-43}$$

$$\equiv 0 \quad \text{if} \quad \frac{\partial}{\partial x} \frac{\partial w}{\partial y} \equiv \frac{\partial}{\partial y} \frac{\partial w}{\partial x}, \quad \text{etc.}
 \tag{IV-44}$$

Therefore, vorticity is solenoidal if and only if the order of differentiation is unimportant, i.e., if the limits are identical regardless of the direction of approach to the point in question. This condition is satisfied if and only

if the function is continuous at the point. The question therefore centers around whether or not velocity profiles (hence velocity gradients) are continuous. If they are discontinuous, i.e., if there is slip, then the vorticity is not solenoidal. If vorticity is not solenoidal then there must be a source or sink of vorticity in the flow field.

Classical hydrodynamic theory assumes continuous functions based on the assumption that volume integrals can be shrunk to a point (i.e., zero volume) without encountering a discontinuity (Aris, 1962).

From the molecular theory it is clear that there are limits to the validity of this assumption. However, it is classically presumed that in most flow problems negligible error has been introduced by the assumption. Certainly for mean flow computations the results seem to substantiate the assumption. The end result is a theory where vorticity is treated as a solenoidal vector function, i.e., there are no sources and sinks within the flow. Furthermore, in this classical theory vorticity is either introduced at the wall or at some "gross" slip surface within the flow, or else it is artificially imbedded within the flow as discrete points of zero volume but infinite vorticity (see Betchov and Criminale; 1967; p. 158).

The classical conclusion generally is that there is no vorticity created within the flow, but that vorticity is created at the wall and diffuses out into the stream to develop a steady-state vorticity distribution normal to the wall (Fig. IV-1).

The alternate viewpoint presented here is that vorticity is created within the flow as well as at the wall, and that the vorticity creation (i.e., source) is directly related to particle and molecular slip in the flow at stations normal to the wall. Furthermore, it is postulated that this slip is directly related to the previously discussed spin field (eqn. IV-31), and "produces" the coupling between vorticity and internal spin.

Consider Figure IV-1. For either accelerated or retarded flow the maximum vorticity occurs at the location of minimum slope of the velocity profile. That is, for streamwise velocity, the maximum vorticity occurs where  $\frac{\partial u}{\partial y}$  is maximum.

When the flow is retarded sufficiently to produce an inflection in the velocity profile the slope  $\frac{\partial u}{\partial y}$  becomes a maximum at a location out in the stream rather than at the wall. This condition gives a vorticity maximum in the stream, and is known to be unstable.

An alternate condition without velocity profile inflection is the fully turbulent profile, and it also provides a vorticity maximum at the wall. Furthermore, the shape of the turbulent profile gives greater velocities near the wall, with the result being a corresponding concentration of vorticity closer to the wall.

The classical theory of vorticity diffusion and self convection has been used to explain the possibility of a vorticity maximum out in the flow. The alternate concept suggested here is that due to friction and slip, a flow with retardation near the wall will naturally develop an inflected velocity

profile representing a local region of slip concentration at and near the inflection, since that is where  $\frac{\partial u}{\partial y}$  is greatest! Furthermore, it is believed that this slip concentration is the very feature that produces the local vorticity maximum in the flow.

At an inflection in the velocity profile, the derivative should be a good approximation to the slope providing the velocity profile is a continuous function. Hence, the error terms would not be expected to contribute significantly in such a region, and should not contribute much even if some characteristic scale prevented shrinking  $\Delta x$ ,  $\Delta y$ , and  $\Delta z$  to zero. However, if the velocity function is a step function (discontinuity caused by slip), then it is precisely at such an inflection point that errors will be greatest due to the slope  $\frac{\Delta u}{\Delta y}$ ; i.e., very small spacings or "lamina" in  $\Delta y$  produce relatively large jumps in  $\Delta u$ .

The classical presumption of course is that the infinitely thin lamina will shear with no slip between lamina, and hence there is no production of vorticity. The alternate concept is that the boundary layer might be described better as a "slip layer" than shear layer.

## V. EXPERIMENTAL OBSERVATIONS

### V.1 Macroscopic Phenomena

This section summarizes macroscopic experimental observations by various investigators regarding gross flow characteristics and correlations needed for engineering design of flight vehicles.

Without concern for molecular and microscopic effects, flow transition usually is detected by evaluation of data from optical or hot-wire techniques. In the latter case the data frequently is plotted as heat transfer (possibly Stanton Number) versus local Reynolds number based on length from the leading edge. The data in Fig. V-1 demonstrate the effect of roughness location. They also show the typical decrease in heat transfer to the plate as laminar flow moves along the plate, then the increase in heat transfer as transition begins, and again a decrease as fully turbulent flow is established. The beginning of transition is taken as the beginning of the rise, and the end of transition is taken as the end of the rise.

Similar data is represented in Fig. V-2, with Stanton number plotted against energy-thickness Reynolds number.

DiCristina (1970) presents heat transfer data from a study of sharp cones at angles of attack, and it is clear that the data varies considerably for different locations around the cone (Figs. V-3). This figure also shows results from optical studies. The optical data was obtained from shadowgraphs and is quite open to interpretation. Shadowgraphs

show the laminar region as a thin white line, increasing in distance from the cone surface at downstream locations. Toward the turbulent region the sharpness of the line diminishes and then it totally disappears in what seems to be a fully developed turbulent region. The end of transition is defined as the most aft position at which this line can be detected, i.e., where it disappears. Obviously this is quite approximate, and the beginning and end of transition is determined more accurately through heat-transfer measurements.

From the heat-transfer measurements, beginning of transition is indicated by increase in heat-transfer over the laminar value, and the end of transition is taken as the peak in the heat-transfer curve. Detailed observations confirmed a region of faint turbulent bursts in the vicinity of the aft location where heat transfer measurements begin to increase.

Figure V-3f is a typical illustration of the effect of Mach number on transition for constant unit Reynolds number. This type of correlation has been accepted practice in the past, but is now known to be very questionable.

Figure V-3f is a replot by DiCristina of data from Softley, et al (1968). Morkovin (1969) also reproduced this data. Morkovin notes that Softley pointed out that the data was obtained from Fig. V-4 (of this document) "by assuming that data obtained at the same unit Reynolds number in different wind tunnels are comparable". Softley and Morkovin emphasize that there is no known substantiation for this assumption, and

the Figure V-3f must be regarded with caution! It is now known that the assumption indeed does not hold unless the tunnels at least are geometrically similar. Unfortunately, such warnings often get dropped. Pate and Schueler (1969) comment further that if an effect of Mach number on transition Reynolds number exists at supersonic and hypersonic speeds, it is doubtful that the trend can be established by comparing transition data at different Mach numbers obtained in wind tunnels with turbulent wall boundary layers because of the influence of radiated aerodynamic noise.

Morkovin focuses attention on the need for determination in general of the environmental disturbances for different facilities and testing techniques, and also on the need for assessment of receptivity of the boundary layer to such disturbances. He emphasizes that the presence of 2D and 3D instability modes suggests that adequate characterization of the disturbances must include specification of the 3D orientation in addition to the intensity and spectral measurements.

Reshotko, Pate and Schueler, and other authors discuss the radiated aerodynamic noise effects on transition in supersonic and hypersonic streams. They show conclusively a significant and continuous increase in transition Reynolds number and decrease in radiated aerodynamic noise (generated by the tunnel wall turbulent boundary layer) with increasing tunnel size. From this, they conclude that a major part of the heretofore unexplained unit Reynolds number effect is probably the result of radiated aerodynamic noise.

Experience in the past with subsonic tunnels showed that transition results were critically dependent on the quality of the flow, and therefore were different from one tunnel to another. However, initial high-speed predictions of transition location disregard this, and were based on the assumption that tunnel freestream disturbances are unimportant at high supersonic and hypersonic speeds. It is now known that these disturbances are important, and that aerodynamic sound radiated from the wall boundary layers is a dominant source. Laufer (1964) showed that the intensity of the radiated pressure fluctuations is proportional to size of the test section, and therefore the simple correlations of the past were missing an important parameter.

Past correlations had exhibited a 0.3-0.4 power variation with unit Reynolds number, as in Fig. V-5 and as exhibited by some of the data in Fig. V-4. However, as Morkovin notes, the Mach 8-9 data in Fig. V-4 fail to follow this anticipated trend. Too often such changes may have been evident in other data but dismissed as "bad points". It is now clear that what has been identified as the unit Reynolds number effect actually does not represent one effect but rather a complex superposition of many functional relationships (probably not all of which are even known yet).

Morkovin calls for further careful microscopic diagnostic measurements in the free stream with simultaneous probing of the boundary-layer response, because this is the only way to clarify the roles of turbulence and sound waves. Of course, even microscopic experiments will contain inaccuracies and regimes of uncertainty. What then constitutes solid evidence, and what indeed is the smallest significant scale?

There also is a question of detection techniques and their relative indications (which also vary with the major parameters). The question is raised by Morkovin, and Figs. V-3 and V-5 and -6 indicate it is important. Figure V-5 shows a large difference in experimentally determined axial location of the beginning of transition from optical data and from heat-transfer measurements. Also, the spatial distribution around the cone is important. At zero angle of attack the distribution is fairly uniform, but at positive angle of attack the pattern is irregular. This is important in engineering design for heat protection because it emphasizes that three-dimensional features of the flow over even a simple sharp cone can be complicated and not subject to good prediction.

An associated question raised by Morkovin concerns the reasoning leading a particular experimenter to focus on the beginning of transition whereas another focuses on the end of the extended transition region. Potter and Whitfield (1960) choose the center of the transition region. This lack of consistency too frequently gets overlooked when experimenters are merely showing trends from a single set of tests, but the matter really is of crucial importance if good usable correlations between a variety of analytical predictions and experiment are ever to be attained.

Figure V-8 illustrates the trend of points for both the beginning and end of transition versus Mach number. This figure also shows the well known "Mach bucket" not shown in Fig. V-3f. As discussed by Morkovin, and Morkovin and Mack (1969), this is a trend reversal in wind tunnel data, and one that is not found in flight data. They discuss several

such trend reversals, and emphasize the need for caution in taking wind tunnel data (which frequently is a composite from several tunnels) and extrapolating the data to very large vehicles in atmospheric flight.

Morkovin discusses many other effects which cannot be considered in this summary. However, one last major parameter cannot be passed over. Morkovin and Mack point out that contrast between any observed growth of disturbances (also its documented dependence on various parameters) and the theoretically expected standard growth and behavior is a key to better understanding and rational correlations. However, they note that before one can approach the instability problem proper, one has to have the tools for computing mean boundary layer profiles for the specific pressure and mass transfer conditions of interest, preferably with different degrees of cooling and possibly real gas effects. This task by itself is a major problem.

Two recent symposia (August 1968 at Stanford, and December 1968 at Langley) provide extensive summaries of present knowledge and techniques for computing turbulent boundary layers, and they provide insight into the difficulty of the overall stability problem. First of all, from these it is clear that computation of velocity profiles is a major task even when the flow is known to be fully turbulent. Now recall that the stability problem starts out with the velocity profile laminar, and continuously progresses thru a transition stage where the profile really is due to a mixture of laminar and turbulent effects. The reason this is important is that disturbance growth rate is of considerable

importance and must be computed to see if the disturbance passes out of the unstable region before the linear theory becomes invalid. Thus, any laminar computation becomes increasingly inadequate before full turbulence is developed, and any reasonably accurate profile computation must be done with equations which include turbulent effects. It will be seen that accurate knowledge of the velocity profile is crucial in the stability analysis, so use of laminar profile with no downstream change is very questionable!

Recall from Section II that a disturbance damps as it moves downstream in what is a stable condition, then it amplifies as it progresses into unstable conditions, and then damps again if it has progressed on through the unstable region before nonlinear effects set in. Now certainly the velocity profile is also changing with distance downstream, and hence a real physical disturbance should be analyzed with a velocity profile that is changing with time, from laminar to turbulent. The simplified classical treatments do not consider this feature, but this is equivalent to a changing of the "spring constant" of the problem and clearly could influence system stability. In the language of flight dynamics, this is the "static stability" of the problem, in contrast to the "dynamic stability" usually analyzed in classical fluid mechanics. It is well known that both can have a significant influence on the resulting amplification or decay of disturbances. The above comments do have to be qualified however, by recognition that rate of change of the profile may be slow compared to some disturbance growth rates, and hence a constant profile may be adequate. This point needs further investigation.

In fluid dynamics the "inviscid" instability is really a static instability of the problem, and arises from the shape of the velocity profile. The "viscous" instability is a "dynamic instability", although dynamic instability by no means is limited to viscosity effects: inertia and elasticity both influence the dynamic stability!

Viscous and inviscid fluid stability effects have been known since the days of Tollmien. What has not been appreciated however is the tremendous effect that velocity profile (static stability) can have on the problem. The importance of this was shown in Figure II-9.

## V.2 Stability Portfolio

Obremski, Morkovin, Landahl, et al (1969) have published a portfolio which emphasizes this point of sensitivity to the profile shape. However, it is difficult enough to compute a single profile, or to conduct a stability analysis for a given profile. How much more difficult it would be to make progressive profile computations and couple these into the stability analysis! A brief discussion of "The Portfolio" will illustrate the importance for boundary-layer stability problems.

To cope with the problem of the difference between critical Reynolds number ( $R_c$ ) and transition Reynolds number ( $R_T$ ), the engineer predicting transition needs information on the amplification characteristics of his particular boundary layer rather than on  $R_c$  alone. The Portfolio points out that stability theory neglects the  $x$ -variation of mean profiles as being of secondary importance and treats  $x$  as

a dormant parameter. This is called a "local-constant-base" or "quasi-parallel flow" assumption. It can be generalized further to time-dependent parameters, where both  $x$  and  $t$  are treated as dormant. Justification is based on the assumption that for many important problems the characteristic times of the Tollmien-Schlichting waves are sufficiently short in comparison with those of the base flow for the approximation to be valid.

The range of validity of the assumption is not considered here, but it is important to note the striking significance of velocity profile as indicated by the study. The Orr-Sommerfeld equation discussed in Section II was used for the Portfolio. Now the only features of the boundary layer which enter the Orr-Sommerfeld equation are mean velocity variation  $U(y)$  and its second derivative  $U''(y)$ . With the boundary conditions  $U(0) = 0$ , and  $U(y) \rightarrow U_e$  as  $y \rightarrow 1$ , the profile curvature  $U''(y)$  fully specifies velocity distribution.

Specification of any profile through its  $U''(y)$  distribution emphasizes the physical role of  $U''(y)$ , i.e., the rate of change of the mean vorticity. This fundamental role can be traced to the fact that in the linearized vorticity equation, the term  $vU''$  is directly related to the mechanism through which the fluctuating disturbance vorticity feeds on the reservoir of mean flow vorticity. Incidentally, it should be noted that this is a "mathematical mechanism" and does not clarify the real "physical mechanism".

Stability theory itself is indifferent to how a given distribution of vorticity, and hence of  $U''(y)$ , arose. Thus, under the hypothesis of local constant base, the instantaneous local  $U''(y)$  distribution (and the boundary conditions) determine the local growth and propagation characteristics of waves.

The Portfolio emphasizes that in comparison of theoretically and experimentally determined stability characteristics of nonsimilar boundary layers, the  $U(y)$  and  $U''(y)$  distributions which are fed into the theory may not correspond to the actual profiles for which amplification,  $Re$ , etc. may have been obtained experimentally. The problem is so sensitive that small inaccuracies in  $U(y)$  and  $U''(y)$  determinations may cause significant discrepancies! For cases demonstrated in the Portfolio, it is noted that differences between the velocity profiles themselves are so small that they could well be missed experimentally without special precaution.

For instance, finite resolution of instruments could easily change the curvature distribution ( $U''$ ) of the profile ( $N_A = 0.075$ ,  $X_\omega = 9.0$ ,  $\Omega = 0$ ) into that of ( $N_A = 0.075$ ,  $X_\omega = 5.06$ ,  $\Omega = 0$ ) as shown in Fig. V-9. This would induce a change in  $\log (R_{\delta^*})_c$  from 2.68 to 3.3 as shown in Fig. V-10. Such an error is very sizable, and yet it arises from an easily undetectable difference in curvature of the velocity profile.

One final point needs to be made; in view of the importance of velocity profile and time development of a disturbance, the transition predictor might wonder what is "the velocity profile" in a transition region of turbulent bursts!

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APPENDIX

Figures

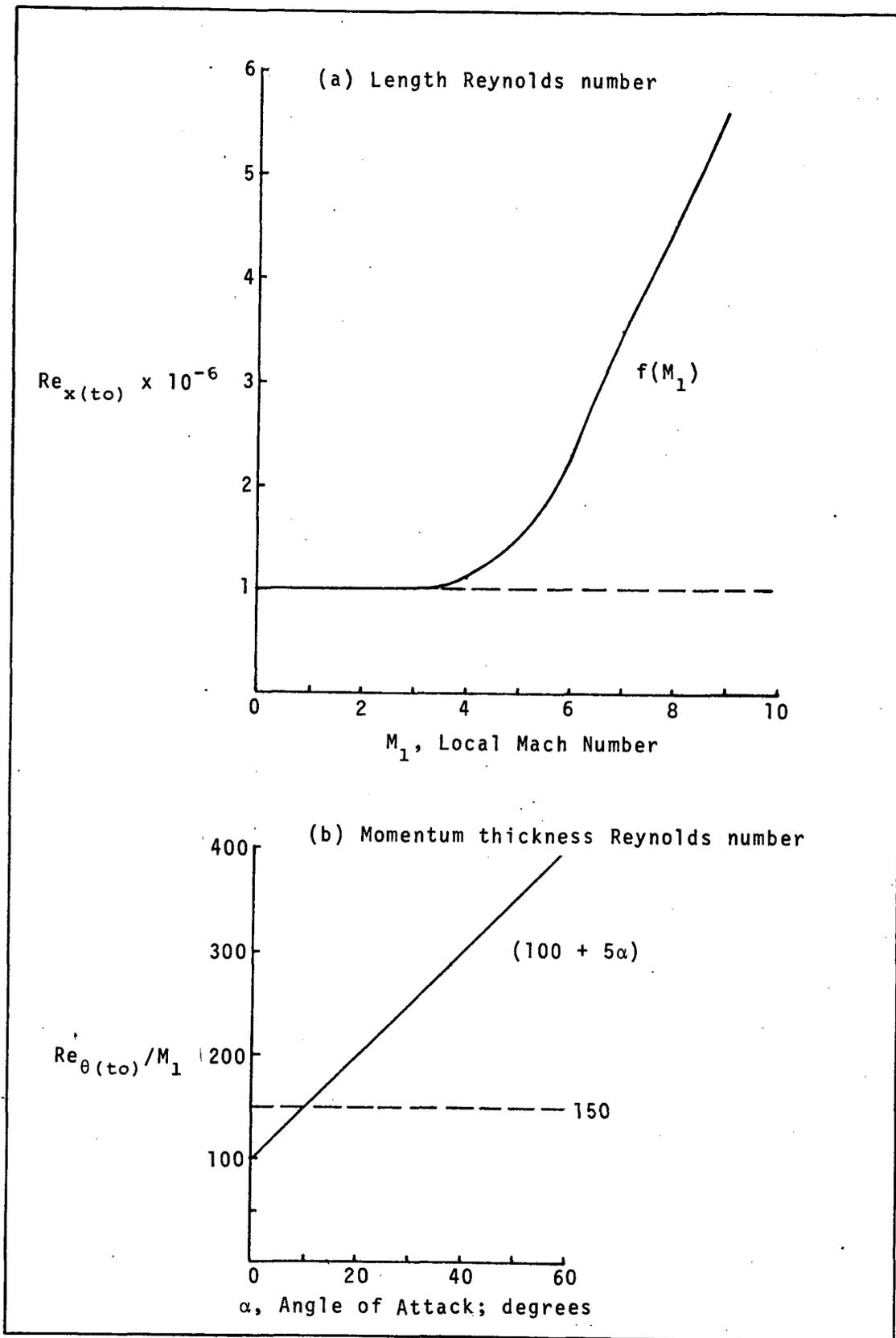


Figure I-1: Transition Onset Reynolds Number Criterion: from Pearce (1970).

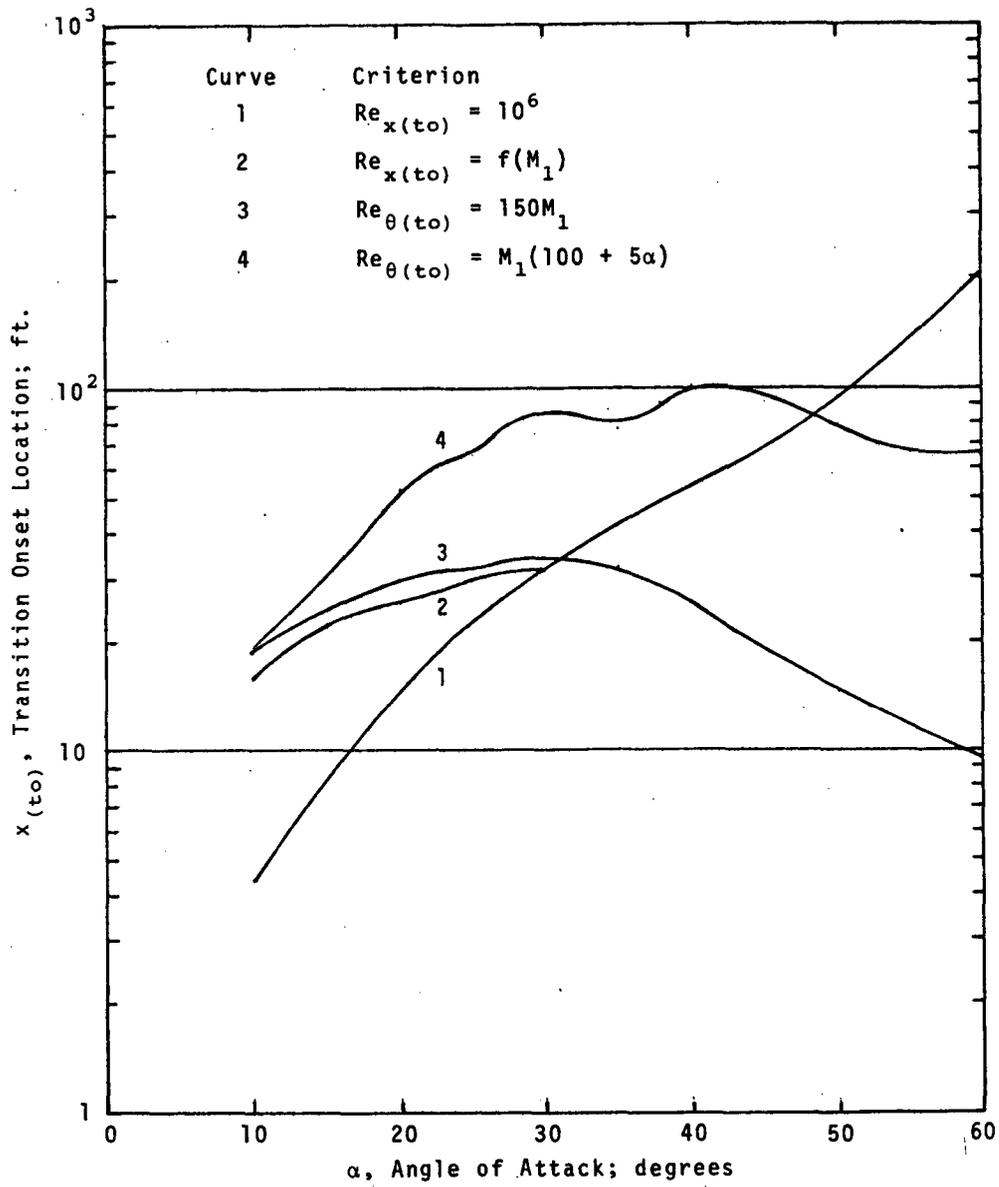
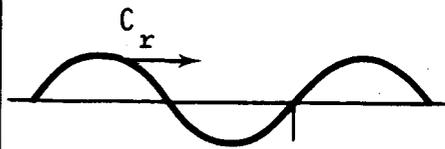


Figure I-2: Transition Onset Lengths; from Pearce (1970).

Synopsis by Mack of linear stability theory for small sinusoidal disturbance of given frequency in boundary layer:



$$\lambda = \frac{2\pi}{\alpha} \quad f = \frac{c_r}{\lambda} = \frac{\omega}{2\pi}; \quad \omega = \alpha c_r$$

f = Frequency,  
 $c_r$  = phase velocity,  
 $\lambda$  = wave length  
 $\alpha$  = Wave number,  
 $\omega$  = circular frequency

I. Theory answers questions:

- (1) Does disturbance damp or amplify?
- (2) If the latter, at what Reynolds numbers, by how much, and for what frequencies?

If the disturbance amplifies enough, it may lead to transition.

Rapid amplification over short distance gives large amplification rate

Large growth of small initial disturbance gives large amplitude ratio

II. Theory relates instability, as measured by amplification rate or amplitude ratio, to parameters governing mean boundary layer.

e.g.: Mach number, Reynolds number

Wall temperature, pressure gradient

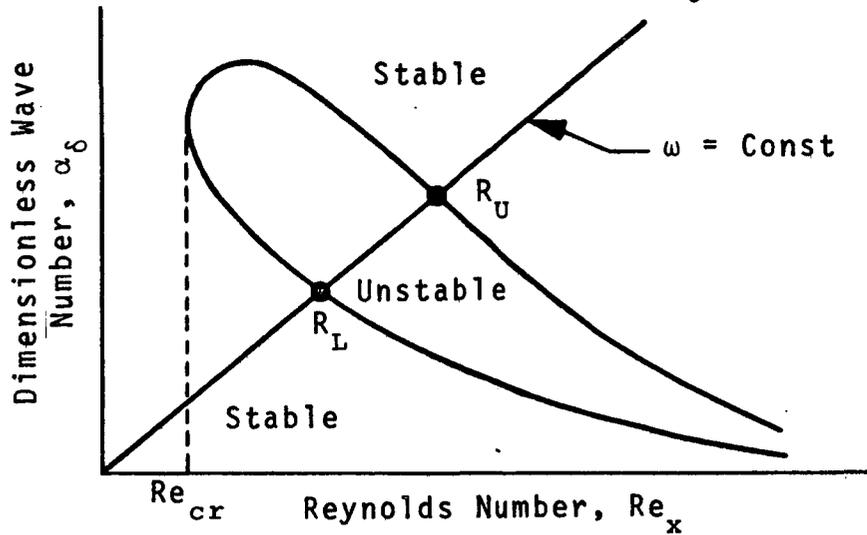
Note: This document uses  $\alpha = \frac{2\pi}{\lambda}$  (dimensional)

and  $\alpha_\delta = \alpha\delta = \frac{2\pi\delta}{\lambda}$  (nondimensional)

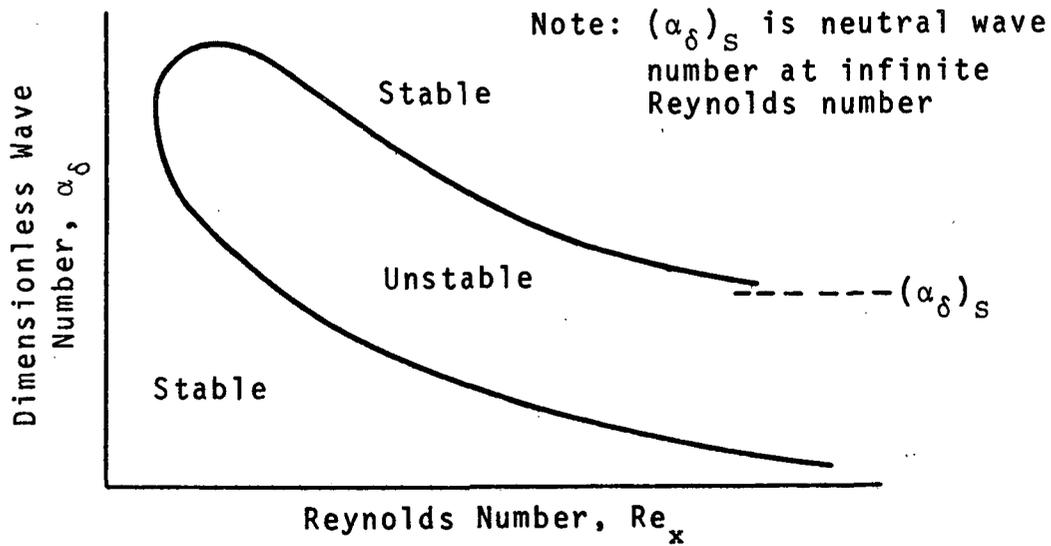
See caution on Fig. II-2.

Figure II-1: Linear Stability Theory: from Mack (1969)

Caution: Literature mixes notation, using  $\alpha$ ,  $\alpha_\delta$ , and  $\alpha_\delta$  interchangeably!



(a) Inviscid stability



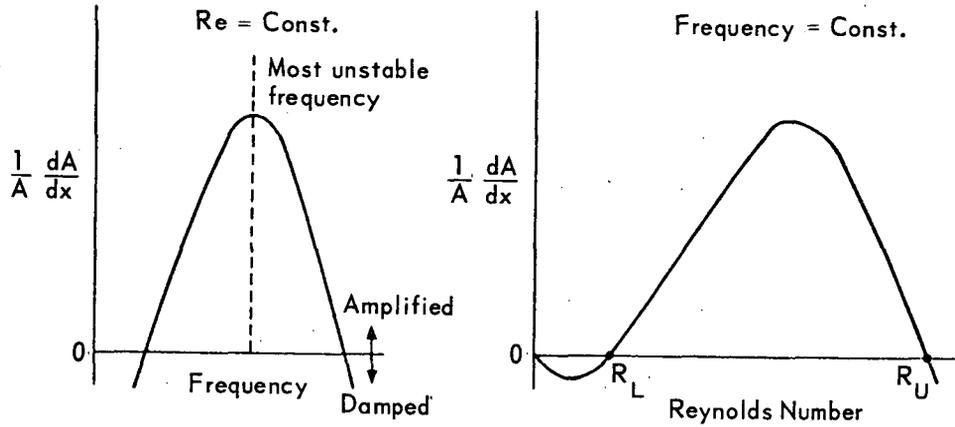
(b) Inviscid instability

Figure II-2: Typical Curves of Neutral Stability: from Mack (1969)

Spatial amplification rate:

$$-\alpha_i = \frac{1}{A} \frac{dA}{dx}$$

A = Amplitude of disturbance,



Spatial amplitude ratio:

$$\frac{A}{A_1} = \exp \int \left( \frac{1}{A} \frac{dA}{dx} \right) dx$$

$A_1$  = some initial amplitude, usually at first neutral point,  $R_L$

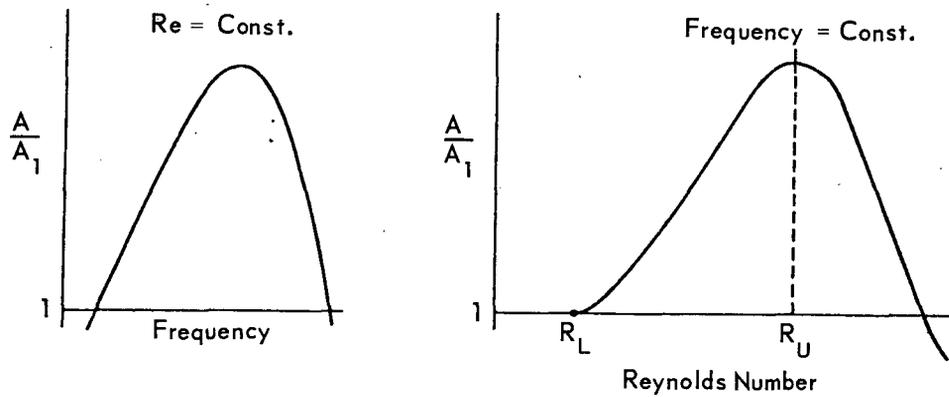


Figure II-3: Typical Results from Stability Analysis; from Mack (1969)

Mack discusses:

(1) Do instability waves exist in a laminar boundary layer?

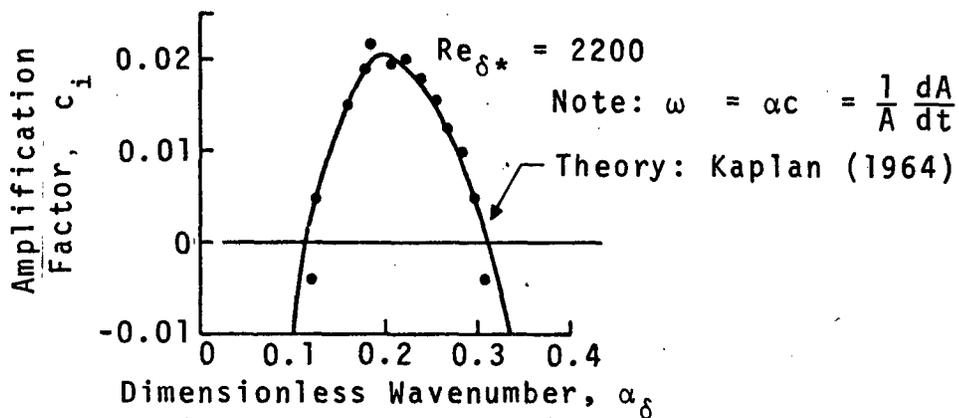
(2) Does theory correctly describe their behavior?

Answer to (1) is YES from numerous observations.

Answer to (2) can be judged from following comparison of theory and experiment.

Incompressible flow, temporal disturbance

• Schubauer - Skramstad (1947)



Compressible flow, spatial disturbance

• Kendall (1967)

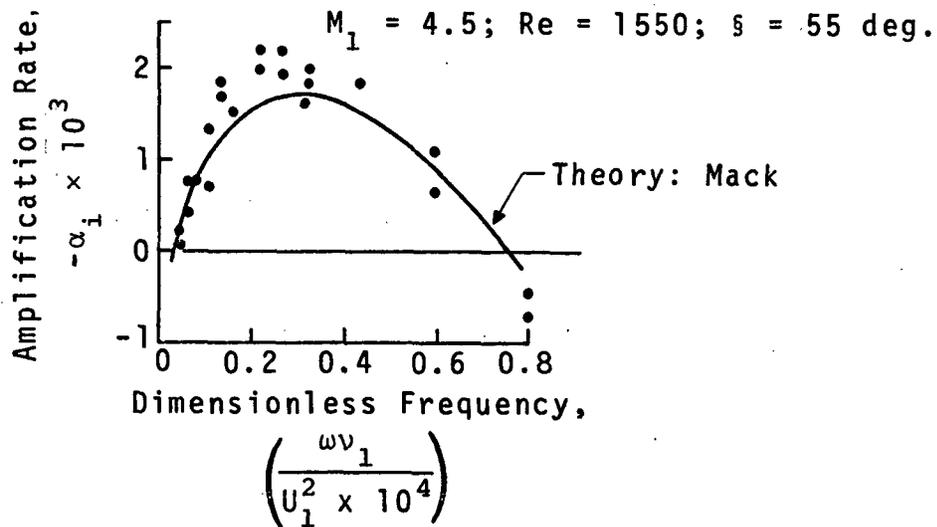
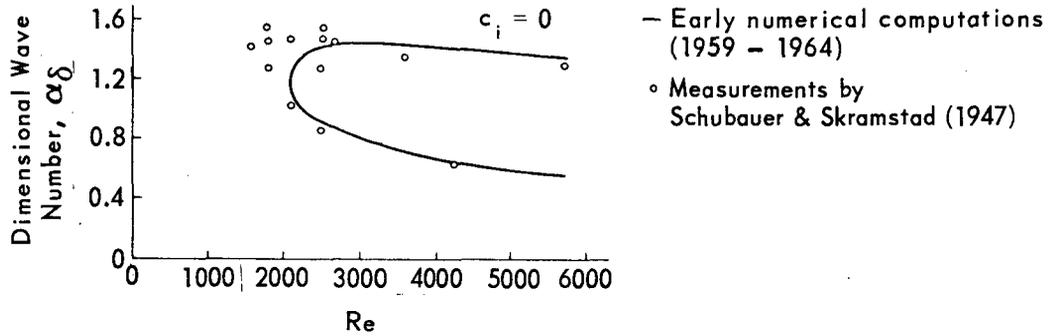
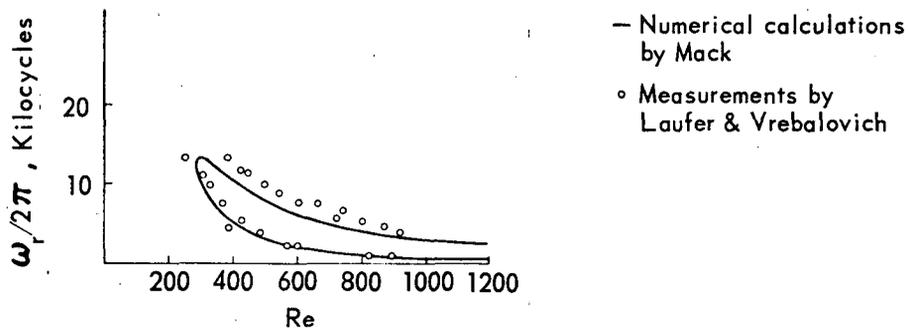


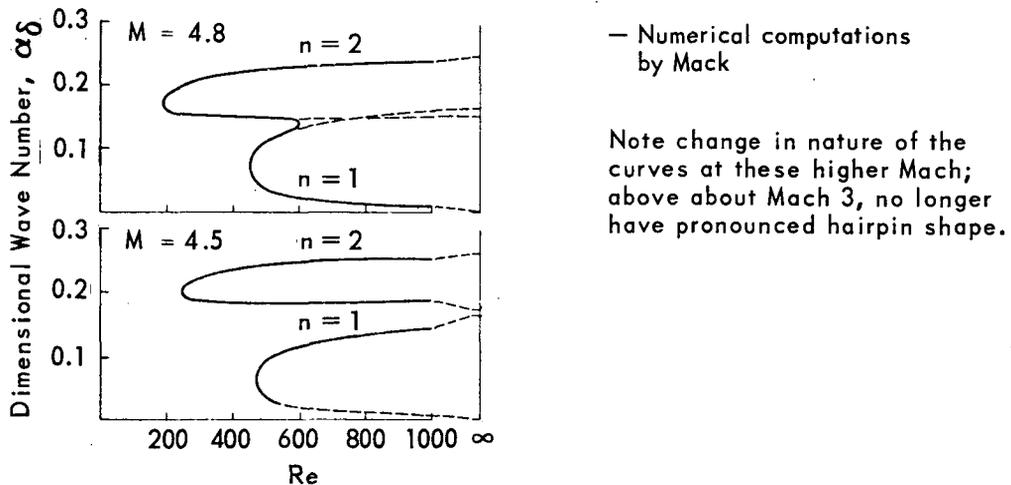
Figure II-4: Experimental Support for Linear Stability Theory: from Mack (1969)



(a) Neutrally stable oscillations of the Blasius layer.



(b) Neutral stability curve for compressible boundary layer at Mach 2.2 (insulated wall).



(c) Neutral stability curves for compressible boundary layer at Mach 4.5 and 4.8 (insulated wall).

Figure II-5: Experimental and Theoretical Neutral Stability Curves: from Betchov and Criminale (1967).

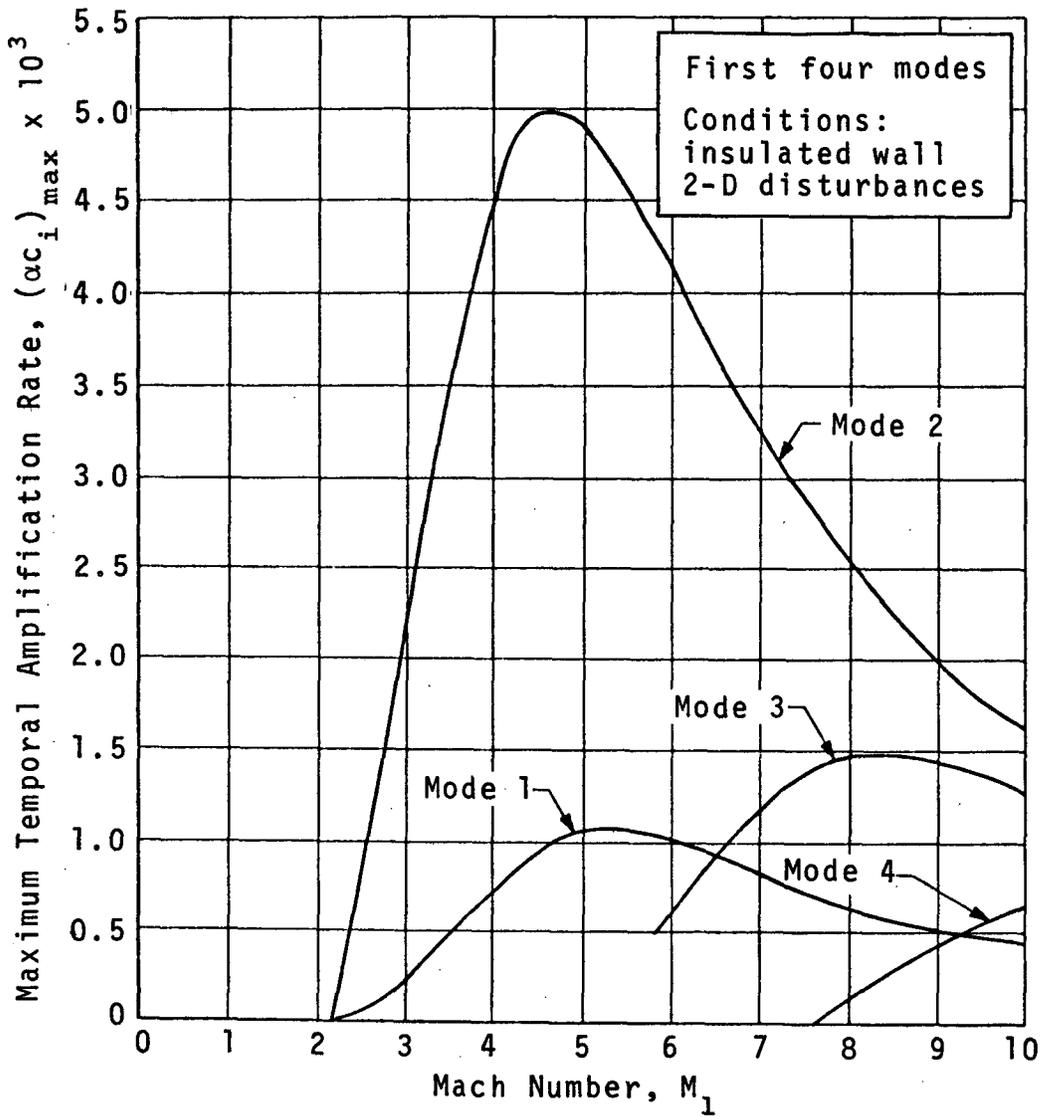


Figure II-6: Effect of Mach Number on Maximum Temporal Amplification; from Mack (1969).

Insulated wall

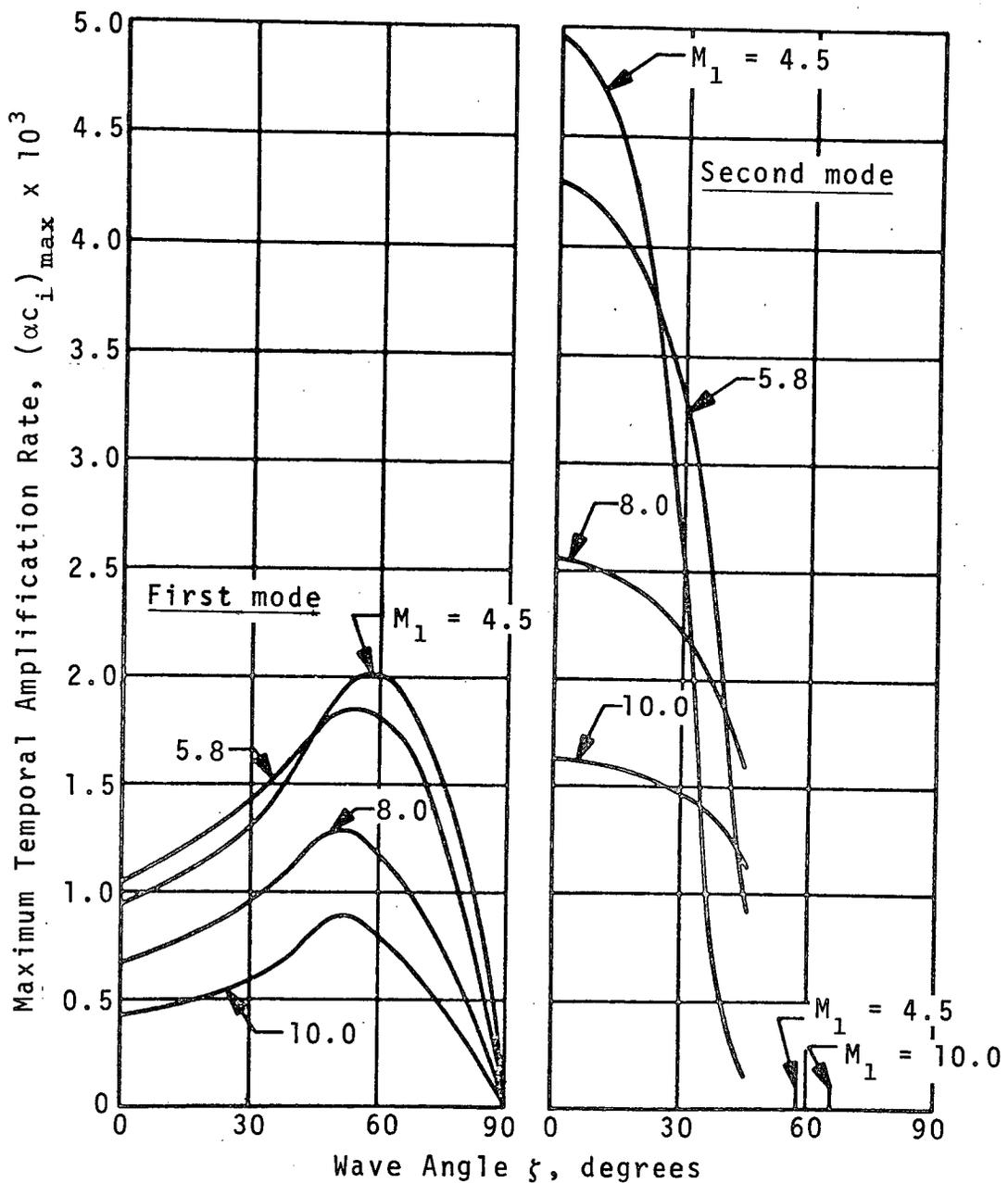


Figure II-7: Effect of Wave Angle on Maximum Temporal Amplification; from Mack (1965b).

A-10

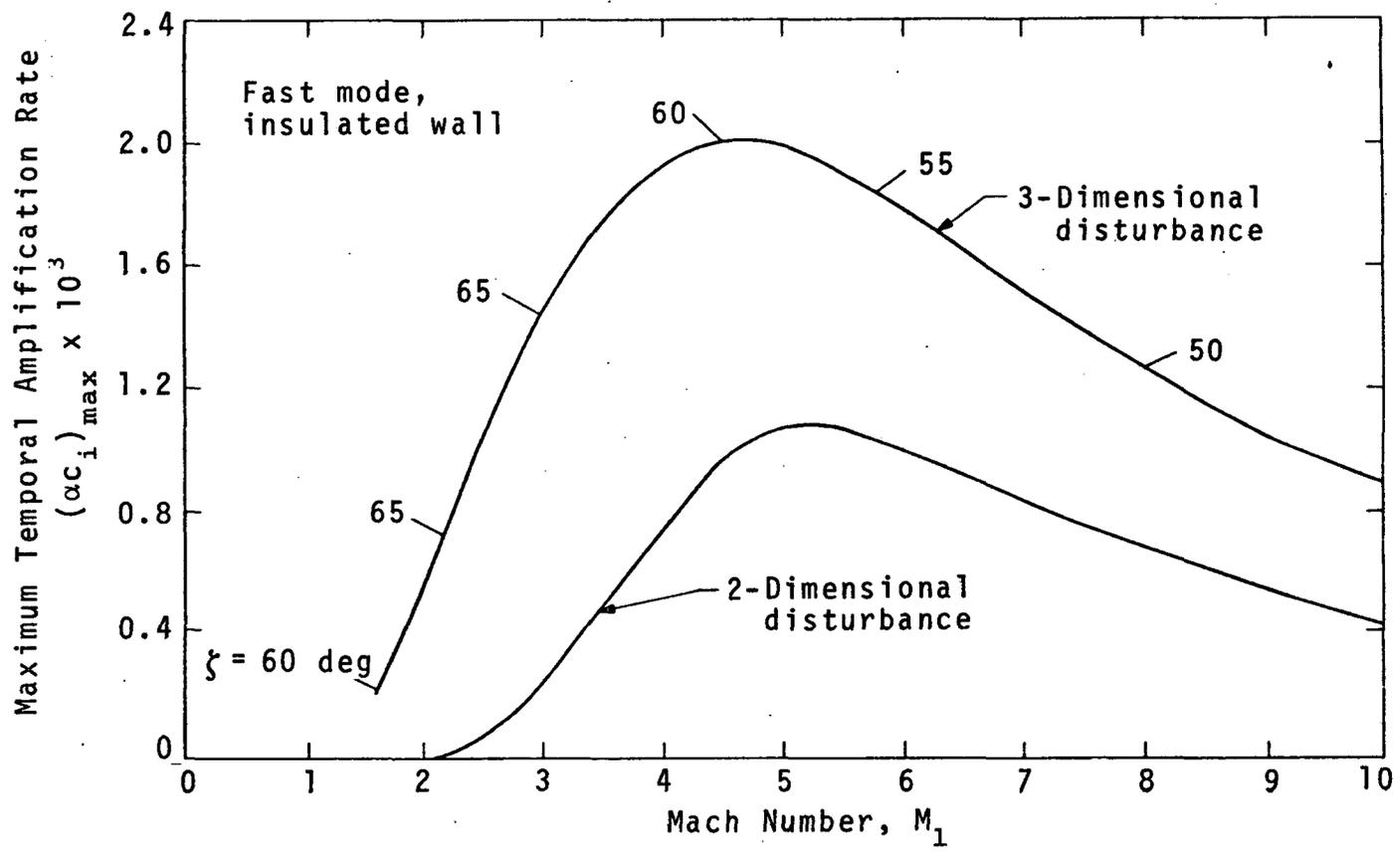


Figure II-8: Effect of Mach Number on Maximum Temporal Amplification Rates; from Mack (1969)

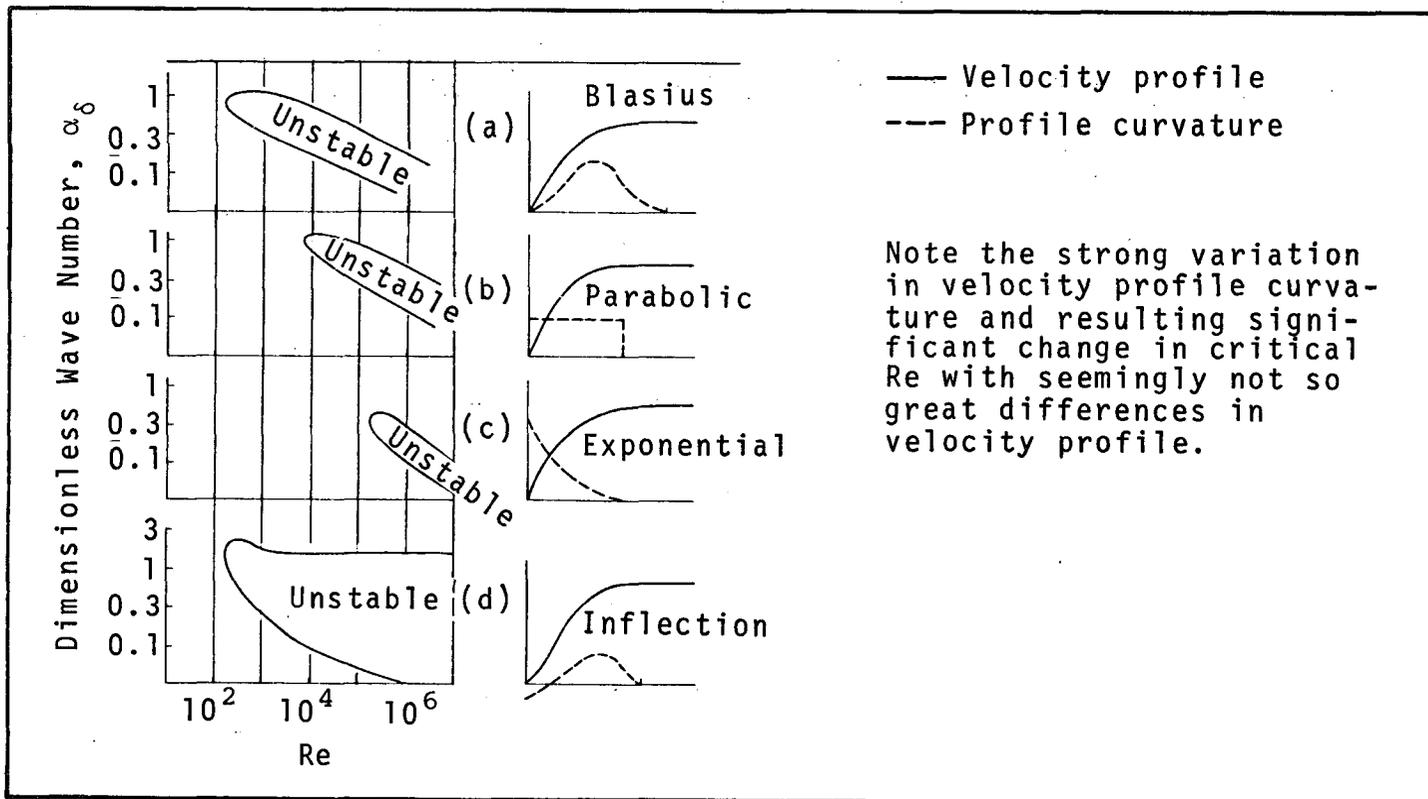


Figure II-9: Stability of Various Velocity Profiles; from Betchov and Criminale (1967).

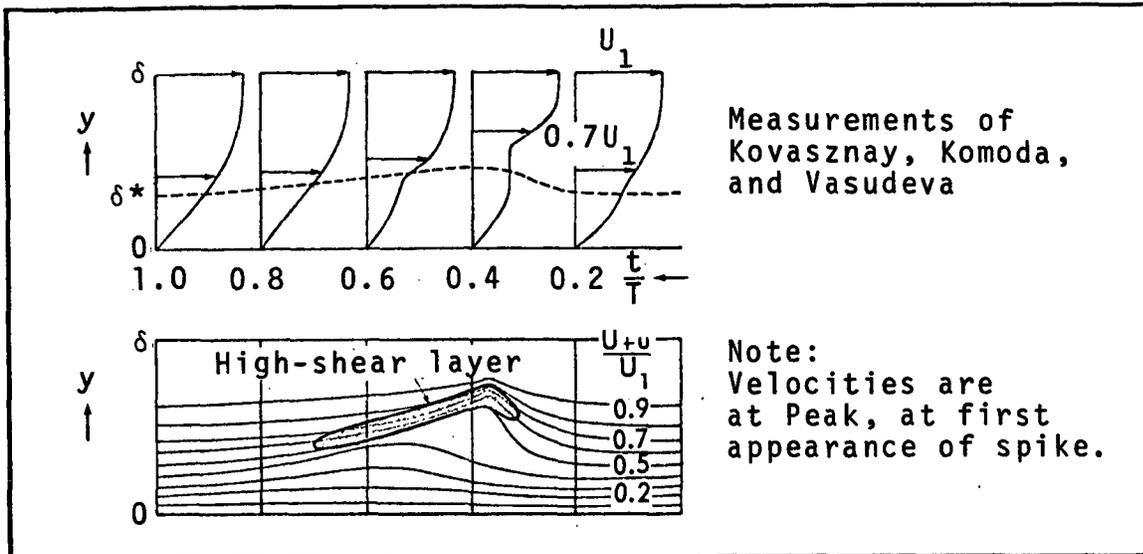


Figure II-10: Instantaneous Velocity Profiles Across Boundary Layer and Contour of High-Shear Layer Superposed Upon Lines of Constant Instantaneous Velocity; from Tani (1967).

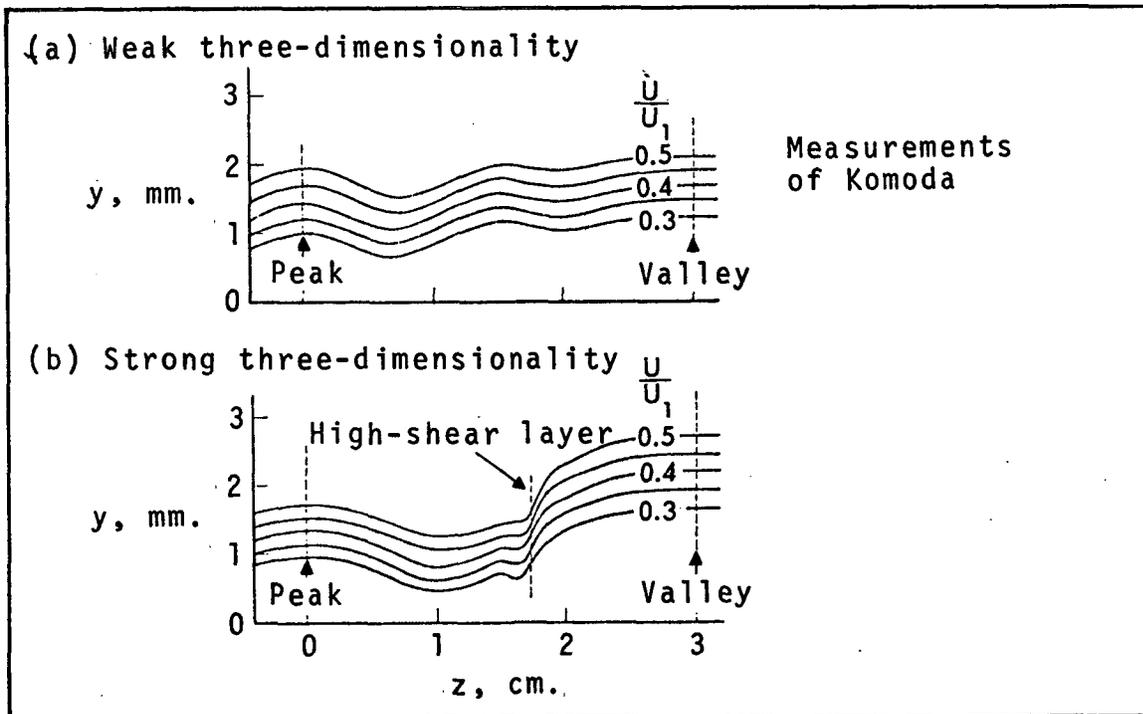


Figure II-11: Contours of Equal Mean Streamwise Velocity; from Tani (1967)

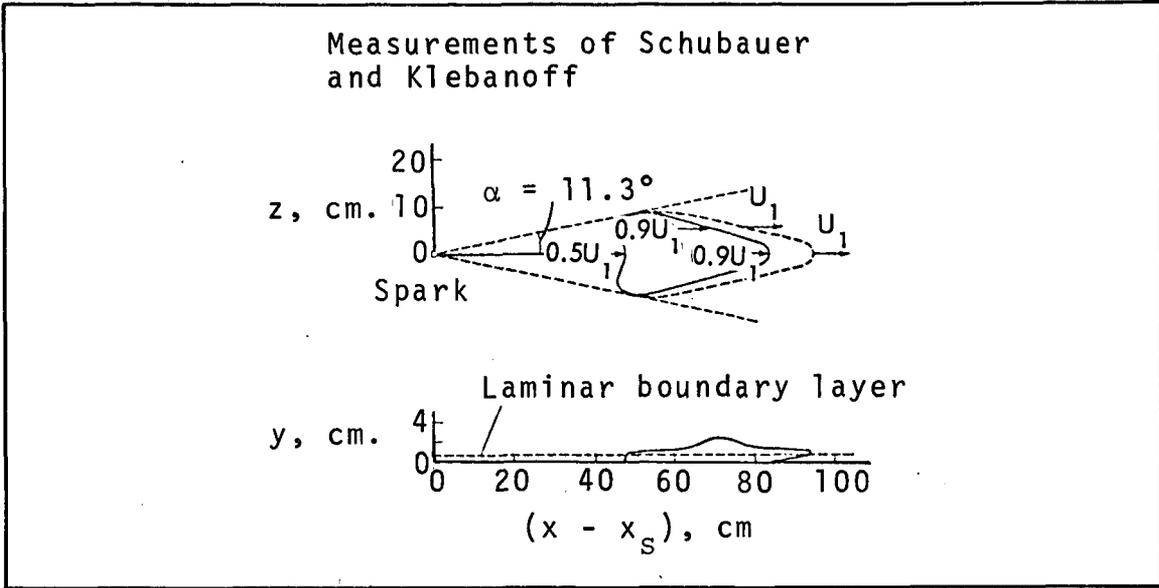


Figure II-12: Plan and Elevation Views of Typical Spark-Induced Turbulent Spot; from Tani (1967)

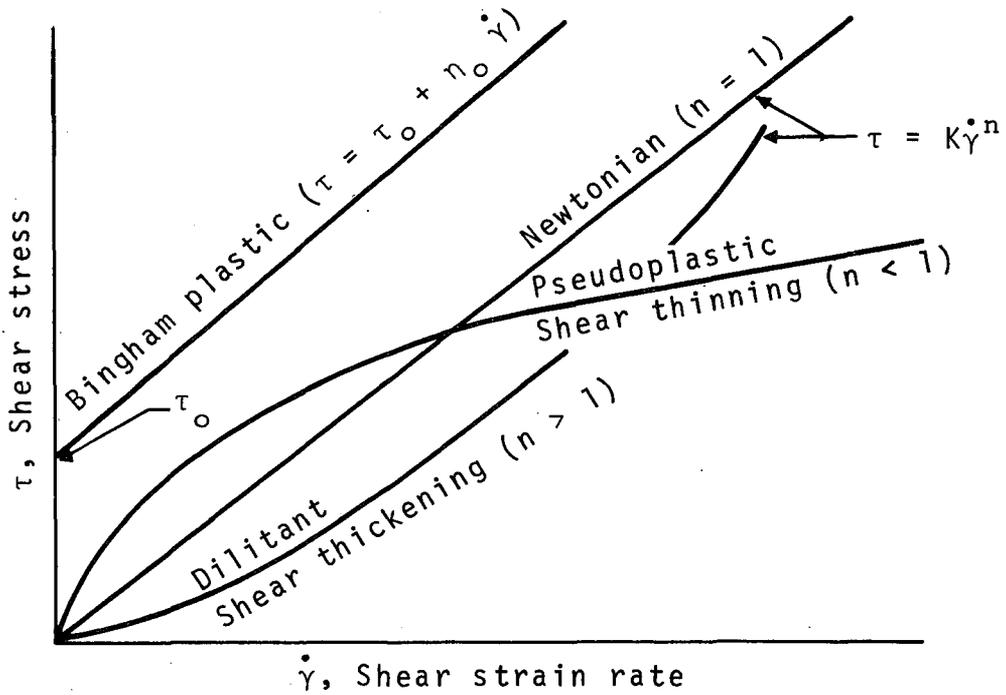


Figure III-1: Non-Newtonian Viscous Behavior; from Bogue & White (1970).

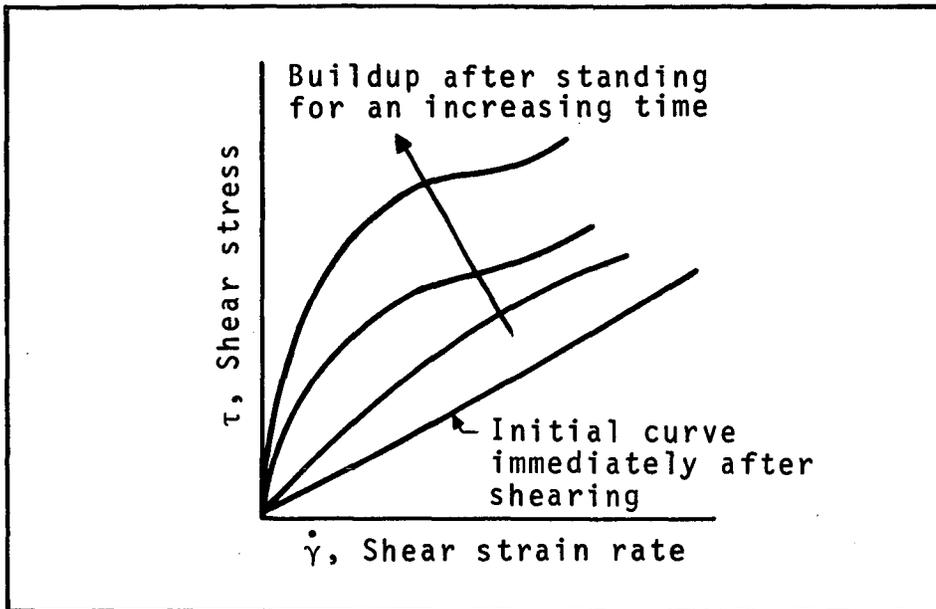


Figure III-2: Thixotropic Fluid Sheared at Different Times; from Bogue & White (1970)

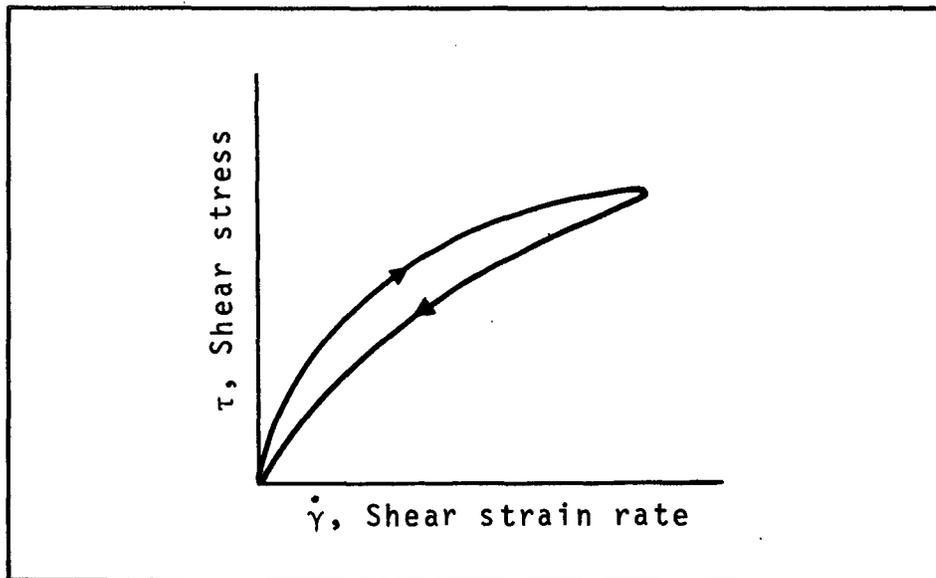
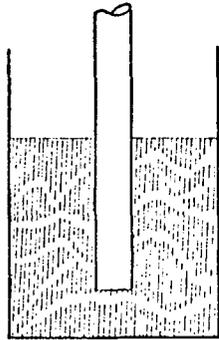
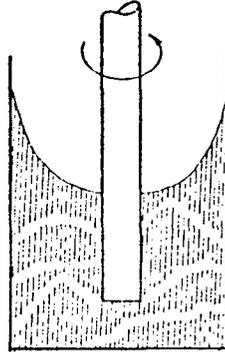


Figure III-3: Hysteresis Loops for a Thixotropic Fluid; from Bogue & White (1970).

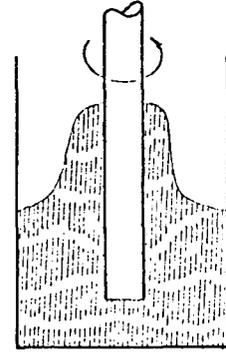
Weissenberg effect



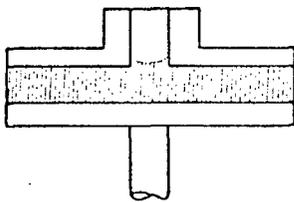
Fluid at rest



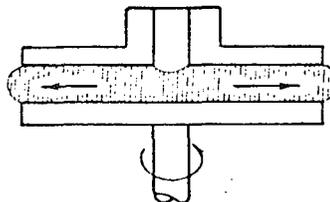
Newtonian fluid in Couette flow



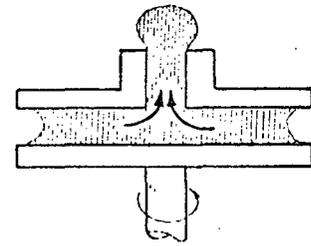
Viscoelastic fluid in Couette flow



Fluid at rest



Newtonian fluid in Torsional flow



Viscoelastic fluid in Torsional flow

Figure III-4: Fluid Behavior in Rotated Devices;  
from Bogue & White (1970)

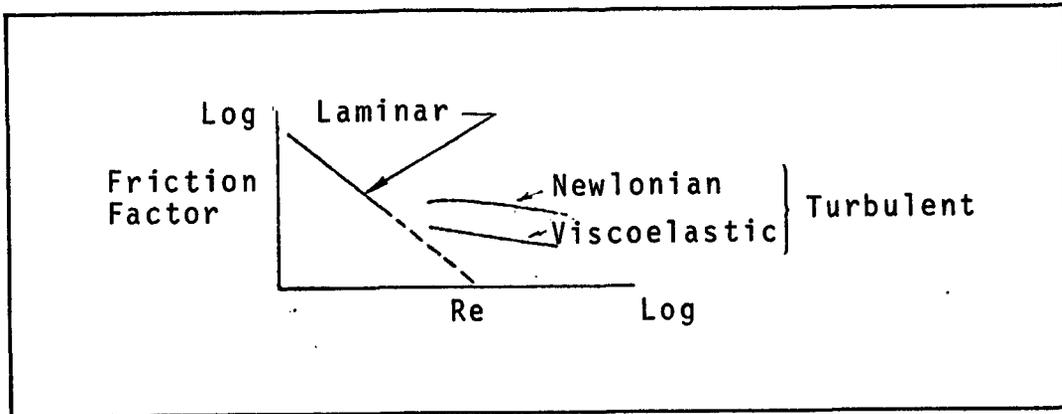


Figure III-5: Viscoelastic Depression of the Turbulent Friction Factor (Toms Effect); from Bogue & White (1970).

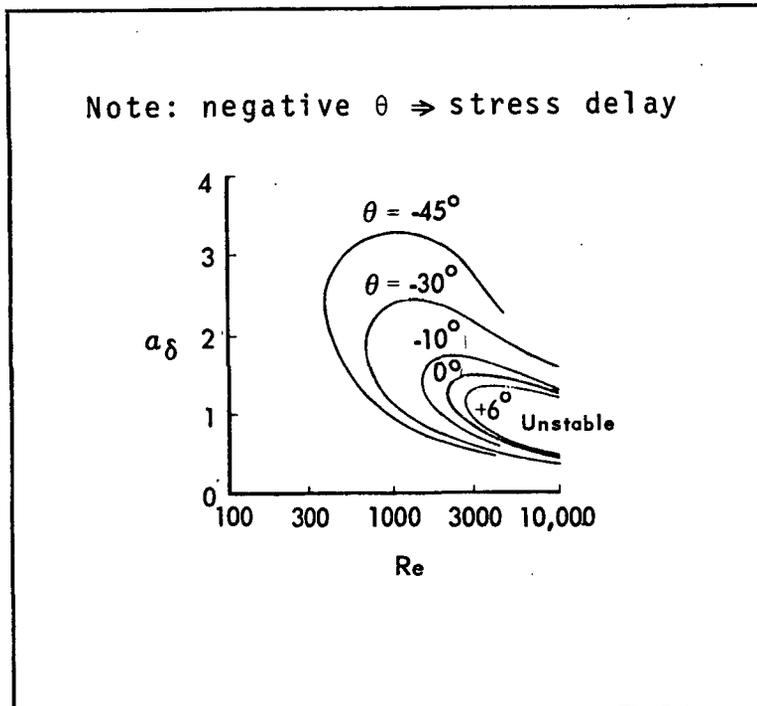


Figure III-6: Neutrally Stable Oscillations of a Blasius Boundary Layer with Viscoelastic Effects; from Betchov & Criminale (1967).

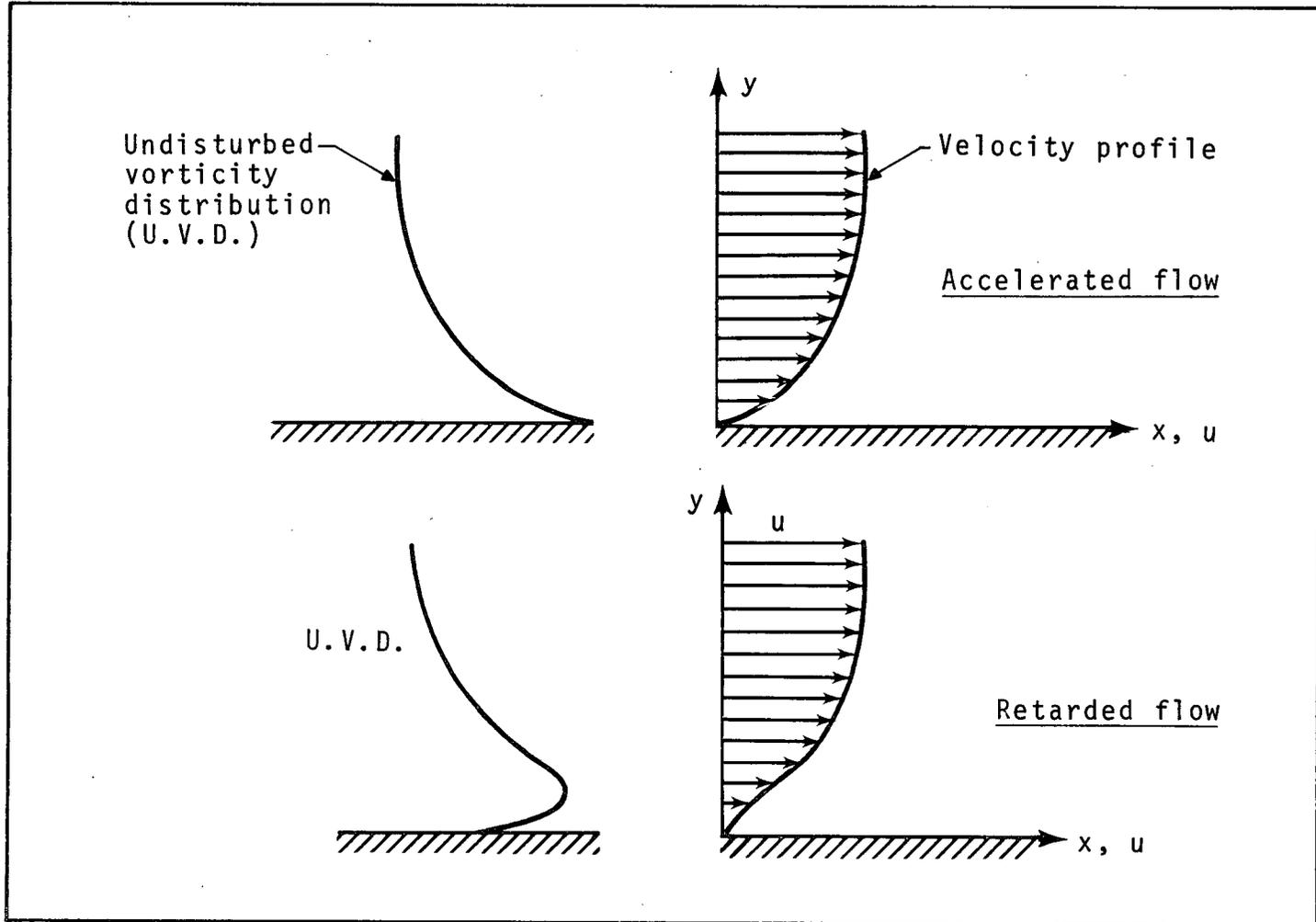


Figure IV-1: Velocity and Vorticity Profiles;  
from Rosenhead (1963).

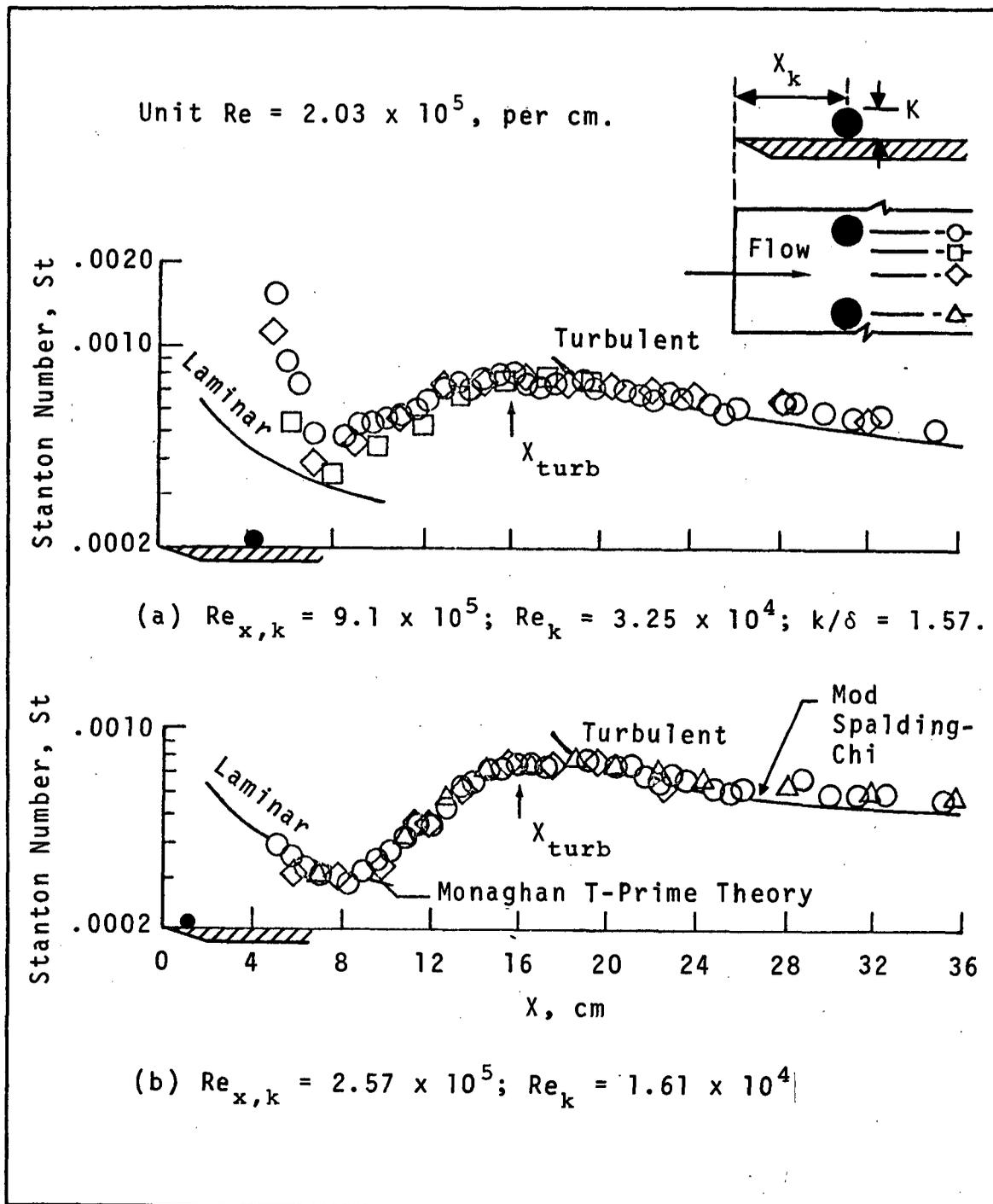


Figure V-1: Effect of Tripped Transition on Heat Transfer; from Morisetti (1968).

Measurement of Mateer, and Polek.  
Ames 3.5 - foot wind tunnel

Model	$M_e$
● 5° Cone	6.6
□ Flat plate	6.8

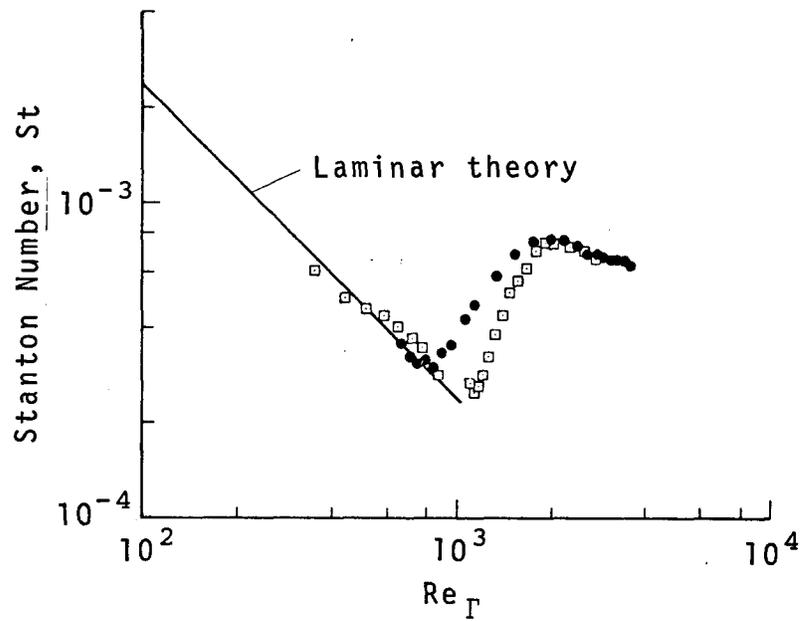
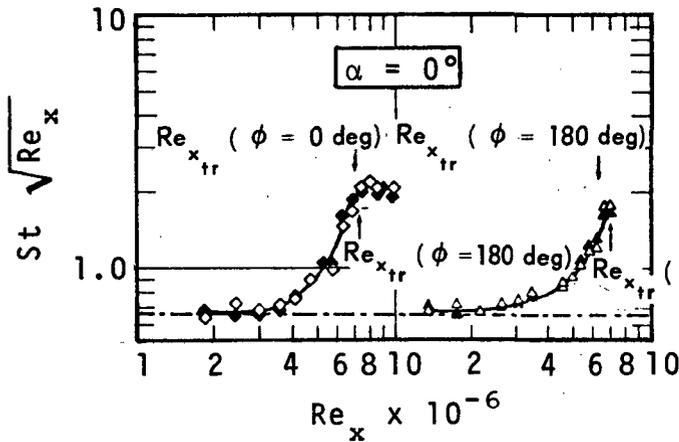


Figure V-2: Effect of Natural Transition on Heat Transfer; from Hopkins (1968)

Note for all figures:

Arrow indicates optical data  
 Sharp cone models  
 Left sides:  $Re_\infty = 2.1 \times 10^6/\text{ft}$   
 Right sides:  $Re_\infty = 1.5 \times 10^6/\text{ft}$

(a)



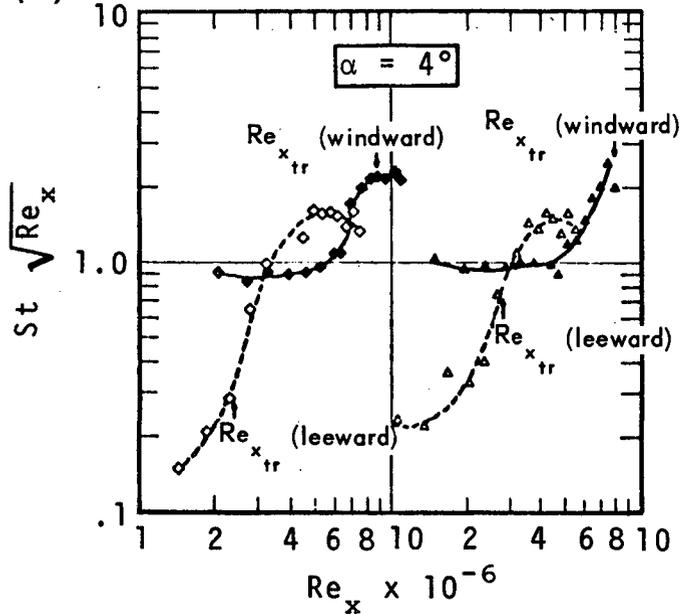
Left side:

- ◇  $\phi = 0 \text{ deg}$
- ◆  $\phi = 180 \text{ deg}$

Right side:

- △  $\phi = 0 \text{ deg}$
- ▲  $\phi = 180 \text{ deg}$

(b)



Left side:

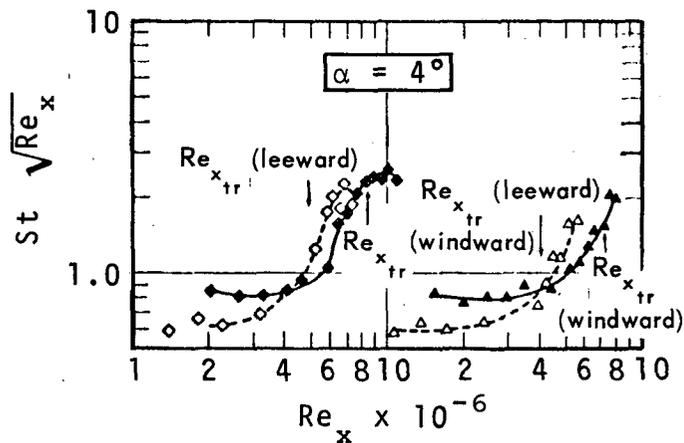
- ◇  $\phi = 0 \text{ deg (leeward)}$
- ◆  $\phi = 180 \text{ deg (windward)}$

Right side:

- △  $\phi = 0 \text{ deg (leeward)}$
- ▲  $\phi = 180 \text{ deg (windward)}$

Figure V-3: Effect of Orientation on Heat Transfer;  
 from DiCristina (1970).

(c)



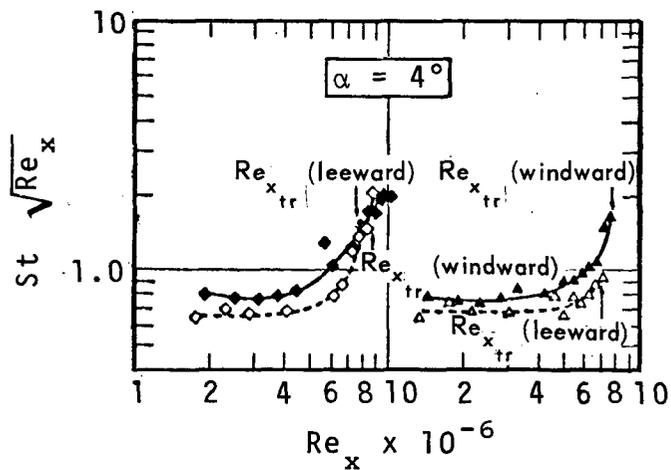
Left side:

- ◊  $\phi = 36$  deg (leeward)
- ◆  $\phi = 216$  deg (windward)

Right side:

- △  $\phi = 36$  deg (leeward)
- ▲  $\phi = 216$  deg (windward)

(d)



Left side:

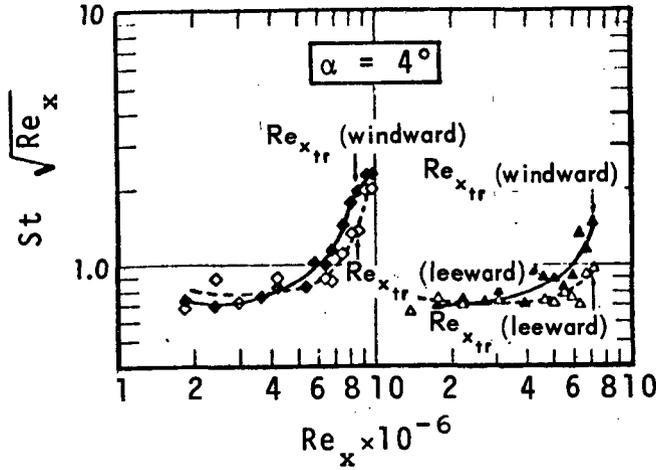
- ◊  $\phi = 72$  deg (leeward)
- ◆  $\phi = 252$  deg (windward)

Right side:

- △  $\phi = 72$  deg (leeward)
- ▲  $\phi = 252$  deg (windward)

Figure V-3 Contd: Effect of Orientation on Heat Transfer.

(e)



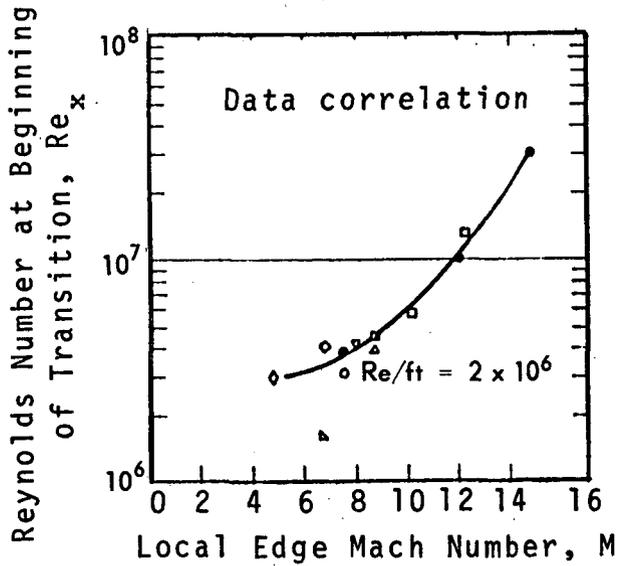
Left side:

$\triangle \phi = 90^\circ$  (leeward)  
 $\triangle \phi = 270^\circ$  (windward)

Right side:

$\diamond \phi = 90^\circ$  (leeward)  
 $\diamond \phi = 270^\circ$  (windward)

(f)



Note: See Morkovin (1969) regarding caution as to this type of correlation.

Figure V-3 Contd: Effect of Orientation on Heat Transfer.



Configuration	M	Facility
◇ Straight tube	6.8	Ames SSFFWT
◆ Straight tube	5.1	Ames SSFFWT
○ Pencil model	7.0	Ames SSFFWT
- - - 25" Cone-cylinder	3.24	NOL pressurized range
◐ Flat plate	1.97	CIT-JPL 20" WT
◑ Flat plate	2.57	CIT-JPL 20" WT
◒ Flat plate	3.70	CIT-JPL 20" WT
◓ Flat plate	4.54	CIT-JPL 20" WT
- + - 20" Cone-ogive-cyl	1.63	CIT-JPL 20" WT
- x - 20" Cone-ogive-cyl	4.09	CIT-JPL 20" WT
□ 10" Cone	3.12	Lewis 1' x 1' WT
△ Hollow cylinder	3.12	Lewis 1' x 1' WT
— Hollow cylinder	3.12	Lewis 1' x 1' WT

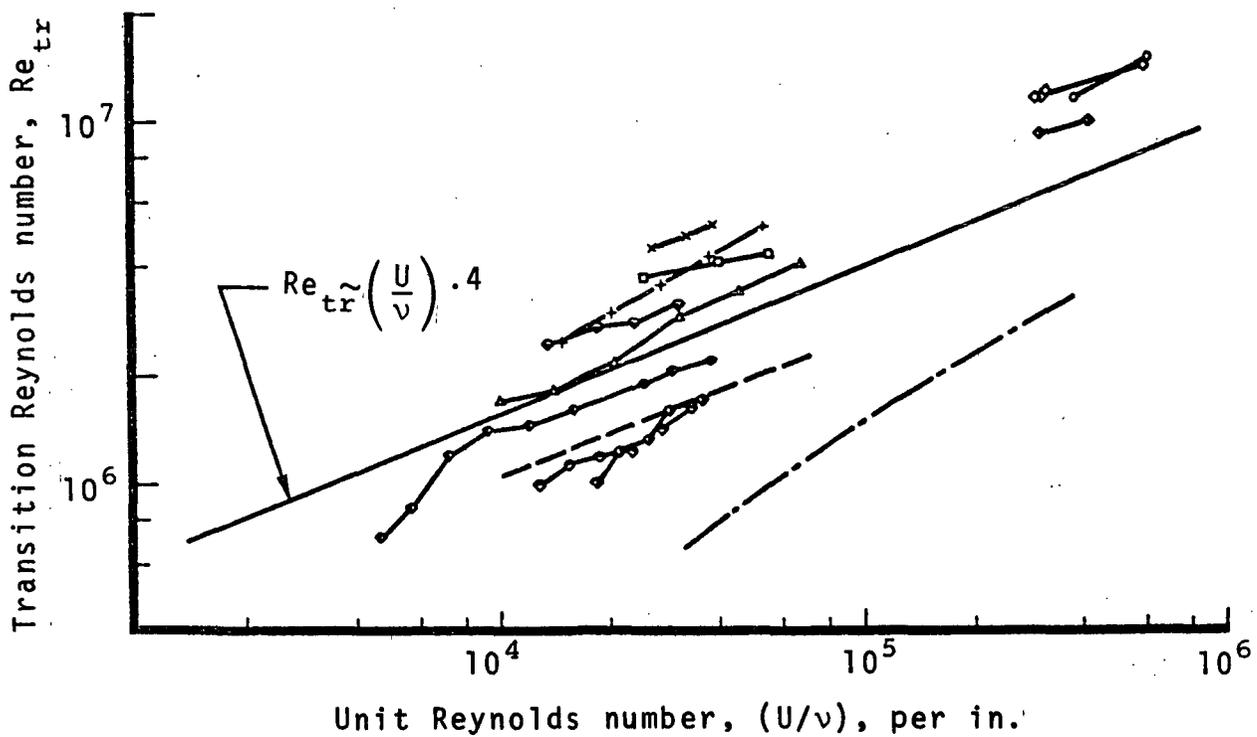


Figure V-5: Typical Comparison of Transition Reynolds Numbers for Variety of Model Configurations; from Nagel (1967).

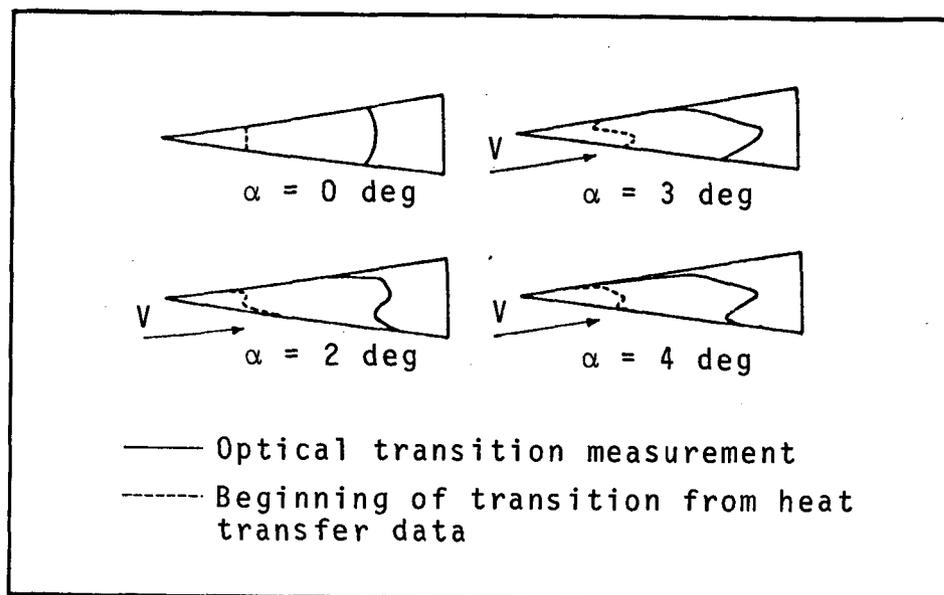


Figure V-6: Comparison of Optical and Gage Measured Boundary-Layer Transition; from DiCristina (1970).

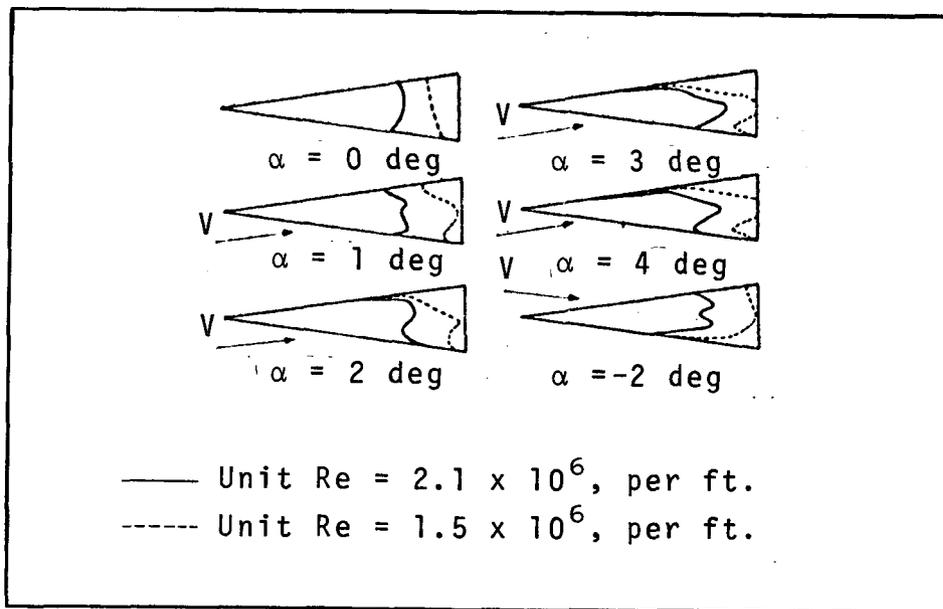


Figure V-7: Spatial Distribution of Boundary Layer Transition with Angle of Attack; from DiCristina (1970).

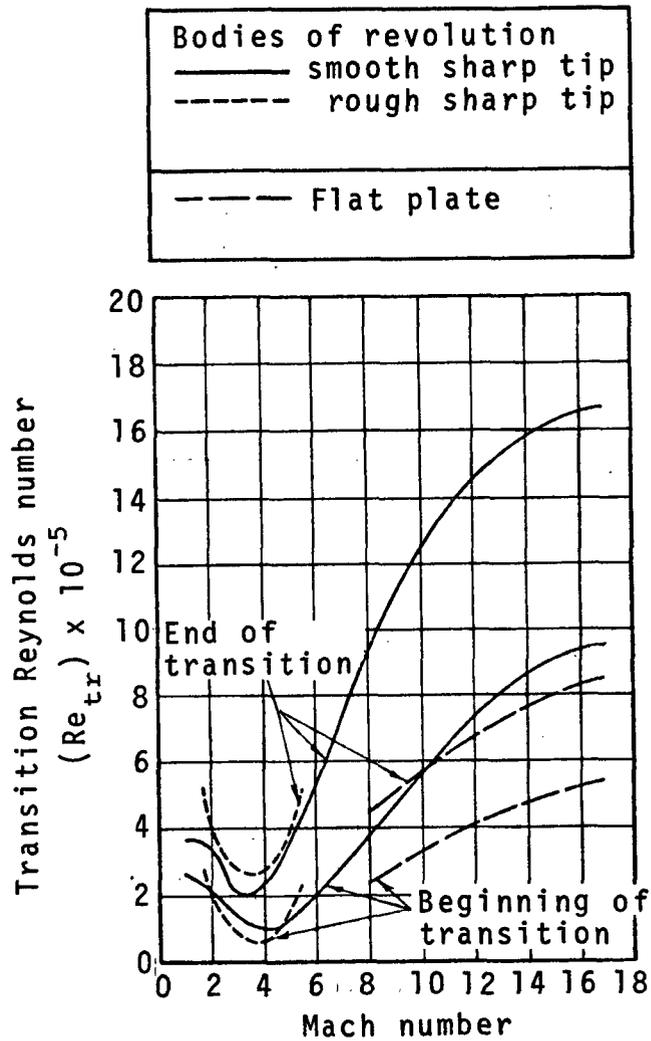


Figure V-8: Mach Bucket Trend Reversal; from Heller (1969).



Freestream unsteadiness  $N_A = 0.075$

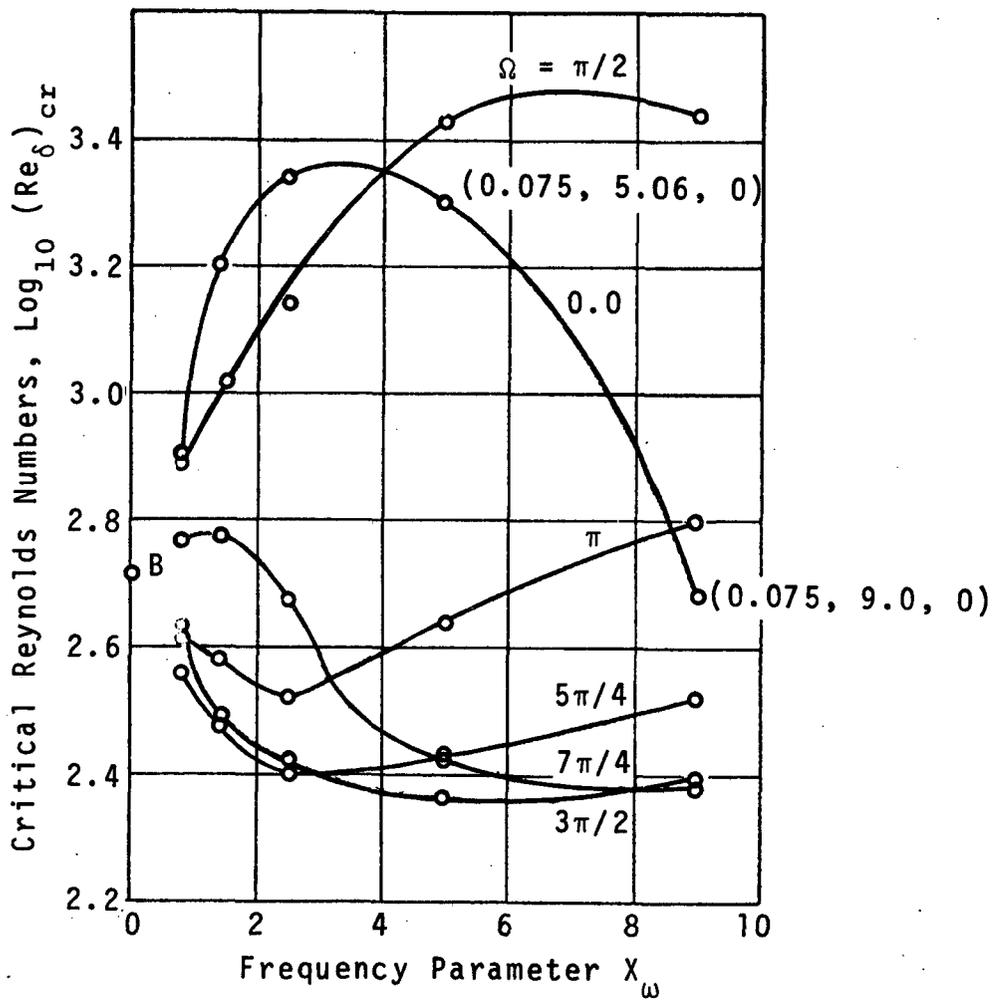


Figure V-10: Critical Reynolds Number Versus Frequency Parameter; from Obremski (1969).