EQUATIONS OF MOTION FOR THE
VARIABLE MASS FLOW—VARIABLE
EXHAUST VELOCITY ROCKET

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Abstract

An equation of motion for a one dimensional rocket is derived as a function of \( f(t) \), the mass flow rate into the acceleration chamber and \( v(x, t) \), the velocity distribution along the chamber, thereby including the transient flow changes in the chamber which have not been explicitly contained in other papers. The derivation of the mass density requires the introduction of the special time coordinate \( \tau \), which satisfies the equation \( 0 = 1 - \frac{\partial \tau}{\partial t} - \frac{\partial v}{\partial x} \). The mass density expression is \( f(t - \tau) \frac{\partial \tau}{\partial x} \).

The equation of motion is derived from both classical force and momentum approaches and is shown to be consistent with the standard equation expressed in terms of flow parameters at the exit to the acceleration chamber.

Introduction

As propulsion systems become more advanced and sophisticated there is a tendency toward exercising more exact control over the mass flow rate into the engine and the acceleration imparted to the mass as it flows through the engine. Hence, the derivation of rocket equations of motion for advanced propulsion systems will have to be based on a rocket model that allows for the mass flow rate into the engine and the velocity distribution along the engine to vary with time.

The objective of the analysis undertaken herein is the acceleration of the rocket expressed in the form:

\[
\frac{du}{dt} = F[M, x_{s}, x_{e}(t), v(x, t), f(t), x_{cp}(t)] \tag{1.1}
\]

where:

- \( M \) = Initial mass of the rocket,
- \( x_{s} \) = Location of the inlet of the acceleration chamber,
- \( x_{e}(t) \) = Location of the end of the acceleration chamber,
- \( v(x, t) \) = Velocity of the mass in the acceleration chamber with respect to the chamber,
- \( f(t) \) = Mass flow rate into the acceleration chamber,
- \( x_{cp}(t) \) = Location of the center of mass of the rocket proper (the rocket minus the acceleration chamber).

The above quantities completely define the rocket herein considered and must be initially specified. The acceleration is derived for the time period before the acceleration chamber is full as well as for the period after mass has stopped entering the chamber. The extensive literature\(^1\) which considers the variable mass flow-variable exhaust velocity rocket does not contain an equation of motion in the form of Eq. (1.1), which would be the parameters normally specified. The approach taken in the literature is to treat the rocket as one entity, without separating the mass in the acceleration chamber from the rocket proper.

The equations of motion which apply to a variable mass system have been a subject of debate among physicists, resulting in no less than eight articles\(^1\) published in the American Journal of Physics during the period 1964-1966 alone. The author finds the following two approaches to be equivalent and applicable to the variable mass problem considered herein.

1. The "momentum" approach: The time rate of change of momentum of all the particles involved must equal zero.
2. The "force" approach: The sum of the forces acting on all the particles involved must equal zero.

II. Derivation of the Mass Density

in the Acceleration Chamber

An analysis of the flow in the acceleration chamber (see Fig. 1) is required before the equation of motion can be obtained in the form of Eq. (1.1). The acceleration chamber in the rocket is used to accelerate the mass inserted into the chamber to the velocity at which it leaves the chamber. It is assumed that the mass enters the chamber with zero relative velocity. The velocity distribution along the chamber is

\[
\frac{dx}{dt} = v = v(x, t) \tag{2.1}
\]

where \( v \) is a specified function of both distance along the chamber and time. The acceleration of the mass within the chamber can be obtained from this velocity distribution.

\[
\tau = \text{Time for particle A to move from } x_{s} \text{ to } x
\]

\[
\Delta \tau = \text{Difference in times that particles 1 and } d \text{ entered chamber}
\]

The amount of time \( \tau \) that the particle at point \( x_0 \) at time \( t_0 \) has been in the acceleration chamber can be found by numerically integrating the equation

\[
\frac{dt}{dx} = \frac{1}{v} \tag{2.2}
\]
between the limits of \( x=x_0 \), \( t=t_0 \) and \( x=x_{s} \), \( t=t_0 + \tau \). \( \tau \) is then obtained from

\[
\tau = t_0 + \int_{t_0}^{x_0} \frac{1}{v} \tag{2.3}
\]
If the variables in Eq. (2.1) are separable, the following equation defines \( r \)
\[
\int_{t_0}^{t} \frac{dx}{x_0} = \int_{t_0}^{t} \frac{dx}{x_0} = \int_{t_0}^{t} \frac{dx}{x_0} = (2.4)
\]
where \( a \) and \( b \) are defined by
\[
v = ab = a(t) b(x)
\]
As \( x_0 \) and \( t_0 \) are completely arbitrary, the general expression for \( r \) is
\[
r = r(x, t)
\]
As time elapses, this particle moves down the chamber. \( r \) increases uniformly with \( t \). Hence, the total differential of \( T \) equals zero.
\[
dT = 0 = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt
\]
Substituting Eq. (2.14) into Eq. (2.15) gives
\[
0 = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial t} dt
\]
As Eq. (2.13) was for an arbitrary \( x \), the partial differential of Eq. (2.16) with respect to \( x \) must equal zero.

III. Derivation of Equation of Motion When Acceleration Chamber is Full

This section contains a derivation of the equation of motion for the variable mass flow-variable exhaust velocity rocket for the time period when the chamber is full. Figure 2 shows the rocket nomenclature. All distances measured relative to the rocket are measured from a frontal position so that an \( x \) coordinate will always be positive in a rearward direction.

The mass flow rate into the acceleration chamber is given by \( f \), a specified function of time. This function, along with the velocity distribution, determines the mass density in the chamber. Consider the mass \( \Delta m \) that occupies a small length \( \Delta x \) centered at \( x \). An expression for this mass is obtained by multiplying the mass flow at the time that the particle at \( x \) left the inlet of the chamber by an interval of time \( \Delta t \).
\[
\Delta m = f \Delta t
\]
The functions \( f \) and \( df/dt \) when starred should be modified by substituting \( t - r \) for \( t \). For example, if \( t = t_0 \), \( df/dt = 3t \), then \( f^* = (t - r)^3 \), \( df/dt \) \( = 3(t - r) \). The time interval \( \Delta t \) is the difference in times that the particles on the right and left sides of \( \Delta x \) entered the inlet of the acceleration chamber and is given by (see Fig. 1)
\[
(2.8)
\]
The mass density \( \rho \) is defined as the mass per unit length, yielding
\[
\rho = \frac{dm}{dx} = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{\Delta m}{\Delta x}
\]
Equation (2.9) may be demonstrated to be valid by using the equation of continuity
\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0
\]
Differentiating Eq. (2.9):
\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial t} = (1 - \frac{\partial \rho}{\partial t}) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial \rho}{\partial t}
\]
Inserting Eq. (2.11) into the equation of continuity gives
\[
0 = (1 - \frac{\partial \rho}{\partial t}) \frac{\partial v}{\partial x} + \frac{\partial v}{\partial x} \frac{\partial \rho}{\partial t}
\]
The easiest way to show that Eq. (2.12) is valid is to introduce a variable \( T \).
\[
T = t - r
\]
\[
(2.13)
\]
\[
(2.14)
\]
As time elapses, this particle moves down the chamber. \( r \) increases uniformly with \( t \). Hence, the total differential of \( T \) equals zero.
\[
(2.15)
\]
Substituting Eq. (2.14) into Eq. (2.15) gives
\[
(2.16)
\]
As Eq. (2.13) was for an arbitrary \( x \), the partial differential of Eq. (2.16) with respect to \( x \) must equal zero.
\[
0 = \frac{\partial^2 r}{\partial x \partial t} + \frac{\partial r}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial^2 r}{\partial x \partial t}
\]
(2.17)
Considering Eq. (2.16) and Eq. (2.17) it is apparent that Eq. (2.12) is valid.

The mass density \( \rho \) is defined as the mass per unit length, yielding
\[
\rho = \frac{dm}{dx} = \lim_{dx \to 0} \frac{\Delta m}{dx} = \frac{\Delta m}{dx}
\]
Equation (2.9) may be demonstrated to be valid by using the equation of continuity
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\]
The easiest way to show that Eq. (2.12) is valid is to introduce a variable \( T \).
\[
T = t - r
\]
\[
(2.13)
\]
\[
(2.14)
\]
where \( v_e \) is the velocity of the mass in the ac-
celeration chamber at \( x \) and \( \rho \) has been set equal
to \( -\frac{dm}{dt} \). Combining Eq. (3.2) and Eq. (3.3) gives
the mass flow rate leaving the rocket
\[
\frac{dm}{dt} + \rho v_e \left( \frac{dx}{dt} - v_e \right) = 0
\]  
(3.4)

The momentum \( P \) of the system (i.e., the momentum of all the particles that were initially
in the rocket) can be expressed as the sum of the
momentum of the particles in the rocket proper plus
the particles that have already left the rocket. The
momentum shall be expressed in an inertial coordinate system which was at rest with respect
to the rocket before it started to move.

The rocket proper, denoted with subscript \( p \),
consists of a group of particles which are loosing
mass at point \( x_p \). The center of mass \( x_c \) of the rocket proper varies with time and is assumed to
be specified. Appendix A contains a derivation
(Eq.(A.9)) of the momentum of this body
\[
p_p = u_m \rho v_e \frac{dx}{dt} = \int x_p (u - v_e) \, dx
\]  
(3.5)

where \( u \) is the velocity of a point fixed to the
acceleration chamber of the rocket. Using the mass
density expression (Eq.(2.9)), the momentum of the
mass in the acceleration chamber is
\[
p_{ac} = \int x_e (u - v_e) \, dx
\]  
(3.6)

Using the mass flow rate (Eq. (3.4)) out of the rocket,
the momentum \( p_e \) of the mass which has left the
rocket can be expressed as
\[
p_e = \int \rho v_e \left( \frac{dx}{dt} - v_e \right) \, dt
\]  
(3.7)

where \( t_e \) is the time at which mass first starts to
leave the rocket. Combining Eqs. (3.5), (3.6) and
(3.7) results in the total momentum of the original
mass of the rocket,
\[
p = u_m \rho v_e \frac{dx}{dt} + \int x_{e_1} \rho (u - v_e) \, dx + \int x_{s_1} \rho (u - v_e) \, dx
\]  
(3.8)

As the total momentum of the system is
conserved, Eq. (3.8) may be differentiated and set
equal to zero. The result is
\[
o = m_r \frac{du}{dt} + u \frac{dm}{dt} - \frac{d^2 m_r \rho v_e}{dt^2} + \frac{d^2 m_r \rho v_e}{dt^2} x_s - \frac{dx}{dt} = 0
\]  
(3.9)

Simplifying the above equation using Eqs. (3.1) and
(3.3) gives
\[
o = m_r \frac{du}{dt} - \frac{d^2 m_r \rho v_e}{dt^2} + \frac{d^2 m_r \rho v_e}{dt^2} x_s - \frac{dx}{dt} = \int \frac{\rho v_e}{dt} \, dx
\]  
(3.10)

The mass of the rocket equals the mass of the rocket
at time \( t_e \) minus what has left the rocket
\[
m_r = m - \int \rho (v_e - \frac{dx}{dt}) \, dt
\]  
(3.11)

The mass of the rocket proper equals
\[
m_p = \int (v_e - \frac{dx}{dt}) \, dt
\]  
(3.12)

The following equation reduces to the continuity
equation when expanded
\[
\frac{\partial v_e}{\partial t} = \frac{\partial^2 v_e}{\partial x^2} \left( \frac{1}{\rho v_e} \frac{dx}{dt} \right)
\]  
(3.13)

Combining Eqs. (3.10)-(3.13) and (2.9) yields the
equation of motion for the variable mass flow-
variable exhaust velocity rocket
\[
o = \left[ M - \int \rho (v_e - \frac{dx}{dt}) \, dt \right] \frac{du}{dt} + (x_{cr} - x_e) \frac{du}{dt}
\]  
(3.14)

where \( f_{cp} \) and \( (\partial v_e/\partial x) \) are computed at \( x=x_{e_1} \). This
is the nearest one can come to deriving a general
expression in the form of Eq. (1.1). The only quantity
in the above equation which is not contained in Eq.
(1.1) is \( p_se \), which can be explicitly obtained only for
certain functions of \( v \) (see Section VIII for an
example).

The above analysis is consistent with the results
obtained by those who treat the variable mass
flow-variable exhaust velocity rocket as a variable
mass problem in which the center of mass of the
whole rocket plays the leading role. The main result
of these approaches is equation B-10, which can be
expressed in terms of the nomenclature of Fig.
2.

\[
o = m_r \frac{du}{dt} + (x_e - x_{cr}) \frac{d^2 m_r}{dt^2} + (v_e - \frac{dx}{dt}) \frac{d^2 m_r}{dt^2}
\]  
(3.15)

where \( m_r = m \), \( v_e = v_{cr} \), \( x_{cr} = x_{cr} \), \( x_e = x_e \), and \( v_e = v_1 \).
\( du/dt \) is the acceleration of the center of mass of the rocket in an inertial coordinate system and
\( x_{cr} \) is the position of the center of mass of the rocket
(see Fig.2).

It will now be shown that Eq. (3.10) is equivalent
to Eq.(3.15). The center of mass of the rocket is
obtained from
Differentiating results in

\[ \frac{d}{dt} \frac{m_r x_{cr}}{m_p x_{cp}} + \int_{x}^{x_e} \frac{dx}{x} \frac{dm}{dx} x_s \]  \hspace{1cm} (3.16)

Differentiating results in

\[ \frac{dm}{dt} x_{cr} = \frac{dm}{dt} x_{cp} + x_e \frac{dm}{dt} x_s + \int_{x}^{x_e} x x_s \frac{dx_s}{dx} \]  \hspace{1cm} (3.17)

The following identity can be established by integration by parts using the continuity equation.

\[ \int x \frac{dx}{dx} x_s \frac{dx_s}{dx} + \int \nu v x dx \]  \hspace{1cm} (3.18)

Combining Eqs. (3.4), (3.17) and (3.18) and setting \( \rho_0 v^2 \) gives

\[ \frac{d}{dt} \frac{x_{cr}}{x_{cp}} + \frac{d}{dt} x_s + \int \nu v x dx \]  \hspace{1cm} (3.19)

Differentiating the above equation results in

\[ \frac{d^2 m}{dt^2} x_{cr} + \frac{d^2 m}{dt^2} x_{cp} + \frac{d}{dt} x_s - \frac{d^2 m}{dt^2} x_s + \int x \frac{dx}{dx} x_s \frac{dx_s}{dx} + \int \nu v^2 x dx \]  \hspace{1cm} (3.20)

Combining Eqs. (3.4), (3.10) and (3.20) gives

\[ 0 = M \frac{dx}{dt} - m p \frac{d^2 x_s}{dt^2} + (x_e - x_s) \frac{d^2 m}{dt^2} x_s + \int v \frac{dx}{dx} \frac{dm}{dx} x_s \]  \hspace{1cm} (3.21)

The inertial acceleration (\( du_{cr}/dt \)) of the center of mass of the rocket is

\[ \frac{du_{cr}}{dt} = \frac{du}{dt} - \frac{d^2 x_{cr}}{dt^2} \]  \hspace{1cm} (3.22)

Combining Eqs. (3.21) and (3.22) yields the desired result, Eq. (3.15).

Equation (3.10) is also consistent with the approach taken in references 5 and 6. Miele applies Newton's Second Law of Motion to the rocket regarded as a continuum in the following manner.

\[ \frac{d}{dt} \left( \int \frac{dx}{dt} \right) dm = m_r \frac{du}{dt} - \int m_r \frac{dm}{dt} dx \]  \hspace{1cm} (3.23)

where \( F \) is the external force, \( du/dt \) is the acceleration of the origin (point fixed on the rocket), \( dv/dt \) is the relative acceleration with respect to the origin and the integral is to be summed over the mass of the rocket. Although Miele extends his approach further, it is easiest to derive Eq. (3.10) directly from Eq. (3.23).

Setting the external force equal to zero and separating the integral into an integral of the mass in the rocket proper and an integral of the mass in the acceleration chamber results in

\[ 0 = m_r \frac{du}{dt} - \int \frac{d}{dt} \frac{dm}{dx} x_s + \int \frac{d}{dt} \frac{dm}{dx} x_e \]  \hspace{1cm} (3.24)

in terms of the notation in Fig. 1 and where \( \rho_p \) is the density in the rocket proper (for \( 0 < x < x_e \)). Appendix A contains a derivation of the equation

\[ \frac{dm}{dt} x_{cr} = \frac{dm}{dt} x_{cp} + \int x \frac{dx}{dx} x_s \frac{dx_s}{dx} + \int \rho_p v x dx \]  \hspace{1cm} (3.25)

Differentiating, setting \( \rho_p v^2 \) equal to zero at \( x = 0 \), and using Eq. (3.13) gives

\[ \frac{d^2 m}{dt^2} x_{cr} + \frac{d^2 m}{dt^2} x_s + \int \rho_p v x dx + \int \rho_p v x dx \]  \hspace{1cm} (3.26)

where it has been assumed that \( \rho_p v^2 \) is equal to zero at \( x = 0 \), \( x_s \) and \( x_e \). Equation (3.24) can be obtained by combining Eqs. (3.10), (3.13), and (3.26).

IV. Derivation of Equation of Motion Before Mass Leaves the Acceleration Chamber

The equation of motion for the time interval before the mass starts to leave from the acceleration chamber is found in a similar manner as was the main equation of motion (Eq.(3.14)). Themomentum expression can be obtained from Eq. (3.6) after eliminating the term representing the expelled mass

\[ p = m_p \frac{d^2 x_{cr}}{dt^2} + \frac{d}{dt} \left( \int x \frac{dx}{dx} \frac{dm}{dx} x_s \right) \]  \hspace{1cm} (4.1)

where \( x_1 \) is the point reached by the first particle that entered the acceleration chamber. \( x_1 \) is obtained by integrating Eq.(2.1). The mass of the rocket proper is given by Eq.(3.12). Differentiating Eq.(4.1), setting equal to zero and using Eq.(2.10) and the derivative of Eq.(3.12) results in

\[ 0 = M \frac{dx}{dt} - \frac{d^2 m}{dt^2} x_{cr} - \frac{d^2 m}{dt^2} x_s - \frac{d^2 m}{dt^2} x_e - \int \frac{dx}{dx} \frac{dm}{dx} x_s - \int \frac{dx}{dx} \frac{dm}{dx} x_e \]  \hspace{1cm} (4.2)

where \( v_1 \) is the velocity at \( x_1 \). Combining Eqs.(3.12) (3.13) and (4.2) gives

\[ 0 = M \frac{dx}{dt} + (x_{cp} - x_s) \frac{dx}{dt} + 2 t \frac{dx}{dt} \]  \hspace{1cm} (4.3)
The only difference between Eq. (3.14) and the above equation besides the obvious ones of interchanging \( x_1 \) for \( x_e \) and the dropping of the integral in the first term is that in the above equation the derivative of \( x_1 \) with respect to time is \( v_1 \), whereas in Eq. (3.14) \( x_e \) is a specified function of time.

V. Derivation of Equation of Motion After Mass Stops Entering The Acceleration Chamber

Once the mass stops entering the acceleration chamber, the mass remaining in the rocket proper remains constant. If \( t_1 \) represents the time that the last particle entered the chamber, at time \( t \) it would have reached point \( x_2 \) found by integrating Eq. (2.1). The integral representing the momentum in the chamber need cover only the range from \( x_2 \) to \( x_e \). Equation (3.8) for this case becomes

\[
\rho = m_r \left( u - \frac{dx_{CP}}{dt} \right) + \int_{x_2}^{x_e} \rho (u - v) dx + \int_{x_2}^{x_e} (v_e - \frac{dx_e}{dt}) (u - v_e) dt
\]

where:

\[
m_r = M \quad \text{if no mass has left the rocket.}
\]

\[
m_r = M - \int_{t_1}^{t} \int_{x_e}^{x_e} (v_e - \frac{dx_e}{dt}) dt
\]

\[
x_a \quad \text{equals} \quad x_e \quad \text{if mass is currently entering the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the inlet.}
\]

\[
x_b \quad \text{equals} \quad x_e \quad \text{if mass is currently leaving the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the exit.}
\]

\[
t_a \quad \text{equals} \quad t \quad \text{if} \quad t < t_1; \quad \text{otherwise} \quad t_a = t_1.
\]

The points \( x_a \) and \( x_b \) when not equal to \( x_s \) or \( x_e \) respectively, are functions of time; they move with a velocity equal to the velocity in the acceleration chamber at that point.

VII. Special Cases

Several rocket models lead to simplified density expressions and corresponding simplified equations of motion. In the following examples both \( x_{CP} \) and \( x_e \) are assumed to be independent of time and \( x_s \) is assumed to occur at \( x_{CP} \). In addition, the reference point in the rocket is assumed to coincide with \( x_s \).

A. Exhaust Velocity that Allows Separation of Variables

Assume that \( v \) is given by

\[
v = ab = a(t) b(x)
\]

Combining Eqs. (3.11), (3.12), (3.13) and (5.3) gives

\[
0 = \left[ M - \int_{t_1}^{t} \left( \frac{dx_{CP}}{dt} \right) \left( v_e - \frac{dx_e}{dt} \right) \right] \frac{dx}{dt}
\]

\[
\frac{1}{2} \left[ \frac{d^2 x_{CP}}{dt^2} - \int_{0}^{t} \frac{d^2 x_{CP}}{dt^2} - \int_{t_a}^{t} \frac{dx_{CP}}{dt} \frac{dx_e}{dt} + \frac{dx}{dt} \frac{dx_e}{dt} \right] dx
\]

The differences between Eqs. (3.14) and (5.4) are the changes in the limits on the integrals and the dropping of the terms containing \( x_s \) and \( f \).

VI. General Equation Of Motion

By comparing the equations of motion (Eqs. (3.14), (4.4) and (5.4)) for the different time periods considered in the previous three sections, the following general equation of motion for the variable mass flow-variable exhaust velocity rocket can be obtained

\[
0 = m_r \frac{du}{dt} + \left( \frac{dx_{CP}}{dt} - x_a \right) \frac{dx_e}{dt} + \frac{dx_{CP}}{dt} \frac{dx_e}{dt}
\]

\[
\frac{1}{2} \left[ \frac{d^2 x_{CP}}{dt^2} - \int_{x_a}^{x_e} \frac{d^2 x_{CP}}{dt^2} - \int_{0}^{t} \frac{dx_{CP}}{dt} \frac{dx_e}{dt} + \frac{dx}{dt} \frac{dx_e}{dt} \right] dx
\]

where:

\[
m_r = M \quad \text{if no mass has left the rocket.}
\]

\[
m_r = M - \int_{t_1}^{t} \int_{x_e}^{x_e} \left( v_e - \frac{dx_e}{dt} \right) dt
\]

\[
x_a \quad \text{equals} \quad x_e \quad \text{if mass is currently entering the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the inlet.}
\]

\[
x_b \quad \text{equals} \quad x_e \quad \text{if mass is currently leaving the acceleration chamber. Otherwise it is the point occupied by the mass in the chamber closest to the exit.}
\]

\[
t_a \quad \text{equals} \quad t \quad \text{if} \quad t < t_1; \quad \text{otherwise} \quad t_a = t_1.
\]

B. Constant Exhaust Velocity Rocket

The constant exhaust velocity rocket implies that the exhaust velocity does not change with time while the mass flow is variable. Hence,

\[
v = b = b(x)
\]

Differentiating Eq. (2.4) with a equal to one gives

\[
\frac{\partial v}{\partial x} = \frac{1}{b} \frac{dx}{dt}
\]
Eqs. (2.9) and (3.14) reduce to

\[ \tau = \int \frac{t}{b} \left( 1 - \frac{x - b}{c} \right) dt \]

(7.7)

where \( \tau = \int \frac{x}{b} \) and \( t_s = \frac{M}{v_c} \text{ where } v_c \text{ is constant.} \)

C. Constant Mass Flow Rate Rocket

The constant mass flow rate rocket implies that the mass flow into the acceleration chamber does not change with time while the exhaust velocity does change with time. Hence,

\[ f = c \]  

(7.8)

where \( c \) is a constant. Eqs. (2.9) and (3.14) reduce to

\[ r = \frac{t}{b} \]

(7.9)

D. Steady State Rocket

Both the mass flow rate and exhaust velocity are independent of time for the steady state rocket. If

\[ f = c, \quad v = b = b(x) \]  

(7.10)

then Eqs. (2.9) and (3.14) reduce to

\[ r = \frac{t}{b} \]

(7.11)

where \( b_e \) is based on \( x = x_e \). Integration of Tsiolkovski's equation gives

\[ u - u_s = b_e \log \left( \frac{M}{M_c} \right) \]

(7.12)

where \( u_s \) is the velocity at time \( t_s \).

VIII. Numerical Example

This section contains an example of when the variables in \( v \) are separable and such that Eq. (2.4) can be integrated. The same assumptions concerning \( x_{\tau_p} \) and \( x_\tau \) are made as was assumed in Section VII. Let the mass flow rate into the acceleration chamber and the velocity be specified by

\[ f = t, \quad v = x^n \]  

(8.1)

The acceleration of the mass in the acceleration chamber is given by

\[ \frac{dv}{dt} = \frac{b}{c} v^n + \frac{a}{c} v^{n-1} \]  

(8.2)

Carrying out the integration of Eq. (2.4) results in

\[ \tau = \left[ t^2 - 2x^{1-n} / (1-n) \right]^{1/2} \]  

(8.3)

Eliminating \( x \) between Eqs. (8.2) and (8.3) gives

\[ \frac{dv}{dt} = \frac{1}{2} \left[ t^2 - (t - \tau)^2 \right]^{1/2} v + \frac{a}{c} v^{n-1} \]  

(8.4)

Hence, to avoid infinite accelerations at the inlet (i.e. when \( \tau \) goes to zero), \( n \) must lie in the range \( 1/2 < n < 1 \).

Differentiating Eq. (8.3) yields

\[ \frac{dx}{dt} = \frac{x^n}{c} \left( 1 - \frac{x - b}{c} \right) \]  

(8.5)

The density (Eq. (2.9) and mass flow rate are

\[ r = \frac{t}{b}, \quad \rho v = t \]  

(8.6)

Equation (4.4) reduces to

\[ 0 = \frac{M}{v_c} \]  

(8.7)

Integrating\( u - u_s = b_e \log \left( \frac{M}{m} \right) \) results in

\[ u = \frac{1}{M} \]  

(8.8)

Inserting into Eq. (8.7) and integrating gives

\[ u = \frac{1}{M} \left[ \frac{1}{2} - \frac{n}{3-n} \right] \left( \frac{1}{2} x - x_e^n \right) \]  

(8.9)

This equation is valid until mass stops entering the chamber at time \( t_s \). This time is found by solving Eq. (8.8) for \( t_s \) after substituting \( x_e \) for \( x_s \).

For times in excess of \( t_s \), Eq. (3.14) reduces to

\[ 0 = (M - t_s^2/2 + x_s^n) \frac{dv}{dt} - x_s^n \]  

(8.11)

Integrating

\[ u - u_s = x_e \left( a - b/c \right) + 2x_e^n \left( x_e + ax/c + bt/bc/2 \right) \]  

(8.12)

where

\[ a = \log \left( \frac{a + t_s}{c - t_s} \right) \]
\[ b = \log \left( \frac{a + t_s}{c - t_s} \right) \]

This equation is valid until mass stops entering the chamber at time \( t_s \). For greater times the equation of motion is found using Eq.(5.4)
where \( x_2 \) is obtained by integrating the velocity.

\[
x_2 = \left( \frac{1-n}{n} \right) \left( \frac{a^2 - t_1^2}{2} \right)^{1/(1-n)}
\]

As Eq. (8.13) is not readily integrable, it can be solved numerically on a computer. Figure 3 is a plot of \( u \) verses \( t \) for the following conditions

\[
v = x^n t, \quad f = t, \quad n = 6, \quad M = 1, \quad m_p = 0.75, \quad m_p = 0.25
\]

The following times are then derived from Eq. (3.12): \( t_s = \sqrt{5}, \frac{t_1}{s} = 1.5 \).

The length of the acceleration chamber equals \( 0.00316 \) and is found by setting \( x_1 \) and \( t_1 \) equal to \( x_2 \) and \( t_2 \) in Eq. (8.8). The time that the last particle leaves the acceleration chamber equals \( \sqrt{2} \) and is determined by setting \( x_2 \) and \( t_2 \) equal to \( x_0 \) and \( t_0 \) in Eq. (8.14). The distance the rocket has moved is found by numerically integrating the equation \( \frac{dy}{dt} = u \) and is also shown in Fig. 3.

**IX. Acknowledgment**

The author would like to express his appreciation to Jim Potter for having derived the mass density expression.

**X. References**


**Appendix A**

**Deprivation of the Momentum of A Variable Mass Body In Terms Of Its Center Of Mass**

The derivation of the momentum of a variable mass body in terms of its center of mass can be based on a discontinuous medium or on a continuous medium. To be consistent with the general approach taken in this paper, a body with a continuous medium will be used. The body is assumed to lose mass at both point \( x_1 \) and \( x_2 \) with \( x_1 \) and \( x_2 \) moving with time (see Fig. 4). The mass leaves with the local velocity \( v \) of the mass in the medium.

The location \( x_C \) of the center of mass is given by

\[
x_C = \int_0^{x_1} x \, dx + \int_{x_1}^{x_2} x \, dx
\]
of Fig. 1 and Eq. (A.5) with \( \rho_0 (dx_2/dt-v_2) \) set equal to zero, Eqs. (A.6) and (A.8) reduce to

\[
\frac{dx_c}{dt} + (x_c - x_0) \frac{dm}{dt} = \int_0^t \rho \, dx + \int_0^t \rho \, v \, dx \\
p = m \frac{dx_c}{dt} + \int_0^t \frac{x_c}{x_0} \, dx + \int_0^t \frac{x_c}{x_1} \, dx \quad \text{(A.9)}
\]

Appendix B.  
Equation of Motion For A Variable Mass Body 
In Terms Of Its Center Of Mass

This appendix contains a derivation of an equation which has been frequently mentioned in the literature as the equation of motion of a variable mass body (of which the rocket is one example). The derivation follows the approach taken in the main section of this paper (differentiating the momentum of the system and setting the resulting expression equal to zero). The body considered is illustrated in Fig. 5.

The mass within the body is

\[
m = \int_0^t \rho \, dx
\]

(B.1)

Assuming that the body loses mass at point \( x_1 \) which varies with time, Eq. (B.1) can be differentiated using Eq. (2.10), resulting in

\[
\frac{dm}{dt} = x_1 \left( \frac{dx_1}{dt} - v_1 \right)
\]

(B.2)

where \( v_1 \) is the velocity of the mass being exhausted.

The momentum of the body plus the momentum of the mass that has left the body equals

\[
p = \int_0^t \rho \, (u - v) \, dx + \int_0^t \rho \, v \, dx \quad \text{(B.3)}
\]

where \( u \) is the velocity of the body and \( t_0 \) is the time that the body started to lose mass. Differentiating Eq. (B.3) using Eqs. (2.10) and (B.2) gives

\[
\frac{dx_1}{dt} + o = m \frac{dx_1}{dt} - x_1 v_1 \quad \int_0^t \frac{x_1}{x_0} \, dx \quad \text{(B.4)}
\]

The location \( x_c \) of the center of mass is given by

\[
x_c = \int_0^t x \, dx
\]

(B.5)

Differentiating gives

\[
\frac{dx_c}{dt} = \int_0^t \frac{x_c}{x_0} \, dx + \int_0^t \frac{x_c}{x_1} \, dx \quad \text{(B.6)}
\]

where \( dx_c/dt \) is the velocity of the center of mass with respect to the body. Using Eqs. (3.18), (B.2) and (B.6) results in
Differentiating once more gives
\[ \frac{d^2 x}{dt^2} = x_1 \frac{d^2 m}{dt^2} + (\frac{dm}{dt} + x_1 v_1) \frac{dx_1}{dt} + \int_0^{x_1} \frac{\partial \nu}{\partial x} \, dx \] (B.8)

The acceleration of the center of mass point is
\[ \frac{d}{dt} \frac{d^2 x}{dx^2} = \frac{d}{dt} \frac{d}{dt} \frac{dx}{dx} \] (B.9)

Combining Eqs. (B.2), (B.4), (B.8) and (B.9) results in
\[ 0 = m \frac{d u}{dt} + (x_1 - x_c) \frac{d^2 m}{dt^2} + (v_1 + \frac{dx_1}{dt} + 2 \frac{dx_c}{dt}) \frac{dm}{dt} \] (B.10)

Introducing the notation \( r_C, r_e \) and \( v \) (absolute velocity) with the following equations (see Fig. 5):
\[ x_1 - x_c = r_C - r_e, \quad v_1 = \frac{dr}{dt} - v \]
\[ \frac{dx_1}{dt} = \frac{dr}{dt} - \frac{dr_e}{dt}, \quad \frac{dx_e}{dt} = \frac{dr}{dt} - \frac{dr_c}{dt} + u_c \] (B.11)

results in the equation of motion as usually specified.
\[ 0 = \frac{dm}{dt} + (r_C - r_e) \frac{d^2 m}{dt^2} + (2 \frac{dr}{dt} - \frac{dr_c}{dt} - v) \frac{dm}{dt} \] (B.12)
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