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Application of Conservative Finite-Difference Methods to Advection Problems

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INTRODUCTION

In the last decade, with the advent of high speed computers, numerical calculations using finite-differencing techniques applied to partial differential equations has been of great interest to gas dynamicists and meteorologists. This has been brought about by the accuracy, capacity and speed that electronic computers offer in the solution of the complex partial differential equations describing the motion of fluids.

Finite-difference equations may be constructed and used in various ways depending on accuracy, stability and important physical considerations, e.g., conservation laws. A "conservation law form" of a system of differential equations may take the form

$$\frac{\partial}{\partial t} \overline{E} + \frac{\partial}{\partial x} \overline{F} = 0 \tag{1}$$

where F and E are conservative variables. Another form of equation (1) may also be considered. This is the "advective form" of equation (1) which is

$$\frac{\partial}{\partial t} \overline{E} + [R] \frac{\partial}{\partial x} \overline{F}_1 = 0 \tag{2}$$

where $\overline{F}_1$ and $\overline{E}$ are vectors and R is a matrix.

In development and use of finite-differencing techniques, meteorologists have favored the advective form of the partial differential equations. This is partly due to the form of the advection equations which are encountered in meteorological studies. Molenkamp (9) and Crowley (4) have extensively applied advective differencing methods to meteorological model problems and have examined the results. However, they have not been able to obtain accurate solutions to the problems considered, except when relatively complicated and time-consuming fourth-order accurate techniques were employed. On the other hand, gas dynamicists have particularly been interested in the application of differencing techniques to "conservative law form" of partial differential equations. Kutler (5) and Anderson and Vogel (1) have successfully applied conservative techniques to sonic-edged, conical, wing-body combinations and flow about a rectangular wing moving supersonically.

It is the purpose of this research to investigate the possibility of application of the most recent conservative and widely applied numerical techniques in gas dynamics to problems encountered in meteorological computations. Solutions of the advection equation are obtained using conservative differencing methods common to gas dynamics. These results are compared to those obtained by Molenkamp, who differenced the advection equation directly. The comparison shows that better results are obtained where conservative form of governing equation is used. This is in agreement with the results obtained by Crowley (4). In addition, better results are obtained with lower order conservative methods as compared with higher order differencing applied to the advection equation.

PROBLEM DEFINITION

In a two-dimensional rectangular coordinate system, the differential advection equation is

$$\frac{\partial A}{\partial t} + u\,\frac{\partial A}{\partial x} + w\,\frac{\partial A}{\partial z} = 0 \tag{3}$$

where A is the quantity being advected. Velocity components u and w are respectively in the x and z directions and t is time. If a steady velocity field is chosen then velocity components are no longer functions of time. The above equation becomes linear and an analytical solution is then possible. Considering the motion of the fluid to be a rotation with constant angular velocity, Ω, an equivalent form of equation (3) in cylindrical coordinates becomes

$$\frac{\partial A}{\partial t} + \Omega\,\frac{\partial A}{\partial \theta} = 0 \tag{4}$$

when θ is the angular coordinate and radial velocity is zero. Equation (4) is the wave equation and its analytical solution is found to be

$$A(r,\theta,t) = A_0(r,\theta-\Omega t) \tag{5}$$

where r is the radial distance from the axis of rotation and A_0 is the initially given distribution of A at time zero. Equation (5) shows that the solution of the wave equation (4) is an angular displacement of the initial distribution A_0.

A conservative form of equation (3) may be obtained considering an incompressible flow. Then the continuity equation is

$$\nabla . \bar{q} = 0 \tag{6}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{7}$$

multiplying each side by A

$$A \frac{\partial u}{\partial x} + A \frac{\partial w}{\partial z} = 0 \tag{8}$$

Now add equation (8) to equation (3), or

$$\frac{\partial A}{\partial t} + u \frac{\partial A}{\partial x} + w \frac{\partial A}{\partial z} + \left[A \frac{\partial u}{\partial x} + A \frac{\partial w}{\partial z}\right] = 0 \tag{9}$$

which may be written in the following form

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x} (Au) + \frac{\partial}{\partial z} (Aw) = 0 \tag{10}$$

This is the conservative form of equation (3). The differencing techniques that are described in the next section are applied to the general form of this equation.

NUMERICAL TECHNIQUES

In the following section those differencing methods considered in the present paper are explained. The form of each equation is given when applied to the general conservative hyperbolic partial differential equation in two dimensions.

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \tag{11}$$

The modified equation for each technique is obtained by applying these methods to the one-dimensional wave equation

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0 \tag{12}$$

In addition to the above, the stability criterion for each technique is given as obtained from the linear stability analysis.

Brailovskaya Method

The first-order predictor-corrector scheme described below was devised by I. Y. Brailovskaya (2) based on central differencing. When Brailovskaya's technique is applied to equation (11), the result is

$$\overline{E}^{n+1}_{j,k} = E^{n}_{j,k} - \frac{\Delta t}{2\Delta x}\left(F^{n}_{j+1,k} - F^{n}_{j-1,k}\right) - \frac{\Delta t}{2\Delta y}\left(G^{n}_{j,k+1} - G^{n}_{j,k-1}\right)$$

$$E^{n+1}_{j,k} = E^{n}_{j,k} - \frac{\Delta t}{2\Delta x} (\bar{F}^{n+1}_{j+1,k} - \bar{F}^{n+1}_{j-1,k})$$

$$- \frac{\Delta t}{2\Delta y} (\bar{G}^{n+1}_{j,k+1} - \bar{G}^{n+1}_{j,k-1}) \qquad (13)$$

The modified partial differential equation (6) for Brailovskaya's method may be found by applying this scheme to one-dimensional wave equation (12)

$$u_t + cu_x = \frac{1}{2} c\nu \, \Delta x \, u_{xx}$$

$$+ \ldots\ldots \qquad (14)$$

where ν is the Courant number.

Brailovskaya's technique is easy to program because of the simplicity of the structure of the scheme and the similarity of the differences in both predictor and corrector. The latter allows the programmer to define only one set of boundary conditions for both of the above steps.

This first-order technique is stable under the following conditions.

$$\left|\sigma_{max} \frac{\Delta t}{\Delta x}\right| \leqslant 1 \;, \quad \left|\sigma_{max} \frac{\Delta t}{\Delta y}\right| \leqslant 1 \qquad (15)$$

where σ_{max} is the maximum eigenvalue of the hyperbolic system under consideration.

For better accuracy of the computation, the modified equation above requires the mesh size to be small such that it decreases the magnitude of the second-order error.

Brailovskaya's technique is not widely used because of its low order of accuracy and also because of the predictor-corrector sequence form which increases the computation time to that of the second order techniques.

Lax-Wendroff Method

A second-order differencing scheme was derived by Lax and Wendroff (7) for which the stability criterion is defined by

$$\left|\sigma_{max}\frac{\Delta t}{\Delta x}\right| \leqslant 1 \ , \quad \left|\sigma_{max}\frac{\Delta t}{\Delta y}\right| \leqslant 1 \tag{16}$$

where again σ_{max} is the maximum eigenvalue of the hyperbolic system under consideration. This technique when applied to equation (11) yields

$$\begin{aligned}
E^{n+1}_{j,k} = E^{n}_{j,k} &- \frac{\Delta t}{2\Delta x}(F^n_{j+1,k} - F^n_{j-1,k}) - \frac{\Delta t}{2\Delta y}(G^n_{j,k+1} - G^n_{j,k-1}) \\
&+ \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 \left[A'^{n}_{j+\frac{1}{2},k}(F^n_{j+1,k} - F^n_{j,k}) - A'^{n}_{j-\frac{1}{2},k}(F^n_{j,k} - F^n_{j-1,k})\right] \\
&+ \frac{1}{2}\left(\frac{\Delta t}{\Delta y}\right)^2 \left[B'^{n}_{j,k+\frac{1}{2}}(G^n_{j,k+1} - G^n_{j,k}) - B'^{n}_{j,k-\frac{1}{2}}(G^n_{j,k} - G^n_{j,k-1})\right] \\
&+ \frac{1}{8}\frac{(\Delta t)^2}{\Delta x . \Delta y}\Big[A'^{n}_{j+1,k}(G^n_{j+1,k+1} - G^n_{j+1,k-1}) - A'^{n}_{j-1,k+1} - G^n_{j-1,k-1}) \\
&+ B'^{n}_{j,k+1}(F^n_{j+1,k+1} - F^n_{j-1,k+1}) - B'^{n}_{j,k-1}(F^n_{j+1,k-1} - F^n_{j-1,k-1})\Big]
\end{aligned} \tag{17}$$

where

$$A' = \frac{\partial F}{\partial E} \quad , \quad B' = \frac{\partial G}{\partial E} \tag{18}$$

and

$$A'^{n}_{j+\frac{1}{2},k}(E) \equiv A'\left[\frac{1}{2}(E^{n}_{j+1,k}+E^{n}_{j,k})\right]$$

$$B'^{n}_{j+\frac{1}{2},k}(E) \equiv B'\left[\frac{1}{2}(E^{n}_{j+1,k}+E^{n}_{j,k})\right] \tag{19}$$

The Lax-Wendroff method applied to the one-dimensional wave equation forms the following modified partial differential equation

$$u_t + cu_x = -\frac{1}{6}\, c\, (1-\nu^2)\Delta x^2\, u_{xxx} + \ldots\ldots \tag{20}$$

The form of the above equation confirms the order of accuracy of the Lax-Wendroff technique. One may note that at a Courant number equal to unity, the above equation reduces to the exact wave equation (12) and thus provides an exact solution.

MacCormack Method

MacCormack (8) developed a second-order predictor-corrector sequence for use in studies involving hypervelocity impact cratering. When applied to equation (11), it yields

$$\overline{E}^{n+1}_{j,k} = E^n_{j,k} - \frac{\Delta t}{\Delta x}\left(F^n_{j+1,k} - F^n_{j,k}\right) - \frac{\Delta t}{\Delta y}\left(G^n_{j,k+1} - G^n_{j,k}\right)$$

$$E^{n+1}_{j,k} = \frac{1}{2}\Big[E^n_{j,k} + \overline{E}^{n+1}_{j,k} - \frac{\Delta t}{\Delta x}\left(\overline{F}^{n+1}_{j,k} - \overline{F}^{n+1}_{j-1,k}\right)$$

$$- \frac{\Delta t}{\Delta y}\left(\overline{G}^{n+1}_{j,k} - \overline{G}^{n+1}_{j,k-1}\right)\Big] \qquad (21)$$

It is interesting to note that this technique is a preferential scheme using a forward predictor and backward corrector. The backward predictor and forward corrector version of MacCormack's technique is also examined and results are reported in this paper. MacCormack's differencing scheme has been applied to gas dynamic problems in recent years and has resulted in accurate solutions comparable to better second-order methods (1, 5).

In this case, the stability bound is again found to be

$$\left|\sigma_{max}\frac{\Delta t}{\Delta x}\right| \leq 1 \quad , \quad \left|\sigma_{max}\frac{\Delta t}{\Delta y}\right| \leq 1 \qquad (22)$$

The following modified partial differential equation is obtained when MacCormack's technique is applied to equation (12)

$$u_t + cu_x = -\frac{1}{6}\, c\, \Delta x^2\, (1-\nu^2)\, u_{xxx} \quad + \ldots\ldots \qquad (23)$$

Note again that at Courant number of unity the above equation reduces to equation (12).

Rusanov Method

In 1969, Rusanov (10) and Burstein and Mirin (3) separately developed a third-order accurate scheme which has been of great interest to gas dynamicists where high-speed computers are available. This three-level predictor-corrector technique provides accurate solutions when applied to gas dynamic equations. The Rusanov method applied to the general equation (11) results in

$$E^{(1)}_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{4}\left[E^n_{j+1,k+1}+E^n_{j,k+1}+E^n_{j+1,k}+E^n_{j,k}\right]$$

$$- \frac{1}{6}\left\{\frac{\Delta t}{\Delta x}\left[F^n_{j+1,k+1}+F^n_{j+1,k}-F^n_{j,k+1}-F^n_{j,k}\right]\right.$$

$$\left.+ \frac{\Delta t}{\Delta y}\left[G^n_{j+1,k+1}+G^n_{j,k+1}-G^n_{j+1,k}-G^n_{j,k}\right]\right\}$$

$$E^{(2)}_{j,k} = E^n_{j,k} - \frac{1}{3}\left\{\frac{\Delta t}{\Delta x}\left[F^{(1)}_{j+\frac{1}{2},k+\frac{1}{2}}+F^{(1)}_{j+\frac{1}{2},k-\frac{1}{2}}-F^{(1)}_{j-\frac{1}{2},k+\frac{1}{2}}-F^{(1)}_{j-\frac{1}{2},k-\frac{1}{2}}\right]\right.$$

$$\left.+ \frac{\Delta t}{\Delta y}\left[G^{(1)}_{j+\frac{1}{2},k+\frac{1}{2}}+G^{(1)}_{j-\frac{1}{2},k+\frac{1}{2}}-G^{(1)}_{j+\frac{1}{2},k-\frac{1}{2}}-G^{(1)}_{j-\frac{1}{2},k-\frac{1}{2}}\right]\right\}$$

$$E^{n+1}_{j,k} = E^n_{j,k} - \frac{1}{4}\left\{\frac{\Delta t}{2\Delta x}(F^n_{j+1,k}-F^n_{j-1,k})-\frac{\Delta t}{3\Delta x}(F^n_{j+2,k}-2F^n_{j+1,k}\right.$$

$$+ 2F^n_{j-1,k}-F^n_{j-2,k}) + \frac{\Delta t}{2\Delta y}(G^n_{j,k+1}-G^n_{j,k-1})$$

$$\left.- \frac{\Delta t}{3\Delta y}(G^n_{j,k+2}-2G^n_{j,k+1}+2G^n_{j,k-1}-G^n_{j,k-2})\right\}$$

$$-\frac{3}{8}\left\{\frac{\Delta t}{\Delta x}\left(F^{(2)}_{j+1,k}-F^{(2)}_{j-1,k}\right)+\frac{\Delta t}{\Delta y}\left(G^{(2)}_{j,k+1}-G^{(2)}_{j,k-1}\right)\right\}$$

$$+\ \gamma_{30}\ \left\{E^{n}_{j+2,k}-4E^{n}_{j+1,k}+6E^{n}_{j,k}-4E^{n}_{j-1,k}+E^{n}_{j-2,k}\right\}$$

$$+\ \gamma_{30}\ \left\{E^{n}_{j,k+2}-4E^{n}_{j,k+1}+6E^{n}_{j,k}-4E^{n}_{j,k-1}+E^{n}_{j,k-2}\right\} \qquad (24)$$

This technique is stable when

$$\left|\sigma_{max}\ \frac{\Delta t}{\Delta x}\right| \leqslant 1 \quad , \quad \left|\sigma_{max}\ \frac{\Delta t}{\Delta y}\right| \leqslant 1 \qquad (25)$$

and

$$4\nu^2 - \nu^4 \leqslant \omega \leqslant 3.0 \qquad (26)$$

where

$$\omega = -\ 24\ \gamma_{30} \qquad (27)$$

When Rusanov's technique is applied to equation (12), the following modified partial differential equation results.

$$u_t + cu_x = -\frac{1}{24}\ c\ \Delta x^3\left[\frac{\omega}{\nu} - 4\nu + \nu^3\right]\ u_{xxxx}$$

$$-\ \frac{1}{120}\ c\ \Delta x^4\ \left[5\omega-4-15\nu^2+4\nu^4\right]\ u_{xxxxx}\ + \ldots \qquad (28)$$

Again it may be noted that when

$$\nu = 1.0 \quad , \quad \omega = 3.0 \qquad (29)$$

an exact solution of wave equation is formed.

Kutler-Warming Method

The most recent third-order differencing scheme developed is the modified version of the Rusanov three-level predictor-corrector sequence developed by Kutler and Warming (11). When applied to equation (11), the Kutler-Warming technique yields

$$E_{j,k}^{(1)} = E_{j,k}^{n} - \frac{2}{3}\frac{\Delta t}{\Delta x}(F_{j+1,k}^{n}-F_{j,k}^{n})-\frac{2}{3}\frac{\Delta t}{\Delta y}(G_{j,k+1}^{n}-G_{j,k}^{n})$$

$$E_{j,k}^{(2)} = \frac{1}{2}\left[E_{j,k}^{n}+E_{j,k}^{(1)}-\frac{2}{3}\frac{\Delta t}{\Delta x}(F_{j,k}^{(1)}-F_{j-1,k}^{(1)})-\frac{2}{3}\frac{\Delta t}{\Delta y}(G_{j,k}^{(1)}-G_{j,k-1}^{(1)})\right]$$

$$\begin{aligned}
E_{j,k}^{n+1} = E_{j,k}^{n} &+ \gamma_{30}\left[E_{j+2,k}^{n}-4E_{j+1,k}^{n}+6E_{j,k}^{n}-4E_{j-1,k}^{n}+E_{j-2,k}^{n}\right] \\
&+ \gamma_{30}\left[E_{j,k+2}^{n}-4E_{j,k+1}^{n}+6E_{j,k}^{n}-4E_{j,k-1}^{n}+E_{j,k-2}^{n}\right] \\
&- \frac{\Delta t}{24\Delta x}\left[-2F_{j+2,k}^{n}+7F_{j+1,k}^{n}-7F_{j-1,k}^{n}+2F_{j-2,k}^{n}\right] \\
&- \frac{\Delta t}{24\Delta y}\left[-2G_{j,k+2}^{n}+7G_{j,k+1}^{n}-7G_{j,k-1}^{n}+2G_{j,k-2}^{n}\right] \\
&- \frac{3}{8}\frac{\Delta t}{\Delta x}\left[F_{j+1,k}^{(2)}-F_{j-1,k}^{(2)}\right]-\frac{3}{8}\frac{\Delta t}{\Delta y}\left[G_{j,k+1}^{(2)}-G_{j,k-1}^{(2)}\right]
\end{aligned} \tag{30}$$

The modified equation which results from the application of this technique to the linear wave equation (12) is

$$\begin{aligned}
u_t+cu_x = &-\frac{1}{24}\, c\, \Delta x^3\left[\frac{\omega}{\nu}-4\nu+\nu^3\right] u_{xxx} \\
&- \frac{1}{120}\, c\, \Delta x^4\left[5\omega-4-15\nu^2+4\nu^4\right] u_{xxxxx} +\cdots
\end{aligned} \tag{31}$$

This technique is modified so that results comparable with the Rusanov scheme are obtained in less computation time. Similarity of the results come from the fact that their modified equations are similar, and, in fact, identical up to the fourth-order. A decrease in computation time is the direct result of simpler equations and less computation in all three levels due to elimination of intermediate grid point calculations. Note again when

$$\nu = 1.0 \quad , \quad \omega = 3.0 \tag{32}$$

the modified equation reduces to equation (12). The usual stability requirement is

$$\left|\sigma_{max} \frac{\Delta t}{\Delta x}\right| \leq 1 \quad , \quad \left|\sigma_{max} \frac{\Delta t}{\Delta y}\right| \leq 1 \tag{33}$$

and

$$4\nu^2 - \nu^4 \leq \omega \leq 3.0 \tag{34}$$

SAMPLE PROBLEM

An initially specified disturbance is considered in rotation about an axis normal to the XZ-plane or the plane of grid points. The angular velocity Ω is taken to be constant for which the stream function is defined as

$$\psi = \frac{\Omega r'^2}{2} \tag{35}$$

where

$$r' = \left[(x-x')^2 + (z-z')^2\right]^{\frac{1}{2}} \tag{36}$$

and x' and z' are the coordinates of the intersection point of the axis of rotation and the grid plane.

An incompressible flow is considered where the governing equation of the fluid flow is

$$\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} + \frac{\partial (Aw)}{\partial z} = 0 \tag{37}$$

Velocity components u and w are determined from the following equations

$$u = -\frac{\partial \psi}{\partial z} \quad , \quad w = \frac{\partial \psi}{\partial x} \tag{38}$$

The initial distribution of A is chosen as

$$A_0(x,z) = \begin{cases} 1 - \frac{1}{4\Delta} r'' & \text{for } r'' < 4\Delta \\ 0 & \text{for } r'' \geqslant 4\Delta \end{cases} \tag{39}$$

where Δ is the grid interval and

$$r'' = \left[(x-x'')^2 + (z-z'')^2\right]^{\frac{1}{2}} \tag{40}$$

where x" and z" are the coordinates of the point of the maximum value of A.

It may be noted that the prescribed initial distribution above describes a cone with its base on the XZ-plane. The peak is placed such that the disturbance is away from the axis of rotation and does not hit the boundaries of the grid plane in the course of its rotation about (x', z'). Also, the values of A are forced to be equal to zero along the boundaries during the sequence of numerical integration.

The following constants were used in all the calculations except where otherwise specified.

n = number of time iterations = 40

Ω = angular velocity = -0.001 radian/second

Δt = time interval = 30 seconds

$\Delta x = \Delta z = \Delta = 1.0$

$x' = 12\Delta$

$z' = 12\Delta$

$x'' = 18\Delta$

$z'' = 12\Delta$

All of the computations were performed on an IBM 360-65 digital computer at the Iowa State University Computation Center.

RESULTS AND CONCLUSION

The numerical techniques described earlier were first applied to the modified Burger's equation in one-dimension

$$\text{Conservative:} \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

$$\text{Advective:} \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

An initial distribution of

$$u(x) = 1.0 \qquad \text{for} \qquad 0 \leq x \leq 50$$

$$u(x) = 0.0 \qquad \text{for} \qquad 50 < x \leq 100$$

was assumed. The results obtained were in complete agreement with Crowley's results indicating that conservative techniques are to be preferred over advective methods. Another important conclusion is also derived. Conservative methods are preferred for problems with continuous and smooth solutions as shown by Crowley and also in problems involving discontinuities such as shock waves. In fact, non-conservative differencing can result in improper wave speed in flows involving discontinuities.

In order to further examine the conservative differencing techniques described earlier, they were applied to the sample problem discussed in the last section with governing equation (37). The relationship of the solution obtained by difference approximations to the analytic solution may be better represented when contour plots of A are examined. Also, the accuracy of each solution in terms of its general approximation, phase and radial displacement can be easily investigated (Figures 1-15). In these figures, the analytical solutions are represented by broken lines, while the solid lines are the solutions obtained by differencing techniques. Table 1 contains a summary of several important features of the results obtained by Molenkamp, while Table 2 contains the results of present conservative solutions.

Examining Figures 1, 2 and 3 with results given in Tables 1 and 2 for the first-order schemes, the superiority of conservative approximations is clearly shown over advective solutions. The advantage is more obvious when higher values of A isolines are considered. Computation time is 30% higher for Brailovskaya's technique than either Upstream N or Upstream N+1, but its higher maximum isoline approximation and lower radial displacement justifies its use. It is to be noted that the accuracy of these first-order techniques may be increased somewhat by decreasing the mesh ratio, $\frac{\Delta t}{\Delta x}$ and $\frac{\Delta t}{\Delta y}$. However, first-order techniques are only simple means of determination of the general behaviour of the solutions and therefore are not recommended for use in solution of complicated partial differential equations.

Second-order techniques (Figures 4-9) resulted in generally better solutions than the first-order methods as expected. The quality of the approximations of Λ isolines, using MacCormack and Lax-Wendroff conservative schemes are comparable to those approximations obtained by Molenkamp using advective Leap-Frog, Arakawa-Euler, and Arakawa-Adams-Bashforth techniques with conservative angular displacement error being 16 to 66 percent less than errors involved in advective solutions.

In general, the MacCormack differencing scheme is a better method overall than any other second-order advective or conservative technique considering the general approximation, computation time, error, and structure of the differencing equations.

Third-order techniques (Figures 10-15) resulted in the most accurate solutions obtained in this investigation. The accuracy of Rusanov-Burstein-Mirin technique had been investigated (1) where accurate solutions were obtained for gas dynamic model equation, i.e., Burger's Equation. The Kutler-Warming method is basically a modified form of the Rusanov technique and, in fact, the similarity of their modified equations suggests a close agreement of the solutions. This proved to be true for the problem under consideration in this report. The approximation obtained by application of these third-order methods closely follows the circular pattern of the analytical solutions proving their advantage over any lower order advective or conservative scheme. Tests were made for different combinations of values of γ_{30} and Δt. When the values

$$\gamma_{30} = -\frac{1}{80} \quad , \quad \Delta t = 60$$

are used, the Kutler-Warming technique yields the best solution to the above problem. It is interesting to note that the above values of γ_{30} and Δt correspond to about one-seventh of the lower bound for γ_{30} prescribed by equation (34). Anderson and Vogel also found that better results are obtained when the γ_{30} values corresponding to lower bound and fractions of the lower bound of the stability equation (34) were used (11). This indicates that linear stability analysis resulting in equation (34) does not define accurate stability bounds for all linear and non-linear problems. It should be noted that γ_{30} may not assume the value of zero, and therefore a limit exists on how small the value of $|\gamma_{30}|$ is to be chosen. This is also shown in Figures 10-15 and Table 2. The quality of the solution is degraded as $|\gamma_{30}|$ assumes values lower than $\frac{1}{80}$. In general, the Kutler-Warming technique is preferred over the Rusanov-Burstein-Mirin method mainly because of the simpler structure of the differencing equations in all three levels and elimination of intermediate grid calculations. A direct result of this is a considerable decrease in computation time. A comparison of the approximations obtained from the application of the above conservative third-order schemes with the results of the advective

Roberts-Weiss method, indicates again, the superiority of the conservative differencing over advective computation. Comparable approximations were obtained in both cases with a ratio of 1.2 to 45 of required computation time in favor of conservative differencing (Tables 1 and 2).

In general, conservative differencing is to be preferred over the advective approximation. Numerical experiments by Crowley and those reported in this paper confirm this fact. An improvement in results is obtained when higher order differencing techniques are applied. This is shown to be independent of whether the equation is in conservative or advective form. Those differencing techniques discussed in this paper are mainly gas dynamic differencing methods, but the results of this investigation in comparison with those by Molenkamp and Crowley indicate that these techniques may be applied to advection equations as well, resulting in better accuracy and more economical computation. The Kutler-Warming version of Rusanov's third-order technique resulted in the most accurate solutions and along with its short computation time presents, at the present time, an optimum differencing method in the solution of meteorological and gas dynamics equations.

Table 1. Maxima, lag, and computation time after 40 iterations (advective)

Type	Order of Accuracy	Maximum at a grid point	Angular displacement error (% of total rotation)	Radial displacement error (% of radius)	Relative computation time
Upstream N	$0(\Delta t, \Delta x)$	0.401	-8.72*	-8.0**	1.0
Upstream N+1	$0(\Delta t, \Delta x)$	0.307	+28.34	-12.0	1.0
Leap-frog (25 x 25)	$0(\Delta t^2, \Delta x^2)$	0.861	-15.26	0	0.7
Leap-frog (50 x 50)	$0(\Delta t^2, \Delta x^2)$	0.838	-8.72	0	2.5
Arakawa-Euler	$0(\Delta t^2, \Delta x^2)$	0.858	-19.62	-2.0	1.5
Arakawa-Adams-Bashforth	$0(\Delta t^2, \Delta x^2)$	0.878	-21.80	0	1.6
Roberts-Weiss	$0(\Delta t^2, \Delta x^4)$	0.896	+2.18	0	45.0

Obtained by C. R. Molenkamp (8)

*Minus sign indicates numerical solution lags analytical solution.

**Minus sign indicates numerical solution has shorter radius than analytical solution.

Table 2. Maxima, lag, and computation time after 40 iterations (conservative)

Type	Order of Accuracy	Maximum at a grid point	Angular displacement error (% of total rotation)	Radial displacement error (% of radius)	Relative computation time
Brailovskaya	$0(\Delta t,\Delta x^2)$				
$\Delta t = 30$		0.768	-8.72*	-4.83**	1.366
$\Delta t = 60$		0.692	-8.72	-4.83	0.783
Lax-Wendroff	$0(\Delta t^2,\Delta x^2)$				
$\Delta t = 30$		0.842	-7.27	-2.44	1.966
$\Delta t = 60$		0.816	-7.27	-2.44	1.078
MacCormack (Forward predictor, backward corrector)	$0(\Delta t^2,\Delta x^2)$				
$\Delta t = 30$		0.816	-7.27	-2.44	1.400
$\Delta t = 60$		0.778	-7.27	-2.44	0.766
(Backward predictor, forward corrector)					
$\Delta t = 30$		0.862	-7.27	-2.44	1.400
$\Delta t = 60$		0.861	-7.27	-2.44	0.766

Table 2 (Continued)

Type	Order of Accuracy	Maximum at a grid point	Angular displacement error (% of total rotation)	Radius displacement error (% of radius)	Relative computation time
Rusanov-Burstein-Mirin	$0(\Delta t^3, \Delta x^3)$				
($\gamma_{30} = -\frac{1}{80}$, $\Delta t = 30$)		0.777	0	-1.60	3.283
Kutler-Warming	$0(\Delta t^3, \Delta x^3)$				
($\gamma_{30} = -\frac{1}{80}$, $\Delta t = 30$)		0.756	0	-1.60	2.441
($\gamma_{30} = -\frac{1}{96}$, $\Delta t = 30$)		0.770	0	-2.46	2.441
($\gamma_{30} = -\frac{1}{112}$, $\Delta t = 30$)		0.781	0	-2.46	2.441
($\gamma_{30} = -\frac{1}{80}$, $\Delta t = 60$)		0.803	0	0	1.316
($\gamma_{30} = -\frac{1}{80}$, $\Delta t = 75$)		0.814	0	-1.6	1.080

Present results

*Minus sign indicates numerical solution lags analytical solution.

**Minus sign indicates numerical solution has shorter radius than analytical solution.

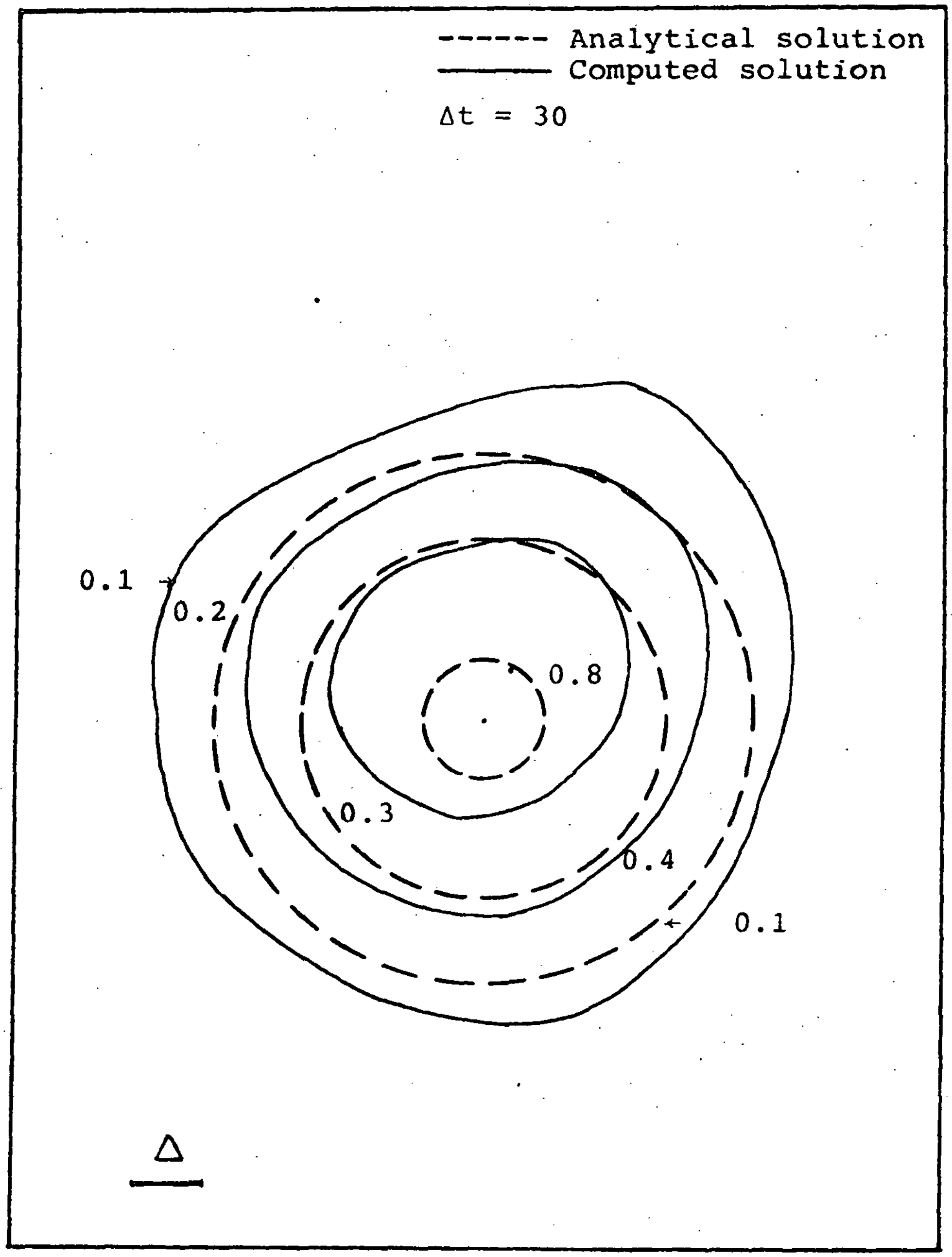

Figure 1. Solution using the Upstream N Method.

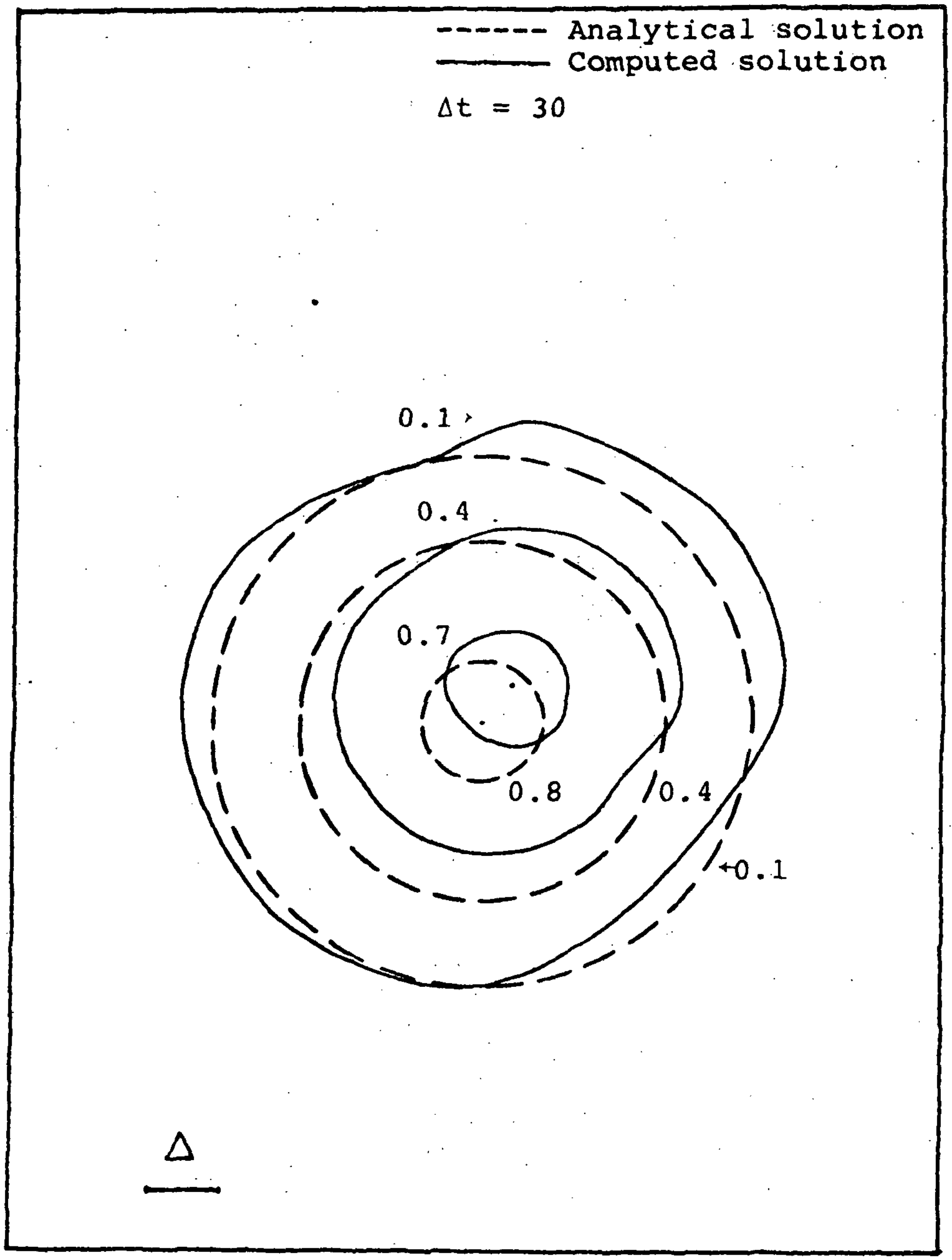

Figure 2. Solution using the Brailovskaya Method.

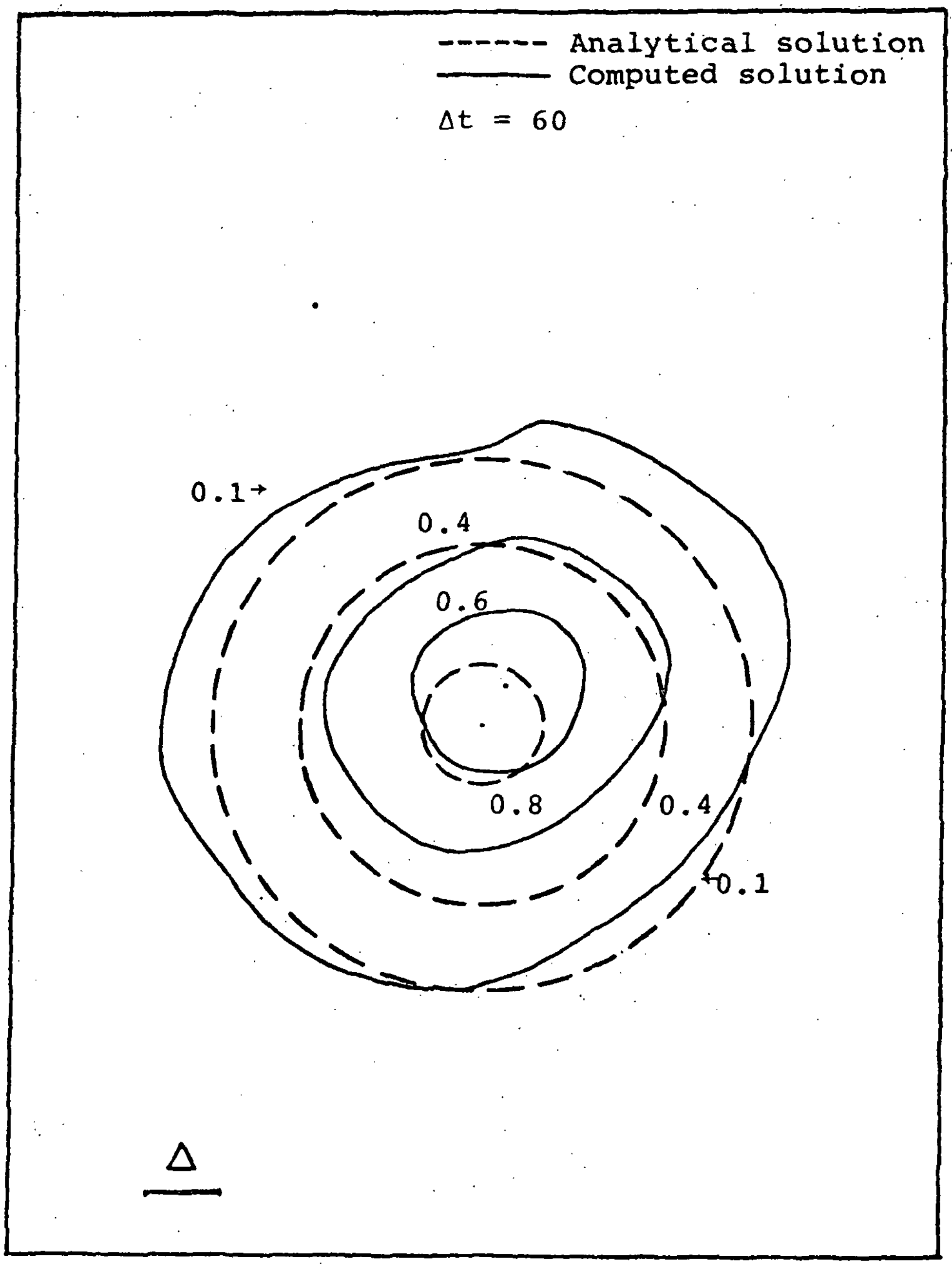

Figure 3. Solution using the Brailovskaya Method.

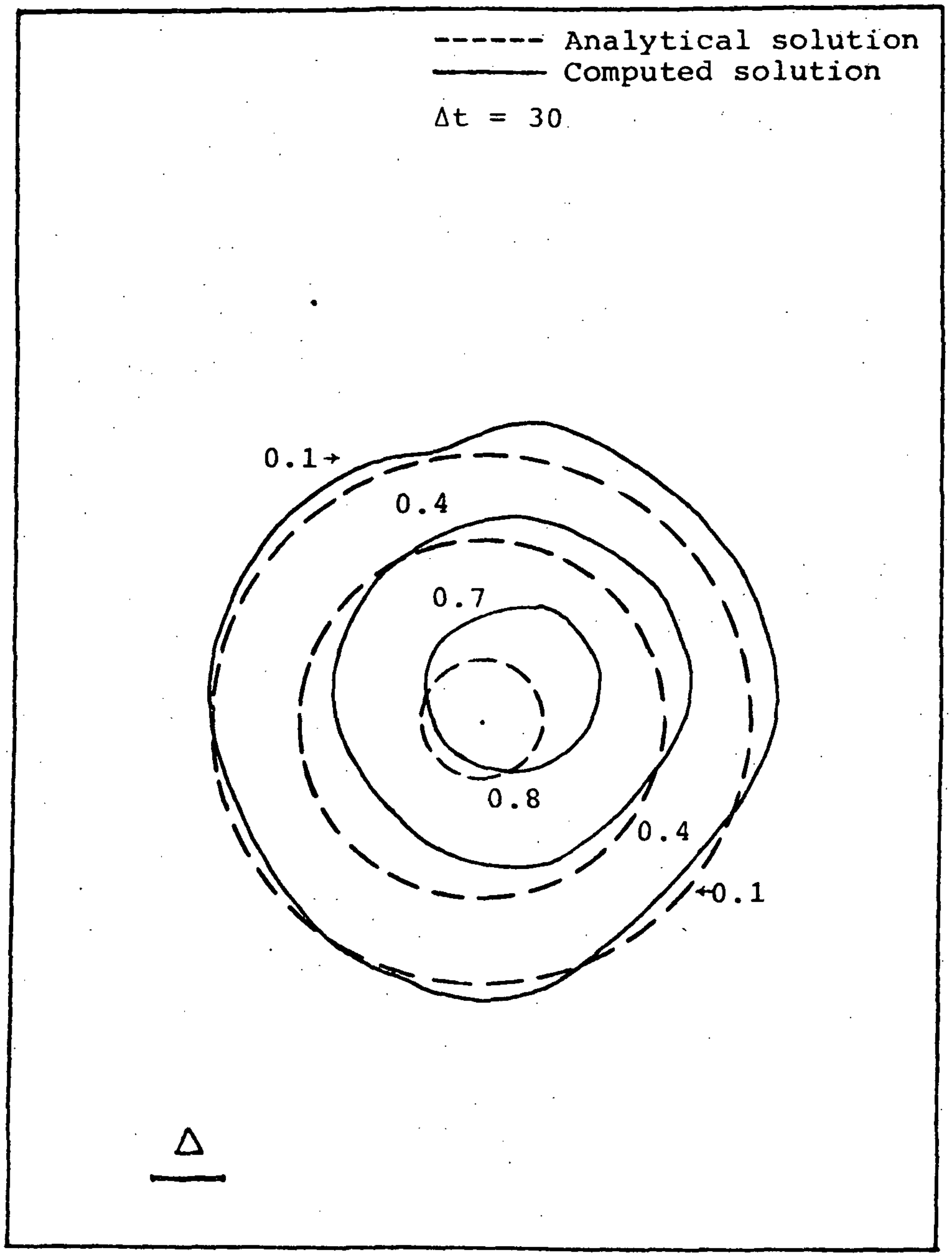

Figure 4. Solution using the Lax-Wendroff Method.

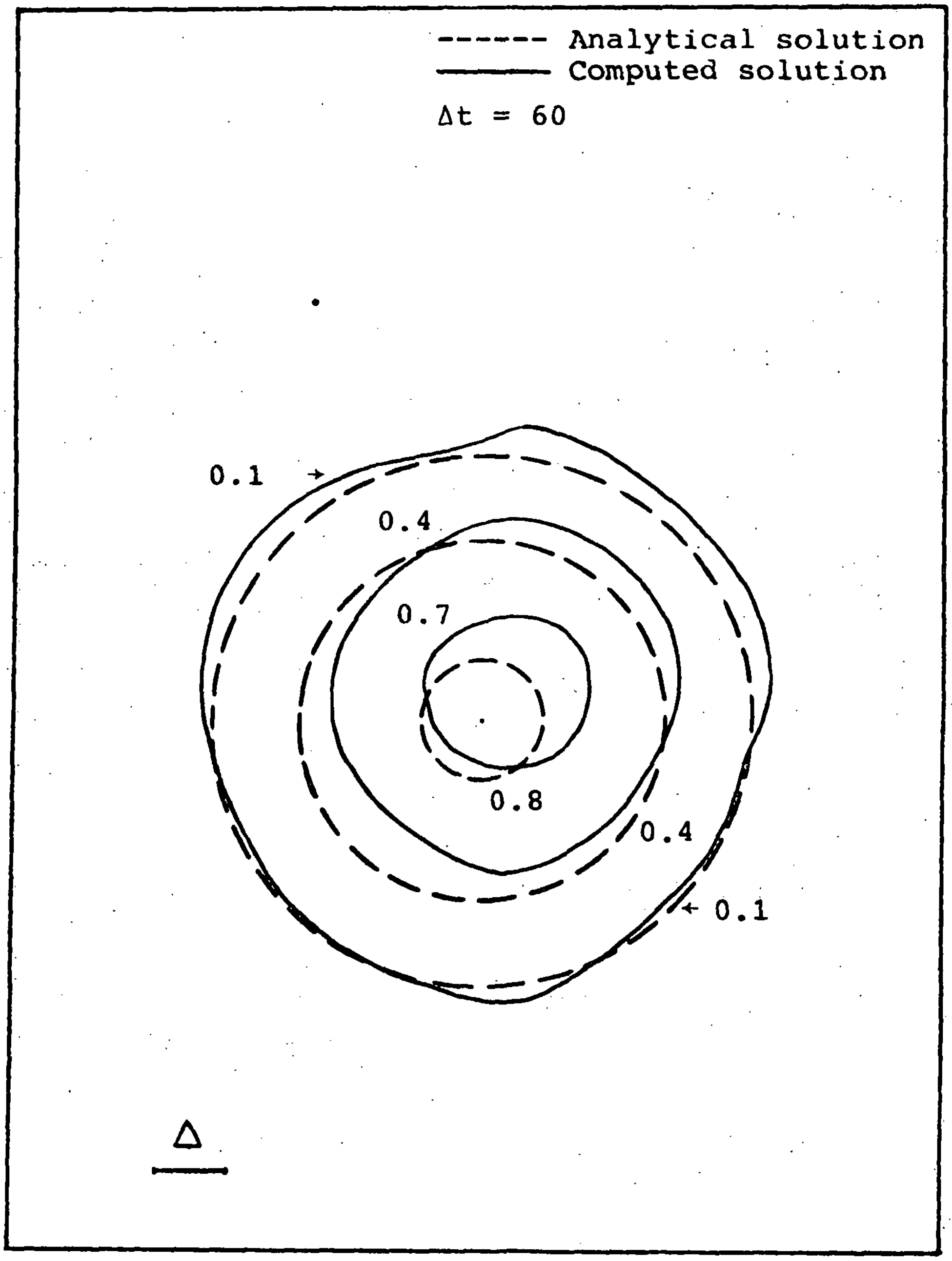

Figure 5. Solution using the Lax-Wendroff Method.

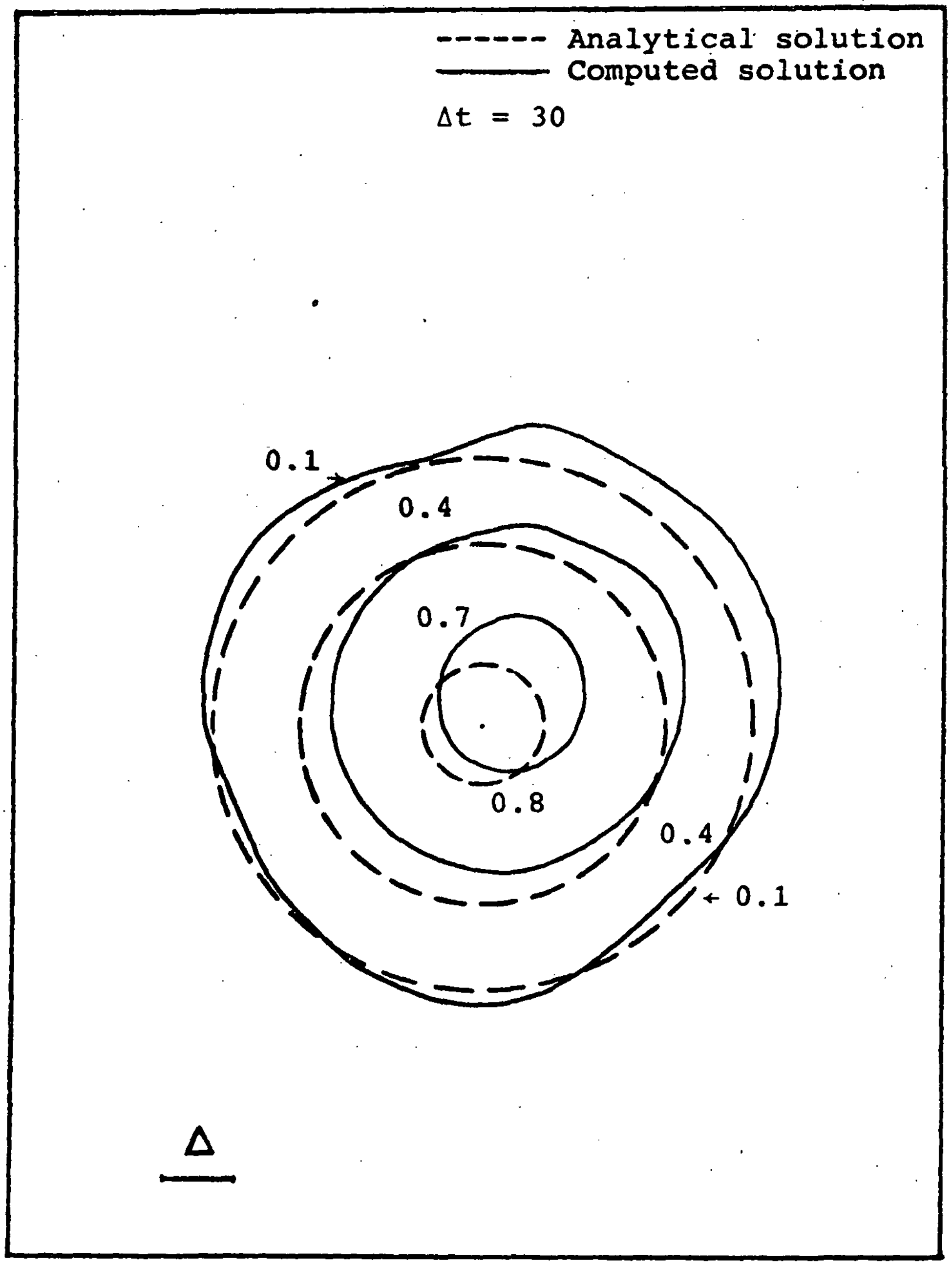

Figure 6. Solution using the MacCormack (forward predictor, backward corrector) Method.

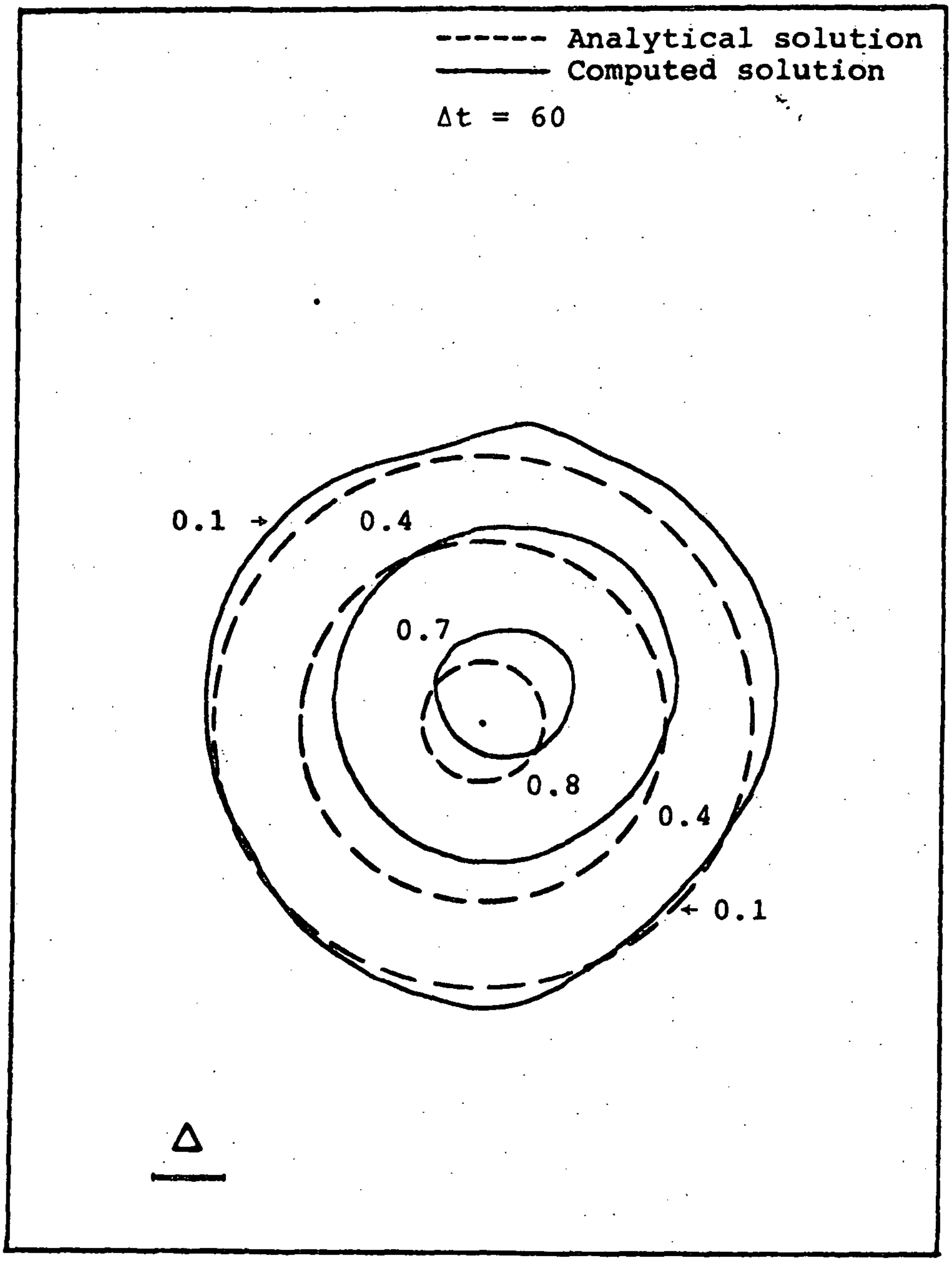

Figure 7. Solution using the MacCormack (forward predictor, backward corrector) Method.

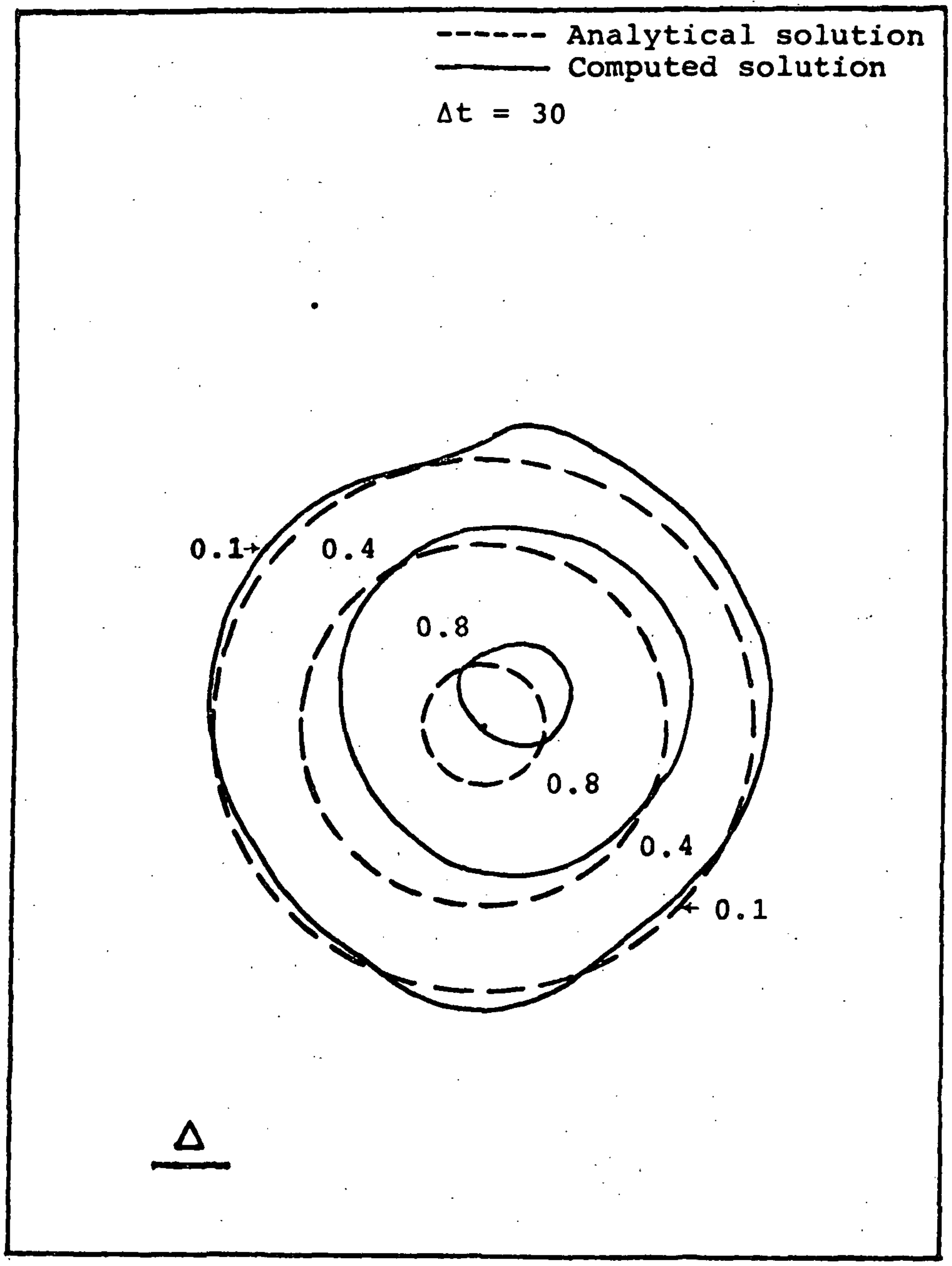

Figure 8. Solution using the MacCormack (backward predictor, forward corrector) Method.

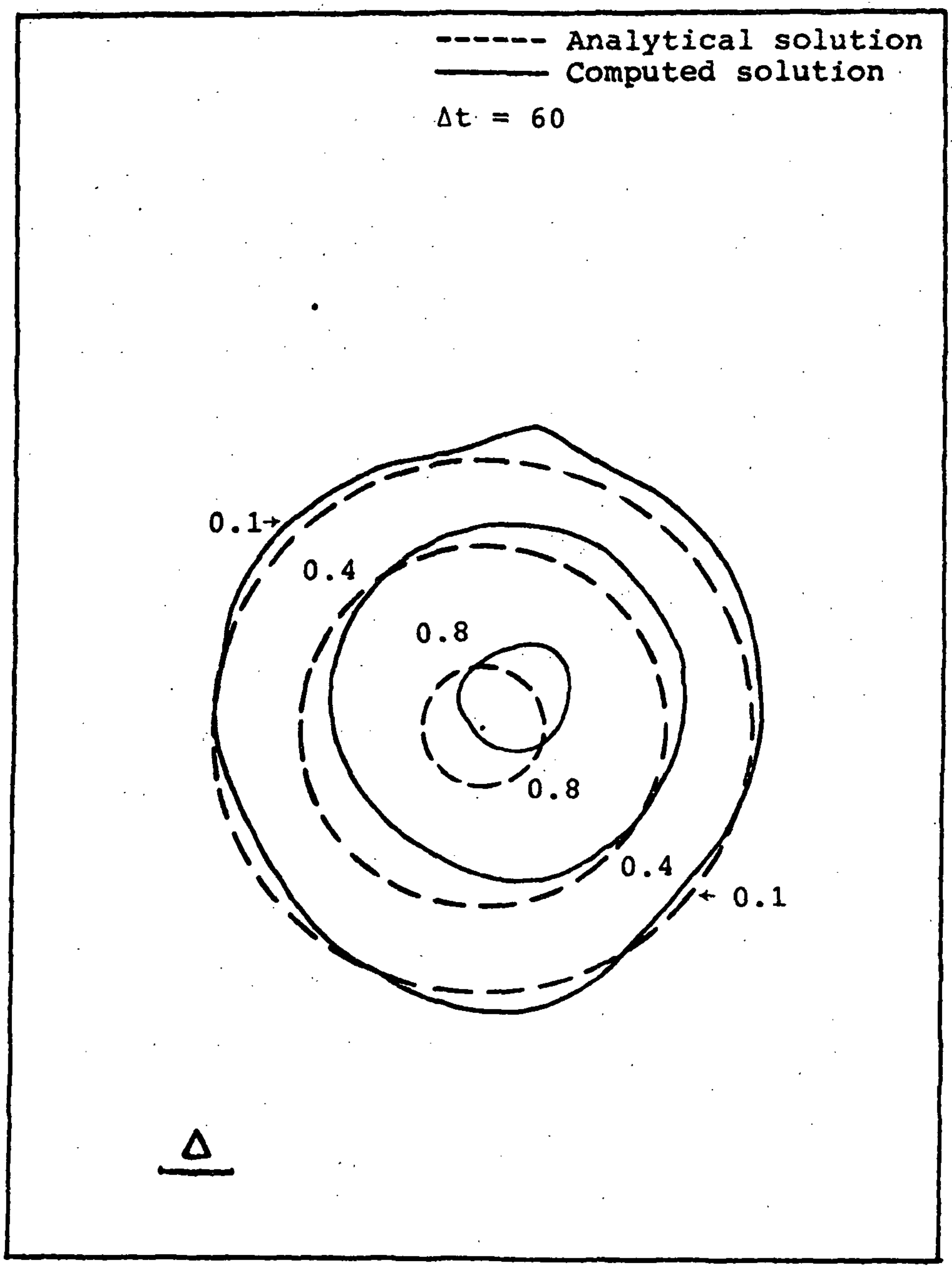

Figure 9. Solution using the MacCormack (backward predictor, forward corrector) Method.

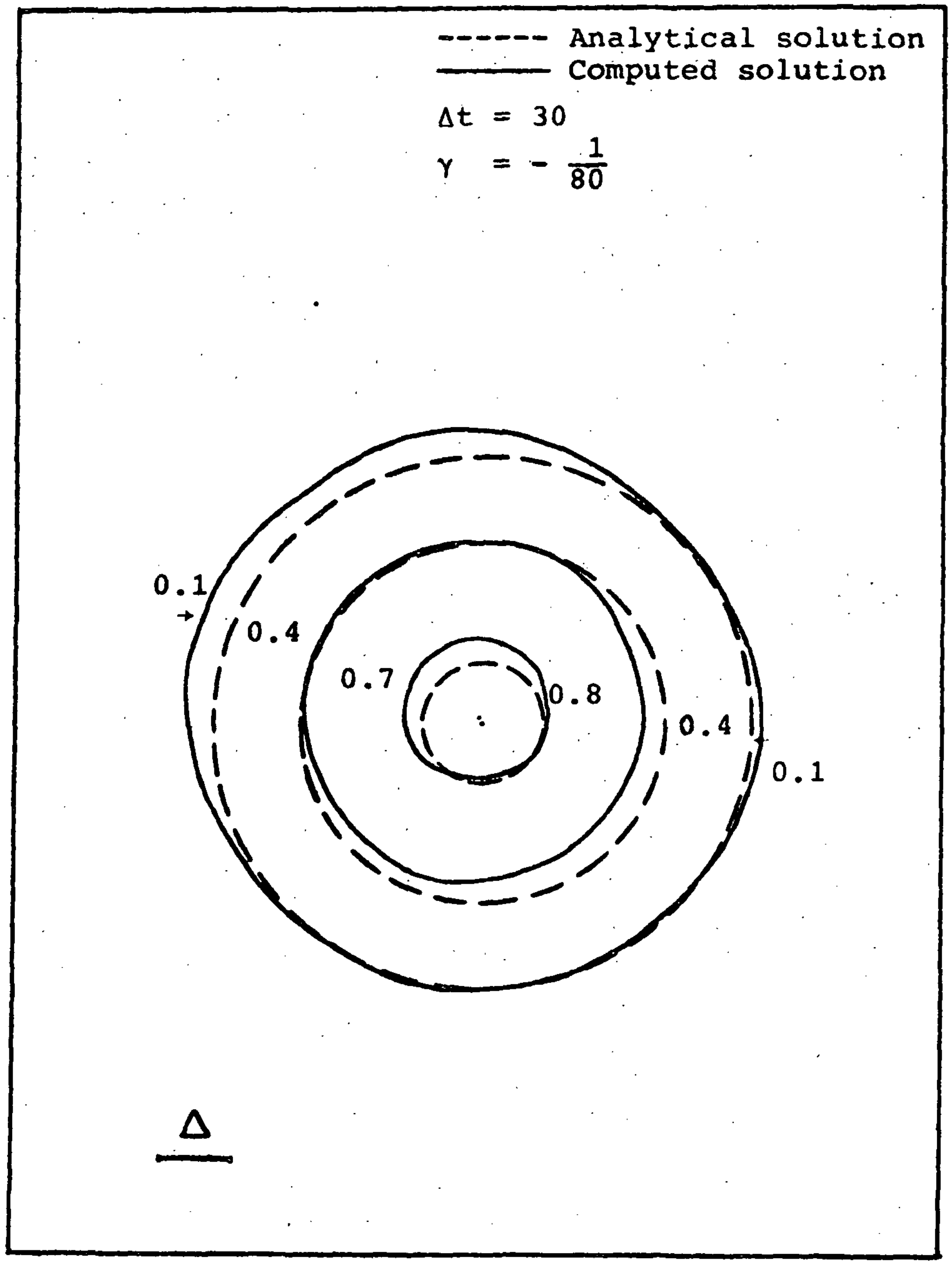

Figure 10. Solution using the Rusanov-Burstein-Mirin Method.

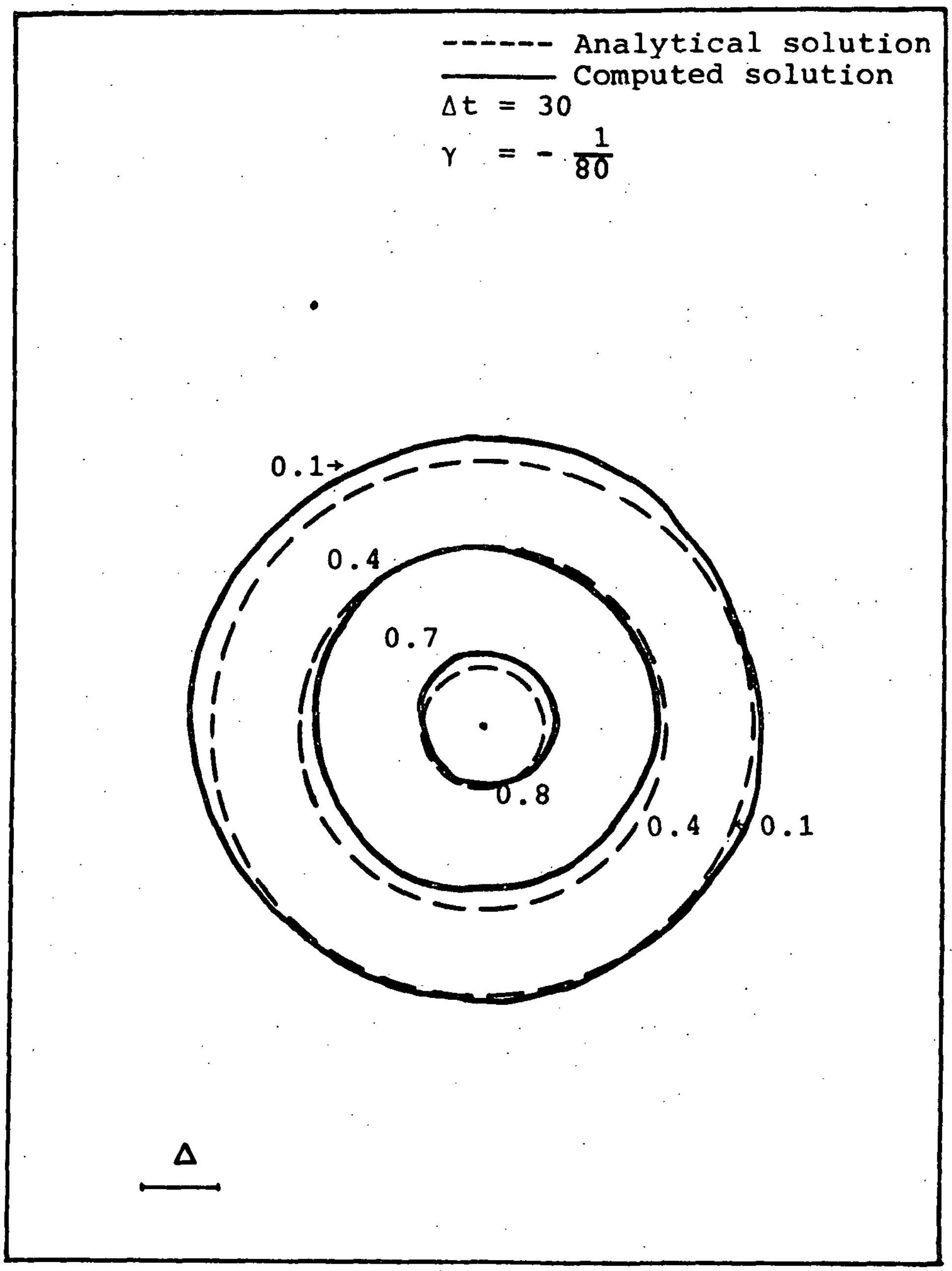

Figure 11. Solution using the Kutler-Warming Method.

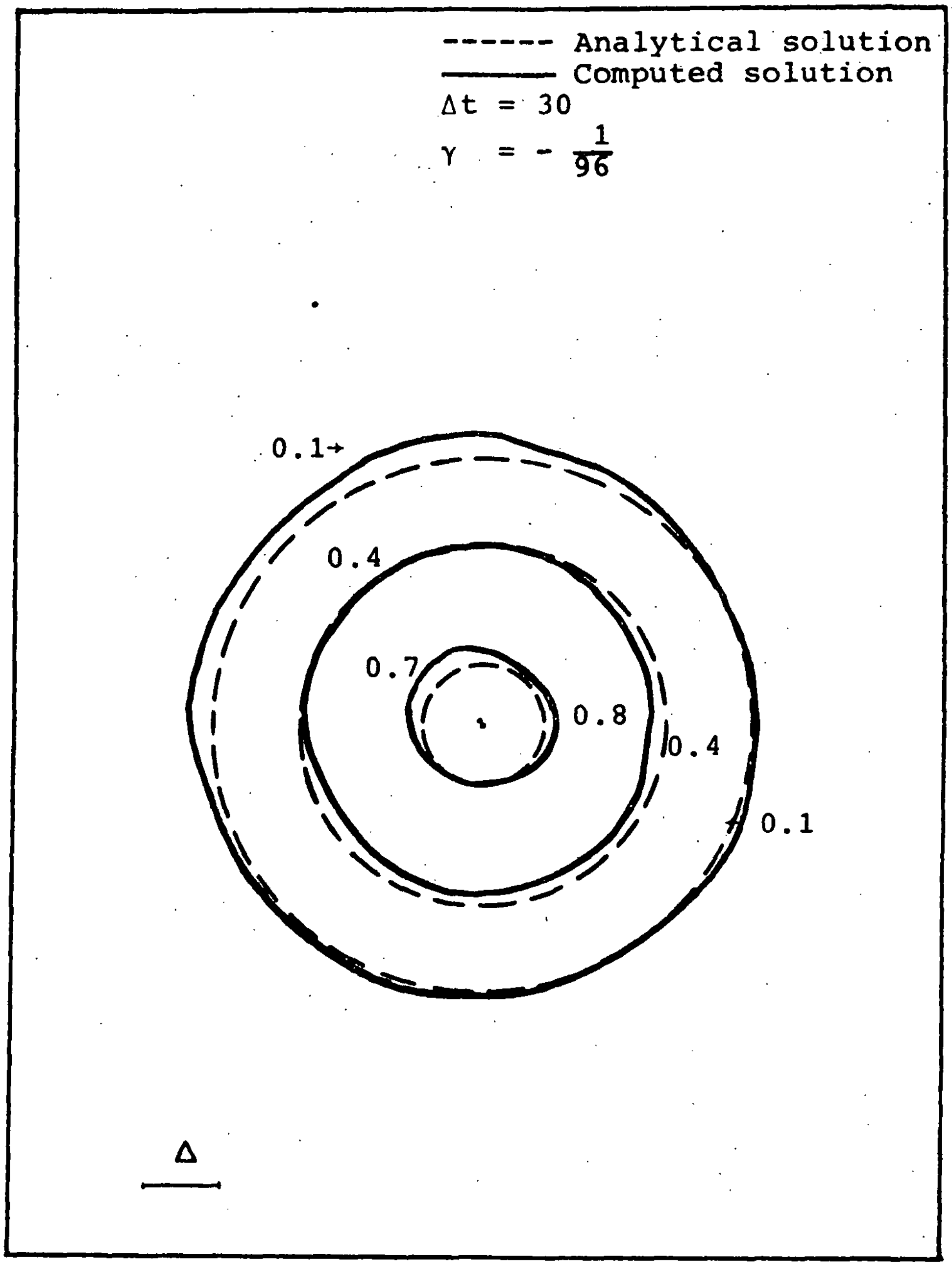

Figure 12. Solution using the Kutler-Warming Method.

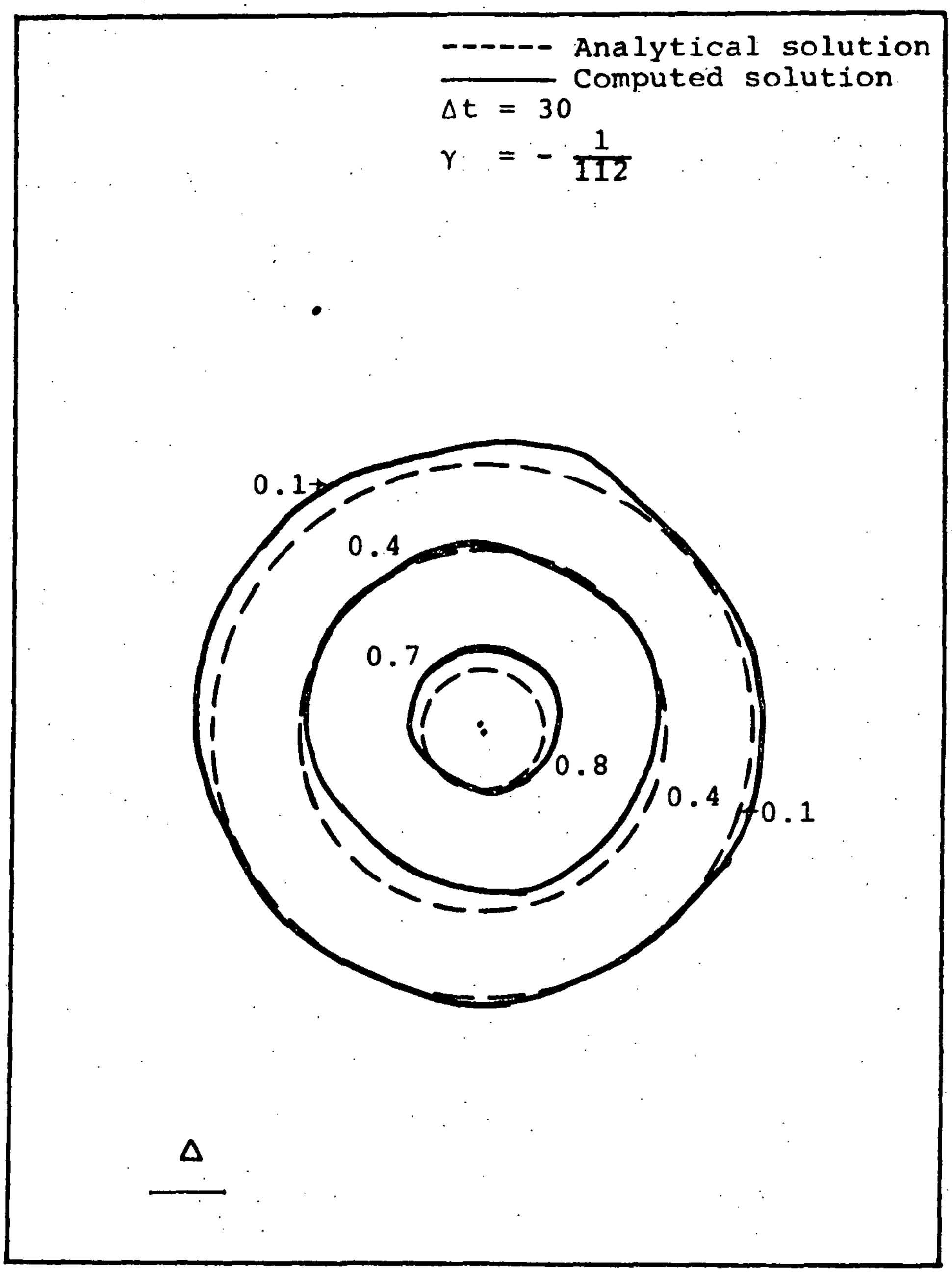

Figure 13. Solution using the Kutler-Warming Method.

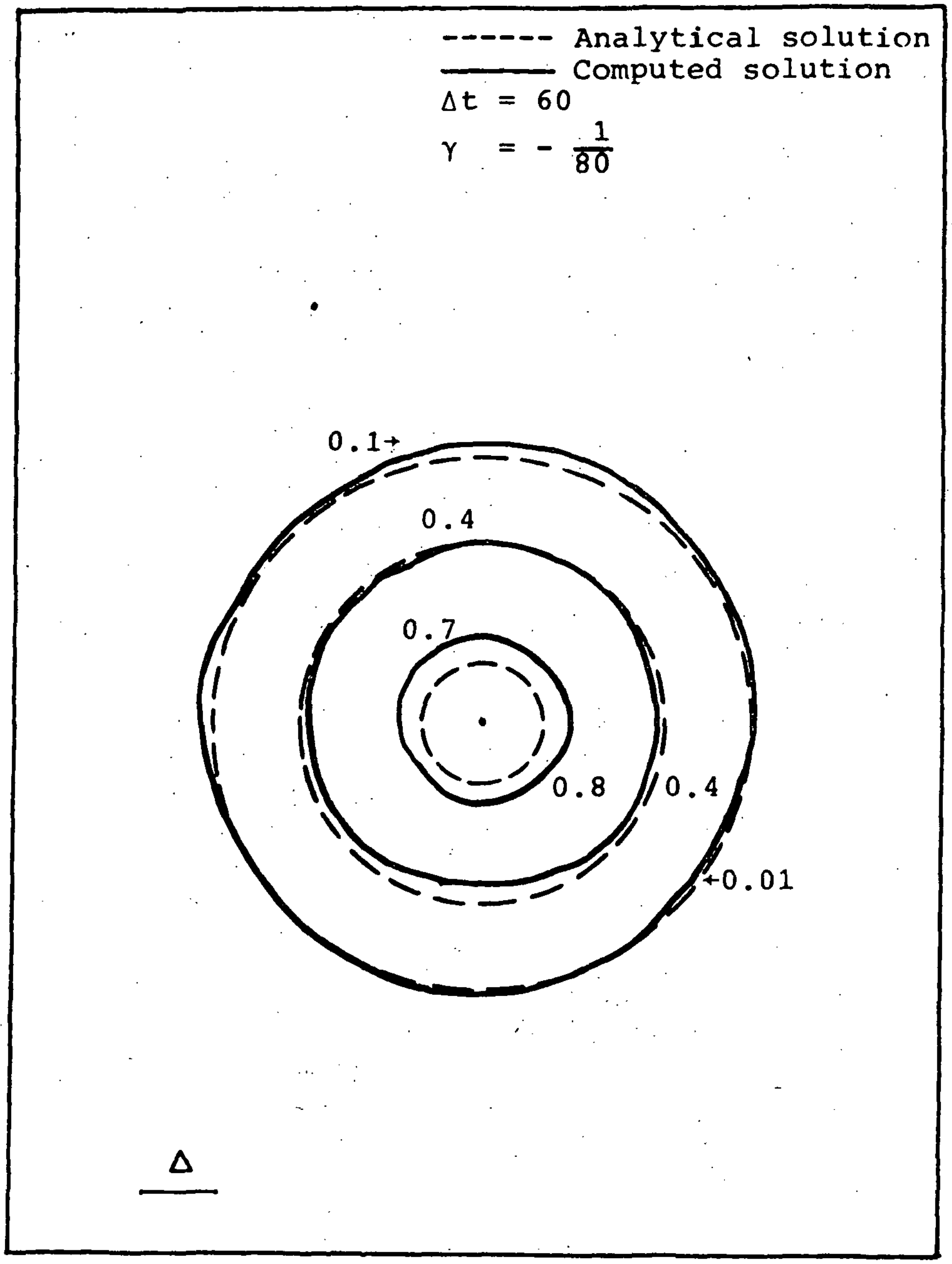

Figure 14. Solution using the Kutler-Warming Method.

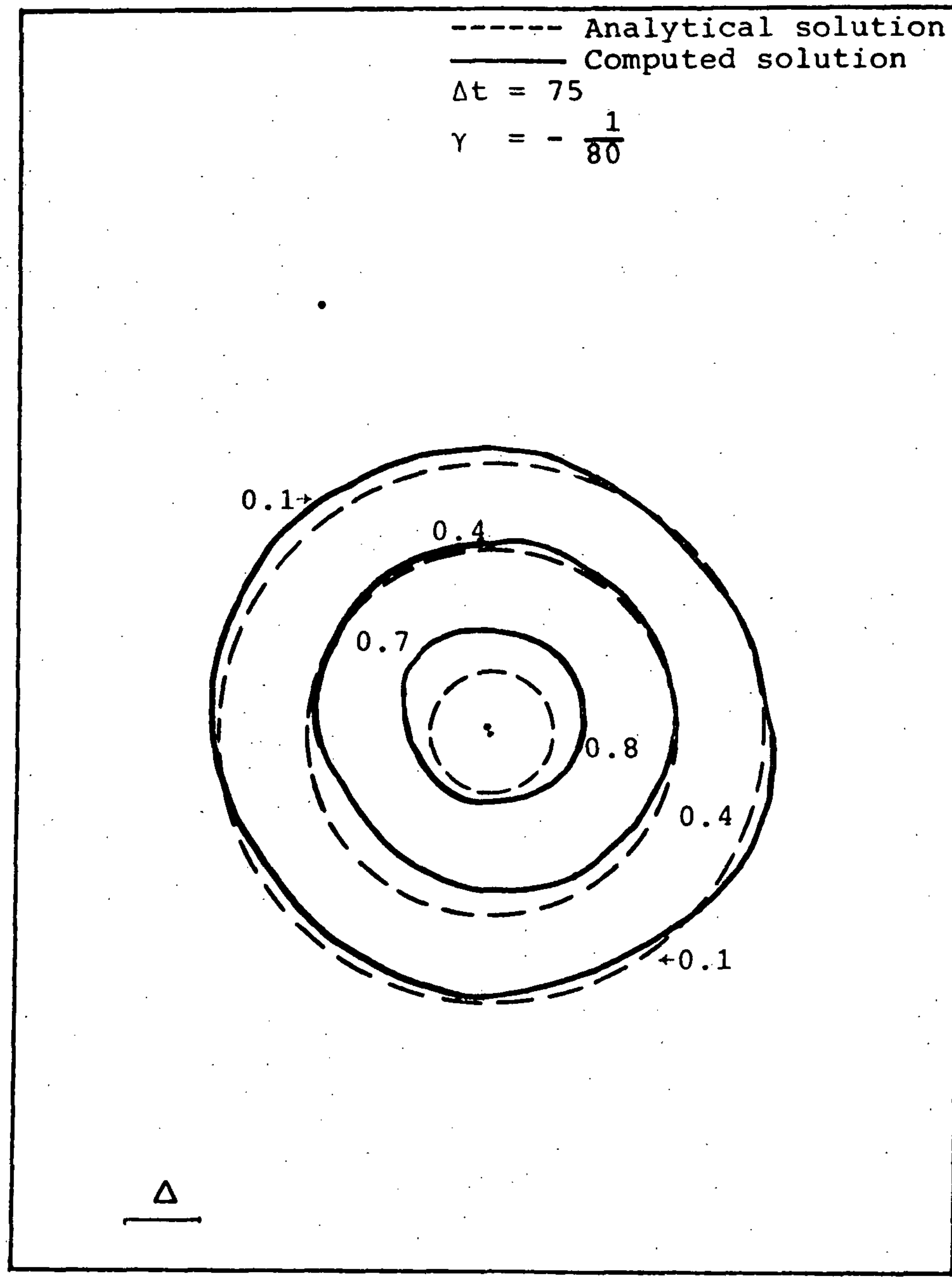

Figure 15. Solution using the Kutler-Warming Method.

REFERENCES

1. Anderson, D. A. and Vogel, J. M. "Numerical Solutions of Flow Fields Behind Rectangular Wings." The Engineering Research Institute, Iowa State University, Ames, Iowa. July 1971.

2. Brailovskaya, I. Y. "A Difference Scheme for Numerical Solution of the Two-Dimensional, Nonstationary Navier-Stokes Equations for a Compressible Gas." Soviet Physics-Daklady 10: 107-110. August 1965.

3. Burstein, S. Z. and Mirin, A. A. "Third-Order Difference Methods for Hyperbolic Equations." Journal of Computational Physics 5: 547-571. 1970.

4. Crowley, W. P. "Numerical Advection Experiments." Monthly Weather Review 96. January 1968.

5. Kutler, P. K. "Application of Selected Finite Difference Techniques to the Solution of Conical Flow Problems." Ph.D. thesis, Iowa State University, Ames, Iowa. 1969.

6. Kutler, P. K. and Lomax, H. "The Computation of Supersonic Flow Fields About Wing-Body Combination by 'Shock-Capturing' Finite Difference Techniques." Ch.8, Lecture Notes in Physics, Springer Verlag. 1971.

7. Lax, P. D. and Wendroff, B. "Difference Schemes for Hyperbolic Equations with High Order of Accuracy." Communications on Pure and Applied Mathematics 17: 381-398. 1964.

8. MacCormack, R. W. "The Effect of Viscosity in Hypervelocity Impact Cratering." American Institute of Aeronautics and Astronautics, Paper No. 69-354: 1-7. 1969.

9. Molenkamp, C. R. "Accuracy of Finite-Difference Methods Applied to the Advection Equation." Journal of Applied Meteorology 7: 160-167. April 1968.

10. Rusanov, V. V. "On Difference Schemes of Third-Order Accuracy for Non-Linear Hyperbolic Systems." Journal of Computational Physics 5: 507-516. 1970.

11. Private communication.

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