L^2-Stability of Distributed Feedback Systems: Singular Perturbation

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Abstract

We consider a continuous time, single input-single output, linear, time-invariant, distributed feedback system F^{ε} containing a small delay of length ε in the loop. Conditions are given under which L^2-stability and L^2-instability of this feedback system can be deduced from those of the reduced model obtained by neglecting the delay.

The two system models associated with F^{ε} are the low-frequency model F^0 and the high frequency model F_H^{ε}. The condition for neglecting the small delay is the L^2-stability of the family of high-frequency models $(F_H^{\varepsilon})_{\varepsilon \geq 0}$, where $\varepsilon \geq 0$ is sufficiently small.

The paper contains a lemma and a theorem. The lemma gives sharp Nyquist-type conditions for the L^2-stability and L^2-instability of the family of high-frequency models $(F_H^{\varepsilon})_{\varepsilon \geq 0}$ for sufficiently small $\varepsilon \geq 0$, while the Theorem gives explicit conditions under which the small delay may or may not be neglected.

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Description of the system model, low-frequency and high-frequency models

We consider a family of continuous-time, scalar linear time-invariant feedback systems $(F^{\varepsilon})_{\varepsilon \geq 0}$ with input u, error e and output y. u, e and y are functions mapping $\mathbb{R}_+$ into $\mathbb{R}$ and satisfy

$$y = T_{\varepsilon}(g * e) \tag{1}$$

$$e = u - y \tag{2}$$

where, * denotes the convolution operator, T_{ε} is a delay of length $\varepsilon \geq 0$; g is a real-valued distribution with support on $\mathbb{R}_+$.

Let $\hat{g}$ denote the Laplace transform of g. We assume that $\hat{g}$ has the following form

$$\hat{g}(s) = \hat{g}_a(s) + \hat{g}_{ap}(s) + \sum_{\alpha=1}^{\ell} \sum_{m=1}^{m_\alpha} r_{\alpha k}/(s-p_\alpha)^m \tag{3}$$

where

$$\left\{ \begin{array}{l} \text{the poles } p_\alpha,\ \alpha = 1, 2, \ldots, \ell \text{ are either real with} \\ \quad \text{real residues} \\ \\ \text{or complex conjugate with complex conjugate residues;} \end{array} \right. \tag{4}$$

$$\text{Re } p_\alpha \geq 0 \quad \text{for} \quad = 1, 2, \ldots, \ell; \tag{5}$$

$$g_a(\cdot) \text{ is a real valued function} \in L^1[0,\infty) \tag{6}$$

$$g_{ap}(t) = \sum_{i=0}^{\infty} g_i \, \delta(t-t_i) \tag{7}$$

$$g_i \in \mathbb{R} \quad \text{for } i = 1, 2, \ldots, \tag{8}$$

$$\sum_{i=1}^{\infty} |g_i| < \infty \tag{9}$$

$$0 = t_0 < t_1 < t_2 < \ldots < t_i < \ldots . \tag{10}$$

Note that $g_a(\cdot) + g_{ap}(\cdot)$ belongs to the convolution-algebra $\mathcal{A}$, [3], if and only if its Laplace transform $\hat{g}_a(\cdot) + \hat{g}_{ap}(\cdot)$ belongs to the algebra $\hat{\mathcal{A}}$ with pointwise product. Moreover, the function $s \longrightarrow \hat{g}_{ap}(s)$ mapping $\{s \in \mathbb{C} : \text{Re } s \geq 0\}$ into $\mathbb{C}$ is almost periodic in Re $s \geq 0$.

The low-frequency model (nominal in this case) is defined to be F^0, i.e., F^0 is the feedback system model governed by

$$y = g * e \tag{11}$$

$$e = u - y \tag{12}$$

The high-frequency model is a member of the family $(F_H^{\varepsilon})_{\varepsilon \geq 0}$ defined by

$$y = T_{\varepsilon}(g_{ap} * e) \tag{13}$$

$$e = u - y \tag{14}$$

Results

We seek conditions under which the L^2-stability and L^2-instability of the family $(F^{\varepsilon})_{\varepsilon \geq 0}$, where $\varepsilon \geq 0$ is sufficiently small, can be deduced from those of F^0 and the family $(F_H^{\varepsilon})_{\varepsilon \geq 0}$, where again $\varepsilon \geq 0$ is sufficiently small.

Lemma

Consider the family of feedback system models $(F_H^{\varepsilon})_{\varepsilon \geq 0}$ defined by (13) and (14). Under these conditions

(A) if $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| < 1$ then F_H^{ε} is L^2-stable for all $\varepsilon > 0$,

(B) if $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| > 1$ then given any $\mu > 0$ there exists an $\bar{\varepsilon} > 0$ such that $0 < \bar{\varepsilon} < \mu$ and $F_H^{\bar{\varepsilon}}$ is L^2-unstable;

(C) if $\hat{g}_{ap}(j\omega)$ is periodic then F_H^{ε} is L^2-stable for all sufficiently small $\varepsilon > 0$ if and only if $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| < 1$.

Theorem:

Consider the two families of feedback system models $(F^{\varepsilon})_{\varepsilon \geq 0}$ defined by (1) and (2) and $(F_H^{\varepsilon})_{\varepsilon \geq 0}$ defined by (13) and (14).

Case 1. Assume $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| < 1$. Under this assumption

(A) if F^0 is L^2-stable then F^{ε} is L^2-stable for all sufficiently small $\varepsilon \geq 0$;

(B) if F^0 is L^2-unstable and if $\begin{cases} 1 + g_0 \neq 0 \\ 1 + \hat{g}(j\omega) \neq 0 \text{ for all } \omega \in \mathbb{R} \end{cases}$

then F^ε is L^2-unstable for all sufficiently small $\varepsilon \geq 0$.

<u>Case 2</u>. Assume

<u>either</u> $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| > 1$

<u>or</u> $\sup_{\omega \in \mathbb{R}} |\hat{g}_{ap}(j\omega)| \geq 1$ and $\hat{g}_{ap}(j\omega)$ is periodic

Under this condition given any $\mu > 0$ there exists an $\bar{\varepsilon} > 0$ such that $0 < \bar{\varepsilon} < \mu$ and $F^{\bar{\varepsilon}}$ is L^2-unstable (whether or not F^0 is L^2-stable).

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