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SUPPLEMENTARY ACTIVE STABILIZATION OF NONRIGID GRAVITY GRADIENT SATELLITES

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ABSTRACT

The paper investigates the use of active control for stability augmentation of passive gravity gradient satellites. The reaction jet method of control is the main interest. Satellite nonrigidity is emphasized. The reduction in the Hamiltonian H is used as a control criteria. The velocities $v_{u\alpha}$, relative to local vertical, of the jets along their force axes are shown to be of fundamental significance. A basic control scheme which satisfies the H reduction criteria is developed. Each jet α is fired when its $v_{u\alpha}$ becomes appropriately large. The jet is de-energized when $v_{u\alpha}$ reaches zero. Firing constraints to preclude orbit alteration may be needed. Control is continued until H has been minimized. This control policy is investigated using impulse and rectangular pulse models of the jet outputs. The impulse model leads to a simple equation for the optimal instantaneous control magnitude. This model, however, is difficult to employ in nonrigid satellite applications due to basic problems that are discussed. The study using the rectangular pulse model includes (1) development of general equations for the pulse duration which yields $v_{u\alpha} = 0$ and (2) investigation of a strategy to override this criteria when necessary to prevent large structural vibrations.

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NOMENCLATURE

a	semi-major axis of orbit
e	orbit eccentricity
$\bar{e}_{b1} - \bar{e}_{b3}$	unit vectors along axes of frame b
$\bar{e}_p$	unit vector parallel to orbit pole
$\bar{e}_R$	unit vector parallel to local vertical
$\bar{e}_{u\alpha}$	unit vector along force axis of jet α
$\bar{G}$	torque vector
$\underline{J}$	inertia dyadic about satellite's c.m.
K_U	arbitrary constant
m	satellite mass
$\underline{N}^{22}_{dm}$	component of $\underline{N}$ due to viscous damping
R	distance from planet's c.m. to satellite's c.m.
T_o	kinetic energy component of zeroth order in $\dot{\underline{q}}$
t_o	time at which a control operation is begun
U	dynamic potential
V	potential energy
V_s	the component of V due to internal stiffness
$\bar{r}_\alpha$	vector from satellite's c.m. to jet α
γ	true anomaly angle
μ	gravitational constant
$\bar{\omega}_b$	angular velocity vector of frame b relative to frames R and r
$\bar{\omega}_R$	true anomaly angular velocity vector

INTRODUCTION

This paper originated in a study performed for the RAE-B satellite. RAE-B will have four 750' gravity gradient booms, a passive damper system, and six freon reaction jets. The jets will be used for spin vector control early in the mission before deployment of the booms and capture by the gravity field. The purpose of the study was to investigate the feasibility of a suggestion, raised by GSFC personnel, to also use the jets for attitude stability augmentation during or after boom deployment if an emergency situation is encountered or if unsatisfactory performance of the damper system is experienced. Feasibility hinged largely on whether or not properly timed force pulses could alleviate attitude librations or tumbling on RAE-B without generating intolerably large boom vibrations. The difficulty arises because the long booms will be extremely flexible, and the force outputs of the jets will be comparatively large.

The material which has been included in the paper is not limited to satellites of the RAE-B configuration. Instead, it is directly applicable to virtually any satellite which is stabilized by the passive gravity gradient method and includes one or more reaction jets. While the analytical methods which are employed do have some potentiality for extension to other types of control, the study is devoted almost entirely to techniques, particularly reaction jets, which provide damping or control by the application of external forces to the system.

The concept of stabilizing satellite attitude solely by gravity gradient forces and a passive damper has received considerable attention during the past 12 years. Satellites which have employed the method, however, often have not performed well.[1-4] Attitude motions of several degrees or more are to be expected even under the best conditions. Failure of the passive damper always is a possibility. Major stability problems have been experienced even when the damper was functioning normally. Particular difficulty has been encountered due to thermal bending of the long booms which usually are needed to obtain adequate moments of inertia. The yaw motion on triaxially-stabilized satellites has been very troublesome. Many systems which are intended to improve the performance by supplementing the gravity forces by semi-passive, semi-active, or active control have been proposed, and some have been built and tested in orbit[5]. Reaction jets are one of the methods by which performance can be augmented. Emergency control of satellites which are strictly passive during normal operation is one potential application; this concept should be potentially attractive on satellites, such as RAE-B, where a jet system already is required for other purposes.

If active control forces are to be applied to satellites with highly nonrigid parts, such as inertia booms, careful consideration must be given to the excitation of the elastic vibration modes. At present, there is considerable interest in the dynamics, stability, and control of nonrigid satellites[6-8]. The present study emphasizes nonrigidity effects. It is applicable to situations in which the purpose of the active control is to attenuate the attitude motions without generating unacceptably large structural vibrations and also to cases where attenuation of both the

initial attitude and structural motions is desired. The work can be applied to rigid satellites by deleting the nonrigidity portions of the mathematical results. The study is devoted to the basic dynamics aspects of the control problem. The related problem of state estimation is given only secondary attention.

The attitude control of rigid satellites by jets has been studied in numerous previous papers. A partial listing is given as Refs. 9 to 13. The present study differs from earlier ones mainly in that (1) it includes nonrigidity of the satellite and (2) it uses the Hamiltonian H to establish the control laws. Previous use of H in satellite studies has been directed mainly toward its application as a Lyapunov function. The first such study was performed by Pringle[14]. Meirovitch, Likins, and Budynas and Poli also have published papers in this area. H also was used by Mackinson and Bainum[15] in analyzing the performance of a magnetically damped, rigid gravity gradient satellite and by Bowers and Williams[16] to optimize the timing criteria for the first boom deployment on RAE-I. Although he did not use the word "Hamiltonian", it also was employed by Watson[17] in a study of the capture of a rigid satellite by the gravity field.

The Hamiltonian used here is the H of the nonrigid satellite's mechanical state $\underline{x}$ relative to the rotating local vertical-orbit pole coordinate frame R under the approximation that the orbit is Keplerian. It will be assumed, as necessary, that the active control forces $\underline{u}$ are balanced so that they do not alter the orbit. Consider first the idealized case of a circular orbit and negligible disturbances and internal damping. Assume that all three moments of inertia are unequal, that the satellite includes no parts -such as free rotors- whose motion is not constrained by

stiffness forces, and that it is passive except for the control jets. When $\underline{u} = \underline{0}$, the satellite then will possess time-invariant stable equilibrium states $\underline{x}_{se}$. H can be defined in such a manner that it is zero when $\underline{x} = \underline{x}_{se}$. H then will be positive for any $\underline{x} \neq \underline{x}_{se}$; it also will be time-invariant when $\underline{u} = \underline{0}$ regardless of the $\underline{x}$ motions. H thus can serve as a metric of the displacement of $\underline{x}$ from $\underline{x}_{se}$. The purpose of the on-board control system can be regarded as being to drive H to the smallest possible value. (H can be driven to zero only if the mathematical model of the dynamics is completely controllable by the thruster systems). H or its reduction ΔH, therefore, can serve as performance measures for developing the control laws. The H criteria must be supplemented by a constraint on the $\underline{x}_{se}$ about which capture is achieved in applications where the satellite is tumbling at the start of the control operation and not all of the stable equilibrium orientations are suitable for the mission. Under some conditions - particularly a highly nonrigid satellite, large control forces, and/or large initial attitude rates - control laws based solely on H can yield unacceptably large structural vibrations; in these cases, structural vibration mode amplitudes impose additional constraints on the control.

H will include cyclic coordinates if two moments of inertia are identical or if the satellite includes elements whose motion is not constrained by stiffness forces. Control criteria based solely on H can control the velocities of these cyclic coordinates, but not their magnitudes. The H viewpoint of active control still is applicable when disturbances, orbit eccentricity e, or internal damping are not negligible. H then, however, will not be time-invariant when $\underline{u} = \underline{0}$. Disturbances and e can produce short term variations in H and, under some conditions, secular growth.

Reaction jet systems, particularly those that are operated with low duty cycles, should be effective mainly against the secular component. Internal damping is beneficial, since it yields $\dot{H} \leq 0$. In some situations, active control operations yield the side benefit of increasing the fraction of H which is contained in structural vibration modes where internal damping is present.

In the following sections, 3 x 1 Cartesian vectors are indicated by an over-bar. Lower case letters with an under-bar are column matrices. Upper case letters with an under-bar are rectangular or square matrices. A prime indicates a matrix transpose or a row vector. A comma signifies differentiation. The double subscript summation convention is employed in portions of the work, but only where specifically indicated. Subscripts i and j span the range 1 to m where m is the total number of generalized coordinates. λ_s, μ_s and ν span 1 to n where n = m-3 is the number of "structural" coordinates. ρ and σ span 1 to 3 to indicate rotation coordinates. ξ spans 1 to 2 m to indicate state variables. α and β span 1 to s where s is the number of on-board jets. The Nomenclature Section includes only symbols which are not defined elsewhere in the paper. Specific definitions are not given for symbols which are believed to be self-defining from their usage in the equations.

BASIC EQUATIONS AND CONTROL POLICY

This section develops an approach to the problem of establishing control laws for active augmentation of the dynamics of nonrigid gravity gradient satellites and derives the necessary analytical model. The following two sections employ this material in studies in which the controller forces first are modeled as impulses and then as rectangular pulses. The Hamiltonian rate $\dot{H}$ under the action of control forces and disturbances is significant to the work. The classical equation[18] $\dot{H} = H_{,t}$ is not applicable because it does not include the effects of nonconservative forces and because it assumes that H is expressed as a function of generalized coordinates q, time t, and generalized momentum rather than q, t, and velocity which will be used here. To obtain $\dot{H}$, consider a mechanical system whose configuration is specified by an mxl coordinate vector $\underline{q}$. Using subscript summation convention, H and the Lagrangian L can be expressed as follows:

$$H = .5\, M_{ij}\, \dot{q}_i\, \dot{q}_j + U \tag{1}$$

$$L = .5\, M_{ij}\, \dot{q}_i\, \dot{q}_j + b_j\, \dot{q}_j - U \tag{2}$$

where

$$U = V - T_o + K_u \tag{3}$$

The operations required to derive the desired $\dot{H}$ equation are (a) substitute Eq. (2) into Lagrange's equation[18], (b) multiply this result by $\dot{q}_i$ and sum, (c) form $\dot{H}$ by differentiating Eq. (1) and simplify using the result of step (b), and (d) convert to matrix notation. The result is:

$$\dot{H} = \dot{\underline{q}}' \{ \underline{f}_Q - \underline{b}_{,t} - .5\, \underline{M}_{,t}\, \dot{\underline{q}} \} + U_{,t} \tag{4}$$

where $\underline{f}_Q$ is the mxl generalized force vector.

In the present satellite application, $\underline{q}$ defines the system's configuration relative to a rotating local vertical reference frame R. The origin of R is the satellite's c.m.. The orbit will be constrained in the modeling to be Keplerian. Equations for $\underline{M}$, $\underline{b}$, and U are given in Appendix A. The partial time derivative terms in Eq. (4) arise from (1) active on-board alteration or control of the satellite's geometry or mass distribution and (2) disturbance phenomena that are not included in f_Q. Often, it is easier to model actively-controlled geometric variables as independently-derived time functions $\underline{\ell}(t)$ than it is to include them in $\underline{q}$. In this case, the partial t derivative terms in Eq. (4) can be the main mechanism through which active control of H is accomplished. Such systems, however, will not be investigated in the present study. The work instead will encompass only scleronomous-coordinate models[18] in which the active control is generated through $\underline{f}_Q$ by point forces $\mathcal{U}_\alpha$ and which otherwise are passive. Assuming that the satellite's mass properties are affected insignificantly by the expenditure of controller fuel, $M,_t$ in Eq. (4) then will be zero. $U,_t$ and $\underline{b},_t$, when nonzero, will affect $\dot{H}$ as disturbances. Eq. (A.3) shows that $\underline{b},_t$ will be zero if orbit eccentricity e is zero. $U,_t$ will be zero if $e = 0$ and if, in addition, phenomena, such as temperature gradient variations, that make V_s vary explicitly with t are negligible.

$\underline{f}_Q$ consists of components $\underline{f}_{Qds}$, $\underline{f}_{Qdm}$, and $\underline{f}_{Qu}$ due to disturbances, internal damping, and the active control forces respectively. In the present application, $\underline{f}_{Qds}$ can include not only non-conservative forces, but also conservative phenomena such as gravity gradient harmonics which act as disturbances and hence should be excluded from H. Let the satellite contain s jets and let $\underline{u}$ (t) be the s x 1 vector comprised of the scaler values of their

force outputs. (In the present terminology, a unit which can apply forces of either polarity is considered to be a single jet). Using the classical equation for generalized forces[18], $\underline{f}_{Qu}$ and $\underline{u}$ can be shown to be related through an equation $\underline{f}_{Qu} = \underline{Y}\,\underline{u}$ where $\underline{Y} = \underline{Y}\,(\underline{q})$. Let $\underline{v}_u$ be the s x 1 vector of the scaler values of the translational velocities of the jets, relative to frame R, along their force axes. It can be shown that $\underline{v}_u$ and $\dot{\underline{q}}$ are related through $\underline{Y}'$.

$$\underline{v}_u = \underline{Y}' \dot{\underline{q}} \tag{5}$$

Lumping all $\dot{H}$ components due to disturbances and orbit eccentricity into a single term $\dot{H}_{ds}$ and letting $\dot{H}_u$ and $\dot{H}_{dm}$ be the components due to active control and internal damping respectively, Eq. (4) now becomes

$$\dot{H} = \dot{H}_u + \dot{H}_{dm} + \dot{H}_{ds} \tag{6a}$$

where

$$\dot{H}_u = \underline{v}_u' \underline{u} = \sum_{\alpha=1}^{s} \dot{H}_{u\alpha} = \sum_{\alpha=1}^{s} v_{u\alpha} u_\alpha \tag{6b}$$

The time integral of $\dot{H}_u$ is the work done, relative to frame R, by $\underline{u}$. A block diagram of the system is given as Figure 1.

The purpose of the active control can be regarded as being to reduce H and/or to maintain it small. The present paper will employ as a basic control principle the tenet that none of the individual jets ever should act directly to increase H. When any jet α is fired, the correct polarity of its force u_α then is opposite to the instantaneous velocity $v_{u\alpha}$. This policy is not optimal in all problems. The solutions of minimum time problems, such as the rigid gravity gradient satellite one considered by Zach[11], frequently yield $\dot{H} > 0$ at times during the operation. The advantage of the present $\dot{H}_{u\alpha} \leq 0$ approach is that it provides a relatively simple basis for developing control laws in nonrigid satellite problems where the dynamics

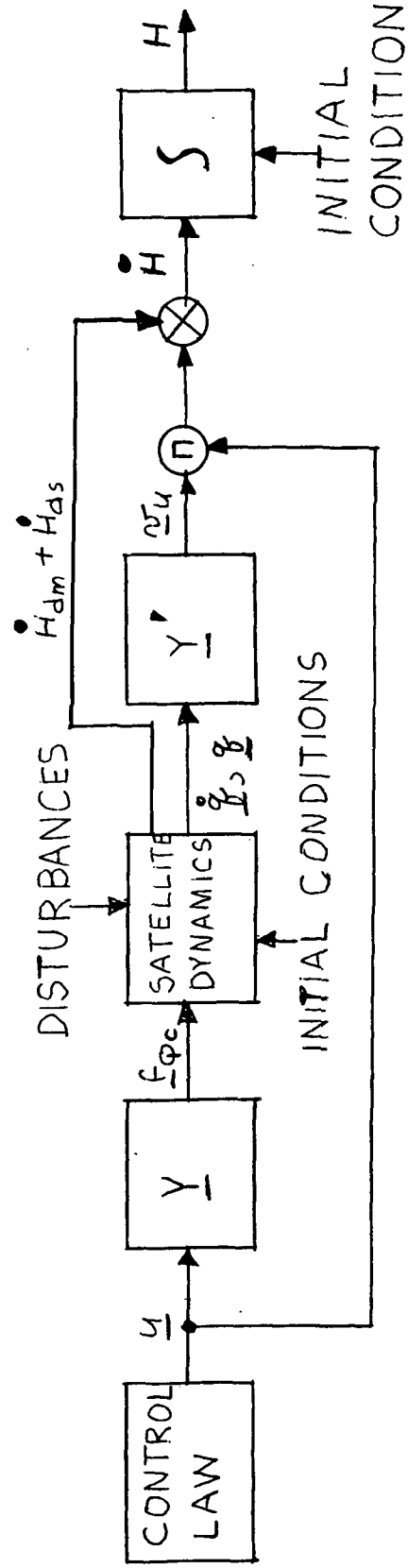

FIGURE 1
SYSTEM BLOCK DIAGRAM

may be nonlinear, high order, and difficult to predict accurately. Depending on the thruster-satellite system and the operational requirements, many control policies are possible within the framework of this $\dot{H}_{u\alpha} \leq 0$ criteria. Eq. (6b) shows that the effectiveness of any jet α in reducing H is directly proportional to the magnitude $|v_{u\alpha}|$ of the instantaneous velocity $v_{u\alpha}$. In applications where fuel utilization efficiency is the main requirement, a simple on-off policy consists of (1) turning each individual jet α on at times when its $|v_{u\alpha}|$ is close to a local maximum and (2) de-energizing α when $|v_{u\alpha}|$ has been reduced to zero or a specified nonzero value. The policy can be modified as necessary to incorporate constraints on the control, such as employing the jets in pairs. The near-maximum $|v_{u\alpha}|$ criteria for energizing the jets is abated in applications where rapid H reduction is more important than fuel economy. By driving the $v_{u\alpha}$'s to zero a sufficient number of times it can be anticipated that, in the absence of disturbances, a condition will be reached wherein the between-pulse values of the $v_{u\alpha}$'s remain negligibly small. This signifies that any remaining motions of the satellite are in uncontrollable modes, and no farther reduction of H by jet firings is possible.

The studies to be presented in the following two sections require dynamical equations for the satellite's response in $\dot{\underline{q}}$, $\underline{q}$, and $\underline{v}_u$. The generalized coordinate method which will be used was employed previously in Ref. 19. The m generalized coordinates are of two types. q_1 to q_3 are a set of Euler angles $\underline{\theta}$ which define the orientation, relative to frame R, of an arbitrarily-selected body frame b of the satellite. The remaining $n = m-3$ coordinates $\underline{\eta}$ specify the satellite's configuration relative to frame b. The term "hybrid coordinates" has been coined by Likins and Wirsching[20] to

indicate "mixed" coordinate sets such as the $\lfloor \underline{\theta}' | \underline{\eta}' \rfloor$ one. The $\underline{\eta}$ portion of $\underline{q}$ is omitted if the satellite is assumed to be rigid. In the equations of motion, it will be convenient to use the angular velocity vector $\bar{\omega}_b$ of frame b relative to frame R instead of $\dot{\underline{\theta}}$. $\bar{\omega}_b$ and $\dot{\underline{\theta}}$ can be related through a transformation $\bar{\omega}_b = \underline{T}_{\omega\theta}\, \dot{\underline{\theta}}$ where $\underline{T}_{\omega\theta} = \underline{T}_{\omega\theta}(\underline{\theta})$. Let $\underline{v}$ be the m x 1 velocity vector $\underline{v} = \lfloor \underline{\omega}_b' | \dot{\underline{\eta}}' \rfloor$. $\underline{v}$ and $\dot{\underline{q}}$ then are related through $\underline{v} = \underline{T}_{vq}\, \dot{\underline{q}}$ where

$$\underline{T}_{vq} = \begin{bmatrix} \underline{T}_{\omega\theta} & \underline{0} \\ \underline{0} & \underline{1} \end{bmatrix}$$

Several rectangular matrices in the dynamics equations must be transformed when switching between $\underline{v}$ and $\dot{\underline{q}}$ velocities. Matrices that are employed with $\underline{v}$ will be indicated by a subscript v. In $\underline{v}$ velocities, Eq. (5) becomes

$$\underline{v}_u = \underline{Y}_v'\, \underline{v} \tag{7}$$

where

$$\underline{Y}_v' = \underline{Y}'\, \underline{T}_{vq}^{-1}$$

The kinetics model which will be used was obtained by modifying the equations developed in Ref. 19 to make them directly applicable to the present application. The resulting equation is

$$\underline{M}_v\, \dot{\underline{v}} + \underline{N}_v\, \underline{v} = \underline{Y}_v\, \underline{u} + \underline{f}_0 + \underline{f}_2 \tag{8}$$

Appendix B presents detailed expressions for the terms appearing in Eq. (8). Similar-appearing equations were used in flexible satellite studies reported in Refs. 20 to 22. The equations in these studies, however, all were linearized, limited to satellites comprised of a rigid central body to which nonrigid auxilliary bodies are attached, and also limited to very specific representations of the nonrigidity. These restrictions have not been imposed on Eq. (8) and the associated equations in Appendix B. The advantage of the

use of $\underline{v}$ rather than $\dot{\underline{q}}$ velocities is largely due to the fact that the method reduces the dependency of the rectangular matrices in Eq. (8) on $\underline{\theta}$. $\underline{M}_v$ and $\underline{Y}_v$ are functions of $\underline{\Omega}$. $\underline{N}_v$ is a function of $\underline{\Omega}$ and $\underline{\theta}$. Unless the satellite is highly nonrigid, however, it usually should be possible to omit the $\underline{\Omega}$ dependency in $\underline{M}_v$, $\underline{Y}_v$, and $\underline{N}_v$. $\underline{M}_v$ is symmetric and positive definite. $\underline{N}_v \underline{v}$ results from (1) gyroscopic forces due to $\bar{\omega}_R$ and (2) viscous damping. $\underline{N}_v$ is skew symmetric if damping is negligible. $\underline{f}_o$ and $\underline{f}_2$ include both actual forces and also "apparent forces" due to kinematic effects. $\underline{f}_o$ is comprised of conservative forces due to internal stiffness and central force field gravity gradient; it also includes the centrifugal potential force. $\underline{f}_2$ consists of apparent forces involving quadratic terms in $\underline{v}$; it also includes disturbance forces.

IMPULSE MODEL OF CONTROL FORCES

This section will use the material developed in the preceding section to investigate the case where the control forces are of sufficiently large magnitude and short time duration that useful information can be obtained by modeling them as impulses. The advantages of the impulse representation are that it is the easiest model to investigate and it yields the simplest result. The study does not require linearization of the equations of motion. The weaknesses of the impulse assumption will be discussed at the end of the section. Let subscripts 0 and 1 indicate conditions at the start and end of a control action which consists of the simultaneous firing of one or more of the s on-board jets. Let $\underline{u}_I$ be the s x 1 impulse vector. $\underline{u}_I$ is the time integral of $\underline{u}$. $\underline{u}_I$ will produce step changes $\Delta \underline{v}$ and ΔH. Since $\underline{q}$ undergoes no change during the impulse, the matrices $\underline{M}_v$, $\underline{N}_v$, and $\underline{Y}_v$ in Eq. (8) will

be constant. $\underline{N}_{\nu\nu}\underline{\nu}$, $\underline{f}_0$, and $\underline{f}_2$ do not affect the $\Delta\underline{\nu}$ step. Eqs. (6) and (8) can be shown to yield

$$\Delta\underline{\nu} = \underline{M}_{\nu}^{-1}\,\underline{Y}_{\nu}\,\underline{u}_I \tag{9a}$$

$$\Delta H = \underline{\nu}_{u0}'\,\underline{u}_I + .5\,\underline{u}_I'\,\underline{D}\,\underline{u}_I \tag{9b}$$

where

$$\underline{D} = \underline{Y}_{\nu}'\,\underline{M}_{\nu}^{-1}\,\underline{Y}_{\nu} \tag{9c}$$

The step change in $\underline{\nu}_u$ is $\Delta\underline{\nu}_u = \underline{D}\underline{u}_I$. Any element $D_{\alpha\beta}$ of $\underline{D}$ thus is the velocity step change along the force axis of jet α which is generated by a unit impulse from jet β. It can be demonstrated that $\underline{D}$ is symmetric and at least positive semidefinite.

Eq. (9b) now will be used to derive the impulse vector $\underline{u}_I^*$ which yields the minimum (i.e., the most negative) value ΔH^* of Δ H. It will be assumed that no amplitude limits are placed on any of the $u_{I\alpha}$'s. The derivation of $\underline{u}_I^*$ then constitutes an unconstrained static minimization problem[24]. The solution given in the following paragraph will demonstrate that $\underline{u}_I^*$ is unique if and only if $\underline{D}$ is nonsingular. $\underline{D}$ will be only positive semidefinite, and hence singular, if the satellite possesses redundant jets. The analytical difficulty which this causes can be overcome by imposing a selected set of s_d constraints on the relations between the $u_{I\alpha}$'s where s_d is the degeneracy[25] of $\underline{D}$. A constraint equation $\underline{u}_I = \underline{C}^c\,\underline{u}_I^c$, therefore, will be assumed. $\underline{u}_I^c$ has dimensions $s_c \times 1$ where $s_c = s - s_d$. The constraints might, for example, be set up to restrict the 3 x 1 Cartesian vector of the total impulse to be zero; this eliminates the potential problem of altering the orbit significantly by the jet firing. The $s_c \times 1$ impulse vector $\underline{u}_I^{\bar{c}}$ can be regarded as the output of a ficticious set of s_c jets which will be referred to here as the "constrained jets." $\underline{u}_I^{c*}$ will be unique, since the

transformed $s_c \times s_c$ matrix $\underline{D}^c = \underline{C}^{c'} \underline{D} \underline{C}^c$ which appears in Eq. (9b) when $\underline{u}_I^c$ is employed is nonsingular. For the present purposes, it will be convenient to impose an additional set of constraints on the jet firing. The discussion in the preceding section indicates that use, in any single firing operation, of all the s_c constrained jets often should be undesirable because it utilizes the available fuel inefficiently. Let $\underline{u}_I^a$ be the $s_a \times 1$ impulse vector from the s_a constrained jets which actually are used in a given firing. $\underline{u}_I^a$ and $\underline{u}_I^c$ can be related by an equation $\underline{u}_I^c = \underline{C}^{ca} \underline{u}_I^a$. $\underline{C}^{ca}$ has dimensions $s_c \times s_a$ where $s_a \leq s_c$. Each row of $\underline{C}^{ca}$ will contain either (1) all zeros or else (2) all zeros except for a single element with a value of unity.

Substitution of the relations specified in the above paragraph into Eq. (9b) yields

$$\Delta H = \underline{v}_{uo}^{a\prime} \underline{u}_I^a + .5\, \underline{u}_I^{a\prime} \underline{D}_a \underline{u}_I^a \tag{10}$$

where $\underline{v}_u^a = \underline{C}' \underline{v}_u$, $\underline{D}^a = \underline{C}' \underline{D} \underline{C}$, and $\underline{C} = \underline{C}^c \underline{C}^{ca}$. $\underline{u}_I^{a*}$ now can be obtained by differentiating Eq. (10) with respect to $\underline{u}_I^a$, setting the left side to zero, and solving. It also is necessary, in principle, to differentiate with respect to $\underline{u}_I^a$ a second time in order to check that the resulting ΔH^* is a minimum rather than a maximum. The result is:

$$\underline{u}_I^{a*} = -\underline{D}_a^{-1} \underline{v}_{uo}^a \tag{11a}$$

$$\Delta H^* = .5\, \underline{v}_{uo}^{a\prime} \underline{u}_I^{a*} = .5 \sum_{\alpha=1}^{s_a} v_{uo\alpha}^a u_{I\alpha}^{a*} \tag{11b}$$

$$\frac{d^2 \Delta H}{d^2 \underline{u}_I^a} = \underline{D}_a \tag{11c}$$

Since $\underline{D}_a$ is positive definite, ΔH^* is a minimum.[24]

$\underline{u}_I^*$ can be obtained from $\underline{u}_I^{a*}$ using $\underline{u}_I^* = \underline{C}\ \underline{u}_I^{a*}$. The equations given in this section can be used to show that $\underline{u}_I^{a*}$ yields $\underline{v}_{u1}^a = \underline{0}$. Thus, the optimal impulse $\underline{u}_I^{a*}$ is the one which drives the force axis velocities, relative to frame R, of the s_a constrained jets to zero. Eq. (11b) shows that the greatest reduction in H in any single firing operation is obtained by using all s_c constrained jets except any whose initial velocities $\underline{v}_{u0\alpha}^a$ are zero. Eq. (11b), however, also shows that the ΔH^* component which is contributed by the optimal impulse $u_{I\alpha}^{a*}$ of any individual constrained jet α is proportional to the magnitude $|\underline{v}_{u0\alpha}^a|$ of its initial velocity. Thus, from the standpoint of fuel efficiency, the most effective firing procedure is to fire each constrained jet only at times when its $|\underline{v}_{u0\alpha}^a|$ is passing through a maximum.

Practical satellites usually should be sufficiently rigid that the variation of $\underline{D}_a^{-1}$ with $\underline{\Omega}$ can be omitted when establishing the criteria for active control. Eq. (11a) shows that $\underline{u}_I^{a*}$ then is not dependent on the complete state $\underline{x}$ of the satellite. Rather, it is dependent only on $\underline{v}_{u0}^a$ If the satellite's sensor system is such that a measure of $\underline{v}_u^a$ can be obtained without generating an estimate of $\underline{x}$, actual knowledge of $\underline{x}$ then is not needed for control.

While the unlimited-amplitude impulse model of the jet outputs yields a simple criteria (Eq. (11a)) for determining the instantaneous optimum control effort, the technique encounters two major difficulties when the satellite is highly nonrigid. First, even if the mathematical model of the satellite and jet outputs were perfect, it is still possible that the $\underline{u}_I^{a*}$ computed by Eq. (11a) might generate unacceptably large structural vibrations. Placing amplitude limits on the impulses is one approach toward alleviating the

potential problem. With this method $\underline{u}_I^{\partial *}$ would be obtained by solving Eq. (10) subject to control constraints $|u_{I\alpha}^c| \leq u_{I\alpha\,max}^c$ ($\alpha = 1$ to s_c). A less fallible, but more difficult, approach is to place the amplitude constraints directly on the magnitudes of the structural vibration modes. This method is used in the following section in the rectangular pulse approximation study.

The second difficulty concerns the compatibility between the approximation that the jet outputs are impulses and the implicit assumption that, for the present purposes, the nonrigidity of the satellite can be modeled adequately by a finite number of coordinates η_ν. The use of a finite number of η_ν's to model a continuous structure requires, in principle, the truncation of an infinite set. The η_ν's which are included are ones which contribute mainly to the lower frequency modes of vibration; the η_ν's which are omitted contribute mainly to the higher modes. Before Eq. (11a) could be used in an actual nonrigid satellite application, it would be necessary to verify, by supplementary analysis, that the excitation of the neglected vibratory modes by the short force pulses will not be great enough to cause difficulties. Also, Eq. (11a) will be accurate only if the time durations of the actual pulses are short enough in relation to the periods of the vibratory modes which are included in the model that the impulse approximation is acceptable. The filtering action which tends to attenuate the effect of non-impulsive inputs on higher vibration modes is not encountered in the present model. The computed $\underline{u}_I^{\partial *}$ therefore, depends on the number of η_ν's that are used.

RECTANGULAR PULSE MODEL OF CONTROL FORCES

In the present section, the active control forces will be approximated as rectangular pulses. This representation alleviates, to a considerable extent, the impulse model difficulties noted at the end of the previous section. The on-off control policy discussed earlier will be used. The problem which will be considered is the computation of the optimum pulse duration τ when the "height" of the force pulse and the satellite's dynamic state $\underline{x}_0$ at the start of the pulse are known. In an actual application, the pulse durations would have to be determined a priori by such a computation if lags or other deficiencies in the sensor system prohibit a "closed loop" determination of the proper jet shut-off points using real-time information.

An analytical investigation using non-impulsive control forces requires, as a practical necessity, a linearized, constant coefficient model of the dynamics. The first step, therefore, will be to linearize Eq. (8). A model which is applicable at least over the time spans of the individual pulses is desired. Linearization of Eq. (8) limits the study to cases where the quadratic $\underline{v}$ terms in $\underline{f}_2$ are negligible over the duration of a pulse. The present work will omit disturbances. $\underline{f}_2$ then can be deleted entirely. The study also must be restricted to cases where the coordinate changes $\Delta \underline{q}$ during a control pulse are sufficiently small that $\underline{M}_v$ and $\underline{N}_v$ can be considered constant and that $\underline{f}_0$ can be approximated by the first two terms in a Taylor series expansion. Most satellites should be sufficiently rigid that no major restrictions on the model's validity are imposed by omission of the higher order effects due to the dynamics of the structural coordinates $\underline{\eta}$. However, when the satellite's initial attitude $\underline{\theta}_0$ is not close to the stable equilibrium attitude, the subsequent $\underline{\theta}$ motions can be large enough to severely degrade the

linearized model's validity. In such cases, the linearized model is applicable only if the control pulses are sufficiently short that only small changes in $\underline{\theta}$ are experienced during their duration. The study then must be limited to large amplitude-short duration pulses such as those which were modeled as impulses in the preceding section.

In linearizing Eq. (8), $\underline{v}$ will be transformed into a velocity vector which can be approximated as the time derivative of an $m \times 1$ coordinate vector $\underline{\beta}$. Consider a structural reference condition $\underline{\eta}_r$ which is close to the initial value $\underline{\eta}_o$. Let β_4 to β_m be chosen as $\underline{\eta} - \underline{\eta}_r$. The condition $\underline{\eta}_{se}$ encountered when the satellite is in equilibrium usually should be a suitable choice for $\underline{\eta}_r$. In establishing the rotation coordinates β_1 to β_3, use will be made of an auxilliary reference frame "r" which has zero angular velocity relative to frame R and which, at t_o, has an orientation close to that of frame b. Let β_1 to β_3 be a set of Euler angles $\underline{\phi}$ which specify the attitude of b relative to r. $\underline{\phi} = \underline{0}$ occurs when b and r are aligned. $\underline{\omega}_b$ and $\dot{\underline{\phi}}$ can be related through an equation $\underline{\omega}_b = \underline{T}_{\omega\phi}\,\dot{\underline{\phi}}$ where $\underline{T}_{\omega\phi} = \underline{T}_{\omega\phi}(\underline{\phi})$ and $\underline{T}_{\omega\phi}(0) = \underline{1}$. Similarly, $\underline{v}$ and $\dot{\underline{\beta}}$ can be related through $\underline{v} = \underline{T}_{v\beta}\,\dot{\underline{\beta}}$ where $\underline{T}_{v\beta} = \underline{T}_{v\beta}(\underline{\phi})$ and $\underline{T}_{v\beta}(\underline{0}) = \underline{1}$; $\underline{T}_{v\beta}$ is defined analogously to $\underline{T}_{vg}$ used previously. Employing this relation in Eq. (8), premultiplying by $\underline{T}'_{v\beta}$ and dropping $\underline{f}_2$ and a quadratic velocity term involving $\dot{\underline{T}}_{v\beta}\,\dot{\underline{\beta}}$ yields

$$\underline{T}'_{v\beta}\,\underline{M}_v\,\underline{T}_{v\beta}\,\ddot{\underline{\beta}} + \underline{T}'_{v\beta}\,\underline{N}_v\,\underline{T}_{v\beta}\,\dot{\underline{\beta}} = \underline{T}'_{v\beta}\,\underline{f}_o + \underline{T}'_{v\beta}\,\underline{Y}_v\,\underline{u} \qquad (12)$$

Eq. (12) will be linearized about the reference (r) condition $\underline{\beta} = \underline{0}$. Superscript r will be used to indicate quantities which are evaluated at $\underline{\beta} = \underline{0}$. When expanded about $\underline{\beta} = \underline{0}$ and linearized, the left side of Eq. (12) reduces to $\underline{M}^r_v\,\ddot{\underline{\beta}} + \underline{N}^r_v\,\dot{\underline{\beta}}$. It can be shown that $\underline{T}'_{v\beta}\,\underline{f}_o = -\,U_{,\underline{\beta}}$

where the symbol $U_{,\underline{\beta}}$ indicates $[U_{,\beta_1} \cdots U_{,\beta_m}]'$. Expanding $U_{,\underline{\beta}}$ in a series about $\underline{\beta} = \underline{0}$ and retaining only the first two terms yields $U_{,\underline{\beta}} = \underline{f}_o^r + \underline{K}_v^r \underline{\beta}$ The elements of $\underline{K}_v^r$ are listed in Appendix B. The $\underline{T}'_{v\beta}$ premultiplication in Eq. (12) was performed so that the resulting stiffness matrix $\underline{K}_v^r$ would be symmetric. Expanding $\underline{T}'_{v\beta}\underline{Y}_v \underline{u}$ and retaining the only the first two terms yields $\underline{Y}_v^r \underline{u} - \underline{P}^r \underline{\beta}$ where the columns $\underline{p}_j^r$ of the unsymmetric $m \times m$ matrix $\underline{P}^r$ are of the form $\underline{p}_j^r = -\underline{B}_j^r \underline{u}$. Detailed expressions for the $\underline{p}_j^r$'s are given in Appendix B. In applications where $\underline{u}$ is constant over the time intervals that the linearized, constant coefficient model is to be used, $\underline{P}^r$ can be included by adding it to $\underline{K}_v^r$ as an additional stiffness effect. It is not certain that $\underline{P}^r$ will be negligibly small with respect to $\underline{K}_v^r$ in all cases. The first three $\underline{P}^r$ rows appear to be potentially comparable in size to the similar $\underline{K}_v^r$ rows whose elements are nonzero solely because of gravity gradient and centrifugal forces. It is believed, however, that omission of $\underline{P}^r\underline{\beta}$ usually should be justifiable largely on the grounds that is much smaller than $\underline{Y}_v^r \underline{u}$. Consequently, $\underline{P}^r\underline{\beta}$ will not be included in the remainder of the paper.

Making the above-noted modifications to Eq. (12) and performing similar operations on Eq. (7) yields the desired linear equations.

$$\underline{v}_u = \underline{Y}_v^{r\prime} \dot{\underline{\beta}} \tag{13a}$$

$$\underline{M}_v^r \ddot{\underline{\beta}} + \underline{N}_v^r \dot{\underline{\beta}} + \underline{K}_v^r \underline{\beta} = \underline{Y}_v^r \underline{u} + \underline{f}_o^r \tag{13b}$$

In order to solve Eq. (13b), conversion to state variables $\underline{x}$ is advantageous. Letting $\underline{x} = [\dot{\underline{\beta}}' \mid \underline{\beta}']'$, Eq. (13) can be converted to the form

$$\underline{v}_u = [\underline{Y}_v^{r\prime} \mid \underline{0}]\, \underline{x} \tag{14a}$$

$$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{g} \tag{14b}$$

where

$$\underline{A} = \left[\begin{array}{c|c} -\underline{M}_{\nu}^{r^{-1}}\,\underline{N}_{\nu}^{r} & -\underline{M}_{\nu}^{r^{-1}}\,\underline{K}_{\nu}^{r} \\ \underline{1} & \underline{0} \end{array}\right] \tag{14c}$$

$$\underline{g} = \left\{\begin{array}{c} \underline{g}_a \\ \underline{0} \end{array}\right\} = \left\{\begin{array}{c} \underline{M}_{\nu}^{r^{-1}}\left(\underline{Y}_{\nu}^{r}\,\underline{u} + \underline{f}_0^{r}\right) \\ \underline{0} \end{array}\right\} \tag{14d}$$

Eqs. (14) next will be transformed into modal coordinates $\underline{z}$. This is considered desirable in the present work because it enables a simpler and more informative solution to be obtained. The $2m$ eigenvalues λ_ξ of $\underline{A}$ will be assumed to be distinct. Let $\underline{x} = \underline{S}\,\underline{z}$ where the $2m$ columns $\underline{s}_\xi$ of $\underline{S}$ are the eigenvectors of $\underline{A}$. Let $\underline{R}$ be $\underline{S}'^{-1}$, and let $\underline{r}_\xi$ indicate the columns of $\underline{R}$. Each $\underline{s}_\xi$ will be separated into two $m\times 1$ components $\underline{s}_\xi = [\underline{s}'_{a\xi} \;\vdots\; \underline{s}'_{b\xi}]'$; this same notational technique will be used later with $\underline{g}$ and $\underline{r}_\xi$. The s columns of $\underline{Y}_{\nu}^{r}$ will be indicated by $m\times 1$ dimension vectors $\underline{y}_{\nu\alpha}^{r}$. Eqs. (14a) and (14b) then can be written in the form

$$v_{u\alpha} = \underline{y}_{\nu\alpha}^{r'} \sum_{\xi=1}^{2m} \underline{s}_{a\xi}\, z_\xi \tag{15a}$$

$$\dot{z}_\xi = \lambda_\xi z_\xi + \underline{r}_\xi' \cdot \underline{g} \tag{15b}$$

The solution of Eqs. (15b) is

$$z_\xi = e^{\lambda_\xi (t-t_0)}\,\underline{r}_\xi' \left[\underline{x}_0 + \int_{t_0}^{t} e^{-\lambda_\xi(\tau - t_0)}\,\underline{g}(\tau)\,d\tau\right] \tag{16}$$

More than one jet may be in use at any given time because of their non-zero firing durations and because of possible constraints on their operation - such as firing them in pairs. Subject to restrictions which such constraints may impose, each jet α should be energized when its $|v_{u\alpha}|$ becomes appropriately large. (The present terminology assumes that each jet can apply forces of either polarity). α is de-energized at, or slightly before, the time that $v_{u\alpha}$ changes sign. Assuming that the jet pulses are rectangular, $\underline{g}$ in Eq. (16) will be constant during "constant control condition" (CCC) time segments during which no jets are turned on or off. Inserting the constant $\underline{g}$ condition into Eq. (16), the z_ξ responses during a CCC are determined to be

$$z_\xi = \underline{h}_\xi' \left[\left(\underline{x}_0 + \frac{1}{\lambda_\xi} \underline{g} \right) e^{\lambda_\xi (t - t_0)} - \frac{1}{\lambda_\xi} \underline{g} \right] \tag{17}$$

Assuming that the CCC is ended by de-energizing a jet because its terminal $v_{u\alpha}$ condition is reached rather than by energizing a new one, the CCC end time can be computed using Eqs. (15a) and (17). The computation consists of a determination (by iteration for example) of the first time point at which the cut off $v_{u\alpha}$ is reached by any of the jets that are in use. An equation for the ΔH due to jet action during the CCC can be derived using Eqs. (6), (15a), and (17). The result is

$$\Delta H = \underline{u}' \underline{V}_v^{h'} \sum_{\xi=1}^{2m} \frac{1}{\lambda_\xi} \underline{b}_{a\xi} \underline{h}_\xi' \left[\left(\underline{x}_0 + \frac{1}{\lambda_\xi} \underline{g} \right) \left(e^{\lambda_\xi (t - t_0)} - 1 \right) - (t - t_0) \underline{g} \right] \tag{18}$$

As was noted previously, a drawback of the pure H reduction criteria for establishing control laws is that it potentially can generate unacceptably large structural vibrations even if the mathematical model can be considered perfect. On some satellites, this problem can be alleviated by employing the thrusters in sets so that the sensitivity of certain of the modes to the firing is reduced or eliminated. The present paper will consider the supplementary technique in which the jets are turned off prior to reaching the $\sigma_{u\alpha} = 0$ condition when necessary to prevent the magnitudes of critical structural modes from exeeding selected limits. This approach encounters some difficulty if the satellite is highly nonrigid and the attitude motions following the control application are sufficiently large that the subsequent structural dynamics are significantly influenced by nonlinear effects. The problem is that energy transfer due to nonlinear coupling, in principle at least, can cause the amplitudes of some of the vibratory modes to increase after the jets are shut off. This phenomena, however, will not be considered in the present work. The study, instead, must be restricted to cases where the previously-developed linearized model is adequate.

It will be assumed that, for the present purposes, internal damping can be considered negligible and that the 2n eigenvalues λ_ρ of the structural modes thus will be imaginary: $\lambda_\rho = j\omega_\rho$. ζ_ρ, $\underline{\Delta}_\rho$ and $\underline{r}_\rho$ will be complex. Since the modes occur in complex conjugate pairs, only n need be included. Using the rectangular control pulse approximation, Eq. (15b) can be integrated once to yield the following equations for the real (R) components of the modal motions.

$$\tilde{v}_{\nu R}^2 + (\tilde{z}_{\nu R} - a_{\nu u})^2 = K_\nu^2 \tag{19a}$$

where

$$\tilde{z}_{\nu R} = z_{\nu R} + \frac{1}{\omega_\nu} \underline{r}'_{a\nu I} \cdot \underline{g}_{af} \tag{19b}$$

$$a_{\nu u} = \frac{-1}{\omega_\nu} \underline{r}'_{a\nu I} \cdot \underline{g}_{au} \tag{19c}$$

$$\tilde{v}_{\nu R} = \frac{1}{\omega_\nu} \dot{\tilde{z}}_{\nu R} \tag{19d}$$

K_ν is a positive constant which depends on the initial conditions. $\underline{g}_{au}$ and $\underline{g}_{af}$ are the components of $\underline{g}_a$ due to $\underline{u}$ and $\underline{f}_o^r$. $\underline{r}_{a\nu I}$ is the imaginary component of $\underline{r}_{a\nu}$. The trajectories defined by Eqs. (19) are circles in the $\tilde{v}_{\nu R} - \tilde{z}_{\nu R}$ phase planes. When $\underline{u} = \underline{0}$, the centers of the loci are at $\tilde{v}_{\nu R}$, $\tilde{z}_{\nu R} = 0$. Firing one or more jets shifts the centers to $z_{\nu R} = a_{\nu u}$.

A suitable choice for the amplitude limit constraint equations is

$$\tilde{v}_{\nu R}^2 + \tilde{z}_{\nu R}^2 \leq L_\nu^2 \tag{20}$$

It is assumed that all modes ν are inside their L_ν boundaries at the start of the control operation. Figure 2 shows a typical response. If, as the result of a control action begun at t_o, a mode ν reaches its L_ν boundary at t_1, a modification to the control is made at t_1 to prevent L_ν from being crossed. This normally would consist of de-energizing the jets that excite mode ν. It would take precedence over the $v_{u\alpha} = 0$ criteria. The point 1 where ν reaches L_ν can be determined by solving Eqs. (19a) and (20). The result is

$$\tilde{z}_{\nu R1} = (L_\nu^2 + a_{\nu u}^2 - K_\nu^2)/2a_{\nu u} \tag{21a}$$

$$\tilde{v}_{\nu R1} = \pm (L_\nu^2 - \tilde{z}_{\nu R1}^2)^{.5} \tag{21b}$$

An imaginary $\tilde{v}_{\nu R1}$ indicates that the L_ν boundary will not be reached. The time $t_1 - t_o$ can be shown to be

$$(t_1 - t_o) = \frac{2}{\omega_\nu} \sin^{-1} \left\{ \cdot \frac{5}{K_\nu} \left[(\tilde{z}_{\nu R1} - \tilde{z}_{\nu R0})^2 + (\tilde{v}_{\nu R1} - \tilde{v}_{\nu R0})^2 \right]^{.5} \right\} \tag{22}$$

CONCLUDING REMARKS

The Hamiltonian provides a useful measure of the displacement of gravity gradient satellites from equilibrium. It can be used as a tool for developing supplementary active control laws for rigid and nonrigid satellites. Such approaches deal with the composite system in a unified manner and can be used in cases the more common methods are difficult to apply. The main drawback appears to be the potentiality of exciting intolerably large structural vibrations when the satellite is nonrigid and the initial attitude motions are large. The velocities $v_{u\alpha}$ of the thrusters along their force axes are of primary concern for control. The impulse model of the jet outputs yields a simple and practical control criteria when the forces are large and the satellite is near-rigid. This approach must be applied with considerable caution, however, if the satellite is highly nonrigid. The rectangular pulse model of the jet outputs has far wider applicability, but leads to more complicated control law mathematics when an "open loop" computation of the jet cut-off times is needed.

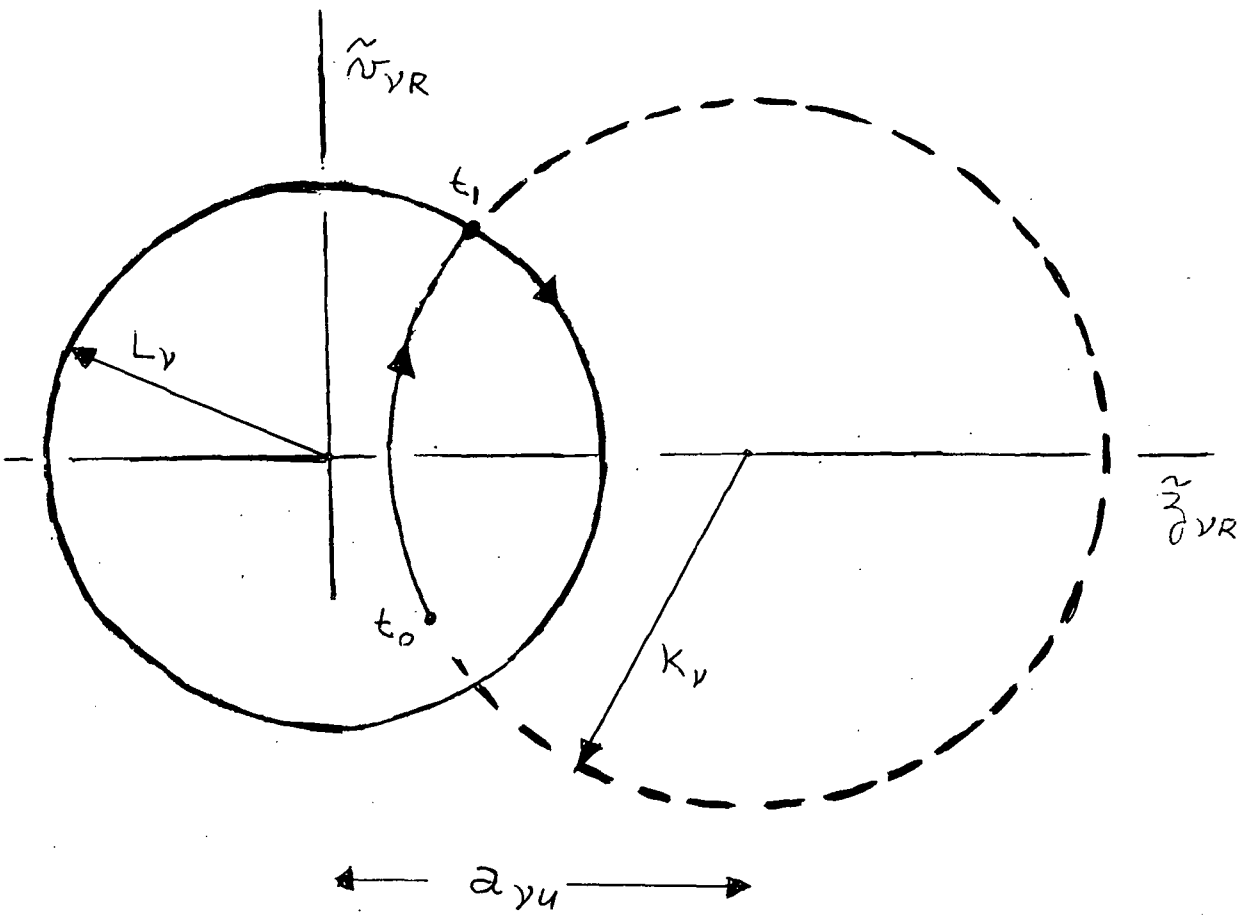

Figure 2. Phase Plane Response of Mode ν

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APPENDIX A

EQUATIONS FOR THE HAMILTONIAN OF NONRIGID GRAVITY STABILIZED SATELLITES

This Appendix and the following one are included to aid in interpreting and employing the material in the main body of the paper. The equations for H can be derived straightforwardly using basic principles. The general form of H was given in Eqs. (1) and (3). Eq. (1) can be converted into matrix notation and can be expressed in $\underline{v}$ velocities by the transformation $\underline{v} = \underline{T}_{vq}\,\dot{\underline{q}}$. In the present application, the terms in Eqs. (1) and (3) can be shown to be

$$\underline{M}_v = \left[\begin{array}{c|c} \underline{J} & -\underline{A}^2 \\ \hline -\underline{A}^{2\prime} & \underline{A}^1 \end{array}\right] \tag{A.1a}$$

$$\underline{M} = \underline{T}_{vq}^{\prime}\,\underline{M}_v\,\underline{T}_{vq} \tag{A.1b}$$

$$U = -\frac{2\mu m e \cos\gamma}{a(1-e^2)^3} + \frac{.5\mu}{R^3}\left[3\bar{e}_R^{\prime}\,\underline{J}\,\bar{e}_R - \mathrm{Trace}\,\underline{J}\right] \tag{A.2}$$

$$-.5\,\bar{\omega}_R\,\underline{J}\,\bar{\omega}_R + V_s - .5\,\dot{\underline{\ell}}^{\prime}\,\underline{A}_{\ell}^{1}\,\dot{\underline{\ell}} + \bar{\omega}_R^{\prime}\,\underline{A}_{\ell}^{2}\,\dot{\underline{\ell}} + K_u$$

The term $\underline{b}$ which is used in Eqs. (2) and (4) is

$$\underline{b}_v^{\prime} = \bar{\omega}_R^{\prime}\left[\underline{J} \,|\, -\underline{A}^2\right] \tag{A.3a}$$

$$\underline{b} = \underline{T}_{vq}^{\prime}\,\underline{b}_v \tag{A.3b}$$

$\underline{A}^1$ and $\underline{A}^2$ are $n \times n$ and $3 \times n$ respectively. The elements $A^1_{\mu\nu}$ of $\underline{A}^1$ and columns $\bar{A}^2_\nu$ of $\underline{A}^2$ are obtained by integrating functions of $\underline{q}$ and $\underline{\ell}$ over the composite satellite; the exact definitions of $A^1_{\mu\nu}$ and $\bar{A}^2_\nu$ are given in Ref. 19. $\underline{\ell}$ is comprised of coordinates which are treated as independently-derived time functions and hence not included in $\underline{q}$. When the satellite is passive, $\dot{\underline{\ell}}$ is zero. $\underline{A}^1_\ell$ and $\underline{A}^2_\ell$ are defined similarly to $\underline{A}^1$ and $\underline{A}^2$.

APPENDIX B

EQUATIONS OF MOTION OF NONRIGID GRAVITY STABILIZED SATELLITES

This Appendix lists detailed equations for the terms in Eqs. (8) and (13). Eq. (8) was obtained by converting the basic nonrigid satellite equations in Ref. 19 into a form which is more convenient for the present study. The main modification was the use of angular rate $\bar{\omega}_b$ relative to frame R in place of $\bar{\omega}$ relative to a nonrotating frame. $\bar{\omega}$, $\bar{\omega}_b$, and their first time derivatives with respect to frame b are related by: $\bar{\omega} = \bar{\omega}_b + \bar{\omega}_R$; $\dot{\bar{\omega}} = \dot{\bar{\omega}}_b + \bar{\omega}_R \times \bar{\omega}_b$. The other modifications are mainly of a notational and algebraic manipulation nature. It might be noted in particular that the symbol η is used here in place of Ref. 19's q. The main restrictions on the equations given in this Appendix are that e and $\dot{\underline{\ell}}$ are assumed zero.

$\underline{M}_\nu$ was defined in Appendix A. The other terms in Eq. (8) are

$$\underline{N}_\nu = \begin{bmatrix} \underline{N}^{11} & \underline{N}^{12} \\ -\underline{N}^{12'} & \underline{N}^{22} \end{bmatrix} \tag{B.1a}$$

$$\underline{N}^{11} = -\left[\left(2\underline{J} - \underline{1}\,\mathrm{Trace}\,\underline{J}\right)\cdot\bar{\omega}_R\right]^S \tag{B.1b}$$

$$\bar{N}^{12}_\nu = -2\,\underline{A}^S_\nu \cdot \bar{\omega}_R \tag{B.1c}$$

$$N^{22}_{\mu\nu} = -2\,\bar{A}^{3'}_{\mu\nu}\,\bar{\omega}_R + N^{22}_{dm\mu\nu} \tag{B.1d}$$

$$\underline{Y}_\nu = \begin{bmatrix} \underline{z} \\ \underline{w} \end{bmatrix} \tag{B.2a}$$

$$\bar{Z}_\alpha = \bar{y}_\alpha \times \bar{e}_{u\alpha} \tag{B.2b}$$

$$W_{\nu\alpha} = \bar{y}_{\alpha,\nu} \cdot \bar{e}_{u\alpha} \tag{B.2c}$$

$$\underline{f}_0 = \begin{Bmatrix} \bar{G}_0 \\ -U_{,\underline{n}} \end{Bmatrix} \tag{B-3a}$$

$$\bar{G}_0 = 3\,\omega_R^2\,\bar{e}_R^S\,\underline{J}\,\bar{e}_R - \bar{\omega}_R^S\,\underline{J}\,\bar{\omega}_R \tag{B.3b}$$

$$f_2 = \begin{Bmatrix} \bar{G}_2 \\ \underline{f}_{2\eta} \end{Bmatrix} \tag{B.4a}$$

$$\bar{G}_2 = \bar{G}_{ds} - \bar{A}^6_{\mu\nu}\,\dot{\eta}_\mu\,\dot{\eta}_\nu - \bar{\omega}_b^S\,\underline{J}\,\bar{\omega}_b + 2\dot{\eta}_\nu\,\underline{A}^5_\nu\,\bar{\omega}_b \tag{B.4b}$$

$$f_{2\eta_\mu} = f_{\varphi ds\mu} - A^4_{\mu\nu\lambda}\,\dot{\eta}_\nu\,\dot{\eta}_\lambda + .5\,\bar{\omega}_b'\,\underline{J}_{3\mu}\,\bar{\omega}_b + 2\dot{\eta}_\nu\,\bar{A}^{3'}_{\mu\nu}\,\bar{\omega}_b \tag{B.4c}$$

$\bar{N}^{12}_\nu$ and $\bar{Z}_\alpha$ are the columns of $\underline{N}^{12}$ and $\underline{Z}$. The double subscript summation convention is used with λ, μ and ν. Superscript S indicates that the Cartesian vector is arranged in its 3 x 3 skew-symmetric form. The A terms are defined in Ref. 19. $U_{,\underline{n}}$ is an n x 1 vector formed by differentiating U with respect to the η elements. Vectors which appear in the first three rows of Eq. (8) are resolved on frame b.

The elements of $\underline{K}^r_\nu$ in Eq. (13) will be derived next. Limiting the work to the $e = 0, \dot{\underline{\ell}} = \underline{0}$ case, Eq. (A.2) can be written as

$$U = V_s + .5\,\omega_R^2\,(3\,\bar{e}_R^{r\prime}\,\underline{C}_{br}'\,\underline{J}^b\,\underline{C}_{br}\,\bar{e}_R^r - \bar{e}_p^{r\prime}\,\underline{C}_{br}'\,\underline{J}^b\,\underline{C}_{br}\,\bar{e}_p^r - \mathrm{Trace}\,\underline{J}^b) + K_u \tag{B.5}$$

Superscripts b and r indicate the resolution frame. $\underline{C}_{br}$ is the direction cosine matrix from r to b. $\bar{e}_R^r$ is constant, $\underline{C}_{br} = \underline{C}_{br}(\underline{\varphi})$, $\underline{J}^b = \underline{J}^b(\underline{\eta})$,

and $V_s = V_s(\underline{\eta})$. Since $\underline{\beta}' = \mathcal{L}\,\underline{\phi}'[\underline{\eta}' - \underline{\eta}'_r]$, we thus have $U = U(\underline{\beta})$. As was noted earlier, $\underline{K}^r_v = [U_{,\underline{\beta\beta}}]^r$. Separating $\underline{K}^r_v$ into submatrices yields

$$\underline{K}^r_v = \begin{bmatrix} U_{,\underline{\phi\phi}} & U_{,\underline{\phi\eta}} \\ U'_{,\underline{\phi\eta}} & U_{,\underline{\eta\eta}} \end{bmatrix} \tag{B.6}$$

The elements $U_{,\mu\nu}$ of the n x n matrix $U_{,\underline{\eta\eta}}$ can be obtained directly by differentiating Eq. (B.5). The columns $U_{,\underline{\phi}\nu}$ of the 3 x n matrix $U_{,\underline{\phi\eta}}$ can be shown to be equal to $-\underline{T}'_{\omega\phi}\,\bar{G}_{o,\nu}$. $\underline{T}'_{\omega\phi}$ will be unity, since it is to be evaluated at the reference condition $\underline{\phi} = \underline{0}$. $\bar{G}_{o,\nu}$ is formed by differentiating Eq. (B.3b) with respect to η_ν; $\underline{J}$ is the only term in Eq. (B.3b) which is a function of $\underline{\eta}$. The elements $U_{,\rho\sigma}$ of the 3 x 3 matrix $U_{,\underline{\phi\phi}}$ are obtained by differentiating Eq. (B.5) with respect to ϕ_σ and ϕ_ρ. As an aid in the derivation, it is advantageous to first specify the sequence of the $\underline{\phi}$ rotations. Let the order in going from frame r to frame b be ϕ_1, ϕ_2, ϕ_3 where the subscripts 1, 2, 3 also indicate axes x_1, x_2, x_3. It then can be shown that $\underline{C}'_{br,\sigma\rho} = \underline{C}'_{br,\rho\sigma} = -\underline{C}'_{br}[\bar{e}^b_{\phi\beta}]^s[\bar{e}^b_{\phi\alpha}]^s$ where α is the larger of the ρ, σ pair and β is the smaller; the $\bar{e}_\phi$'s are unit vectors along the instantaneous axes of Euler angle rotation. Using this relation and Eq. (B.5), $U_{,\rho\sigma}$ can be shown to be

$$\begin{aligned} U_{,\rho\sigma} = -\omega_R^2\,\bar{e}'_{b\beta}\,[\,3\,\bar{e}^s_R\,(\underline{J}\,\bar{e}^s_R + \{\underline{J}\,\bar{e}_R\}^s) \\ - \bar{e}^s_p\,(\underline{J}\,\bar{e}^s_p + \{\underline{J}\,\bar{e}_p\}^s)]\,\bar{e}_{b\alpha} \end{aligned} \tag{B.7}$$

The m x m matrix $\underline{P}^T$ generated when expanding $\underline{T}'_{\nu\beta}\,\underline{Y}_\nu\,\underline{u}$ in Eq. (12) will be considered next. Its columns $\underline{p}_j^T$ are

$$\underline{p}_j^T = -\left[\frac{\partial}{\partial \beta_j}\left(\underline{T}'_{\nu\beta}\,\underline{Y}_\nu\right)\right]^T \underline{u} \tag{B.8}$$

Let the $\underline{\phi}$ rotations be taken in the order noted in the preceding paragraph. Noting that $\underline{Y}_\nu^T\,\underline{u}$ is the control torque $\bar{G}_u^T$, it then can be shown that

$$[\underline{p}_1\ \underline{p}_2\ \underline{p}_3] = \begin{bmatrix} 0 & G_{u3} & -G_{u2} \\ 0 & 0 & G_{u1} \\ 0 & 0 & 0 \\ & \underline{0} & \end{bmatrix} \tag{B.9}$$

where $G_{u\sigma}$ is the torque component along axis b_σ. The remaining $\underline{p}$'s are

$$\underline{p}_\nu = -\left[\underline{Y}_{\nu}\,\underline{\nu}\right]^T \underline{u} \tag{B.10}$$