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SEMILINEAR (TOPOLOGICAL) SPACES

AND APPLICATIONS*

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0. INTRODUCTION

In this paper we define semivector spaces and develop some of their algebraic aspects including some structure theory; then topopogize these spaces to obtain semilinear topological spaces for which we identify a hierarchy of local convexity axioms; establish a number of fixed point and minmax theorems for spaces with various local convexity properties; illustrate how the spaces of concern arise naturally as various hyperspaces of linear and semilinear (topological) spaces; and, finally, indicate briefly how all the above are applied in socio-economic analysis and optimization.

In contrast to vector spaces, we build semivector spaces upon underlying commutative <u>semi</u>-groups, so that the space need not have an origin (identity element) and its elements need not possess inverses. In the same contrast, a consequent weakness of semivector spaces is the ability for the distribution $(\lambda + \mu)s = \lambda s \oplus \mu s$ to fail altogether or for certain pairs (λ, μ) of field elements, where s is a generic semivector, λs and μs are results of "scalar multiplication" and $\oplus$ is "semivector addition." Neither is $0s$ required to equal the origin e even if e belongs to the semivector space (0 denotes the additive identity of the field). Also, in the case where the reals are taken as the field, singleton sets need not be convex. These are some of the features distinguishing semivector spaces from vector spaces. Section 1 treats these and other algebraic matters pertaining to semivector spaces. As a result, some ("gross") structure theory is also developed. Examples are provided to illustrate a variety of semivector spaces.

In Section 2, semivector spaces are topologized by requiring their algebraic operations ("scalar multiplication" and "semivector addition") to be continuous. It is asked when the topology of the mob underlying a semilinear topological space can be strengthened to yield translates of open sets by points open without destroying the continuity of either of the algebraic operations. After a hierarchy of local convexity axioms is identified for real semilinear topological spaces, product invariance is investigated for spaces of the various types of local convexity.

Section 3 deals with the real semilinear topological spaces in which singleton sets are convex ("pointwise convexity"). In this section we generalize or extend central fixed point results due to S. Kakutani [194], H. F. Bohnenblust and S. Karlin [1950] and K. Fan [1952], which in turn are generalizations of results due to L. E. J. Brouwer [1912], J. Schauder [1930] and A. Tychonoff [1935], respectively. In particular, our Theorem 3.1 generalizes Kakutani's FPT (Fixed Point Theorem), and it is used to establish Theorem 3.2 where Tychonoff's FPT is generalized by means of procedure used by Fan [1952]. Theorem 3.3 and the Minmax Theorems 3.4 and 3.5 are applications which we believe to have significance for economic theory and the theories of games and optimization. Theorem 3.6 extends Fan's FPT. Theorem 3.7 and the Minmax Theorem 3.8 stand in relation to it as do 3.3 and 3.4 to 3.2. Using the fixed point theory of this section, elsewhere, in [Prakash, 1971] and [Sertel, 1971], we show the existence of dynamic equilibria for certain rather general social and economic systems. J. L. Kelley has indicated the importance of convexity arguments as the basis for results distinguishing the theory of linear topological spaces from that of topological groups [1955; p. 110]. The results presented here may be taken to illustrate that much of the power gained from convexity properties does not require as strong a structure as that of a linear topological space.

Section 4 constructs a variety of semivector and topological semivector spaces as hyperspaces. Many of these, e.g., the semivector spaces consisting of the non-vacuous compact and convex sets in a vector space, are important for social and economic analysis and the theories of games and optimization. In models used in these areas, the behavors in a social system or consumers and producers in an economic system or the players in a game or, finally, optimizing agents in general are restricted in their choice typically to a compact and convex region in a linear topological space. For example, in the case of a consumer, the choice of commodity bundles (vectors) which can be enjoyed is the set of commodity bundles which can be acquired without exceeding a budget constraint. In the case of a player in a game, the choice of strategy may be limited to the set of probability measures defined on some sigma-algebra of "actions" (see [Sertel, 1969, 1971]), where this set of probability measures is convex, but also compact when suitable topologies are used.

Section 5 indicates, albeit briefly, how the material of the earlier sections is applied in social and economic equilibrium analysis and optimization. We intend to make such applications the subject of a soon forthcoming separate paper.

1. SEMIVECTOR SPACES

1.0 Definition: Let $(S, \oplus)$ be an Albelian semigroup and let $(F, +, .)$ be a field,[1] denoting its additive and multiplicative identity by 0 and 1, respectively. Then S together with a map $\Psi: F \times S \to S$, where we denote $\Psi(\lambda, s) = \lambda s$, will be called a semivector space over F iff the following are satisfied:

Axiom 1: $1s = s$

Axiom 2: $\lambda(\mu s) = (\lambda \cdot \mu)s \qquad s, t \in S; \lambda, \mu \in F.$

Axiom 3: $\lambda(s \oplus t) = \lambda s \oplus \lambda t$

The elements of S will be called semivectors. A subset $T \subset S$ will be called a semivector subspace of S iff it is a semivector space under the restrictions to T of the algebraic operations of S. If $F = R$, the field of reals, then S will be called a real semivector space.

One may note that Axioms 1 and 2 yield a special type of automaton further particularized by Axiom 3.

Given a semivector space S over a field F, for each $\lambda \in F$ let Ψ_λ denote the restriction of Ψ to $\lambda \times S$. Then, for each $\lambda \neq 0$, Ψ_λ is an automorphism of S onto S; furthermore, if S has an origin (or a null element, or identity) e defined by the property that $e \oplus s = s$ for all $s \in S$, then $\lambda e = e$.

[1] The reader will soon notice, as M. P. Schützenberger already has, that the full set of field axioms for F is unnecessary for much of the development to follow. In ignoring the fact that these axioms can be relaxed at various points in the development below, we are exchanging some (easily obtainable) generality for uniformity in exposition.

On the other hand, Ψ_0 is an endomorphism of S under which $\Psi_0(S)$ is a semivector subspace which is "unscaled," in the sense that $\mu n = n$ for all $n \varepsilon \Psi_0(S)$ and all $\mu \varepsilon F$. We will denote $\Psi_0(S)$ by N.

As with an ordinary semigroup, if S does not have an identity, one may pass from S to $S \cup \{e\}$ by adjunction of an identify e (where e does not stand for any element in S). In the case where S is a semivector space, this is done by extending $\oplus$ and Ψ to $S \cup \{e\}$, setting $e \oplus e = e$, $e \oplus s = s \oplus e = s (s \varepsilon S)$, and $\lambda e = e (\lambda \varepsilon F)$. Henceforth we will adopt the convention of denoting by S^e a semivector space with identity e, obtained by adjunction if necessary.

From here on Λ_n will denote the simplex $\{(\lambda_0, \ldots, \lambda_n) \varepsilon E^{n+1} | \sum_{i=0}^{n} \lambda_i = 1; \lambda_i \geq 0, i=0, \ldots, n\}$. Given any two semivectors x,x' in a real semivector space S, their segment $[x:x']$ will be defined as $\{s = \lambda x \oplus \lambda' x' | (\lambda,\lambda') \varepsilon \Lambda_1\}$. A subset $T \subset S$ will be called convex iff $[x:x'] \subset T$ whenever $x, x' \varepsilon T$. The following are plain: if A is convex in S, then $\mu A = \{\mu a | a \varepsilon A\}$ is convex $(\mu \varepsilon R)$; if B, too, is convex in S, then so are $A \oplus B = \{a \oplus b | a \varepsilon A, b \varepsilon B\}$ and all convex combinations $\lambda A \oplus \lambda' B$ $((\lambda,\lambda') \varepsilon \Lambda_1)$. It is important to note that, unlike in vector spaces, in semivector spaces there is no guarantee that x or x' belongs to $[x:x']$ or even that $x \varepsilon [x:x]$.

For example, consider the commutative semigroup[2] $([R], \oplus)$ of the set $[R]$ of all non-empty subsets, $A, B \subset R$, and obtain a real semivector space with origin $e = \{0\}$ by setting $\lambda A = \{\lambda a |\ a \in A\}$ if $\lambda \neq 0$, and $\lambda A = R$ otherwise.

In further contrast to vector spaces, a semivector space S need not have an identity (merely delete all singletons in the above example), nor need $0s = e$ even if S has an identity e (readmit singleton $\{0\}$ as e). To continue, any of the examples above illustrates that S need not be cancellative. Of course, S need not have an inverse s^{-1} for each $s \in S$, nor need $s^{-1} = \Psi(-1, s)$ even if it has - for example, form a real semivector space by taking the additive group G of reals and setting $\Psi(\lambda, g) = |\lambda| g$ $(\lambda \in R, g \in G)$. A property conspicuously missing in all of the above examples is the distributivity $(\lambda + \mu)s = \lambda s \oplus \mu s$, but the following definition enables us to study the structure of semivector spaces with regard to various localized versions of this property.

1.1 Definition: Let $(S, \oplus, \Psi)$ be a semivector space over a field F. The set $D(S) \subset F \times F$ defined by $D(S) = \{(\lambda, \mu) |\ (\lambda + \mu)s = \lambda s \oplus \mu s$ for all $s \in S\}$ will be called the region of distributivity of S, and S will be said to distribute at (λ, μ) iff $(\lambda, \mu) \in D(S)$. A real semivector

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[2] The semigroup operation $\oplus$ being defined by $A \oplus B = \{a + b | a \in A, b \in B\}$ for all $A, B \in [R]$.

space S will be called <u>pointwise convex</u> iff $\Lambda_1 \subset D(S)$; i.e., $[s:s] = s (s \in S)$.

Since S and F are both commutative, it follows directly from the definition of $D(S)$ that $D(S)$ is symmetric, i.e., that $(\lambda, \mu) \in D(S)$ if $(\mu. \lambda) \in D(S)$. Furthermore, by Axioms 2 and 3, $(\lambda + \mu)s = \lambda s \oplus \mu s$ implies $\nu.(\lambda + \mu)s = (\nu. \lambda)s \oplus (\nu.\mu)s$, so that $(\lambda, \mu) \in D(S) \Rightarrow F(\lambda, \mu) \subset D(S)$, where $F(\lambda, \mu)$ denotes the set $\{(\nu.\lambda, \nu.\mu) | \nu \in F\}$.

1.2 <u>THEOREM</u>: Let S be a semivector space over a field F. Then (denoting $\Psi_0(S) = N$) the following are equivalent:

1. $(0, 0) \in D(S)$;
2. N is a commutative band (in fact an "unscaled" semivector subspace);
3. $\{T_n = \Psi_0^{-1}(n) | n \in N\}$ is a partition of S into a semilattice of semivector subspaces T_n each of which distributes at $(0, 0)$. Furthermore, if F is the field of reals, then each of the above is equivalent to:
4. N is pointwise convex.

<u>Proof</u>: [It was noted earlier in this section that N is an "unscaled" semivector subspace of S].

<u>ad (1 => 2)</u>: For any $0s = n \in N$, $0s \oplus 0s = (0 + 0)s = 0s = n$, so that N consists of idempotents.

<u>ad (2 => 3)</u>: [The T_n's clearly partition S].

Writing $\mathrm{Sup}(m, n) = m \oplus n$ in (the semilattice) N, the blocks T_n form a semilattice by setting $\mathrm{Sup}(T_m, T_n) = T_{\mathrm{Sup}(m, n)}$. Defining $T_m \oplus T_n = \{t_m \oplus t_n \mid t_m \in T_m, t_n \in T_n\}$, also note $T_m \oplus T_n \subset T_{\mathrm{Sup}(m,n)}$.] Let $t_m, t'_m \in T_m$. T_m is a semivector subspace, as $0(\lambda t_m) = m$, whereby $\lambda t_m \in T_m$; and as $0(t_m \oplus t'_m) = m$ by idempotence of m, so that T_m is closed under $\oplus$. Finally, T_m distributes at $(0, 0)$, as $(0 + 0)t_m = m = m + m = 0t_m \oplus 0t_m$.

ad (3 => 1): If each T_n distributes at $(0, 0)$, then so does $s = \bigcup_N T_n$. Now assume $F = R$.

ad (2 => 4): Use the unscaledness of N and the idempotence of each of its members.

ad (4 => 1): Given any $s \in S$, $0s = n$ and $(\lambda, \lambda') \in \Lambda_1$, we have $0s \oplus 0s = \lambda(0s) \oplus \lambda'(0s)$, and pointwise convexity of N equates this to $0s = n$.

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1.3 THEOREM: Let $\{(T_a, \oplus^a, \Psi^a) \mid a \in A\}$ be a semilattice of disjoint semivector spaces over F such that, for each $a \in A$, $\Psi_0{}^a(T_a) = \{a\}$ and T_a distributes at $(0, 0)$. Denote $\bigcup_A T_a = S$. Then there exists a semivector space $(S, \oplus, \Psi)$ over F of which each $(T_a, \oplus^a, \Psi^a)$ is a semivector subspace, and S distributes at $(0, 0)$.

Proof: For example, define $\oplus$ and Ψ as below. Order A so that $a \geq b$ iff $T_a \geq T_b$, and define $a \oplus b = \mathrm{Sup}(a, b)$ $(a, b \in A)$. [Replace 'Sup' by 'Inf' throughout if $\{T_a \mid a \in A\}$

is a lower rather than an upper semilattice.] Finally, set

$$t_a \oplus t'_b = \begin{cases} \mathrm{Sup}(a, b), & \text{if } a \neq b; \\ t_a \oplus t'_b, & \text{if } a = b; \end{cases}$$

($t_a \in T_a$, $t_b \in T_b$; $a, b \in A$); and define $\Psi(s) = \Psi^a(s)$ iff $s \in T_a (a \in A)$.

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We now turn to structural aspects of distribution at $(0, \lambda) \neq (0, 0)$.

1.4 <u>THEOREM</u>: Using the notation of 1.2, among the following statements, the first two are equivalent; furthermore, if $F = R$, then all three are equivalent.

1. S distributes at a point $(0, \lambda)$ distinct from $(0, 0)$.
2. $\{T_n \mid n \in N\}$ is a partition of S into a semilattice of semivector subspaces T_n (each of which distributes at $(0, \lambda)$) such that n is the origin of $T_n (n \in N)$.
3. For each $s \in S$, s is the first element of $[s:s]$.

<u>Proof</u>: <u>ad (1 => 2)</u>: As shown immediately after 1.1, if $(0, \lambda) \in D(S)$, then $(\mu.0, \mu.\lambda) \in D(S)$ for all $\mu \in F$. Setting $\mu = 0$ and using 1.2, it remains only to prove that n is the origin of T_n, which now follows from the fact that S distributes also at $(0, 1)$: for any $t_n \in T_n$, $n \oplus t_n = 0t_n \oplus 1t_n = (0 + 1)t_n = t_n$.

<u>ad (2 => 1)</u>: Trivial.

Now assume $F = R$.

ad (2 => 3): Let $s \in T_n$. Then $0s \oplus 1s = n \oplus s = s$.

ad (3 => 1); Obvious.

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1.5 THEOREM: In the statement of Theorem 1.3, replace (0, 0) by $(0, \lambda)$ for some $\lambda \neq 0$ and strengthen the hypothesis so that a is the origin of T_a $(a \in A)$.

Proof: Same as that of Theorem 1.3.

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We close this section by giving some exercises and examples in illustration of some facts which follow easily from the above.

1.6 Exercise: Let S^e be a cancellative semivector space.[3] Then, among the following statements, the first three are equivalent, 4 implies 5, and all follow from 6.

1. S^e distributes at (0, 0).
2. S^e distributes at (0, 1).
3. $0s = e$ for all $s \in S^e$.
4. For all $s \in S^e$, $(-1)s \oplus s = e$.
5. If S^e distributes at (a, b), then it distributes at (a + b, -b) and (a + b, -a).
6. S^e distributes at some (c, d) and (c + d, -d) for which $d \neq 0$.

Proof: 3 => 2 => 1 even without cancellation. Also, 1 => 3, for $0s \oplus 0s = 0s$ cancels to $0s = e$. Given 4, if $(a + b)s$

[3] A semivector space S is cancellative iff $\oplus s = \oplus t$ implies $s = t$ for all $r,s,t \in S$.

$= as \oplus bs$, then $(a + b)s \oplus (-b)s = as \oplus bs \oplus (-b)s = as$, whereby S^e distributes at $(a + b, -b)$; the rest of 5 follows similarly from 4. Obviously 6 implies 1, i.e., 1-3. To see that it implies 4, hence 5, merely note that $\frac{1}{d} e = e$, since $d \neq 0$, and that $cs = (c + d - d)s = (c + d)s \oplus (-d)s = cs \oplus ds \oplus (-d)s$, cancelling to $e = ds \oplus (-d)s$.

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1.7 Exercise: A pointwise convex space S^e is a vector space iff $(-1)s \oplus s = e$ for all $s \in S$.

1.8 Remark: We should caution the reader that a pointwise convex space S need not contain an origin, and that, even if it does, $0s = e$ need not be satisfied by all $s \in S^e$. These deficiencies are illustrated in the order of mention by the examples below.

1.9 Example: Let $(X, \oplus^1, \Psi^1)$ and $(Y, \oplus^2, \Psi^2)$ be disjoint pointwise convex spaces containing their respective origins $e^1 = \Psi^1{}_0(X)$ and $e^2 = \Psi^2{}_0(Y)$. Define a pointwise convex space $(X \cup Y, \oplus, \psi)$ as follows:

(i) $x \oplus x' = x \oplus^1 x'$ $\qquad (x, x' \in X)$,

(ii) $y \oplus y' = y \oplus^2 y'$ $\qquad (y, y' \in Y)$,

(iii) $x \oplus y = e^1$ $\qquad (x \in X, y \in Y)$;

(iv) $$\Psi_\lambda(s) = \begin{cases} \Psi_\lambda{}^1(s), & \text{if } s \in X; \\ \Psi_\lambda{}^2(s), & \text{if } s \in Y. \end{cases}$$

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Evidently, $X \cup Y$ contains no origin.

1.10 <u>Example</u>: Let $(X, \oplus^1, \Psi^1)$ and $(Y, \oplus^2, \Psi^2)$ be as above, except that if $x \in X$ has an inverse in X then $x = e^1$ is now required. Define a pointwise convex space $(X \cup Y, \oplus, \Psi)$ with identity e^1 by leaving (i), (ii) and (iv) of 1.9 unchanged and modifying (iii), for $x \in X$ and $y \in Y$, to

$$\text{(iii)}' \quad x \oplus y = \begin{cases} y, \text{ if } x = e^1; \\ x, \text{ if } x \neq e^1. \end{cases}$$

Evidently, $0y = e^2 \neq e^1$ for all $y \in Y$.

1.11 <u>Exercise</u>:[4] Let X be a semivector space over a field F and $(Y, \oplus)$ an Abelian semigroup. In particular, X and Y may be objects in the category of semivector spaces over a field F. Let $\mathrm{Hom}(X, Y)$ denote the set of all morphisms from X to Y. Define "scalar multiplication" and "semivector addition" in $\mathrm{Hom}(X, Y)$ by setting $\lambda f(x) = f(\lambda x)$ and $(f \oplus g)(x) = f(x) \oplus g(x)$ for all $x \in X$, all $\lambda \in F$ and all $f, g \in \mathrm{Hom}(X, Y)$. Then, $\mathrm{Hom}(X, Y)$ is a semivector space over the field F. These facts are not altered even if X is strengthened to be a vector space or Y to be an Abelian group.

[4] This example was suggested by some of the constructions of Keimel [1967a and 1967b].

2. TOPOLOGICAL SEMIVECTOR SPACES

2.0 Definition: Let S and F be topological spaces such that F is a field and $(S, \oplus, \Psi)$ is a semivector space over F. Then $(S, \oplus, \Psi)$ is called a semilinear topological space (or topological semivector space) over F iff $\oplus$ and Ψ are continuous.

In a semilinear topological space, it is immediate from the continuity of Ψ that each Ψ_λ is continuous. Hence, writing $\mu = \frac{1}{\lambda}$, so long as $\lambda \neq 0$, we have Ψ_μ continuous, whereby Ψ_λ is both open and a closed map. It follows also that Ψ is an open map when restricted to $(F \setminus \{0\}) \times S$.

Strengthenability of the Topology[5]

If $(S^e, \oplus)$ is a commutative mob with Hausdorff topology on S^e, it is possible to strengthen the topology on S^e without destroying the continuity of $\oplus$ and in such a way that (i) the nbd system of e is unaltered, while (ii) $U \oplus s$ is now open whenever U is open in S^e and $s \in S^e$ [Paalman-De Miranda, 1964; Theorem 3.2.13]. Given a semilinear topological space $(S^e, \oplus, \Psi)$ over a field F with S^e Hausdorff, by a "strengthened" topology on S^e we will mean one which satisfies (i) and (ii) as just stated. We may ask now whether there exists a strengthened topology on S^e under which $(S^e, \oplus, \Psi)$ remains a semilinear topological space. (Of course, in a linear topological space the topology is already a strengthened version of itself).

[5] We are grateful to Prof. T. Bowman for pointing out an error in an earlier draft of this section.

Having Paalman-De Miranda's result as stated above, the question clearly boils down to whether the continuity of Ψ can be preserved under a strengthened topology on S^e. Although we are unable to assert in general when this can be done and when it cannot, we recognize a research problem here and offer the following as an example where it cannot be done even though the space whose topology is to be strengthened is, as the reader may check, a pointwise convex semilinear topological space with identity and with a topology which is locally compact, metrizable, 3° locally convex (see 2.2 and 4.4), etc.

Example: Let F be the real field with usual topology, and define a real semilinear topological space $(\mathcal{KQ}(R), \oplus, \Psi)$ over F, where $\mathcal{KQ}(R)$ is the set of all nonempty compact and convex subsets of R (the topological group of the reals with usual topology), where $[a, b] \oplus [c, d] = \{x + y \mid (x, y) \in [a, b] \times [c, d]\}$ and $\lambda[a, b] = [\lambda a, \lambda b]$ for all $a, b, c, d \in R$, $\lambda \in F$, with $a \leq b$ and $c \leq d$, and where the finite topology is taken on the hyperspace $S^e = \mathcal{KQ}(R)$. Fix attention to any non-singleton $t \in S^e$, and consider the restriction $\Psi^t : F \times \{t\} \to S^e$ of Ψ. Strengthen the topology on S^e by declaring the translates $U \oplus s$ to be (basic) open for each $s \in S^e$ and each "originally" open $U \subset S^e$. The fact is that Ψ^t is not continuous under the strengthened topology on S^e. For let U be an open nbd of e, and consider the (basic) open nbd $U \oplus \lambda t$ of λt for some $\lambda > 0$. Now the inverse image Ω of $U \oplus \lambda t$ under Ψ^t

contains (λ, t), but it contains no (μ, t) such that $0 < \mu < \lambda$. For suppose that $0 < \mu < \lambda$ and that $\mu t = \lambda t \oplus u$ for some $u \in U$. Then $t = \frac{\lambda}{\mu} t \oplus \frac{1}{\mu} u$, which is impossible since diam $(\frac{\lambda}{\mu} t \oplus \frac{1}{\mu} u) \geq$ diam $(\frac{\lambda}{\mu} t)$ and, as $\frac{\lambda}{\mu} > 1$, diam $(\frac{\lambda}{\mu} t)$ > diam (t), whereby diam (t) < diam $(\frac{\lambda}{\mu} t \oplus \frac{1}{\mu} u)$, contradicting that $\mu t = \lambda t \oplus u$. Thus, $\lambda = \mathrm{Inf}(\Omega)$, so that Ω is not open showing that Ψ^t is not continuous. Thus, the semilinear topological space just considered, despite all its properties, does not remain a semilinear topological space when its topology is strengthened in the fashion sought.

Real Semilinear Topological Spaces and Axioms of Local Convexity

The following intuitively pleasing fact is a natural one early to check.

2.1 Lemma: In real semilinear topological spaces topological closure (Cl) preserves convexity.

Proof: Let Q be convex in S, a real semilinear topological space. If $Q = \phi$ there is nothing to prove, so let q, q' be adherent points of Q. Suppose $\lambda q \oplus \lambda' q' = \bar{q} \notin \mathrm{Cl}(Q)$ for some $(\lambda, \lambda') \in \Lambda_1$. Then there exists a nbd V of $\bar{q}$ disjoint from $\mathrm{Cl}(Q)$. The map $\Omega: S \times S \to S$, defined by $\Omega(x,x') = \lambda x \oplus \lambda' x'$, being continuous, there is a nbd $U \times U'$ of (q, q') such that $\Omega(U \times U') \subset V$. Since q and q' are adherent points of Q, however, there exists $(y, y') \in U \times U'$ such that $y, y' \in Q$. Then, by convexity of Q, $\Omega(y, y') \in Q$, a contradiction.

#

Apart from preparation for their use in the fixed point theory of Section 3, our motivation for stating the following "axioms of local convexity" derives from the fact that, although for a topological subspace X of a linear topological (Hausdorff) space the first three are always equivalent and all four are equivalent when X is convex, we are able to assert only weaker relationships between them in the case of semilinear topological spaces. Given a subset X in a real semilinear topological space, we have the following

2.2 <u>Axioms</u>:

0. For any $x \in X$ and any nbd V of x, in the subspace topology of X there exists a convex nbd U of x such that $U \subset V$;
1. There exists a quasi-uniformity $E = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in E$, there exists a closed $E_\beta \in E$ with $E_\beta \subset E_\alpha$ and $E_\beta(x)$ convex for each $x \in X$;
2. There exists a quasi-uniformity $E = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in E$, there exists a closed $E_\beta \in E$ with $E_\beta \subset E_\alpha$ and $E_\beta(K)$ convex for each compact and convex subset $K \subset X$;
3. X is convex and there exists a uniformity $E = \{E_\alpha \subset X \times X \mid \alpha \in A\}$ of X inducing its subspace topology, such that, for each $E_\alpha \in E$, there exists a convex $E_\beta \in E$ with $Cl(E_\beta) \subset E_\alpha$.

X will be called 0°/1°/2°/3° locally convex (l.c.) accordingly as it satisfies 0/1/2/3 among these axioms. Thus, 0° local convexity is the familiar local convexity.

2.3 Proposition: Given a subset X of a real semilinear topological space,

1. If X is 1° l.c., then it is 0° l.c.;
2. If X is 2° l.c. and pointwise convex, then it is 1° l.c.; and
3. If X is 3° l.c., then it is 2° l.c.

2.4 Proposition: Every 0° l.c. T_1 space is pointwise convex.

Proof: Let X be a 0° l.c. T_1 space, and let $x \in X$. As X is 0° l.c., there is a local base $\mathcal{B} = \{B_\alpha | \alpha \in A\}$ at x consisting of convex nbds. Thus, $x \in B = \bigcap_A B_\alpha$, and B is convex. In fact, $B = \{x\}$. For, supposing $y \in B$ for some $y \neq x$, as X is T_1, there exists a nbd U of x to which y does not belong, whereby $y \notin B_\alpha \subset U$ for some $B_\alpha \in \mathcal{B}$, contradicting that $y \in B$. Thus $\{x\}$ is convex. #

Products of Semilinear Topological Spaces

Given a family $\{(S_\alpha, \oplus^\alpha, \Psi^\alpha) | \alpha \in A\}$ of semivector spaces over a field F, their product $(S = \prod_A S_\alpha, \oplus, \Psi)$ is the semivector space the algebraic operations $\oplus$ and Ψ of which are defined coordinatewise as follows:

$$\{s_\alpha\}_{\alpha \in A} \oplus \{t_\alpha\}_{\alpha \in A} = \{s_\alpha \oplus^\alpha t_\alpha\}_{\alpha \in A}$$

$$\Psi_\lambda(\{s_\alpha\}_{\alpha \in A}) = \{\Psi_\lambda^\alpha(s_\alpha)\}_{\alpha \in A}$$

$(s_\alpha, t_\alpha \in S_\alpha;\ \alpha \in A;\ \{s_\alpha\}_{\alpha \in A}, \{t_\alpha\}_{\alpha \in A} \in S;\ \lambda \in F)$. Of course, $e = \{e_\alpha\}_{\alpha \in A}$ is the origin of S iff e_α is the origin of S_α for each $\alpha \in A$. Furthermore, S distributes wherever all of its factors do so. Turning to semilinear topological spaces, the product, taking the product topology on S, is, of course, a semilinear topological space iff each of its factors is.

2.5 __Lemma:__ Let $\{X_\alpha \mid \alpha \in A\}$ be a family of 2° l.c. spaces of which all but finitely many X_α are convex, and let E be a quasi-uniformity inducing the product topology on $X = \prod_A X_\alpha$. Then, for every $F \in E$, there exists a closed $E \in E$ such that $E \subset F$ and $E(K)$ is convex whenever K is the product $K = \prod_A K_\alpha$ of compact and convex subsets $K_\alpha \subset X_\alpha$.

__Proof:__ Contained in F, find a basic $H \in E$ which restricts a finite set $N \subset A$ of coordinates, including (w.l.g.) the set $M \subset A$ of indices m for which X_m is not convex. Now

$$H = \prod_N H_n \times \prod_{A\setminus N} (X_\alpha \times X_\alpha),$$

where H_n belongs to the quasi-uniformity E_n of $X_n (n \in N)$. For each $n \in N$, using the 2° l.c. of X_n, find a closed $E_n \in E_n$ such that $E_n \subset H_n$ with $E_n(K_n)$ convex for each compact and convex $K_n \subset X_n$. Write $E = \prod_N E_n \times \prod_{A\setminus N} (X_\alpha \times X_\alpha)$. #

2.6 __Lemma:__ The product of a family of 1° l.c. spaces is 1° l.c. if all but a finite number of the factor spaces are convex.

<u>Proof</u>: Imitate the last proof. #

Upon noting that projections preserve convexity, and assuming that the product space is compact, it is easily verified also that the factor spaces are 1°/2° l.c.. Finally, the proof of the following stronger proposition for the case of 3° l.c. spaces is omitted being straightforward.

2.7 <u>Proposition</u>: The product of a family of spaces is 3° l.c. iff each of the factor spaces is 3° l.c..

3. FIXED POINT AND MINMAX THEOREMS

Throughout this section we will be concerned only with pointwise convex real semilinear topological (pcrst) spaces, not necessarily containing an origin; these spaces will be assumed Hausdorff, and the real field will be taken with the usual topology.

Given topological spaces X and Y and a mapping f of X into the set of non-empty subsets of Y, when we say that f is upper semicontinuous (usc), we will mean that, for each $x \in X$, given a nbd $V \subset Y$ of $f(x)$, there exists a nbd U of x such that $f(U) \subset V$.

For the composition of two binary relations $F \subset A \times B$ and $E \subset C \times D$, we will write $E \circ F$ for the set (binary relation)

$$\{(a, d) \mid \exists x \in B \cap C \text{ such that } (a, x) \in F \text{ and } (x, d) \in E\}.$$

In the sequel, '$CQ(X)$' should be read as "the set of non-empty closed and convex subsets of X".

3.1 THEOREM (Fixed Point): Let S be a pcrst space, let X be the closed convex hull $X = \{x = \lambda_o a_o \oplus \ldots \oplus \lambda_n a_n \mid \lambda = (\lambda_o, \ldots, \lambda_n) \in \Lambda_n\}$ of $\{a_o, \ldots, a_n\} \subset S$, and let $f\colon X \to CQ(X)$ be an upper semi-continuous transformation of X into the set $CQ(X)$ of non-empty, closed and convex subsets of X. Then there exists a (fixed) point $x \in X$ such that $x \in f(x)$.

Proof: Let $\psi\colon \Lambda_n \to X$ be the map defined by $\psi(\lambda) = \lambda_o a_o \oplus \ldots \oplus \lambda_n a_n$, and let $\Psi\colon \Lambda_n \times \Lambda_n \to X \times X$ be the map defined by $\Psi(\lambda, \mu) = (\psi(\lambda), \psi(\mu))$. Since the algebraic operations of S are continuous, so are ψ and Ψ.

Let $g \subset X \times X$ be the graph of f and let $G \subset \Lambda_n \times \Lambda_n$ be the graph of the map $F: \Lambda_n \to \Lambda_n$ defined by $F(\lambda) = \psi^{-1}(f(\psi(\lambda)))$. Thus $G = \psi^{-1}(g)$. Since Λ_n is compact, by continuity of ψ, $X = \psi(\Lambda_n)$ is compact, hence regular. Thus g is closed, since f is usc. Hence, by continuity of ψ, G is closed, whereby F is usc by compactness of Λ_n.

Clearly, for each $\lambda \in \Lambda_n$, $F(\lambda)$ is non-empty; also, it is closed, since $f(\psi(\lambda))$ is closed and ψ is continuous. Furthermore, $F(\lambda)$ is convex. For let $\mu, \mu' \in F(\lambda)$, i.e., for some $y, y' \in f(\psi(\lambda))$, let $y = \mu_o a_o \oplus \ldots \oplus \mu_n a_n$ and $y' = \mu'_o a_o \oplus \ldots \oplus \mu'_n a_n$; for $\beta' = (1 - \beta) \in [0, 1]$ let $\bar{\mu} = \beta\mu \oplus \beta'\mu'$. Then $\bar{\mu} \in F(\lambda)$, since, denoting $\bar{y} = \bar{\mu}_o a_o \oplus \ldots \oplus \bar{\mu}_n a_n$, by pointwise convexity of S, $\bar{y} = \beta y \oplus \beta' y'$; and, by convexity of $f(\psi(\lambda))$, $\beta y \oplus \beta' y' \in f(\psi(\lambda))$.

Hence, by Kakutani's fixed point theorem (1941), there exists a $\lambda^* \in \Lambda_n$ such that $\lambda^* \in F(\lambda^*)$, implying that $\psi(\lambda^*) \in f(\psi(\lambda^*)) \subset X$. #

3.2 <u>THEOREM (Fixed Point)</u>: Let $f: X \to X$ be a continuous transformation of a 1° l.c., compact and convex subset X of a pcrst space, Then there exists a (fixed) point $x \in X$ such that $x = f(x)$.

<u>Proof</u>: Since X is compact, there exists a unique uniformity on X compatible with its subspace topology. Since X is 1° l.c. we assume that $\{E_\alpha \subset X \times X \mid \alpha \in A\}$ is a fundamental system of closed entourages of this uniformity such that $E_\alpha(x)$ is

(closed and) convex for all $x \in X$. Define $Y_\alpha = \{x \mid x \in E_\alpha(f(x))\}$. We will show that Y_α is non-empty and closed for each $\alpha \in A$. Then, as the intersection of any finite collection of Y_α's is non-empty, compactness of X will imply that $\bigcap_{\alpha \in A} Y_\alpha \neq \phi$, thus proving the theorem, for $x \in \bigcap_{\alpha \in A} Y_\alpha$ implies $x = f(x)$.

To show that Y_α is non-empty, let $\{D_\alpha \mid \alpha \in A\}$ be a family of open symmetric entourages such that $D_\alpha \subset E_\alpha$ $(\alpha \in A)$. Thus, for any given $\alpha \in A$, $\{D_\alpha(x) \mid x \in X\}$ is an open cover of X, so that there exist $x_0, \ldots, x_n \in X$ with $X \subset \bigcup_{i=0}^{n} D_\alpha(x_i)$. Denote the closed convex hull of $\{x_0, \ldots, x_n\}$ by $P = \{p = \lambda_0 x_0 \oplus \ldots \oplus \lambda_n x_n \mid \lambda = (\lambda_0, \ldots, \lambda_n) \in \Lambda_n\}$. Define the map G_α on P by $G_\alpha(p) = E_\alpha(f(p)) \cap P$. Then, by symmetricity of $D_\alpha \subset E_\alpha$, for all $p \in P$, $G_\alpha(p)$ is non-empty; clearly it is also closed and convex. Thus G_α maps P into $\mathcal{CQ}(P)$. Denoting the graph of $E_\alpha \circ f$ by Ω_α, the graph of G_α is simply $\Gamma_\alpha = \Omega_\alpha \cap P \times P$. Since E_α is usc (by the closedness of E_α in the compact $X \times X$) and since f is continuous, $E_\alpha \circ f$ is usc, i.e., Ω_α is closed, as X is regular (in fact, compact). Hence, Γ_α is closed and, by compactness of the range X, G_α is usc. Thus, by Theorem 3.1, there exists $p \in P$ such that $p \in G_\alpha(p)$, i.e., $p \in Y_\alpha$, showing that Y_α is non-empty. Y_α is obviously closed, since it is nothing but the projection $\pi_X(\Omega_\alpha \cap \Delta)$ of the compact set $\Omega_\alpha \cap \Delta$ where Δ is the diagonal in $X \times X$.

#

3.3 THEOREM (Fixed Point): Let $\{X_\alpha | \alpha \in A\}$ be a family of 1° l.c., compact and convex subsets of (pcrst) spaces, and let $\{f_\alpha: X \to X_\alpha | \alpha \in A\}$ be a family of continuous functions on $X = \prod_A X_\alpha$. Define $F: X \to X$ by $F(x) = \prod_A f_\alpha(x)$. Then there exists a (fixed) point $x \in X$ such that $x = F(x)$.

Proof: Clearly, X is a non-empty, compact and convex subset of a pcrst space. Since each X_α is 1° l.c., so is X. Furthermore, F is continuous, as each f_α is so. Hence, the result follows readily by application of Theorem 3.2. #

3.4 THEOREM (Minmax): Let X_1 and X_2 be 1° l.c., compact and convex. Let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$f_1(x_2) = \{x_1 | u(x_1, x_2) = \underset{y \in X_1}{\text{Max}}\, u(y, x_2)\}$$

$$f_2(x_1) = \{x_2 | u(x_1, x_2) = \underset{z \in X_2}{\text{Min}}\, u(x_1, z)\}$$

define functions $f_1 : X_2 \to X_1$ and $f_2 : X_1 \to X_2$. Then

$\underset{X_1}{\text{Max}}\,\underset{X_2}{\text{Min}}\, u(x_1, x_2) = \underset{X_2}{\text{Min}}\,\underset{X_1}{\text{Max}}\, u(x_1, x_2)$.

Proof: It is obvious that, for all $(\bar{x}_1, \bar{x}_2) \in X$,

$\underset{X_1}{\text{Max}}\, u(x_1, \bar{x}_2) \geq \underset{X_2}{\text{Min}}\,\underset{X_1}{\text{Max}}\, u(x_1, x_2) \geq \underset{X_1}{\text{Max}}\,\underset{X_2}{\text{Min}}\, u(x_1, x_2) \geq$

$\underset{X_2}{\text{Min}}\, u(\bar{x}_1, x_2)$.

Clearly, the functions f_1 and f_2 are continuous, so that the function $F: X \to X$ defined by $F(x_1, x_2) = (f_1(x_2), f_2(x_1))$ is continuous. Then by Theorem 3.3, there exists an $x^* \in X$

such that $x^* = (x_1^*, x_2^*) = F(x^*)$. Hence, $\underset{X_1}{\text{Max}}\ u(x_1, x_2^*)$ $= \underset{X_2}{\text{Min}}\ u(x_1^*, x_2)$, thus proving the desired equality. #

3.5 THEOREM (Minmax): Let A_1 and A_2 be non-empty but finite sets, each lying in a pcrst space, and let X_1 and X_2 be the closed convex hull of A_1 and A_2, respectively. Let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$f_1(x_2) = \{x_1 \mid u(x_1, x_2) = \underset{z \varepsilon X_1}{\text{Max}}\ u(z, x_2)\}$$

$$f_2(x_1) = \{x_2 \mid u(x_1, x_2) = \underset{z \varepsilon X_2}{\text{Min}}\ u(x_1, z)\}$$

define maps $f_1 : X_2 \to CQ(X_1)$ and $f_2 : X_1 \to CQ(X_2)$. Then $\underset{X_1}{\text{Max}}\ \underset{X_2}{\text{Min}}\ u(x_1, x_2) = \underset{X_2}{\text{Min}}\ \underset{X_1}{\text{Max}}\ u(x_1, x_2)$.

Proof: Use Theorem 3.1.

3.6 THEOREM (Fixed Point): Let $f: X \to CQ(X)$ be an upper semi-continuous transformation of a 2° l.c., compact and convex subset X of a pcrst space into the set $CQ(X)$ of non-empty, closed and convex subsets of X. Then there exists a (fixed) point $x \varepsilon X$ such that $x \varepsilon f(x)$.

Proof: As in the proof of Theorem 3.2, it suffices to show that the sets $Y_\alpha = \{x \mid x \varepsilon E_\alpha (f(x))\}$ are non-empty and closed, where, in this case, $\{E_\alpha \mid \alpha \varepsilon A\}$ is a fundamental system of closed entourages of the space X such that $E_\alpha(K)$ is (closed and) convex for each non-empty, compact and convex subset $K \subset X$. The proof is the same as that of Theorem 3.2 except that appeal is now made to the upper semi-continuity, rather than the continuity, of f. #

3.7 <u>THEOREM (Fixed Point)</u>: Let $\{X_\alpha | \alpha \varepsilon A\}$ be a family of 2° l.c., compact and convex subsets of pcrst spaces, and let $\{f_\alpha: X \to CQ(X_\alpha) | \alpha \varepsilon A\}$ be a family of upper semi-continuous transformations, where $X = \prod_A X_\alpha$. Define $F: X \to \prod_A CQ(X_\alpha)$ by $F(x) = \prod_A f_\alpha(x)$ $(x \varepsilon X)$. Then there exists a (fixed) point $x \varepsilon X$ such that $x \varepsilon F(x)$.

<u>Proof</u>: Clearly, F is an usc transformation of the non-empty, compact and convex space X into $CQ(X)$. Although X need not be 2° l.c., by the 2° local convexity of each X_α, the uniformity on X allows a fundamental system $\{E_i | i \varepsilon I\}$ of closed entourages such that, whenever K is the product $k = \prod_A K_\alpha$ of compact and convex subsets $K_\alpha \subset X_\alpha$, $E_i(K)$ is closed and convex (See Lemma 2.4). Notice that F(x) is such a product of compact and convex sets $f_\alpha(x) \subset X_\alpha$. Thus, as in Theorem 3.6 defining $Y_i = \{x | x \varepsilon E_i(F(x))\}$, it is clear that Y_i is non-empty and closed for each $i \varepsilon I$, implying that $\bigcap_I Y_i \neq \phi$ and proving the theorem. #

3.8 <u>THEOREM (Minmax)</u>: Let X_1 and X_2 be 2° l.c., compact and convex, each lying in some pcrst space, and let u be a continuous real-valued function on $X = X_1 \times X_2$, such that

$$f_1(x_2) = \{x_1 | u(x_1, x_2) = \max_{y \varepsilon X_1} u(y, x_2)\}$$

$$f_2(x_1) = \{x_2 \mid u(x_1, x_2) = \underset{z \in X_2}{\mathrm{Min}}\ u(x_1, z)\}$$

define point-to-set transformations $f_1\colon X_2 \to \mathcal{CQ}(X_1)$ and $f_2\colon X_1 \to \mathcal{CQ}(X_2)$, respectively. Then $\underset{X_2}{\mathrm{Min}}\ \underset{X_1}{\mathrm{Max}}\ u(x_1, x_2) = \underset{X_1}{\mathrm{Max}}\ \underset{X_2}{\mathrm{Min}}\ u(x_1, x_2)$.

<u>Proof</u>: Straightforward #

4. HYPERSPACES AS EXAMPLES

In this section we show how (topological) semivector spaces arise naturally as certain hyperspaces of certain (topological) vector and semivector spaces. In topologising hyperspaces, we use the upper semifinite, finite or, when applicable, uniform topology, regarding all of which we adopt E. Michael [1951] as standard reference.[6]

4.0 Standing Notation: Given a set X, [X] will denote the set of non-empty subsets of X. If X is a topological space, $\mathcal{C}(X)$, $\mathcal{O}(X)$ and $\mathcal{K}(X)$ will denote the set of non-empty subsets of X which are closed, open and compact, respectively. If X lies in a real semivector space, $\mathcal{Q}(X)$ will denote the set of non-empty convex subsets of X. Finally, we will denote $\mathcal{CQ}(X) = \mathcal{C}(X) \cap \mathcal{Q}(X)$, $\mathcal{OQ}(X) = \mathcal{O}(X) \cap \mathcal{Q}(X)$ and $\mathcal{KQ}(X) = \mathcal{K}(X) \cap \mathcal{Q}(X)$.

4.1 Lemma: Let X be a subset of a real semilinear (Hausdorff) topological space, and let $\mathcal{H}(X)$ denote the hyperspace $\mathcal{C}(X)$ if X is regular, and $\mathcal{K}(X)$ otherwise. Equipping $\mathcal{H}(X)$ with the finite topology, $\mathcal{HQ}(X) = \mathcal{H}(X) \cap \mathcal{Q}(X)$ is then closed in $\mathcal{H}(X)$.

Proof: For some directed set D, let $\{P_n \in \mathcal{HQ}(X) \mid n \in D\}$ be a net in $\mathcal{H}(X)$ converging to P_*. (Of course, $P_* \in \mathcal{H}(X)$). Let $p_*, p'_* \in P_*$, and take an arbitrary convex combination $\bar{p}_* = \lambda p_* \oplus \lambda' p'_*$. We need to show that $\bar{p}_* \in P_*$.

There exist nets $\{p_n \in P_n \mid n \in D\}$ and $\{p'_n \in P_n \mid n \in D\}$ converging, respectively, to p_* and p'_*.[7] It suffices to show

[6] For definitions, see the Appendix at the end.

[7] The existence of such nets is guaranteed in virtue of Lemma 1 of the Appendix.

that the net $\{\bar{p}_n \in P_n | n \in D\}$, defined by $\bar{p}_n = \lambda p_n \oplus \lambda' p'_n$, converges to $\bar{p}_*$.[8] Define Ω: $X \times X \to X$ by $\Omega(x, x') = \lambda x \oplus \lambda' x'$. By continuity of Ω, for each nbd V of $\bar{p}_*$, there exists a nbd $U \times U' \subset X \times X$ of (p_*, p'_*) such that $\Omega(U \times U') \subset V$. Clearly, $\{\bar{p}_n \in P_n | n \in D\}$ is eventually in V. #

Let $(S, \oplus, \Psi)$ be a semivector space over F and, for any A, B S, define $A \oplus B = \{a \oplus b | a \in A, b \in B\}$ and $\lambda A = \Psi(\lambda, A) = \{\lambda a | a \in A\}$. Then $([S], \oplus, \Psi)$ is a semivector space, and if F = R, then $(\mathcal{Q}(S), \oplus, \Psi)$ is a semivector subspace of [S]; furthermore, in this case, S is a semivector subspace of $\mathcal{Q}(S)$ iff S is pointwise convex, and S is pointwise convex only if $\mathcal{Q}(S)$ is so.

A topological semivector space $(S, \oplus, \Psi)$ is a topological semivector subspace of $(\mathcal{K}(S), \oplus, \Psi)$, which, in turn, is a topological semivector subspace of $([S], \oplus, \Psi)$, giving the finite topologies to $\mathcal{K}(S)$ and [S]. Furhtermore, $\mathcal{K}(S)$ is Hausdorff iff S is so. If $(S, \oplus, \Psi)$ is a semilinear topological space with a "strong" topology, (cf, Section 2), i.e., a topology in which $U \oplus s$ is open whenever $U \subset S$ is open and $s \in S$ (such as in linear topological spaces), then $(\mathcal{O}(S), \oplus, \Psi)$ is a topological semivector subspace of $([S], \oplus, \Psi)$, giving the finite topology to $\mathcal{O}(S)$ and to [S]. It follows that $\mathcal{Q}(S)$ and $\mathcal{KQ}(S)$ are topological semivector subspaces of [S] whenever [S] is a topological semivector space, and $\mathcal{OQ}(S)$ is a (topological) semivector space whenever $\mathcal{O}(S)$ is a (topological)

[8] For, then, Lemma 2 of the Appendix guarantees that $\bar{p}_* \in P_*$.

semivector space.

4.2 <u>Proposition</u>: Let S^e be a real semilinear topological space, and let $X \subset S^e$ be convex. Then $KQ(X)$ is convex. Assume that S^e has a strong topology (i.e., a topology which is a strengthened version of itself, as described above) and equip $KQ(X)$ with the upper semifinite topology. If X is 0° l.c., then so is $KQ(X)$ - although it need not be Hausdorff even if X is Hausdorff. Furthermore, if X is pointwise convex (so that $KQ(X)$, too, is pointwise convex), then $KQ(X)$ is 0° l.c. only if X is 0° l.c..

<u>Proof</u>: The rest being clear, we only prove that $KQ(X)$ with the upper semifinite topology is 0° l.c. when X is so. Let $A \in KQ(X)$, and let $W \subset KQ(X)$ be a nbd of A. Find a basic nbd $\langle V \rangle$ of A such that $\langle V \rangle \subset W$. Then $V \subset X$ is a nbd of $A \subset X$. By continuity of $\oplus$, for each $a \in A$ there exist open nbds U_a of e and W_a of a such that $U_a \oplus W_a \subset V$, while the 0° local convexity of S^e allows us to assume each U_a to be convex and the strong topology assures us that each $U_a + W_a$ is open. $\{U_a \oplus W_a \mid a \in A\}$ thus being an open cover of the compact A, it has a finite subcover $\{U_{a_i} \oplus W_{a_i} \mid i \in I\}$. Denoting $U = \bigcap_I U_{a_i}$ and $W = \bigcup_I W_{a_i}$, we see that $A \subset U \oplus A \subset U \oplus W \subset V$ and that $U \oplus A$ is convex. Furthermore, $U \oplus A$ is open in the strong topology, so that $\langle U \oplus A \rangle = KQ(U \oplus A)$ is an open convex nbd of $A \in KQ(X)$, while $\langle U \oplus A \rangle \subset \langle V \rangle \subset W$,

as desired. #

4.3 Corollary: If X is convex in a (0°) locally convex real linear topological space (not necessarily Hausdorff), then $KQ(X)$ is convex, pointwise convex and, with the upper semi-finite topology, 0° l.c. as well.

Proof: The topology of a linear topological space being strong, the last proposition applies. #

4.4 Corollary: Let X be convex and T_1 in a semilinear topological space with strong topology. Then X is 0° l.c. iff $KQ(X)$ with the upper semifinite topology is 0° l.c.

Proof: "Only if" follows from 4.2. As X is pointwise convex by 2.4, "if" also follows from 4.2. #

4.5 Proposition: Let L be a (0°) locally convex linear topological space. The $KQ(L)$, with the uniform (or, equivalently, the finite) topology, is 3° l.c..

Proof: Let $\{W_\alpha \mid \alpha \in A\}$ be a fundamental system of convex and symmetric nbds of the origin $e \in L$, so that $\{E_\alpha \subset L \times L \mid \alpha \in A\}$ is a fundamental system of entourages of the uniform structure of L, having defined $E_\alpha(x) = x \oplus W_\alpha$ for each $\alpha \in A$ and $x \in L$. For any $P \in KQ(L)$, $E_\alpha(P) = P \oplus W_\alpha \subset L$ is a nbd of $P \subset L$. By definition, the uniform structure on $KQ(L)$ is the one generated by A and collections $\mathcal{F}_\alpha(P) = \{T \in KQ(L) \mid P \subset E_\alpha(T)$ and $P \cap E_\alpha(t) \neq \phi$ for all $t \in T\}$, that is to say, $\mathcal{F}_\alpha(P) = \{T \in KQ(L) \mid P \subset E_\alpha(T)$ and $T \subset E_\alpha(P)\}$ $(P \in KQ(L))$.

It suffices to show that each F_α is convex. To see this, fix α and note that $(P, Q) \in F_\alpha$ iff $P \subset Q \oplus W_\alpha$ and $Q \subset P \oplus W_\alpha$. Let (P, Q), $(P', Q') \in F_\alpha$, and consider an arbitrary convex combination $(\bar{P}, \bar{Q}) = (\lambda P \oplus \lambda' P', \lambda Q \oplus \lambda' Q')$, recalling that $KQ(L)$ is a pointwise convex (topological) semivector space, so that $\bar{P}$, $\bar{Q} \in KQ(L)$. Since W_α is convex, we have $\bar{P} = \lambda P \oplus \lambda' P' \subset \lambda(Q \oplus W_\alpha) \oplus \lambda'(Q' \oplus W_\alpha) = \lambda Q \oplus \lambda' Q' \oplus W_\alpha = \bar{Q} \oplus W_\alpha$. Similarly, $\bar{Q} \subset \bar{P} \oplus W_\alpha$. Hence, $(\bar{P}, \bar{Q}) \in F_\alpha$, as was to be shown. #

5. APPLICATIONS IN SOCIO-ECONOMIC ANALYSIS AND OPTIMIZATION

Fixed point methods have been a traditional means of demonstrating the existence of economic equilibria, which can be looked upon as fixed points of certain economic adjustment processes. For example, given a finite number of commodities which are exchanged in an economy, consier a vector of prices with a price for each commodity. By choice of definition for 'commodity' the prices can be assumed non-negative without loss of generality, and they can be scaled so as to add up to unity as long as at least one commodity is scarce in the sense of having a positive price. Modelling the (exchange) economy to resemble an auction hall, whenever a price vector is proposed to the participants, each declares the amount of each commodity which he wants and, given his present endowment of commodities to be used in exchange, he can afford at the proposed prices. As a result, there may be (positive, zero, or negative) excess demand for a given commodity. If there is positive (negative) excess demand for some commodity, its price is bid up (down). An equilibrium here is a price vector for which the excess demand in each commodity is zero, so that no commodity price is bid up or down while all markets are "cleared". One asks whether such an equilibrium exists. Under rather reasonable conditions on the preferences of the individuals, using a suitable fixed point theorem, one can show that indeed there does exist an equilibrium and, in fact, one can say more concerning the stability, optimality, etc, of such equilibria.

The above hints at how fixed point methods may be used in the equilibrium analysis of the simplest of economic systems. To do the same for optimization theory, suppose one is trying to find a point x^* in a compact and convex set X lying in a real linear topological space, such that x^* maximizes some real-valued function $f:X \to R$, where we simplify matters by assuming that f is quasi-concave and upper sime-continuous. Suppose one can design an algorithm consisting of a point-to-set transformation $T:X \to \mathcal{CQ}(X)$ such that $f(y) > f(x)$ for all $y \in T(x)$ $(x \in X)$, unless x maximizes f, in which case $f(x) = \operatorname{Sup}_X f$ and $x \in T(x)$ is a fixed point of T, solving the maximization problem. In problems such as this, fixed point theorems tell us which monotonically improving maps T have fixed points, so that we may design one which works. Also, before trying to optimize, one may wish to feel assured that there exists an optimum. In proving the existence of such optima, fixed point methods again become very useful.

To come to the particular contribution of semilinear topological spaces and their fixed point properties as presented here, imagine the case where the "feasible region" to which choices are constrained is altered by the very choice of point in that region. (For instance, whether or not one has chosen to learn mathematics when young affects what one may or may not do when older. Also, unless one invests some of one's resources in building a spaceship, one is not able to go to the Moon.) Allow it to happen, furthermore, that the constraints operating on one's future choice of behavior depend also on other agents' choices and constraints. Given a set A of choice-making agents $\alpha \varepsilon A$, assume that the feasible region for each α is always a point $k_\alpha \varepsilon CQ(X_\alpha)$, where $X_\alpha \varepsilon KQ(L_\alpha)$ for some (0°) locally convex real linear topological space L_α. From 4.5 we know that, taking the finite topology on $Y_\alpha = KQ(X_\alpha)$, Y_α is 3° l.c.; we also know that it is compact and convex, lying in a pointwise convex real semilinear topological space $KQ(L_\alpha)$. Furthermore, all these properties are shared by $Z_\alpha = X_\alpha \times Y_\alpha$. Denote $X = \prod_A X_\alpha$, $X^\alpha = \prod_{A\setminus\{\alpha\}} X_\beta$ and $Z = \prod_A Z$. Assume that each α's feasible region k_α is determined by a continuous "feasibility transformation" $t_\alpha: Z \to Y_\alpha$. Given a feasible region k_α and a point $x^\alpha \varepsilon X^\alpha$ representing how all the other agents chose to behave, α computes the set $\alpha(k_\alpha, x^\alpha)$ of "best" choices $x_\alpha \varepsilon k_\alpha$ such that $f_\alpha(x_\alpha, x^\alpha)$ $= \sup_k f_\alpha(\cdot, x^\alpha)$, given some continuous and quasi-concave real-valued function $f_\alpha : X \to R$ representing a complete preordering of X according to α's preferences. Then $\alpha(k_\alpha, x^\alpha)$ will be a nonempty compact and convex subset of k_α X_α, where (k_α, x^α) is also the projection $\pi_{Y_\alpha \times X_\alpha}(z)$. Hence, $\{t_\alpha(z)\} \times \alpha(k_\alpha, x^\alpha)$ is nonempty, compact and convex in $Y_\alpha \times X_\alpha = Z_\alpha$. Thus, $\prod_A(\{t_\alpha(z)\} \times \alpha(k_\alpha, x^\alpha))$ is nonempty, compact and convex in Z. If α is usc, as it will be in this case, then we have an usc map of Z into $KQ(Z)$. It is easy to see how our fixed point theory now yields a fixed point $z \varepsilon Z$, which in this case is a "dynamic social equilibrium" - "dynamic," because the feasible regions k_α were allowed to change as a function of z, not necessarily being constant.

The simplicity brought by the method above is that feasible regions are treated as points in a semilinear topological (hyper) space, without being restricted to consideration as sets *per se*. Many quite general forms of "feasibility transformations" now become easy to work with, giving simplicity, as well as generality, to the analysis of "dynamic social systems" (where feasible regions are not necessarily fixed, but are endogenous to the adjustment processes in the system).

We have indicated only how one uses the methods presented here in proving the existence of economic and social equilibria. Equilibrium analysis, of course, is meant to do more than just prove equilibria to exist. Nevertheless, the existence question has to be settled in the affirmative for a sufficiently unrestrictive set of conditions before one can proceed on sound footing. In [Prakash, 1971] and [Sertel, 1971] such a footing is offered in more detail and generality and with rigor.

APPENDIX

Definition: Let X and Y be topological spaces, and let $[X]$ denote the set of all non-empty subsets of X. The upper semifinite (u.s.f.) topology on $[X]$ is the one generated by taking as a basis for open collections in $[X]$ all collections of the form $\langle U \rangle = \{A \in [X] \mid A \subset U\}$, and the lower semifinite (l.s.f.) topology on $[X]$ is the one generated by taking as a sub-basis for open collections in $[X]$ all collections of the form $\langle U \rangle^{-} = \{A \in [X] \mid A \cap U \neq \emptyset\}$, where U is an open subset in X. The finite topology on $[X]$ is the one generated by taking as basis for open collections in $[X]$ all collections of the form $\langle U_1, \ldots, U_n \rangle = \{A \in [X] \mid A \subset \bigcup_1^n U_i\ ,\ A \cap U_i \neq \emptyset \text{ for } i = 1, \ldots, n\}$. A mapping $f\colon Y \to [X]$ is called upper semi-continuous (u.s.c.) [resp. lower semi-continuous (l.s.c.)] iff it is continuous with respect to the u.s.f. [resp. l.s.f.] topology on $[X]$.

Remark: It follows that f is continuous with the finite topology on [X] iff it is both u.s.c. and l.s.c.

LEMMA # 1: Let X be any topological space and let $\mathcal{C}(X)$, the set of all nonempty closed subsets of X, carry the finite topology. For some directed set $(D, \geq)$, let $\{P_n \in \mathcal{C}(X) \mid n \in D\}$ be a net in $\mathcal{C}(X)$ converging to some point $P^* \in \mathcal{C}(X)$. Then, for every $p^* \in P^*$, there exists a net $\{p_n \in P_n \subset X \mid n \in D\}$ in X, converging to p^*.

Proof. Let $p^* \in P^*$ be such that there exists no net $\{p_n \in P_n \mid n \in D\}$ which converges to p^*. Then there exists a nbd U of p^*, $U \subset X$, such that for every $m \in D$, $\exists\, n \in D$, $n \geq m$, for which $P_n \cap U = \emptyset$. Define $\mathcal{U} = \langle X, U \rangle = \{A \in \mathcal{C}(X) \mid A \subset X,\ A \cap U \neq \emptyset\}$. Clearly, $\mathcal{U}$ is a nbd of P^* such that if $P_n \cap U = \emptyset$, then $P_n \notin \mathcal{U}$. Then $\mathcal{U}$ is a nbd of P^* such that the net $\{P_n \mid n \in D\}$ is not eventually in $\mathcal{U}$. This is a contradiction. #

LEMMA # 2: Let X be a Hausdorff topological space, and let $\mathcal{H}(X)$ denote the hyperspace $\mathcal{C}(X)$ if X is regular, and $\mathcal{K}(X)$ otherwise, where $\mathcal{K}(X)$ denotes the set of all nonempty compact subsets of X and where $\mathcal{H}(X)$ is equipped with the finite topology. For some directed set $(D, \geq)$, let $\{P_n \in \mathcal{H}(X) \mid n \in D\}$ be a net in $\mathcal{H}(X)$ converging to some $P^* \in \mathcal{H}(X)$. Then, for every net $\{p_n \in P_n \subset X \mid n \in D\}$ in X which converges to a point $p^* \in X$, $p^* \in P^*$.

Proof. Let a net $\{p_n \in P_n \mid n \in D\}$ converge to some point $p^* \in X$ and suppose $p^* \notin P^*$. Then there exist nbds U of the subset P^* and V of the point p^*, $U, V \subset X$, such that $U \cap V = \emptyset$. Since $\{P_n \mid n \in D\}$ converges to P^*, there exists $m \in D$ such that for every $n \geq m$, $n \in D$, $P_n \in \mathcal{U} = \langle U \rangle = \{A \in \mathcal{H}(X) \mid A \subset U\}$ implying that $P_n \cap V = \emptyset$, for $\mathcal{U}$ is a nbd of $P^* \in \mathcal{H}(X)$. Thus the net $\{p_n \in P_n \mid n \in D\}$ is eventually not in V. This is a contradiction. #

REFERENCES

1. (Bohnenblust & Karlin, 1950)

   H. F. Bohnenblust and S. Karlin, "On a Theorem of Ville," Contributions to the Theory of Games, I, pp. 155-160, Annals of Mathematics Studies No. 24, Princeton University Press, Princeton, 1950.

2. (Brouwer, 1912)

   L. E. J. Brouwer, "Über Abbildungen von Mannigfaltigkeiten," Mathematische Annalen, 71, 1912, pp. 97-115.

3. (Fan, 1952)

   Ky Fan, "Fixed Point and Minmax Theorems in Locally Convex Topological Linear Spaces," Proceedings of the National Academy of Sciences, U. S., 38, 1952, pp 121-126.

4. (Kakutani, 1941)

   Shizuo Kakutani, "A Generalization of Brouwer's Fixed Point Theorem," Duke Mathematical Journal, 8, 1941, pp. 457-459.

5. (Keimel, 1967a)

   Klaus Keimel, "Eine Exponentialfunktion für kompakte abelsche Halbgruppen", Mathematische Zeitschrift, 96, 1967, pp. 7-25.

6. (Keimel, 1967b)

   Klaus Keimel, "Lokal kompakte Kegelhalbgruppen und deren Einbettung in topologische Vektorräume", Mathematische Zeitschrift, 99, 1967, pp. 405-428.

7. (Kelley, 1955)

   John L. Kelley, General Topology, D. Van Nostrand Company, Inc., Princeton, N. J., 1955.

8. (Michael, 1951)

   Ernest Michael, "Topologies on Spaces of Subsets," Transactions of the American Mathematical Society, 71, 1951, pp. 152-182.

9. (Paalman - De Miranda, 1964)

   A. B. Paalman - De Miranda, Topological Semigroups, Mathematical Centre Tract No. 11, Amsterdam, 1964.

10. (Prakash, 1971)

Prem Prakash, Foundations of Systems: For Decision, Planning, Control and Social/Economic Analysis, Ph. D. Dissertation, M. I. T., 1971.

11. (Prakash & Sertel, 1970)

Prem Prakash and Murat R. Sertel, "Fixed Point and Minmax Theorems in Semi-Linear Spaces," Sloan School of Management, M. I. T., Working Paper 484-70, September, 1970.

12. (Schauder, 1930)

J. Schauder, "Der Fixpunktsatz in Funktionalraumen," Studia Mathematica, II, 1930, pp. 171-180.

13. (Sertel, 1969)

Murat R. Sertel, "A Formal Framework and Fundamental Results for Social Analysis," Sloan School of Management, MIT, Working Paper 429-69, October, 1969.

14. (Sertel, 1971)

Murat R. Sertel, Elements of Equilibrium Methods for Social Analysis, Ph. D. Dissertation, M. I. T., 1971.

15. (Tychonoff, 1935)

A. Tychonoff, "Ein Fixpunktsatz," Mathematische Annalen, III, 1935, pp. 767-776.