The Resolution of Point Sources of Light
as Analyzed by Quantum Detection Theory

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ABSTRACT

The resolvability of point sources of incoherent light is analyzed by quantum detection theory in terms of two hypothesis-testing problems. In the first, the observer must decide whether there are two sources of equal radiant power at given locations, or whether there is only one source of twice the power located midway between them. In the second problem, either one, but not both, of two point sources is radiating, and the observer must decide which it is. The decisions are based on optimum processing of the electromagnetic field at the aperture of an optical instrument. In both problems the density operators of the field under the two hypotheses do not commute. The error probabilities, determined as functions of the separation of the points and the mean number of received photons, characterize the ultimate resolvability of the sources.
Two point sources of light appear as one when they are very close; one function of a telescope or a microscope is to separate their images to the point of distinguishability. How well it does so is measured by its resolving power. When all the aberrations of the lens system have been eliminated, diffraction of the light at the aperture of the instrument remains to spread the images and cause them to overlap. According to the commonly accepted Rayleigh criterion, the two images are said to be resolved when the peak illuminance of the diffraction pattern of one falls on the first minimum of the diffraction pattern of the other. Alternative measures of image resolvability and instrumental resolving power have been proposed, and methods such as apodization for maximizing them have been studied.\[1,2\] These strive essentially to raise as far as possible the likelihood that an observer will see two close point sources as indeed two rather than one. However ingeniously his optical system may be designed, his perception is subject to error because of the stochastic, quantum-mechanical nature of the light. By studying resolution from the standpoint of hypothesis testing, we can determine how the observer can decide most reliably whether there are two sources or only one, and in this way we can bring out the fundamental limitations on the resolvability of two luminous points.

Two such decision problems will be studied in this paper. In the first, the observer is to decide whether a single point source of known power is present in an object plane, or whether two sources are present, each emitting half the power. In the second, he is to decide which of two point sources is radiating during a certain interval, only one being allowed to radiate at a time. The sources, at known locations, emit incoherent, quasimonochromatic light of given spectral density. The decisions are to be based on the electro-
magnetic field at the aperture \( A \) of an optical instrument during a fixed observation interval \((0, T)\). The instrument is to process that field in such a manner that the decisions can be made most reliably. The minimum attainable probability of error characterizes the resolvability of the sources.

The optimum processing of the aperture field is determined by quantum detection theory [3-5]. It requires us to find the eigenvectors and eigenvalues of the operator \( \rho_1 - \lambda \rho_0 \), where \( \rho_0 \) and \( \rho_1 \) are the quantum-mechanical density operators of the aperture field under the two hypotheses in question, and \( \lambda \) is a constant. In neither of our problems do the density operators commute. Except for choices between pure states, there are few physically significant pairs of noncommuting density operators for which exact sets of eigenvalues and eigenvectors have been calculated. In order to obtain exact solutions, it has been necessary to assume the absence of background light, errors in the decisions arising only because of the quantum nature of the light from each source. We determine in this way an absolute limit to their resolution.

In attacking each problem we must first find an expansion of the aperture field in terms of which the density operators take the most convenient forms. [6,7] Then the eigenvalue equation is written down in the coherent-state representation, [8] which leads to especially simple eigenvectors, constructed from the eigenvectors of \( \rho_0 \) and \( \rho_1 \) individually. The vanishing of a determinant of the coefficients of these eigenvectors gives us a quadratic equation for the eigenvalues. From the eigenvalues and eigenvectors the probabilities of error can be calculated. They depend on the separation \( \varepsilon \) of the two sources, on the form and size of the aperture, and on the total average number \( N_s \) of photons received during the observation interval.
The observer is to choose between two hypotheses: \((H_0)\) a single point source with a total radiant power \(P_s\) is present at the origin \(u = 0\) of an object plane at distance \(R\), and \((H_1)\) two point sources, each with radiant power \(\frac{1}{2} P_s\), are present at points \(u = -\frac{1}{2} \xi\) and \(u = +\frac{1}{2} \xi\) in that plane. The sources radiate quasimonochromatic incoherent light with a mean angular frequency \(\Omega\), wavelength \(\lambda = 2\pi c/\Omega\) (\(c\) is the velocity of light), and a spectral density \(X(\omega - \Omega)\) whose width \(2\pi W\) is much less than \(\Omega\).

The decision is to be based on observations of the electromagnetic field at the aperture \(A\) of an optical instrument during an interval \((0, T)\). The aperture plane lies parallel to the object plane. The field can be treated as a scalar function \(\psi(\mathbf{r}, t)\) of aperture coordinates \(\mathbf{r}\) and time \(t\), provided—as we assume—that the sources are close enough together that the rays from them are paraxial, \(|\xi|/R \ll 1\), and \(W \ll \Omega\). The field is broken into its positive-frequency part \(\psi^+(\mathbf{r}, t)\) and its negative-frequency part \(\psi^-(\mathbf{r}, t)\),

\[
\psi(\mathbf{r}, t) = \psi^+(\mathbf{r}, t) + \psi^-(\mathbf{r}, t).
\]

Classically, because the sources radiate natural, incoherent light, \(\psi^+(\xi, t)\) is a complex Gaussian spatio-temporal random process. Quantum-mechanically the field is described by its density operators \(\rho_0\) and \(\rho_1\) under the two hypotheses. In the coherent-state or \(P\)-representation\(^8\) these have Gaussian forms depending only on the mutual coherence functions of the light, which are the quantum-mechanical averages\(^9\)

\[
\frac{1}{2} \text{Tr} \rho_1 \psi(-)(\xi_1, t_1) \psi(\xi_2, t_2) = \int \int \chi(t_1 - t_2) \exp[-i\Omega(t_1 - t_2)] d\xi_1 d\xi_2 d\xi_1 d\xi_2,
\]

\[
\psi_s^{(i)}(\xi_1, \xi_2) = \chi(t_1 - t_2) \exp[-i\Omega(t_1 - t_2)], \quad i = 0, 1. \tag{1}
\]
Now \( \psi^{(+)}(x, t) \) is an operator in the Hilbert space of states of the field, \( \psi^{(-)}(x, t) \) is its Hermitian conjugate, and \( \text{Tr} \) stands for the trace; \( \chi(t) \), the temporal coherence function, is the Fourier transform of \( \chi(\omega) \), normalized so that \( \chi(0) = 1 \).

The spatial coherence function \( \varphi_S^{(i)}(x_1, x_2) \) is proportional to the spatial Fourier transform of the radiance distribution \( B(\nu) \) in the object plane. \(^9\) Under hypothesis \( H_0 \) \( B(\nu) = P_S \delta(\nu) \), and the spatial coherence function

\[
\varphi_S^{(0)}(x_1, x_2) = \left( \frac{E_S}{AT} \right) F_{r_1} F_{r_2}^* \tag{2}
\]

is constant except for the ubiquitous Fresnel factors \( F_{r_1} \) and \( F_{r_2}^* \), which are defined as

\[
F_{r_i} = \exp(i k r_{i2}/2R), \quad k = 2\pi/\lambda, \quad i = 1, 2. \tag{3}
\]

Here \( E_S = P_S AT/4\pi R^2 \) is the total average energy received at the aperture, whose area is \( A \), during the observation interval. Under hypothesis \( H_1 \)

\[
B(\nu) = \frac{1}{2} P_S [\delta(\nu - \frac{1}{2} \xi) + \delta(\nu + \frac{1}{2} \xi)],
\]

and the spatial coherence function is

\[
\varphi_S^{(1)}(x_1, x_2) = \frac{1}{2} \left( \frac{E_S}{AT} \right) \{\exp[i \xi \cdot (x_1 - x_2)/2R] + \exp[-i \xi \cdot (x_1 - x_2)/2R]\} F_{r_1} F_{r_2}^* \\
= \left( \frac{E_S}{AT} \right) \cos \mu(x_1 - x_2) F_{r_1} F_{r_2}^*, \tag{4}
\]

where \( x_{j1} = (x_j, y_j), \quad j = 1, 2, \quad \mu = k\xi/2R, \quad \xi = |\xi| \).

The \( x \)-axis has been set parallel to the line between the two sources.

The optimum strategy for processing the incident light and making the decision is most expediently derived from a decomposition of the aperture field \( \psi^{(+)}(x, t) \) into a countable set of spatio-temporal modes, \(^6,7\) represented by products of spatial mode functions \( \eta_p(x) \) and temporal mode functions \( \gamma_m(t) e^{-i\Omega t} \).

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The $Y_m(t)$ are taken as the eigenfunctions of the temporal coherence function $\chi(t)$, as given by the integral equation

$$\chi_m Y_m(t_1) = T^{-1} \int_0^T \chi(t_1 - t_2) Y_m(t_2) \, dt_2; \quad (5)$$

they are orthonormal with respect to the observation interval $(0, T)$. The eigenvalues $\chi_m$, which sum to 1, give the average fraction of the light going into each temporal mode.\[9\] When $WT \gg 1$ there are roughly $WT$ such modes containing a significant portion of the light. By virtue of (5) the temporal modes are statistically independent, and the density operators $\rho_0$ and $\rho_1$ can be factored into products, of which each factor refers to a different temporal mode; we have here a spatio-temporal counterpart of the Karhunen-Loève expansion. For simplicity we assume at first that only a single temporal mode is excited; its mode function can be taken as $Y_1(t) = T^{-\frac{1}{2}}$, with $\chi_1 = 1$. Later we shall without difficulty extend our results to a large number ($WT \gg 1$) of statistically independent temporal modes.

The spatial mode functions $\eta_p(\xi)$ are orthonormal over the aperture,

$$\int_A \eta_q^*(\xi) \eta_p(\xi) \, d^2\xi = \delta_{pq}. \quad (6)$$

In the simplest binary detection problems\[9\] these can be simultaneously eigenfunctions of both spatial coherence functions $\varphi_s^{(i)}(\xi_1, \xi_2)$, $i = 0, 1$, and the spatial modes are statistically independent under both hypotheses $H_0$ and $H_1$, but that is impossible here.

In order to limit the number of spatial modes that must be considered, we postulate that the aperture $A$ is symmetrical with respect to the $x$- and $y$-axes. Our results will be illustrated for a rectangular aperture having one side of length $a$ in the $x$-direction and centered at the origin. The two eigenfunctions of $\varphi_s^{(1)}(\xi_1, \xi_2)$, which is proportional to $\cos \mu x_1 \cos \mu x_2 + \sin \mu x_1 \sin \mu x_2$, are
\[ \eta_1(x) = C_1 \cos \mu x \ Fr, \]
\[ \eta_2(x) = C_2 \sin \mu x \ Fr, \]

where Fr is again a Fresnel Factor, and \( C_1 \) and \( C_2 \) are normalization constants.

We call \( \eta_1(x) \) the cosine mode, \( \eta_2(x) \) the sine mode. The associated eigenvalues \( h_1 \) and \( h_2 \), defined by

\[ h_p \eta_p(x) = \frac{A^{-1}}{A} \int_A \varphi_s^{(1)}(x_1, x_2) \eta_p(x_2) \ dx_2, \]

are for the rectangular aperture

\[ h_1 = \frac{1}{2}(1 + \text{sinc} \sigma), \quad h_2 = \frac{1}{2}(1 - \text{sinc} \sigma), \]
\[ \sigma = \frac{\mu a}{\pi} = \frac{ea}{\lambda R}. \]

They determine what fraction of the light goes into each mode under hypothesis \( H_1 \), and \( h_1 + h_2 = 1 \). The remaining eigenfunctions of \( \varphi_s^{(1)}(x_1, x_2) \) are orthogonal to \( \eta_1(x) \) and \( \eta_2(x) \) and have zero eigenvalues.

Under hypothesis \( H_0 \) all the light goes into the planar mode \( \eta_0'(x) = A^{-1/2} \ Fr \), but this is not orthogonal to \( \eta_1(x) \). We therefore introduce what we shall call the zero mode,

\[ \eta_0(x) = (C_0' + C_0'' \cos \mu x) \ Fr, \]

choosing \( C_0' \) and \( C_0'' \) so that \( \eta_0(x) \) is both orthogonal to \( \eta_1(x) \) and normalized. As an even function of \( x \) it is also orthogonal to \( \eta_2(x) \), which is odd in \( x \).

The remaining spatial modes are generated by the Gram-Schmidt procedure in such a way as to be orthogonal to \( \eta_0(x) \), \( \eta_1(x) \), and \( \eta_2(x) \). They are eigenfunctions of both \( \varphi_s^{(0)}(x_1, x_2) \) and \( \varphi_s^{(1)}(x_1, x_2) \), but with zero eigenvalues. As none of them is excited by the incident light under either hypothesis, we can disregard them henceforth.

The aperture field \( \psi^{(+)}(x, t) \) is expanded in a series of spatio-temporal modes \( \eta_p(x) \gamma_m(t) \) with expansion coefficients proportional to
\[ a_{pm} = \left(2\hbar\pi\right)^{-1/2} \int_0^T \int_A \eta_p^*(\xi) \gamma_m^*(t) \psi^{(+)\dagger}(\xi, t) \ d^2\xi \ dt, \quad (12) \]

where \( \hbar \) is Planck's constant \( h/2\pi \). Quantum-mechanically \( a_{pm} \) is the annihilation operator for photons in the \( (pm) \) mode. The density operators \( \rho_0 \) and \( \rho_1 \) of the field under the two hypotheses depend only on the covariance matrices \( \varphi^{(i)} \) of these coefficients; the elements of the covariance matrices can be expressed through (1) in terms of the mutual coherence functions of the field,

\[ \varphi^{(i)}_{pq,mn} = \text{Tr} \rho_i a_{qn}^+ a_{pm} = \]

\[ (T/\hbar\pi) \chi_m \delta_{mn} \int_A \int_A \eta_p^*(\xi_1) \varphi^{(i)}_{pq}(\xi_1, \xi_2) \eta_q(\xi_2) \ d^2\xi_1 \ d^2\xi_2, \quad (13) \]

where we have used (5). Because we are for the time being treating only a single temporal mode, \( \gamma_1(t) \), we put \( \chi_1 = 1 \) and drop the subscripts \( m \) and \( n \) referring to the temporal modes. If we now use our mode functions from (7), (8), and (11) and the spatial covariance functions in (2) and (4), we find that the covariance matrices are, with \( p \) and \( q \) equal to 0, 1, or 2,

\[ \varphi^{(0)}_{pq} = N_S y_p y_q (1 - \delta_{p2}) (1 - \delta_{q2}), \quad (14) \]

\[ \varphi^{(1)}_{pq} = N_S h_p \delta_{pq} (1 - \delta_{p0}) \quad (15) \]

where \( \delta_{pq} \) is the Kronecker delta, equal to 1 for \( p = q \) and to zero for \( p \neq q \),

\[ N_S = E_S/\hbar \pi \quad (16) \]

is the mean number of photons received,

\[ y_i = A^{-1/2} \int_A \eta_i(\xi) \ d^2\xi, \]

and \( h_1 \) and \( h_2 \) are the eigenvalues defined by (9). For the rectangular aperture,

\[ y_1^2 = 2 \text{sinc}^2(\xi_0 \sigma)/(1 + \text{sinc} \sigma), \]

\[ y_0^2 = 1 - y_1^2, \quad \sigma = \epsilon a/\lambda R, \quad (17) \]
and $h_1$ and $h_2$ are given in (10). The density operators in the $P$-representation are [8]

$$\rho_1 = \iiint p_1(\alpha) \prod_{k=0}^{2} |a_k\rangle \langle a_k| \left( \frac{d^2 a_k}{\pi} \right),$$

with

$$a = (a_0, a_1, a_2), \quad a_k = a_{kx} + ia_{ky}, \quad d^2 a_k = da_{kx} da_{ky},$$

and

$$P_1(\alpha) = \pi^{-3} |\text{det} \varphi(i)|^{-1} \exp \left[ -\sum_{p=0}^{2} \sum_{q=0}^{2} \alpha_p^* (\varphi(i)^{-1})_{pq} \alpha_q \right],$$

$$i = 0, 1. \quad (18)$$

Each integral is taken over the entire complex $\alpha$-plane. The density operators $\rho_0$ and $\rho_1$ do not commute.
2. Two Sources or One? The Optimum Strategy.

According to quantum detection theory the optimum strategy for choosing between two hypotheses requires measuring the projection operator \[ P = \sum_{n,r} |\eta_{nr}\rangle\langle\eta_{nr}| U(\eta_{nr}), \] (19)

where \( U(x) \) is the unit step function, and \( |\eta_{nr}\rangle \) is an eigenvector of the operator \( \rho_1 - \Lambda \rho_0 \) with eigenvalue \( n_{nr} \) given by the operator equation

\[ (\rho_1 - \Lambda \rho_0) |\eta_{nr}\rangle = n_{nr} |\eta_{nr}\rangle; \] (20)

the sum in (19) is over all the eigenstates. (The paired subscripts, which are non-negative integers, are used for later convenience.) The result of measuring \( P \) is either the number 0, which brings the decision for hypothesis \( H_0 \), or the number 1, which brings the decision for \( H_1 \). Here \( \Lambda \) is a parameter depending on the decision criterion. If the average error probability is to be minimum, \( \Lambda \) is the ratio \( \zeta/(1 - \zeta) \) of the prior probabilities of hypotheses \( H_0 \) and \( H_1 \). If the Neyman-Pearson criterion has been adopted, \( \Lambda \) is set to yield a pre-assigned false-alarm probability. The average probability of error is

\[ P_e = \zeta Q_0 + (1 - \zeta) (1 - Q_d) \]

\[ = (1 - \zeta) \left[ 1 - \sum_{n,r} n_{nr} U(\eta_{nr}) \right], \] (21)

the false-alarm probability that \( H_1 \) is chosen when \( H_0 \) is true is

\[ Q_0 = \sum_{n,r} \langle\eta_{nr}|\rho_0|\eta_{nr}\rangle U(\eta_{nr}), \] (22)

and the detection probability—the probability of saying there are two sources when there really are two—is
\[ Q_d = \sum_{n,r} \langle n_r | \rho_1 | n_r \rangle \ U(n_{nr}). \]  

Both \( Q_0 \) and \( Q_d \) are functions of \( \Lambda \).

Our principal task is to solve (20). By expressing the eigenvectors \( |\eta_k \rangle \) in terms of the coherent states as \[ F_{nr}(q^*) = \langle q | n_r \rangle \exp(i_2 | q |^2), \]

\[ |q \rangle = |a_0 \rangle |a_1 \rangle |a_2 \rangle, \quad |q |^2 = |a_0 |^2 + |a_1 |^2 + |a_2 |^2, \]  

we can write (20) as an integral equation \[ \int \left[ R_1(\bar{\theta}^*, \gamma) - \Lambda R_0(\bar{\theta}^*, \gamma) \right] \exp(-|\gamma|^2) F_{nr}(\gamma^*) \prod_{i=0}^{2} (d^2 \gamma_i / \pi) \]

\[ = n_{nr} F_{nr}(\bar{\theta}^*), \]  

where

\[ R_1(\bar{\theta}^*, \gamma) = \langle \bar{\theta} | \rho_1 | \gamma \rangle \exp \left( i \left( |\bar{\theta}|^2 + |\gamma|^2 \right) \right) = \]

\[ \det(I + \varphi^{(i)})^{-1} \exp \sum_{p=0}^{2} \sum_{q=0}^{2} \beta_p |(I + \varphi^{(i)})^{-1} \varphi^{(i)}|_{pq} \gamma_q, \]

\[ i = 0, 1, \]  

with \( I \) the 3 x 3 identity matrix. Here, by (14) and (15),

\[ R_1(\bar{\theta}^*, \gamma) = (1 - v_1)(1 - v_2) \exp(v_1 \beta_1^* \gamma_1 + v_2 \beta_2^* \gamma_2), \]

\[ v_j = N_j / (N_j + 1), \quad N_j = h_j \ N_s, \quad j = 1, 2, \]  

and

\[ R_0(\bar{\theta}^*, \gamma) = (1 - v_0) \exp[v_0(y_0 \beta_0^* + y_1 \beta_1^*)(y_0 \gamma_0 + y_1 \gamma_1)], \]

\[ v_0 = N_s / (N_s + 1). \]  

A typical eigenvalue of the density operator \( \rho_1 \) is the probability

\[ p_{nr}^{(1)} = (1 - v_1)(1 - v_2) \ v_1^n \ v_2^r \]  

of finding \( n \) photons in the cosine mode \( \eta_1(\tau) \) and \( r \) photons in the sine mode.
the associated eigenvector is, in the coherent-state representation, proportional to $\gamma_1^* \gamma_2^*$. For $r > 0$ this is also an eigenvector of $\rho_1 - \Lambda \rho_0$,

$$F_{nr}(\gamma^*) = (n! r!)^{-\frac{1}{2}} \gamma_1^* \gamma_2^*,$$

$$\eta_{nr} = p_{nr}^{(1)}, \quad r > 0,$$  \hspace{1cm} (30)

because $\rho_0 | \eta_{nr} \rangle = 0$. This means that the observer should measure the number of photons, or the energy, in the sine mode, which can be done without disturbing the other modes; and if he finds any photons there at all, $r > 0$, he chooses hypothesis $H_1$. We need, therefore, to consider further only the strategy to be followed when no photons are found in the sine mode ($r = 0$).

The density operator $\rho_0$ has the eigenvalues

$$p_n^{(0)} = (1 - v_0) v_0^n,$$  \hspace{1cm} (31)

which are the probabilities of finding various numbers of photons in the planar mode $\eta_0'(\xi)$; the associated eigenvectors are in the coherent-state representation proportional to $(\gamma_0 \gamma_0^* + \gamma_1 \gamma_1^*)^n$. As a solution of the integral equation (25) we therefore try

$$F_{n0}(\gamma^*) = x_0 (\gamma_0 \gamma_0^* + \gamma_1 \gamma_1^*)^n + x_1 \gamma_1^* \gamma_1^*,$$  \hspace{1cm} (32)

with $x_0$ and $x_1$ constants yet to be evaluated. After substituting it and carrying out the integration, we find by the method outlined in the appendix

$$(x_1 + y_1^n x_0) p_{n0}^{(1)} \beta_1^{*n} - \Lambda (x_1 y_1^n + x_0) p_n^{(0)} (y_0 \beta_0^* + y_1 \beta_1^*)^n$$

$$= \eta_{n0} [x_0 (y_0 \beta_0^* + y_1 \beta_1^*)^n + x_1 \beta_1^{*n}].$$  \hspace{1cm} (33)

Equating the coefficients of the two types of terms, we obtain the linear homogeneous equations

$$(p_{n0}^{(1)} - \eta_{n0}) x_1 + y_1^n p_{n0}^{(1)} x_0 = 0,$$

$$- \Lambda y_1^n p_n^{(0)} x_1 - (\Lambda p_n^{(0)} + \eta_{n0}) x_0 = 0,$$  \hspace{1cm} (34)
and setting the determinant of the coefficients of $x_0$ and $x_1$ equal to zero yields a quadratic equation for the eigenvalues $\eta_{n0}$:

$$
\eta_{n0}^2 - \left( p_{n0}^{(1)} - \Lambda p_{0}^{(0)} \right) \eta_{n0} - \Lambda p_{n0}^{(1)} p_{0}^{(0)} (1 - y_1^{2n}) = 0. \tag{35}
$$

For $n = 0$ the two eigenvalues are zero and

$$
\eta_{00} = p_{00}^{(1)} - \Lambda p_{0}^{(0)}. \tag{36}
$$

Whether $\eta_{00}$ is positive or negative depends on $\Lambda$; for $\Lambda = 1$, $\eta_{00} < 0$. Of the remaining pairs of eigenvalues one is positive and one is negative, $\eta_{n0}^{(+)} > 0$, $\eta_{n0}^{(-)} < 0$. Putting these values back into one or the other part of (34) enables the ratio $x_1/x_0$ to be determined. The values of $x_0$ and $x_1$ are then found from the normalization requirement

$$
\langle \eta_{n0} | \eta_{n0} \rangle = 1 = \int \int \int F_{n0}(\gamma^*) |\gamma|^2 \exp(-|\gamma|^2) \prod_{i=0}^{2} (d^2 \gamma_i/\pi)
$$

$$
= n!(x_1^2 + 2y_1^n x_0 x_1 + x_0^2). \tag{37}
$$

By using (30) and

$$
\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} p_{nr}^{(1)} = 1, \tag{38}
$$

we can write the average error probability from (21) as

$$
P_e = \zeta Q_0 + (1 - \zeta) Q_d =
$$

$$
(1 - \zeta) \left\{ \min[p_{00}^{(1)}, \Lambda p_{0}^{(0)}] + \sum_{n=1}^{\infty} (p_{n0}^{(1)} - \eta_{n0}^{(+)}) \right\}. \tag{39}
$$

To find the false-alarm probability we use (22), in which the sum can be restricted to terms with $r = 0$ because

$$
\langle \eta_{nr} | \rho_0 | \eta_{nr} \rangle = 0, \quad r > 0. \tag{40}
$$

We find for each term

$$
\langle \eta_{n0} | \rho_0 | \eta_{n0} \rangle =
$$
\[
\int \int \int F_{n0}^*(\beta^*) R_0(\beta^*, \gamma) F_{n0}(\gamma^*) \exp(-|\beta|^2 - |\gamma|^2) \prod_{i=0}^{2} (d^2 \beta_i d^2 \gamma_i / n^2) \\
= n!(x_0 + x_1 y_1)^n p_n^{(0)}, \tag{41}
\]

whereupon the false-alarm probability becomes

\[
Q_0 = \sum_{n>0} p_n^{(0)} \frac{(x_0 + x_1 y_1)^n}{x_1^2 + 2x_0x_1 y_1 + x_0^2} + p_0^{(0)} \frac{1}{U(P_0 - \lambda P_0^{(0)})}
\]

\[
= \sum_{n>0} p_n^{(0)} [\eta_{n0}^{(+)} - p_n^{(1)} (1 - y_1^{2n})] / R_n + p_0^{(0)} \frac{1}{U(P_0 - \lambda P_0^{(0)})}, \tag{42}
\]

where

\[
R_n = [(p_n^{(1)} - \lambda p_n^{(0)})^2 + 4\lambda p_n^{(1)} p_n^{(0)} (1 - y_1^{2n})]^{1/2}. \tag{43}
\]

By using (39), (42), and rather much algebra, we find for the detection probability

\[
Q_d = 1 - \sum_{n>0} p_n^{(1)} \frac{x_0^2 (1 - y_1^{2n})}{x_0^2 + 2x_0x_1 y_1 + x_1^2} - p_0^{(1)} \frac{1}{U(\lambda p_0^{(0)} - p_0^{(1)})}
\]

\[
= 1 - \sum_{n>0} p_n^{(1)} [\lambda p_n^{(0)} y_1^{2n} - (p_n^{(1)} - \eta_{n0}^{(+)})] / R_n \\
- p_0^{(1)} \frac{1}{U(\lambda p_0^{(0)} - p_0^{(1)})}. \tag{44}
\]

If thermal background light is present, the covariance matrices in (14) and (15) have additional terms \(\mathcal{N}_{pq}\), where \(\mathcal{N}\) is the mean number of background photons per mode. The eigenfunctions \(F_{nr}(\gamma^*)\) are sums of trinomials in \(\gamma_0^*, \gamma_1^*, \gamma_2^*\) with various powers, but the equations for their coefficients and the determinantal equation for the eigenvalues are much more complicated.
Multiple Spatio-temporal Modes

Now we suppose that the field is divided among a great many statistically
independent temporal modes, each of which bears three spatial modes \( \eta_0(r) \),
\( \eta_1(r) \), and \( \eta_2(r) \). We shall show that (39), (42), and (44) still hold, provided
that the probabilities \( p_n^{(1)} \) and \( p_n^{(0)} \) are replaced by Poisson probabilities,

\[
p_n^{(0)} = N_s^n \exp(-N_s)/n!, \quad (45)
\]

\[
p_n^{(1)} = h \cdot p_n^{(0)}. \quad (46)
\]

Let us consider the finite number \( v \gg \omega T \) of temporal modes having as temporal
eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_v \), the \( v \) largest eigenvalues of \( \chi(t) \). Later we let
\( v \) and \( \omega T \) go to infinity.

The first step in the decision strategy is to determine whether there
are any photons in any of the \( v \) sine modes \( \eta_2(r) \gamma_m(t) e^{-i\omega t} \). If so, hypothesis
H_1 is chosen at once, for we know that this would be impossible under hypothesis
H_0. The only eigenvectors of \( \rho_1 - \omega \rho_0 \) that need to be considered further, as
before, are those in which there are no photons in any of the \( v \) sine modes.

We add a subscript \( m \) to the complex variables \( \gamma \) and \( \xi \) to indicate the
temporal mode they refer to, and we recognize that the coherent-state representa-
tions \( R_0 \) and \( R_1 \) of the density operators \( \rho_0 \) and \( \rho_1 \) are now products of functions
like those in (27) and (28), with a factor for each temporal mode. The eigen-
functions of the integral equation now have the form, replacing (32), of

\[
F([n_m'; 0]; \gamma^*) = x_0 \prod_{m=1}^{\nu} (y_0 \gamma_{m0}^* + y_1 \gamma_{ml}^*)^{n_m} + x_1 \prod_{m=1}^{\nu} \gamma_{ml}^{n_m}, \quad (47)
\]

where we have replaced \( F_n[\gamma^*] \) by \( F([n_m, r_m]; \gamma^*), \gamma = \{\gamma_{m0}, \gamma_{ml}\} \), and shall replace
\( \eta_{nr} \) by \( \eta([n_m, r_m]) \). When this eigenfunction is substituted into the multimode
version of the integral equation (25), we obtain, much as before,

\[
\begin{align*}
(x_1 + y_1^n x_0) \prod_m P_n^{(1)}(n_m) \beta_{m1}^n m \\
- \lambda(x_1 y_1^n + x_0) \prod_m P_n^{(0)}(n_m) (y_0 \beta_{m0}^n + y_1 \beta_{m1}^n)^n m \\
= \eta([n_m, 0]) F([n_m, 0]; \gamma^*),
\end{align*}
\]

where

\[
n = \sum_m n_m
\]

is the total number of photons in the states contributing to the given eigenfunction. By equating coefficients of like terms on both sides of (48), we obtain a pair of linear equations like those in (34), except for the replacements

\[
\begin{align*}
P_n^{(1)}(n_m, 0) &= \prod_m P_n^{(1)}(n_m) = \prod_m (1 - \nu_{1m})(1 - \nu_{2m})\nu_{1m}^{n_m}, \\
P_n^{(0)}(n_m) &= \prod_m P_n^{(0)}(n_m) = \prod_m (1 - \nu_{0m})\nu_{0m}^{n_m},
\end{align*}
\]

\[
\begin{align*}
\nu_{jm} &= h_j x_m / (h_j x_m + 1), j = 1,2, \\
\nu_{0m} &= \chi_m / (\chi_m + 1).
\end{align*}
\]

As the time-bandwidth product WT increases, the temporal eigenvalues \( \chi_m \) become smaller and smaller, and we can approximate the denominators in the expressions for \( \nu_{0m}, \nu_{1m}, \) and \( \nu_{2m} \) by 1. Then for \( \nu \gg WT \gg 1, \)

\[
\begin{align*}
P_n^{(0)}(n_m) &\rightarrow \prod_m \exp(-\chi_m n_m) (\chi_m n_m)^{n_m} = \exp(-N_s) \prod_m (\chi_m n_m)^{n_m}, \\
P_n^{(1)}(n_m, 0) &\rightarrow \prod_m \exp(-h_1 x_m n_m) \exp(-h_2 x_m n_m) (h_1 x_m n_m)^{n_m} \\
&= h_1^n \exp(-N_s) \prod_m (\chi_m n_m)^{n_m} = h_1^n P_n^{(0)}(n_m),
\end{align*}
\]
since \( h_1 + h_2 = 1 \) and the eigenvalues \( \gamma_m \) sum to 1.

From (35) we see that the eigenvalue \( \eta((n_m, 0)) \) is now proportional to \( p^{(0)}(n_m) \) through a factor \( \eta_n' \) depending only on the sum \( n \) of the \( n_m \)'s,

\[
\eta((n_m, 0)) = \eta_n' p^{(0)}(n_m);
\]

that factor is a root of the equation

\[
\eta_n'^2 - (h_1^n - \Lambda)\eta_n' - \Lambda h_1^n (1 - y_1^n) = 0.
\]

Furthermore, by the new version of (34) arising from (48), the coefficients \( x_0 \) and \( x_1 \) stand in a ratio depending only on the sum \( n \) and not on the individual \( n_m \)'s. If we go through the rest of the calculation of the error probabilities, we find that the sums in (39), (42), and (44) consist of terms with \( p^{(1)}(n_m, 0) \) or \( p^{(0)}(n_m) \) multiplied by factors depending only on the sum \( n \). We can therefore combine terms having the same value of \( n = \sum n_m \), replacing \( p^{(0)}(n_m) \) when \( WT \gg 1 \) by the total probability of \( n \) photons under hypothesis \( H_0 \), which is the Poisson probability \( p_n^{(0)} \) in (45), and by virtue of (52) replacing the terms \( p^{(1)}(n_m, 0) \) by \( h_1^n p_n^{(0)} \).

We have used a scalar theory of the electromagnetic field, which seems to require that the light be linearly polarized. Unpolarized light can be divided into two statistically independent linearly polarized components, each of which is then broken up into temporal modes. The only change in our analysis is a doubling of the number of temporal modes, and as the number of such modes eventually goes to infinity, our results for \( WT \gg 1 \) must hold for unpolarized light as well.
The Error Probabilities

We evaluate all the probabilities of error for observation over a rectangular aperture. In Fig. 1 is plotted, as a function of the parameter \( \sigma = \varepsilon a/\lambda R \) and for various values of \( N_s \), the average error probability for choices between hypotheses \( H_0 \) and \( H_1 \), equal prior probabilities being assigned to each. As the two point sources separate, this error probability approaches \( \frac{1}{2} \exp(-N_s) \). When the sources are very far apart, the observer can count the numbers of photons in the planar mode \( n_0' \) and in all modes orthogonal to it. Finding any at all in the planar mode, he chooses \( H_0 \); finding any at all in the rest, he chooses \( H_1 \). The only possibility of an error arises when no photons at all are counted, whereupon he chooses \( H_0 \) and \( H_1 \) at random with equal probabilities \( 1/2 \). The probability of this event is \( \exp(-N_s) \). The limiting value of \( \frac{1}{2} \exp(-N_s) \) is reached when \( \sigma = 2 \), with only small deviations above it for \( \sigma > 2 \). If we say that when \( \sigma = 2 \) the two point sources are resolved as well as they ever can be for a given average number \( N_s \) of received photons, we require a separation of \( 2\lambda R/a \), which is twice that specified by the Rayleigh criterion.

In Fig. 2 is plotted the detection probability \( Q_d \) versus the false-alarm probability \( Q_0 \) for \( N_s = 2 \) and various values of \( \sigma = \varepsilon a/\lambda R \). These curves, along each of which \( A \) is a parameter, are sometimes called the operating characteristics of the system. The portions between the dashed lines, where \( A = 1 \), are straight lines and represent the use of a randomized strategy each time no photons at all are observed, hypothesis \( H_1 \) then being chosen with a certain probability \( f \) and \( H_0 \) with probability \( 1 - f \). As \( f \) varies from 0 to 1, the straight lines are traversed from left to right. To the right of the dashed lines is the region \( A < 1 \) (\( \zeta < 1/2 \)); to the left is the region \( A > 1 \) (\( \zeta > 1/2 \)).
Zero false-alarm probability can be attained with a finite detection probability

\[ Q_d = 1 - \exp[-N_s(1 - h_1 y_1^2)] = 1 - \exp[-N_s(1 - \text{sinc}^2 \frac{1}{2} \sigma)]; \quad Q_0 = 0, N = \infty. \quad (53) \]

The strategy achieving this chooses hypothesis \( H_1 \) if any photons are observed in any mode orthogonal to the planar mode \( n_0'(r) \). Since this is impossible when a single source is present \( (H_0) \), \( Q_0 = 0 \). An error occurs under hypothesis \( H_1 \) only when no photons happen to appear in those orthogonal modes. The average number of photons in the planar mode under \( H_1 \) is

\[
\mathbb{E}(\eta_0|H_1) = N_s \int_A \int \cos \mu(x_1 - x_2) d^2r_1 d^2r_2 = N_s h_1 y_1^2. \quad (54)
\]

The average number in the rest of the modes is \( N_s (1 - h_1 y_1^2) \), and the probability that no photons are observed in them and \( H_0 \) is chosen is \( \exp[-N_s(1 - h_1 y_1^2)] \), whence we obtain (53). The complementary probability \( 1 - Q_d \) has been plotted in Fig. 3 versus \( \sigma \) for various mean numbers \( N_s \) of received photons. A larger detection probability can be attained by accepting a positive false-alarm probability and adopting a much more complicated strategy.

Infallible detection \( (Q_d = 1) \) can be achieved by accepting a false-alarm probability

\[ Q_0 = \exp(-N_s y_0^2), \quad Q_d = 1, \quad N = 0, \]

where \( y_0^2 = 1 - y_1^2 \) is given by (17). For \( \sigma \ll 1, y_0^2 \sim (\pi \sigma)^4/720; \) thus the false-alarm probability is close to 1 unless \( N_s \) is very large or the sources are well separated. To achieve this pair \( (Q_0, Q_d = 1) \) hypothesis \( H_1 \) is chosen when there are no photons in the zero mode \( n_0(r) \) given by (11); otherwise \( H_0 \) is chosen. The mean number of photons in the zero mode under hypothesis \( H_0 \) is
given by (14) as \( N_s y_0^2 \), and the probability that there is none is \( \exp(-N_s y_0^2) \). Under \( H_1 \) there will never be any photons in the zero mode, and it will always be correctly chosen.

The Circular Aperture

We list here for reference the formulas for the constants appearing in our solution when the aperture is circular, centered at the origin and having radius \( a \). With \( m = \mu a = \kappa a/2R = \pi a/\lambda R \),

\[
y_1^2 = 8m^{-1}[m + J_1(2m)]^{-1} [J_1(m)]^2,
\]

\[
h_1 = \frac{1}{2} [1 + m^{-1} J_1(2m)],
\]

\[
y_0^2 = 1 - y_1^2, \quad h_2 = 1 - h_1,
\]

\[
h_1 y_1^2 = 4m^{-2}[J_1(m)]^2 ; \quad (55)
\]

the last describes the Airy pattern for a circular aperture. Here \( J_1(x) \) is the Bessel function of order 1.
3. One Point Source or the Other?

An optical communication system could transmit binary information by turning on one or the other of two point sources for intervals of duration $T$. If the sources are very close, a distant observer will make errors in deciding which one is on during a given interval. Both sources can emit radiant power $P_s$, and they are separated by $\varepsilon$. We can put one at the origin $u = 0$ without loss of generality when the sources are close to the optic axis normal to the aperture at its center ($\varepsilon \ll R$). Under hypothesis $H_0$ the emitting source is at $u = 0$, and the spatial covariance of the aperture field is that given by (2); under $H_1$ it is at $u = \varepsilon$, and the spatial covariance function is

$$\varphi_s^{(1)}(\xi_1, \xi_2) =$$

$$(E_s/AT) \exp[i k \varepsilon \cdot (\xi_1 - \xi_2)/R] Fr_1 Fr_2^*.$$  \hspace{1cm} (56)

Otherwise the sources are as before. The first task is to find suitable spatial modes for the aperture field.

The planar mode $\eta_0'(\xi) = A^{-1/2} Fr$ is an eigenfunction of $\varphi_s^{(0)}(\xi_1, \xi_2)$; an eigenfunction of $\varphi_s^{(1)}(\xi_1, \xi_2)$ is similarly

$$A^{-1/2} \exp(i k \varepsilon \cdot \xi/R) Fr .$$

We therefore take as our basic modes $\eta_0'(\xi)$ and

$$\eta_1'(\xi) = [C_1' + C_1'' \exp(i\mu' x)] Fr ,$$

$$\mu' = k\varepsilon/R, \varepsilon = |\xi| ,$$  \hspace{1cm} (57)

choosing $C_1'$ and $C_1''$ so that $\eta_1'(\xi)$ and $\eta_0'(\xi)$ are orthogonal and properly normalized. We find

$$C_1'' = [(1 - q_0^2)A]^{-1/2}, C_1' = -C_1'' q_0 ,$$

$$q_0 = A^{-1} \int_A \exp(i\mu' x) d^2 x .$$  \hspace{1cm} (58)
The aperture $A$ may have an arbitrary shape, and without loss of generality $q_0$ can be taken as real. The remaining spatial modes are made orthogonal to $\eta_0'(r)$ and $\eta_1'(r)$ by the Gram-Schmidt process. They are unexcited by the source under either hypothesis and can be disregarded. As before we start by assuming that $\omega T << 1$ and only a single temporal mode needs to be considered. Later we again extend our formulas to cover a great many statistically independent temporal modes.

The two spatio-temporal modes are represented quantum-mechanically by harmonic oscillators, and the density operators $\rho_0$ and $\rho_1$ under the two hypotheses have Gaussian P-representations with mode correlation matrices given by (13) as

$$\begin{align*}
\varphi_{pr}(0) &= N_s \delta_p \delta_r 0, \\
\varphi_{pr}(1) &= N_s q_p q_r, \\
(p, r) &= (0, 1), q_1 = (1 - q_0^2)^{1/2};
\end{align*}$$

(Ns) is the mean number of photons received from either source during $(0, T)$.

The optimum strategy is again specified by the eigenvectors of $\rho_1 - \Lambda \rho_0$, which can be determined from an integral equation like (25), except that only two modes instead of three are involved, and a single subscript suffices to label the eigenvalues and eigenvectors. Here

$$\begin{align*}
R_0(\beta^*, \gamma) &= (1 - v_0) \exp v_0 \beta_0^* \gamma_0, \\
R_1(\beta^*, \gamma) &= (1 - v_0) \exp[v_0(q_0 \beta_0^* + q_1 \beta_1^*)(q_0 \gamma_0 + q_1 \gamma_1)], \\
v_0 &= N_s/(N_s + 1).
\end{align*}$$

The eigenfunctions in the coherent-state representation of (24) are now

$$F_n(\gamma^*) = x_0 \gamma_0^{*n} + x_1(q_0 \gamma_0^* + q_1 \gamma_1^*)^n,$$

which when substituted into (25) yields
\[ p_n[(x_0 q_0^n + x_1)(q_0 \beta_0^* + q_1 \beta_1^*)]^n \]
\[ -\Lambda(x_0 + x_1 q_0^n) \beta_0^{*n} = \eta_n F_n(\beta^*) , \]
\[ p_n = (1 - v_0) v_0^n , \] (63)

and equating coefficients of like terms gives the homogeneous equations
\[ p_n q_0^n x_0 + (p_n - \eta_n) x_1 = 0 , \]
\[ -(\Lambda p_n + \eta_n) x_0 - \Lambda p_n q_0^n x_1 = 0 , \] (64)

the determinant of which, when set equal to zero, provides the quadratic equation for the eigenvalues,
\[ \eta_n^2 - (1 - \Lambda) p_n \eta_n - \Lambda(1 - q_0^{2n}) p_n^2 = 0 . \] (65)

When the prior probabilities of the two hypotheses are equal, \( \Lambda = 1 \), and
\[ \eta_n = \pm p_n(1 - q_0^{2n})^{1/2} . \] (66)

The error probability, given by (21), is now
\[ P_e = \frac{1}{2} \left\{ P_0 + \sum_{n=1}^{\infty} p_n[1 - (1 - q_0^{2n})^{1/2}] \right\} . \] (67)

An argument like that in Section 2 permits us to include multiple temporal modes in the limit \( WT >> 1 \) by simply replacing \( p_n \) by the Poisson probability
\[ p_n = N_n \exp(-N_s) / n! . \] (68)

In Fig. 4 we have plotted the resulting error probability, postulating a rectangular aperture of width \( a \) in the direction parallel to the line between the sources. The separation \( \epsilon \) is embodied in the parameter \( \sigma = ca/\lambda R \), and \( q_0 = \text{sinc} \sigma \). The limiting value is again \( \frac{1}{2} \exp(-N_s) \), attained when \( \sigma = 1 \), which corresponds to the image separation prescribed by the Rayleigh criterion when a diffraction-limited optical system is used. Now, however, the image of only one point source will be present at a time.
Appendix

Some Coherent-State Calculus

Let \( \vec{a} = (a_0, a_1, \ldots) \) be a column vector of coherent-state amplitudes, let \( \vec{a}^+ = (a_0^*, a_1^*, \ldots) \) be its Hermitian conjugate row vector, and let \( \mathbb{K} \) be an \( n \times n \) Hermitian matrix. Then the multivariate Gaussian integral in its most convenient form for calculations with the coherent-state representation is

\[
\int \ldots \exp(-\vec{a}^+ \mathbb{K} \vec{a} + \vec{\beta}^+ \vec{a} + \vec{\alpha}^+ \vec{\gamma}) \prod_{i=1}^{n} (d^2 a_i / \pi) = 
\]

\[
(\text{det} \ \mathbb{K})^{-1} \exp(\vec{\beta}^+ \mathbb{K}^{-1} \vec{\gamma}),
\]

where \( d^2 a_i = da_{ix} da_{iy} \), \( \vec{\beta} \) and \( \vec{\gamma} \) are constant column vectors, and the integration is carried out over the entire \( 2n \)-dimensional space of \( \{a_{ix}, a_{iy}\} \).

In order to derive the coherent-state representation of the density operator (18), as given in (26), one applies equation (9.11) of Glauber's paper\[8\] to (18), with \( \varphi^{(1)} = \varphi \),

\[
R_{i}(\vec{\beta}, \vec{\gamma}) = |\text{det} \ \varphi|^{-1} \int \ldots \int \exp(-\vec{a}^+ \varphi^{-1} \vec{a} + \vec{\beta}^+ \vec{a} + \vec{\alpha}^+ \vec{\gamma} - \vec{\alpha}^+ \vec{\gamma}) \prod_{i} (d^2 a_i / \pi)
\]

\[
= |\text{det} (\varphi + I)|^{-1} \exp[\vec{\beta}^+(\varphi^{-1} + I)^{-1} \vec{\gamma}]
\]

by (69) with \( \mathbb{K} = \varphi^{-1} + I \), where \( I \) is the identity matrix.

The integrals required when substituting the eigenfunctions of (32) into (25) are most easily derived from the generating function

\[
f_{i}(\vec{\beta}^*, \kappa) = \int \ldots \int R_{i}(\vec{\beta}^*, \vec{\gamma}) \exp \left[ \sum_{i=0}^{2} (\kappa_1 \gamma_i^* - |\gamma_i|^2) \right] \prod_{i=0}^{2} (d^2 \gamma_i / \pi)
\]

\[
= |\text{det} (\varphi^{(1)} + I)|^{-1} \int \ldots \int \exp[\vec{\beta}^+(\varphi^{(1)}+I)^{-1} \vec{\gamma} + \gamma^+ \kappa - \gamma^+ \gamma] \prod_{i} (d^2 \gamma_i / \pi)
\]

\[24\]
\[ \det (\varphi^{-1} + 2)^{-1} \exp[\varphi^{-1} (\varphi^{-1} + 2)^{-1} \chi], \quad i = 0, 1. \] (71)

We see that the effect of the integration is to replace \( \varphi \) by \( \chi \). Thus

\[ f_1(\xi^*, \chi) = (1 - v_1)(1 - v_2) \exp(v_1 \beta_1^* \kappa_1 + v_2 \beta_2^* \kappa_2), \quad (72) \]

\[ f_0(\xi^*, \chi) = (1 - v_0) \exp[v_0(y_0 \beta_0^* + y_1 \beta_1^*)(y_0 \kappa_0 + y_1 \kappa_1)]. \quad (73) \]

Putting \( \kappa_0 = \varphi_0 y_0, \kappa_1 = \varphi_1 y_1, \kappa_2 = 0 \) in these, we get the generating function

\[ f_1'(\xi^*, \varphi) = \int \int \int R_1(\xi^*, \varphi) \exp[\varphi(y_0 \gamma_0^* + y_1 \gamma_1^*) - |\gamma|^2] \prod_i (d^2 \gamma_i / \pi), \]

\[ i = 0, 1, \quad (74) \]

with

\[ f_1'(\xi^*, \varphi) = (1 - v_1)(1 - v_2) \exp(\varphi v_1 y_1 \beta_1^*) \quad (75) \]

\[ f_0'(\xi^*, \varphi) = (1 - v_0) \exp[\varphi v_0(y_0 \beta_0^* + y_1 \beta_1^*)], \quad (76) \]

since \( y_0^2 + y_1^2 = 1 \). The required multivariate integrals are then \( i = 0, 1 \)

\[ \int \int \int \gamma_1^{n} R_1(\xi^*, \gamma) \exp(-|\gamma|^2) \prod_i (d^2 \gamma_i / \pi) \]

\[ = \partial^n f_1(\xi^*, \chi) / \partial \kappa_1^n \bigg|_{\kappa=0}, \quad (77) \]

\[ \int \int \int (y_0 \gamma_0^* + y_1 \gamma_1^*)^{n} R_1(\xi^*, \gamma) \exp(-|\gamma|^2) \prod_i (d^2 \gamma_i / \pi) \]

\[ = \partial^n f_1'(\xi^*, \varphi) / \partial \varphi^n \bigg|_{\varphi=0}, \quad (78) \]

from which (33) follows immediately. Equation (63) is similarly derived.
Acknowledgment

This research has been carried out under Grant NGL 05-009-079 from the National Aeronautics and Space Administration.

References


10. Reference 5, eqs. (4.3) and (4.11), pp. 331, 333.

Figure Captions

Fig. 1. Average error probability $P_e$ in deciding whether two sources or one is present, versus the separation parameter $\sigma = \epsilon a/\lambda R$. The curves are indexed with the mean number $N_s$ of received photons.

Fig. 2. Operating characteristics of the optimum strategy for deciding whether two sources or one is present. The curves are indexed with the separation parameter $\sigma = \epsilon a/\lambda R$; $N_s = 2$.

Fig. 3. False-dismissal probability $(1 - Q_d)$ in decisions whether two sources are present, versus the separation parameter $\sigma = \epsilon a/\lambda R$. The false-alarm probability is zero. The curves are indexed with the mean number $N_s$ of received photons.

Fig. 4. Average error probability $P_e$ in deciding which of two sources separated by $\epsilon$ is radiating, versus the separation parameter $\sigma = \epsilon a/\lambda R$. The curves are indexed with the mean number $N_s$ of received photons.
Figure 1

\[ \sigma = \frac{\varepsilon a}{\lambda R} \]
Figure 2

$N_S = 2$