

NTIS: HC# 325

X-553-72-286

PREPRINT

NASA TM X-66122

THE VLBI TIME DELAY FUNCTION FOR SYNCHRONOUS ORBITS

B. ROSENBAUM

(NASA-TM-X-66122)	THE VLBI TIME DELAY	N73-13887
FUNCTION FOR SYNCHRONOUS ORBITS	B.	
Rosenbaum (NASA)	Jul. 1972 24 p CSCL	
	22C	Unclas
	G3/30	50357

JULY 1972



GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

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X-553-72-286

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FOR SYNCHRONOUS ORBITS

B. Rosenbaum
Geodynamics Branch
Trajectory Analysis and Geodynamics Division

July 1972

Goddard Space Flight Center
Greenbelt, Maryland

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FOR SYNCHRONOUS ORBITS

ABSTRACT

VLBI is a new satellite tracking technique that to date has been applied largely to the tracking of synchronous orbits. These orbits are favorable for VLBI in that the remote satellite range allows continuous viewing from widely separated stations. The primary observable, geometric time delay, is the time difference for signal propagation between satellite and baseline terminals. Extraordinary accuracy in angular position data on the satellite can be obtained by observation from baselines of continental dimensions. In satellite tracking though the common objective is to derive orbital elements. A question arises as to how the baseline vector bears on the accuracy of determining the elements. Our approach to this question is to derive an analytic expression for the time delay function in terms of Kepler elements and station coordinates. The analysis, which is for simplicity based on elliptic motion, shows that the resolution for the inclination of the orbital plane depends on the magnitude of the baseline polar component and the resolution for in-plane elements (semi-major axis, mean orbital longitude and eccentricity) depends on the magnitude of a projected equatorial baseline component. An application is made to obtain orbital elements from VLBI tracking of the ATS-3 synchronous satellite.

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THE VLBI TIME DELAY FUNCTION FOR SYNCHRONOUS ORBITS

INTRODUCTION

Very Long Baseline Interferometry (VLBI) is a recently developed RF technique that is coming into use for the tracking of spacecraft. In this technique the spacecraft radiates noise over a broad frequency band and the amplitude variations of the signals are simultaneously recorded at two widely separated observing stations. Their incoming signals are essentially the same except for a difference in arrival time (geometric time delay, τ_g) and a relative doppler frequency shift (fringe rate, ν_F). The stations are each instrumented with VLBI backend systems which include independent atomic frequency and timing standards and magnetic tape recorders for the processing and recording of the signals. A detailed discussion of the methods of VLBI whereby signals are processed and reduced to tracking data is given in Ref. 1. Here we are concerned with the analytic properties of the VLBI tracking data as related to satellite orbital elements.

The time delay measures the time difference for signal propagation between satellite and the terminals of the baseline formed by the two stations. Thus

$$\tau_g = \frac{R_1 - R_2}{c} \quad (1)$$

R_1 and R_2 denoting the range to the stations and c the speed of light. Fringe rate is defined as a delay rate. The equation is

$$\nu_F = f \dot{\tau}_g \quad (2)$$

f being the signal frequency.

When tracking at remote range these observables essentially yield angular data. The resolution for determining position will depend not only on instrumental factors such as the stability of the atomic standards, signal recording bandwidth, and SNR, but also on the station locations. An electrical interconnection between station terminals is not utilized so the baseline can be of continental dimensions. It is then possible to achieve extraordinary accuracy in the measurement of angles. This is the well known feature of VLBI.

For satellite tracking there is a further interest to see how the baseline vector, both in magnitude and orientation, appear in the problem of determining orbital elements. The orbits tracked by VLBI to the present time are largely of the near-geostationary type. This report then derives for these orbits an analytical expression for τ_g in terms of station coordinates and Kepler elements and examines its dependence on the baseline components. The analysis assumes elliptic motion and is limited to arc length such that secular drift can be treated as a small quantity. An application of the analysis is made to the tracking data of the ATS-3 satellite.

ANALYSIS

Series expansion of τ_g . Let \vec{r} denote the satellite position relative to the Earth center of mass, O, and \vec{S}_1 and \vec{S}_2 denote the location of the baseline terminals. Then by Eq. 1

$$\tau_g = \frac{|\vec{r} - \vec{S}_1|}{c} - \frac{|\vec{r} - \vec{S}_2|}{c} \quad (3)$$

or

$$\tau_g = \frac{r \sqrt{1 - \left(\frac{2 \hat{r} \cdot \vec{S}_1}{r} - \frac{S_1^2}{r^2} \right)}}{c} - \frac{r \sqrt{1 - \left(\frac{2 \hat{r} \cdot \vec{S}_2}{r} - \frac{S_2^2}{r^2} \right)}}{c} \quad (4)$$

The parameter for the series expansion of the radicals is a ratio on the order of the Earth radius to the satellite geocentric radial distance. A series with a stronger convergence can be formed by utilizing the property of the synchronous orbits. This is accomplished by taking as a point of reference a geostationary position near the satellite.

Let t_a be a time for the satellite crossing of the right ascending node and let \vec{a} denote an Earth-fixed point P in the equatorial plane such that OP equals the semi-major axis (Figure 1). The direction of \vec{a} is taken to coincide with the right ascending node at $t = t_a$. Then at this instant \vec{a} and the satellite position \vec{r} are colinear.

The separation vector of the satellite from P is

$$\vec{\rho} = \vec{r} - \vec{a} \quad (5a)$$

and that of station locations similarly are

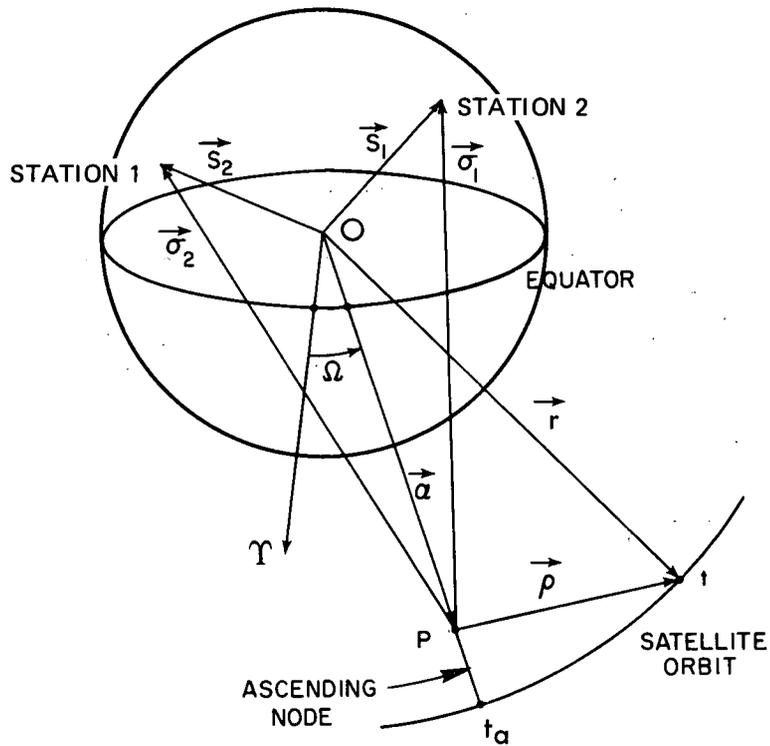


Figure 1. Satellite and Station Vector Geometry.

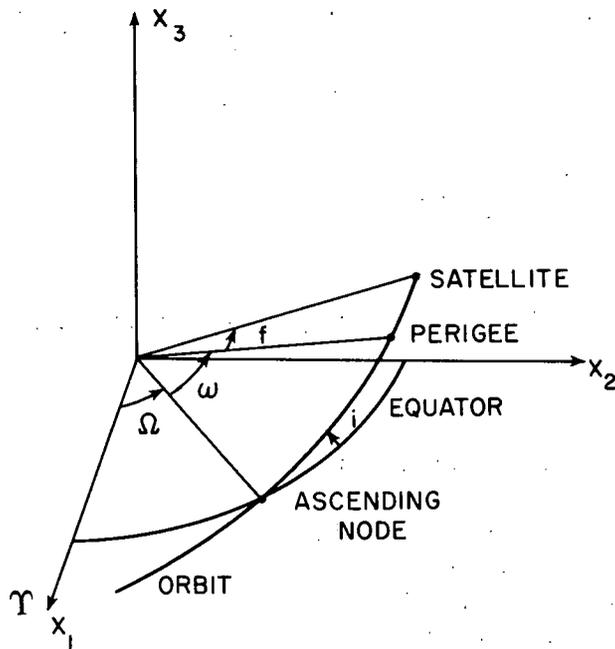


Figure 2. Geometry of Orbital Elements.

$$\begin{aligned}\vec{\sigma}_1 &= \vec{S}_1 - \vec{a} \\ \vec{\sigma}_2 &= \vec{S}_2 - \vec{a}\end{aligned}\tag{5b}$$

In terms of these vectors the time delay is rewritten as

$$\tau_g = \frac{\sigma_1 \sqrt{1 - \left(\frac{2 \vec{\rho} \cdot \hat{\sigma}_1}{\sigma_1} - \frac{\rho^2}{\sigma_1^2} \right)}}{c} - \frac{\sigma_2 \sqrt{1 - \left(\frac{2 \vec{\rho} \cdot \hat{\sigma}_2}{\sigma_2} - \frac{\rho^2}{\sigma_2^2} \right)}}{c}\tag{6}$$

Then using the expansion

$$\sqrt{1 - \eta} = 1 - \frac{\eta}{2} - \frac{\eta^2}{8} - \frac{\eta^3}{16} - \dots, \quad |\eta| < 1\tag{7}$$

Eq. 6 becomes

$$\begin{aligned}c \tau_g &= \sigma_1 - \sigma_2 - \vec{\rho} \cdot (\hat{\sigma}_1 - \hat{\sigma}_2) + \frac{1}{2} \rho^2 \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \\ &\quad - \frac{1}{2} \left(\frac{(\vec{\rho} \cdot \hat{\sigma}_1)^2}{\sigma_1} - \frac{(\vec{\rho} \cdot \hat{\sigma}_2)^2}{\sigma_2} \right) + \frac{\rho^2}{2} \left(\frac{(\vec{\rho} \cdot \hat{\sigma}_1)}{\sigma_1^2} - \frac{(\vec{\rho} \cdot \hat{\sigma}_2)}{\sigma_2^2} \right) \\ &\quad - \frac{\rho^4}{8} \left(\frac{1}{\sigma_1^3} - \frac{1}{\sigma_2^3} \right) + \dots\end{aligned}\tag{8}$$

The leading members of this series are to be developed in terms of orbital elements.

Notation for the orbital elements is as follows:

- a semi-major axis
- e eccentricity
- i inclination
- ω argument of perigee
- Ω right ascension of the ascending node
- M mean anomaly

f true anomaly
 E eccentric anomaly
 n mean motion.

Reference frame. In the Earth-centered inertial reference frame (Figure 2) the satellite Cartesian coordinates are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & P_3 \\ Q_1 & Q_2 & Q_3 \\ R_1 & R_2 & R_3 \end{pmatrix} \begin{pmatrix} r \cos f \\ r \sin f \\ 0 \end{pmatrix} \quad (9)$$

where

$$\begin{aligned} P_1 &= \cos \Omega \cos \omega - \cos i \sin \Omega \sin \omega \\ P_2 &= -\cos \Omega \sin \omega - \cos i \sin \Omega \cos \omega \\ P_3 &= \sin i \sin \Omega \\ Q_1 &= \sin \Omega \cos \omega + \cos i \cos \Omega \sin \omega \\ Q_2 &= -\sin \Omega \sin \omega + \cos i \cos \Omega \cos \omega \\ Q_3 &= -\sin i \cos \Omega \\ R_1 &= \sin i \sin \omega \\ R_2 &= \sin i \cos \omega \\ R_3 &= \cos i \end{aligned} \quad (10)$$

In order to refer the satellite position to P a transformation is first made to an Earth-fixed Cartesian system u, v, z where the positive direction of the u-axis is directed through P. The relation is

$$\begin{pmatrix} u \\ v \\ z \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (11)$$

the rotation angle θ being of the form

$$\theta = \omega_e t + \alpha \quad (12)$$

where t is measured from epoch, ω_e is the Earth's rotation rate, and α is a parameter to be determined.

From Eqs. 9-11 we have

$$\begin{aligned} u &= r [\cos (\Omega - \theta) \cos (\omega + f) - \cos i \sin (\Omega - \theta) \sin (\omega + f)] \\ v &= r [\sin (\Omega - \theta) \cos (\omega + f) + \cos i \cos (\Omega - \theta) \sin (\omega + f)] \\ z &= r \sin i \sin (\omega + f) \end{aligned} \quad (13)$$

At time t_a the crossing of the nodal line

$$u(t_a) = r, \quad v(t_a) = z(t_a) = 0 \quad (14)$$

For a positive crossing the conditions are

$$f(t_a) = -\omega, \quad \alpha = \Omega - \omega_e t_a \quad (15)$$

Then Eq. 12 becomes

$$\theta = \omega_e (t - t_a) + \Omega \quad (16)$$

The longitude of the u -axis is clearly

$$L = \Omega - \alpha_g(t_a) \quad (17)$$

α_g denoting the right ascension of the Greenwich meridian.

To determine t_a we need the relation between the mean anomaly and true anomaly which to first order in the eccentricity is

$$M = f - 2e \sin f \quad (18a)$$

or

$$n t + M_0 = f - 2e \sin f \quad (18b)$$

where M_0 is the mean anomaly at epoch. Evaluating this at $t = t_a$ and using Eq. 15 gives the desired relation for t_a

$$n t_a + M_0 = -\omega + 2 e \sin \omega \quad (19)$$

Now introducing Eq. 19 into 17 yields

$$L = \Omega + \omega + M_0 + n t_a - \alpha_g(t_a) - 2 e \sin \omega \quad (20a)$$

or

$$L = \Omega + \omega + M_0 - \alpha_g(0) - A \quad (20b)$$

where

$$A = (\omega_e - n) t_a + 2 e \sin \omega \quad (21)$$

In Eq. 20b L is expressed in terms of $\Omega + \omega + M$, known as the mean orbital longitude, and α_g both at epoch; and A which is a higher order term.

The parameter L is a reference longitude for the satellite position. In the case of a geostationary orbit it is the one parameter determined from time delay measurements.

Separation vector, $\vec{\rho}$. The near-geostationary orbits which we analyze are characterized by the conditions, $e \ll 1$, $i \ll 1$, and $|(\omega_e - n) t| \ll 1$. Restrictions on these parameters, form a limit on the magnitude of $\vec{\rho}$.

By Eq. 5a

$$\vec{\rho} = \begin{pmatrix} \rho_u \\ \rho_v \\ \rho_z \end{pmatrix} = \begin{pmatrix} u - a \\ v \\ z \end{pmatrix} \quad (22)$$

The components of $\vec{\rho}$ are to be expressed in terms of the mean anomaly. But first the relation between the true and eccentric anomaly

$$r \cos f = a (\cos E - e)$$

$$r \sin f = a \sqrt{1 - e^2} \sin E \quad (23)$$

is introduced in Eq. 9. Then the components of $\vec{\rho}$ in Eq. 22 can be written as

$$\begin{aligned}
\rho_u/a &= (\cos E - e) [\cos (\Omega + \omega - \theta) + (1 - \cos i) \sin (\Omega - \theta) \sin \omega] \\
&\quad - \sqrt{1 - e^2} \sin E [\sin (\Omega + \omega - \theta) - (1 - \cos i) \sin (\Omega - \theta) \cos \omega] - 1 \\
\rho_v/a &= (\cos E - e) [\sin (\Omega + \omega - \theta) - (1 - \cos i) \cos (\Omega - \theta) \sin \omega] \\
&\quad + \sqrt{1 - e^2} \sin E [\cos (\Omega + \omega - \theta) - (1 - \cos i) \cos (\Omega - \theta) \cos \omega] \\
\rho_z/a &= \sin i [(\cos E - e) \sin \omega + \sqrt{1 - e^2} \sin E \cos \omega] \tag{24}
\end{aligned}$$

The magnitude squared of $\vec{\rho}$ appears in one of the terms in the expansion of τ_g given in Eq. 8. This is readily calculated from Eq. 24 giving

$$\begin{aligned}
\rho^2/a^2 &= 1 + (1 - e \cos E)^2 \\
&\quad - 2 \{(\cos E - e) [\cos (\Omega + \omega - \theta) + (1 - \cos i) \sin (\Omega - \theta) \sin \omega] \\
&\quad - \sqrt{1 - e^2} \sin E [\sin (\Omega + \omega - \theta) - (1 - \cos i) \sin (\Omega - \theta) \cos \omega]\} \tag{25}
\end{aligned}$$

The mean anomaly is introduced by the Fourier series expansion of $\cos E$ and $\sin E$

$$\begin{aligned}
\cos E - e &= -\frac{3e}{2} + \left(1 - \frac{3e^2}{8}\right) \cos M + \left(\frac{e}{2} - \frac{e^3}{3}\right) \cos 2M \\
&\quad + \frac{3e^2}{8} \cos 3M + \dots \tag{26a}
\end{aligned}$$

$$\sin E = \left(1 - \frac{e^2}{8}\right) \sin M + \left(\frac{e}{2} - \frac{e^3}{6}\right) \sin 2M + \frac{3e^2}{8} \sin 3M + \dots \tag{26b}$$

and

$$\begin{aligned} \sqrt{1-e^2} \sin E = & \left(1 - \frac{5e^2}{8}\right) \sin M + \left(\frac{e}{2} - \frac{5e^3}{12}\right) \sin 2M \\ & + \frac{3e^2}{8} \sin 3M + \dots \end{aligned} \quad (26c)$$

Combining Eqs. 25, 26a, and 26b and using the approximation, $i \ll 1$ we have after rearranging terms

$$\begin{aligned} \frac{\rho^2}{2a^2} = & 1 - \cos(M + \Omega + \omega - \theta) \\ & + e \left[-\cos M + \frac{3}{2} \cos(\Omega + \omega - \theta) - \frac{1}{2} \cos(2M + \Omega + \omega - \theta) \right] \\ & + \frac{e^2}{2} \left[-\frac{1}{2} + \cos 2M - \frac{3}{4} \cos(M + \Omega + \omega - \theta) + \frac{3}{4} \cos(3M + \Omega + \omega - \theta) \right. \\ & \left. + \frac{1}{2} \sin M \sin(\Omega + \omega - \theta) \right] \\ & - \frac{i^2}{2} \sin(M + \omega) \sin(\Omega - \theta) + O(e^3, i^2 e) \end{aligned} \quad (27)$$

This equation appears to have terms of the zeroth and first order. A closer inspection, however, reveals that the lower order terms cancel. To show this we need to rewrite the form of the arguments. The relation used is obtained by combining Eqs. 16 and 19 giving

$$pM + \Omega + \omega - \theta = A - (\omega_e - pn)t + (p-1)M_0 \quad (28a)$$

or alternatively

$$pM + \Omega + \omega - \theta = (p-1)M + A - (\omega_e - n)t \quad (28b)$$

where p is a positive or negative integer.

For the first two terms of Eq. 27 we have using Eq. 28a

$$1 - \cos (M + \Omega + \omega - \theta) = 1 - \cos [A - (\omega_e - n) t]$$

$$\doteq \frac{[A - (\omega_e - n) t]^2}{2} \quad (29)$$

This is a second order term since we consider arc length such that $|(\omega_e - n) t| \ll 1$.

Next the coefficient of e in Eq. 27 is

$$I = -\cos M + \frac{3}{2} \cos (\Omega + \omega - \theta) - \frac{1}{2} \cos (2M + \Omega + \omega - \theta) \quad (30a)$$

or rearranging the arguments by Eq. 28b

$$I \doteq -\cos M + \frac{3}{2} \cos [M + (\omega_e - n) t - A] - \frac{1}{2} \cos [M - (\omega_e - n) t + A]$$

Then using the approximation $\cos (M + \eta) \doteq \cos M - \eta \sin M$ we have

$$I \doteq -2 [(\omega_e - n) (t - t_a) - 2e \sin \omega] \sin M \quad (30b)$$

Thus by a cancellation of the zeroth order terms the expression of Eq. 30a becomes of the first order.

We now rewrite Eq. 27 using Eq. 28b

$$\frac{\rho^2}{2a^2} = 1 - \cos [A - (\omega_e - n) t] + e \left\{ -\cos M + \frac{3}{2} \cos [M + (\omega_e - n) t - A] \right.$$

$$\left. - \frac{1}{2} \cos [M - (\omega_e - n) t + A] \right\} + \frac{e^2}{2} \left\{ -\frac{1}{2} - \frac{3}{4} \cos [A - (\omega_e - n) t] \right.$$

$$\left. + \cos 2M + \frac{3}{4} \cos [2M + A - (\omega_e - n) t] - \frac{1}{4} \cos [A - (\omega_e - n) t] \right\}$$

$$\begin{aligned}
& + \frac{1}{4} \cos [2M - A + (\omega_e - n)t] \Big\} + \frac{i^2}{4} \left\{ \cos [A - (\omega_e - n)t] \right. \\
& \left. - \cos [2M + 2\omega - A + (\omega_e - n)t] \right\} + O(e^3, i^2 e)
\end{aligned} \tag{31}$$

bearing in mind that the coefficient of e can be approximated by the expression of Eq. 30b.

Thus ρ^2 is second order in e , i , and $(\omega_e - n)t$. Now the term in the expansion of τ_g (Eq. 8)

$$\rho^2 \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right)$$

is still another order of magnitude lower since $\sigma_1 \doteq \sigma_2$. It is generally small compared to the linear term, $\vec{\rho} \cdot (\hat{\sigma}_1 - \hat{\sigma}_2)$. To analyze the latter we need to consider the vector difference, $\hat{\sigma} \equiv \hat{\sigma}_1 - \hat{\sigma}_2$.

Sigma vectors. By Eq. 5b

$$\vec{\sigma}_1 = \begin{pmatrix} \sigma_{1,u} \\ \sigma_{1,v} \\ \sigma_{1,z} \end{pmatrix} = \begin{pmatrix} S_{x,1} \cos L + S_{y,1} \sin L - a \\ -S_{x,1} \sin L + S_{y,1} \cos L \\ S_{z,1} \end{pmatrix} \tag{32a}$$

and

$$\vec{\sigma}_2 = \begin{pmatrix} \sigma_{2,u} \\ \sigma_{2,v} \\ \sigma_{2,z} \end{pmatrix} = \begin{pmatrix} S_{x,2} \cos L + S_{y,2} \sin L - a \\ -S_{x,2} \sin L + S_{y,2} \cos L \\ S_{z,2} \end{pmatrix} \tag{32b}$$

$S_{x,1}$, $S_{y,1}$, $S_{z,1}$ and $S_{x,2}$, $S_{y,2}$, $S_{z,2}$ denoting the Earth-centered Cartesian station coordinates. The magnitudes are

$$\sigma_1 = \sqrt{a^2 + S_1^2 - 2a(S_{x,1} \cos L + S_{y,1} \sin L)} \tag{33a}$$

$$\sigma_2 = \sqrt{a^2 + S_2^2 - 2a(S_{x,2} \cos L + S_{y,2} \sin L)} \quad (33b)$$

where

$$S_1 = \sqrt{S_{x,1}^2 + S_{y,1}^2 + S_{z,1}^2}$$

$$S_2 = \sqrt{S_{x,2}^2 + S_{y,2}^2 + S_{z,2}^2}$$

The difference of the unit vectors is denoted by

$$\hat{\sigma}_1 - \hat{\sigma}_2 \equiv \vec{\sigma} = \begin{pmatrix} \sigma_u \\ \sigma_v \\ \sigma_z \end{pmatrix} \quad (34)$$

the components being

$$\begin{aligned} \sigma_u &= \bar{S}_x \cos L + \bar{S}_y \sin L - \bar{a} \\ \sigma_v &= -\bar{S}_x \sin L + \bar{S}_y \cos L \\ \sigma_z &= \bar{S}_z \end{aligned} \quad (35)$$

where

$$\bar{S}_x = \frac{S_{x,1}}{\sigma_1} - \frac{S_{x,2}}{\sigma_2}, \quad \bar{S}_y = \frac{S_{y,1}}{\sigma_1} - \frac{S_{y,2}}{\sigma_2}, \quad \bar{S}_z = \frac{S_{z,1}}{\sigma_1} - \frac{S_{z,2}}{\sigma_2}$$

and

$$\bar{a} = a \left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2} \right) \quad (36)$$

The magnitude of $\vec{\sigma}$ is then

$$\sigma = \sqrt{\bar{a}^2 + \bar{S}_x^2 + \bar{S}_y^2 + \bar{S}_z^2 - 2\bar{a}(\bar{S}_x \cos L + \bar{S}_y \sin L)} \quad (37)$$

It will be useful to express the components of $\vec{\sigma}$ through angles μ, ν defined by the equations

$$\begin{aligned}\sigma_u &= \sigma \cos \mu \cos \nu \\ \sigma_v &= \sigma \cos \mu \sin \nu \\ \sigma_z &= \sigma \sin \mu\end{aligned}\tag{38}$$

Then

$$\tan \nu = \frac{\sigma_v}{\sigma_u}\tag{39}$$

and

$$\sin \mu = \frac{\sigma_z}{\sigma}\tag{40}$$

The leading term in the expansion of τ_g in Eq. 8 is $\sigma_1 - \sigma_2$. Its differential is readily determined by the above equations to be

$$\Delta(\sigma_1 - \sigma_2) = -\sigma_u \Delta a - a \sigma_v \Delta L\tag{41}$$

The geometrical significance of the components of $\vec{\sigma}$ is clarified when $\sigma_1 = \sigma_2$. This is a reasonable approximation when the angle between \hat{a} and the station directions is about 50° or less. Then

$$\vec{\sigma} = \frac{\vec{\sigma}_1 - \vec{\sigma}_2}{\sigma_1}\tag{42}$$

But by Eq. 5b

$$\vec{\sigma}_1 - \vec{\sigma}_2 = \vec{S}_1 - \vec{S}_2 = \vec{D}\tag{5c}$$

where \vec{D} is the baseline vector. Hence

$$\begin{aligned}\sigma_1 \sigma_u &\doteq D_u \\ \sigma_1 \sigma_v &\doteq D_v \\ \sigma_1 \sigma_z &\doteq D_z\end{aligned}\tag{43}$$

and

$$\tan \nu \doteq \frac{D_v}{D_u}, \quad \sin \mu \doteq \frac{D_z}{D}, \quad \cos \mu \doteq \frac{D_{eq}}{D} \quad (44)$$

where $D_{eq} = \sqrt{D_u^2 + D_v^2}$ is the projection of \vec{D} on the equatorial plane.

Finally another useful relation is

$$\sigma \cos \mu \doteq \frac{D \cos \mu}{\sigma_1} = \frac{D_{eq}}{\sigma_1} \quad (45)$$

Expansion of $\vec{\rho} \cdot \hat{\sigma}$. The term of τ_g (Eq. 8) linear in $\vec{\rho}$ is of the form $\vec{\rho} \cdot (\hat{\sigma}_1 - \hat{\sigma}_2) = \vec{\rho} \cdot \vec{\sigma}$. From Eqs. 24 and 38 we have that

$$\begin{aligned} \vec{\rho} \cdot \vec{\sigma} &= a \sigma \cos \mu \{ -\cos \nu + (\cos E - e) (\cos (\Omega + \omega - \theta - \nu)) \\ &+ (1 - \cos i) \sin (\Omega - \theta - \nu) \sin \omega - \sqrt{1 - e^2} \sin E (\sin (\Omega + \omega - \theta - \nu)) \\ &- (1 - \cos i) \sin (\Omega - \theta - \nu) \cos \omega \} \\ &+ a \sigma \sin \mu \sin i [(\cos E - e) \sin \omega + \sqrt{1 - e^2} \sin E \cos \omega] \end{aligned} \quad (46)$$

Then eliminating E by Eqs. 24a and 24c and using the approximation $i \ll 1$

$$\begin{aligned} \vec{\rho} \cdot \vec{\sigma} &= a \sigma \cos \mu \left\{ -\cos \nu + \cos (F + M) + \frac{i^2}{4} (\cos (F - M - 2\omega)) \right. \\ &- \cos (F + M) + e \left(-\frac{3}{2} \cos F + \frac{1}{2} \cos (F + 2M) \right) \\ &+ \frac{e i^2}{8} [\cos (F - 2M - 2\omega) - \cos (F + 2M) - 3 \cos (F - 2\omega) \\ &+ 3 \cos F] + \frac{e^2}{8} [-4 \cos (F + M) + 3 \cos (F + 3M) + \cos (F - M)] \left. \right\} \end{aligned} \quad (47)$$

$$a \sigma \sin \mu \sin i \left\{ -\frac{3e}{2} \sin \omega + \left(1 - \frac{3e^2}{8}\right) \sin (M + \omega) \right. \\ \left. + \frac{e}{2} \sin (2M + \omega) - \frac{e^2}{4} \sin M \cos \omega + \frac{3e^2}{8} \sin (3M + \omega) \right\}$$

where

$$F = \Omega + \omega - \theta - \nu \quad (48)$$

To show the dependence of the arguments in Eq. 47 on the mean anomaly, Eq. 48 can be rewritten using Eq. 28b

$$F + p M = (p - 1) M + A - (\omega_e - n) t \quad (49)$$

where the right member is now a sum of an integral multiple of M plus higher order terms.

It is useful to write $\vec{\rho} \cdot \vec{\sigma}$ in a form that shows the harmonic of each term explicitly. From Eq. 49 we readily have

$$F + p M = -(\omega_e - p n) t + A + (p - 1) M_0 \quad (50)$$

Introducing this into Eq. 47, yields

$$\vec{\rho} \cdot \vec{\sigma} = a \sigma \cos \mu \left\{ -\cos \nu + \cos (A - \nu) \cos (\omega_e - n) t + \sin (A - \nu) \sin (\omega_e - n) t \right. \\ \left. + \frac{e}{2} \left[-3 \cos (A - \nu - M_0) \cos \omega_e t - 3 \sin (A - \nu - M_0) \sin \omega_e t \right. \right. \\ \left. \left. + \cos (A - \nu + M_0) \cos (\omega_e - 2n) t + \sin (A - \nu + M_0) \sin (\omega_e - 2n) t \right] \right. \\ \left. + \frac{e^2}{8} \left[-4 \cos (A - \nu) \cos (\omega_e - n) t - 4 \sin (A - \nu) \sin (\omega_e - n) t \right. \right. \\ \left. \left. + 3 \cos (A - \nu + 2M_0) \cos (\omega_e - 3n) t + 3 \sin (A - \nu + 2M_0) \sin (\omega_e - 3n) t \right. \right. \\ \left. \left. + \cos (A - \nu - 2M_0) \cos (\omega_e + n) t + \sin (A - \nu - 2M_0) \sin (\omega_e + n) t \right] \right\}$$

$$\begin{aligned}
& + \frac{i^2}{4} \left[\cos (A - \nu - 2 M_0 - 2 \omega) \cos (\omega_e + n) t + \sin (A - \nu - 2 M_0 - 2 \omega) \sin (\omega_e + n) t \right. \\
& \left. - \cos (A - \nu) \cos (\omega_e - n) t - \sin (A - \nu) \sin (\omega_e - n) t \right] \\
& + \frac{e^2 i}{8} \left[3 \cos (A - \nu - M_0) \cos \omega_e t + 3 \sin (A - \nu - M_0) \sin \omega_e t \right. \\
& \left. - 3 \cos (A - \nu - M_0 - 2 \omega) \cos \omega_e t - \sin (A - \nu - M_0 - 2 \omega) \sin \omega_e t \right. \\
& \left. + \cos (A - \nu - 3 M_0 - 2 \omega) \cos (\omega_e + 2 n) t + \sin (A - \nu - 3 M_0 - 2 \omega) \sin (\omega_e + 2 n) t \right. \\
& \left. - \cos (A - \nu + M_0) \cos (\omega_e - 2 n) t - \sin (A - \nu + M_0) \sin (\omega_e - 2 n) t \right] \\
& + a \sigma \sin \mu \sin i \left\{ \sin (\omega + M_0) \cos n t + \cos (\omega + M_0) \sin n t \right. \\
& \left. + \frac{e}{2} \left[- 3 \sin \omega + \sin (\omega + 2 M_0) \cos 2 n t + \cos (\omega + 2 M_0) \sin 2 n t \right] \right. \\
& \left. - \frac{e^2}{4} \left[\frac{3}{2} \sin (\omega + M_0) \cos n t + \frac{3}{2} \cos (\omega + M_0) \sin n t \right. \right. \\
& \left. \left. - \frac{3}{2} \sin (\omega + 3 M_0) \cos 3 n t - \frac{3}{2} \cos (\omega + 3 M_0) \sin 3 n t \right. \right. \\
& \left. \left. + \cos \omega \cos M_0 \sin n t + \cos \omega \sin M_0 \cos n t \right] \right\} \quad (51)
\end{aligned}$$

Discussion. The above equation is written as the sum of two parts, one having the common factor $a \sigma \cos \mu$ and the other having the factor $a \sigma \sin \mu \sin i$. Within the approximation $\sigma_1 \doteq \sigma_2$ it can, by Eqs. 44, 45, and 51, be put in the form

$$\vec{p} \cdot \vec{\sigma} = \frac{a}{\sigma_1} D_{\text{eq}} \left\{ \dots \right\} + \frac{a}{\sigma_1} D_z \sin i \left\{ \dots \right\} \quad (52)$$

the ratio $a/\sigma_1 \sim 0.9$. Thus the members on the right side depend on the equatorial component of the baseline, D_{eq} , and the polar component, D_z , respectively. This expression affords some incite on how these components bear on determining orbital elements.

Consider first τ_g measured from a polar oriented baseline. With reference to Eqs. 51 and 52 we can see that the resolution for determining the inclination depends on the baseline according to the formula $\Delta i \sim (D_z \cos i)^{-1}$. This is an analogue for the orbital element of the more familiar relation in VLBI where resolution in measuring angular position depends on the reciprocal projected baseline length. The determination of elements other than the inclination varies as $(D_z \sin i)^{-1}$ which is much degraded for small inclination. The polar oriented baseline is then primarily for accurate measurement of the inclination of the orbital plane.

When measured from an equatorial oriented baseline τ_g has a term linear in time. By Eq. 44 this term can be rewritten

$$\begin{aligned}
 a \sigma \cos \mu \sin (A - \nu) \sin (\omega_e - n) t \\
 & \doteq \frac{a}{\sigma_1} D_{eq} [\sin (A - \nu)] (\omega_e - n) t \\
 & \doteq \frac{a}{\sigma_1} D_v (\omega_e - n) t \qquad (53)
 \end{aligned}$$

since $|A| \ll 1$ and $|(\omega_e - n)t| \ll 1$. The difference $\omega_e - n$ then can be measured as D_v^{-1} , that is the reciprocal of the projected equatorial baseline component normal to the OP line. This term is the basis for a precise determination of n since it appears differenced with ω_e . Further examination of the terms of Eq. 51 corresponding to D_{eq} shows that the eccentricity is of the first order and inclination is of the second order.

The elements Ω , ω , and M_0 arise in the arguments of trigonometric terms having coefficients e , i , or $(\omega_e - n) t$ to the first order. Consequently their resolution is degraded. For the limiting case of a geostationary orbit these elements become indeterminate.

The mean orbital longitude is however well defined. This appears in the leading term of the expansion of τ_g , namely $\sigma_1 - \sigma_2$. By Eqs. 41 and 43

$$\Delta (\sigma_1 - \sigma_2) \cong - \frac{D_u}{\sigma_1} \Delta a - \frac{a D_v}{\sigma_1} \Delta (\Omega + \omega + M_0) \quad (54)$$

The semi-major axis can be considered determined from the mean motion as described above. Then

$$\Delta (\Omega + \omega + M_0) \sim - \Delta (\sigma_1 - \sigma_2) / D_v$$

It is possible then to determine the summed elements, $\Omega + \omega + M_0$, with a higher precision than its individual components.

TRACKING OF ATS-3

VLBI satellite tracking observations yield geometric time delay data as a function of time. By identifying this numerical function with the analytical expression, $\tau_g(t)$, the Kepler orbital elements can be deduced. This has been done using data from a VLBI tracking experiment on the ATS-3 communication satellite.

The C-band noise signals of the spacecraft were observed at the ATS sites of Rosman, North Carolina and Mojave, California during June 3, 1971 to June 10, 1971 when the satellite longitude was approximately 79°W, a few degrees east of Rosman. The difference in the latitude of the two stations differs by scarcely more than 8', the geodetic coordinates being

	<u>Latitude</u>	<u>Longitude (W)</u>	<u>Height (m)</u>
Mojave	35°19'53"78	116°53'16"14	887
Rosman	35°11'56"07	82°52'32"81	828

The equatorial component is 3050 km and polar component is 12 km.

Figure 3 shows the distribution of tracking periods superimposed on a curve representing the time delay function. The periods are each of three minute duration. On day numbers 152, 153, 154, and 155 observations were for a restricted part of the day but for day 159 observations were distributed over a twelve hour period. The precision of the time delay data is 5 to 10 nanoseconds corresponding to 1.5 to 3 meters.

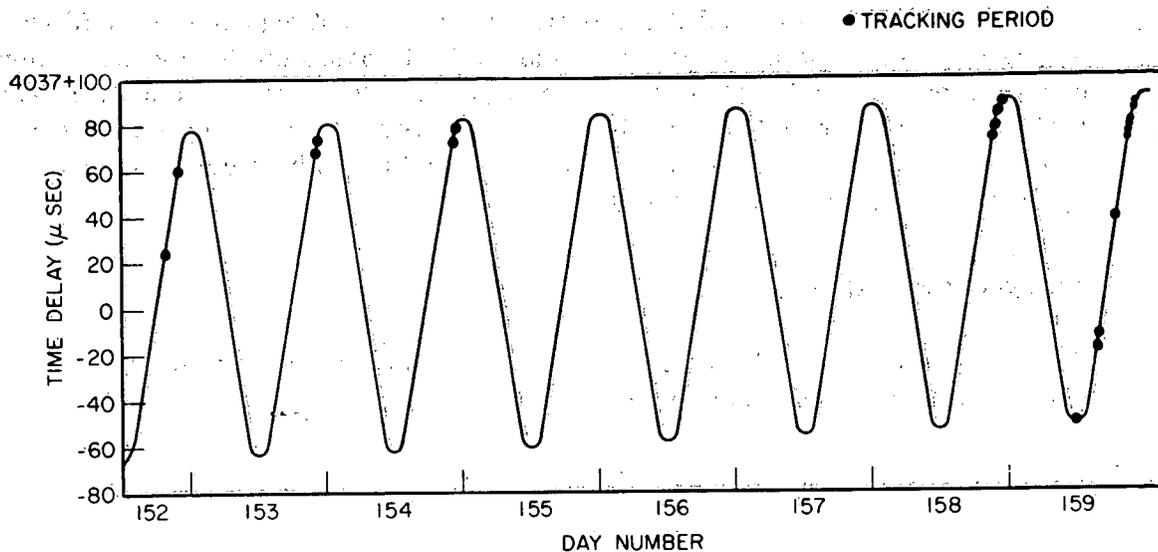


Figure 3. Time delay curve showing the VLBI tracking periods of ATS-3, days 152-159, 1971.

As a starting point for calculating the orbital elements the terms of the time delay function were evaluated using a priori elements (based on range and range rate tracking from Mojave and Rosman) issued by the ATS project. Terms greater than 30 nanoseconds (10 meters) were retained for which the functional form was essentially

$$c \tau_g = c_0 + c_1 t + c_2 \cos \omega_e t + c_3 \sin \omega_e t + c_4 \cos 2 \omega_e t + c_5 \sin 2 \omega_e t \quad (55)$$

The coefficients were evaluated by the method of least squares, the standard deviation in time delay being 0.66 microseconds. One of the coefficients was smaller than the st. dev. and therefore was not regarded as significant. Five elements were determined and are compared with those issued by the ATS project. The epoch is May 31, 1971, 00^h.

<u>Element</u>	<u>ATS Project</u>	<u>VLBI Time Delay</u>	<u>Difference</u>
a (km)	42,165.43	42,165.54	0.11
e	2.914×10^{-3}	2.96×10^{-3}	5×10^{-5}
i	1°706	1°6	-0°1
M ₀	87°150	95°	8°
$\Omega + \omega + M_0$	168°652	168°61	-0°04
ω	-0°712	-	-

It should be understood that the method here is not expected to yield accurate values for the elements since the model is based on the approximation of elliptic motion. To derive a best estimate of the ATS-3 orbit from VLBI observations we are using GEODYN (Ref. 1). This program has an optimum force field model and employs rigorous numerical methods which are designed for application to geodynamic investigations.

ACKNOWLEDGMENTS

The author gratefully acknowledges many interesting discussions on this work with Drs. J. Ramasastry and S. Ross.

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