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THE ENTROPY RATE ADMISSIBILITY CRITERION FOR SOLUTIONS OF HYPERBOLIC CONSERVATION LAWS

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by

Constantine M. Dafermos+

Center for Dynamical Systems
Division of Applied Mathematics
Brown University
Providence, R.I. 02912

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LATIN MINISCULE

b, d, f, j, k, m, q, t, u, v, w, x

LATIN CAPITAL

A, B, C, D, F, H, L, T

GREEK MINISCULE

ζ	zeta	ξ	xi
η	eta	σ	sigma
λ	lambda	τ	tau
μ	mu	χ	chi
ν	nu	ω	omega

GREEK CAPITAL

Σ sigma

SCRIPT

$\mathscr{D}$(D script)

OTHER SYMBOLS

ϵ, ∞, $\int$, $\times$, $+$, $-$, $<$, $\leq$, $>$, $\geq$, $=$

1. Introduction

Consider a hyperbolic conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

where $u = (u_1, \ldots, u_m)$, $f = (f_1, \ldots, f_m)$. A convex function $\eta(u)$ is called an <u>entropy</u> for (1.1) with <u>entropy flux</u> $q(u)$ if

$$\eta(u)_t + q(u)_x = 0 \tag{1.2}$$

holds identically for any smooth vector field $u(x,t)$ which satisfies (1.1). For differentiable $f(u), \eta(u)$ and $q(u)$, (1.2) follows from (1.1) if

$$\sum_{j=1}^{m} \frac{\partial \eta}{\partial u_j} \frac{\partial f_j}{\partial u_k} = \frac{\partial q}{\partial u_k}, \quad k = 1, \ldots, m. \tag{1.3}$$

For $m = 1$, every convex function $\eta(u)$ is an entropy for (1.1) with entropy flux $q(u) = \int_0^u \eta'(w)\,df(w)$. For $m = 2$, Lax [1] shows that a strictly convex entropy exists, at least locally. For $m > 2$, however, (1.3) is overdetermined and the existence of a non-trivial(*) entropy may be attributed only to a happy coincidence. Nevertheless, conservation laws that result from Continuum Mechanics and Thermodynamics are endowed with a

(*) That is, other than a linear function of u.

"natural" entropy having a physical interpretation. Moreover, Lax [1] observes that $\eta(u) = \sum_{j=1}^{m} u_j^2$ is an entropy for (1.1) if gradf is symmetric.

It is known that the initial value problem for (1.1) does not have, in general, a global classical solution even if the initial data are smooth. On the other hand, in the class of weak solutions (bounded measurable functions which satisfy (1.1) in the sense of distributions) uniqueness is lost. A number of criteria motivated by mathematical and/or physical considerations have been proposed in order to single out an admissible weak solution (for a survey see, e.g., [2]). In the equations of gas dynamics, the requirement that entropy should increase across shocks rules out nonadmissible solutions. Motivated by this occurence, Lax [1] postulates the following

Entropy Admissibility Criterion: Let $\eta(u)$ be an entropy for (1.1). A weak solution u of (1.1) will be admissible if it satisfies

$$\eta(u)_t + q(u)_x \leq 0 \tag{1.4}$$

in the sense of distributions.

The following interesting results are established in [1]: The entropy criterion is compatible with the viscocity criterion. Furthermore, if $\eta(u)$ is strictly convex, the entropy criterion is equivalent (at least for moderately strong shocks) to Lax's shock admissibility condition [3]. Thus, it is reasonable to conjecture that in the class of genuinely nonlinear hyperbolic conservation laws [3] the entropy criterion characterizes completely the admissible solutions.

The situation is different, however, if (1.1) is not genuinely nonlinear; in this case the entropy criterion for any particular entropy does not rule out all solutions that are disqualified by the viscocity criterion. In order to circumvent this difficulty, Lax [1] makes the requirement that (1.4) should be satisfied for every entropy associated with (1.1). This singles out the admissible solution in the case $m = 1$ where every convex function is an entropy.

We are not completely satisfied with this approach for two reasons: First, it seems to be effective only in the case of a single equation $(m = 1)$ since systems are not endowed with so rich a collection of entropies. Second, its physical motivation is not very clear because, even when several entropies are associated with a system (1.1) resulting from thermodynamics, only one of them enjoys a physical interpretation(*). It is plausible that this "natural" entropy is sufficient to characterize admissible solutions.

The intent of this work is to put forward an alternative

(*) For example, every convex $\eta(u)$ is an entropy for the model equation $u_t + uu_x = 0$ of gas dynamics but only $\eta(u) = \frac{1}{2} u^2$ has a physical interpretation (mechanical energy).

criterion in which admissibility is dictated by a single entropy. We consider the initial value problem for a hyperbolic conservation law (1.1) with a strictly convex entropy $\eta(u)$ and we restrict our attention to bounded solutions $u(x,t)$ such that $u(\cdot,t) \in C^{0}([0,T);[L^{1}(-\infty,\infty)]^{m})$ (*). We then introduce the total entropy at $t \in [0,T)$

$$H_u(t) = \int_{-\infty}^{\infty} \eta(u(x,t))\,dx \tag{1.5}$$

and we postulate the following

Entropy Rate Admissiblity Criterion: A solution $u(x,t)$ will be called admissible if there is no solution $v(x,t)$ with the property that for some $\tau \in [0,T)$, $u(x,t) = v(x,t)$ on $(-\infty,\infty) \times [0,\tau]$ and $D_+H_v(\tau) < D_+H_u(\tau)$.

Roughly speaking, we require that for the admissible solution the total entropy decreases with the highest possible rate. The physical motivation of the above criterion is clear and its relationship to Lax's entropy criterion is obvious; we attempt here to establish its equivalence to other accepted admissibility criteria. The investigation is hampered by the scarcity of test cases (systems that are not genuinely nonlinear for which the problem of admissibility of solutions has been solved). In any case, we are able to justify the entropy rate criterion in the following two cases:

(*) The experience with $m = 1$ indicates that this is the natural class of functions in which solutions should be sought.

(i) The single equation $(m = 1)$ in the class of piecewise smooth solutions(*).

(ii) The system

$$u_t - v_x = 0$$
$$v_t - f(u)_x = 0, \tag{1.6}$$

$f' > 0$, in the class of piecewise constant solutions(**).

This article by no means exhausts the subject. Further investigation is necessary in many directions. The entropy rate criterion should be tested on broader classes of solutions, in the case of several space variables, as well as on larger systems. (The system of equations of gas dynamics $(m = 3)$ is the natural next candidate.) Possible implications on existence, uniqueness and stability of solutions should be explored. Finally, the criterion should be tested on non-hyperbolic conservation laws. The equations of nonlinear thermoelasticity provide an example of this nature.

(*) As shown by Ballou [4], this class generates the entire class of weak solutions via completion.

(**) We have to confine our investigation in this class because the solution of the Riemann problem [5, 6, 7] is virtually all the information available at the present for the system (1.6) in the case it is not genuinely nonlinear. There are indications (see [7]), that the class of piecewise constant solutions generates general weak solutions via completion.

2. The Single Equation

We test here the entropy rate criterion on (1.1), with $m = 1$, in the class of piecewise smooth solutions with smooth shocks. A function $u(x,t)$ in this class is a weak solution if (1.1) is satisfied at every point (x,t) where u is smooth while the Rankine-Hugoniot condition

$$\dot{X}(t) = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \tag{2.1}$$

holds across each shock $x = X(t)$ (*). Furthermore, the solution is admissible if Oleinik's E-condition [8]

$$\frac{f(u) - f(u_-)}{u - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \tag{2.2}$$

for all u between u_- and u_+, is satisfied across every shock.

We compute the rate of entropy decay for a solution $u(x,t)$. Although every strictly convex function $\eta(u)$ can serve equally well as an entropy in our investigation, we select $\eta(u) = \frac{1}{2}u^2$ which is physically motivated (see the footnote on page 3)

(*) As is usual, we employ the notation $u_- = u(X(t) - 0, t)$, $u_+ = u(X(t) + 0, t)$. For definiteness we assume that for each t, $u(x,t)$ is continuous from the left. Moreover, shocks $x = X(t)$ are defined on intervals closed from below and open from above.

and, in addition, leads to a rate of entropy decay with an interesting geometric interpretation.

Fix $\tau \geq 0$. Assume, for simplicity, that $u(x,\tau)$ has compact support in $(-\infty,\infty)$. Then

$$D_+H_u(\tau) = \int_{-\infty}^{\infty} uu_t dx - \frac{1}{2} \sum_{\text{shocks}} \dot{\chi}(\tau)(u_+^2 - u_-^2)$$

where the summation extends over all shocks that intersect the line $t = \tau$. In the above equation we substitute $\dot{\chi}(\tau)$ from (2.1) and u_t from (1.1). After some computations we arrive at

$$D_+H_u(\tau) = \sum_{\text{shocks}} A(u_-,u_+) \tag{2.3}$$

where we employ the notation

$$A(v,w) = \frac{1}{2}[f(v) + f(w)](w-v) - \int_v^w f(\sigma)d\sigma. \tag{2.4}$$

We observe that $A(v,w)$ is the (signed) area between the graph of f and the chord that joins $(v,f(v))$ with $(w,f(w))$.

We intend to prove that every solution which satisfies the entropy rate criterion satisfies also Oleinik's E-condition. To this end, we assume that $u(x,t)$ is a solution such that (2.2) fails at a point $(\chi(\tau),\tau)$ of a shock $x = \chi(t)$ and we construct another solution with higher rate of entropy decay at τ. Without loss of generality, let $(\chi(\tau),\tau) = (0,0)$. For definiteness, assume $u_- < u_+$ at $(0,0)$ (an analogous argument holds if $u_- > u_+$). Then there is $u_o \in (u_-, u_+)$ such that

$$\frac{f(u_o) - f(u_-)}{u_o - u_-} < \frac{f(u_+) - f(u_-)}{u_+ - u_-} \tag{2.5}$$

which is equivalent to

$$A(u_-, u_+) > A(u_-, u_o) + A(u_o, u_+). \tag{2.6}$$

We now define

$$w(x,t) = \begin{cases} u(x,t) & \text{for } x \leq f'(u_-)t, \quad t \geq 0 \\ u_- & \text{for } x > f'(u_-)t, \quad t \geq 0 \end{cases}$$

$$\omega(x,t) = \begin{cases} u_+ & \text{for } x \leq f'(u_+)t, \quad t \geq 0 \\ u(x,t) & \text{for } x > f'(u_+)t, \quad t \geq 0 \end{cases}$$

and we consider the initial value problems

$$\dot{\xi}(t) = \frac{f(u_o) - f(w(\xi(t),t))}{u_o - w(\xi(t),t)}, \quad \xi(0) = 0, \tag{2.7}$$

$$\dot{\zeta}(t) = \frac{f(u_o) - f(\omega(\zeta(t),t))}{u_o - \omega(\zeta(t),t)}, \quad \zeta(0) = 0. \tag{2.8}$$

By (2.1) and (2.5), $\dot{\xi}(0) < \dot{X}(0) < \dot{\zeta}(0)$ so that (local) solutions of (2.7) and (2.8) exist and $\xi(t) < X(t) < \zeta(t)$, t positive small. We now set

$$v(x,t) = \begin{cases} w(x,t) & \text{for} \quad x \leq \xi(t) \quad , \ t \geq 0 \\ u_o & \text{for} \quad \xi(t) < x \leq \zeta(t), \ t > 0 \\ \omega(x,t) & \text{for} \quad \zeta(t) < x \quad , \ t \geq 0. \end{cases}$$

It is easily verified that v is a local weak solution of (1.1) with $v(x,0) = u(x,0)$. Furthermore, using (2.3) and (2.6), $D_+H_v(0) < D_+H_u(0)$ which shows that u does not satisfy the entropy rate criterion.

3. A System of Two Equations

In this section we test the entropy rate criterion on the system (1.6) which arises in gas dynamics and nonlinear elasticity. In this connection, the "natural" entropy (mechanical energy) and entropy flux are given by

$$\eta(u,v) = \frac{1}{2} v^2 + \int_0^u f(\sigma)\,d\sigma, \quad q(u,v) = -vf(u). \tag{3.1}$$

We consider piecewise smooth solutions $(u,v)(x,t)$ with smooth shocks. For definiteness we assume that for each t $(u,v)(x,t)$ is continuous from the left. Moreover, shocks $x = X(t)$ are defined on intervals closed from below and open from above.

The Rankine-Hugoniot conditions across a shock $x = X(t)$ here read

$$\begin{aligned} (u_+ - u_-)\dot{X}(t) + v_+ - v_- &= 0 \\ (v_+ - v_-)\dot{X}(t) + f(u_+) - f(u_-) &= 0. \end{aligned} \tag{3.2}$$

Applying the viscocity criterion, Wendroff [5] shows that a solution is admissible if across every shock $x = X(t)$ the following analog of Oleinik's E-condition is satisfied for every u between u_- and u_+:

$$\frac{f(u) - f(u_-)}{u - u_-} \leq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \quad \text{if } \dot{X} > 0$$
$$\frac{f(u) - f(u_-)}{u - u_-} \geq \frac{f(u_+) - f(u_-)}{u_+ - u_-} \quad \text{if } \dot{X} < 0. \tag{3.3}$$

We compute the rate of entropy decay. Fix $\tau \geq 0$. Assume for simplicity that $(u,v)(x,\tau)$ has compact support in $(-\infty,\infty)$. Then

$$D_+H_{(u,v)}(\tau) = \int_{-\infty}^{\infty} [vv_t + f(u)u_t]dx - \sum_{\text{shocks}} \dot{X}(\tau)[\tfrac{1}{2}(v_+^2 - v_-^2) + \int_{u_-}^{u_+} f(\sigma)d\sigma].$$

Substituting in the above equation u_t and v_t from (1.6) and integrating,

$$D_+H_{(u,v)}(\tau) = - \sum_{\text{shocks}} [v_+f(u_+) - v_-f(u_-) + \tfrac{1}{2}\dot{X}(\tau)(v_+^2 - v_-^2) + \int_{u_-}^{u_+} f(\sigma)d\sigma].$$

Using (3.2) and after some computations we arrive at

$$D_+H_{(u,v)}(\tau) = \sum_{\text{shocks}} \dot{X}(\tau)A(u_-,u_+) \tag{3.4}$$

where $A(\cdot,\cdot)$ is again defined by (2.4).

We now confine our attention to piecewise constant solutions. We intend to prove that every solution in this class which satisfies the entropy rate criterion satisfies also the admissibility criterion (3.3). To this end, starting from a solution for which (3.3) fails, we construct another solution with higher rate of entropy decay. The construction is based on the solution of the Riemann problem for (1.6) described in [6].

We will simplify considerably the construction, without affecting its essential features, by treating here only the special case where $f(u)$ is piecewise linear. After this is done, the reader will have no difficulty to envisage the modifications that are necessary in order to carry over the construction to general $f(u)$.

Assume that (u,v) is a piecewise constant solution of (1.6) such that (3.3) fails at a point $(X(\tau),\tau)$ of a shock $x = X(t)$. Without loss of generality, $(X(\tau),\tau) = (0,0)$. For definiteness, let $u_- < u_+$, $v_- < v_+$ (a similar argument goes through in the remaining cases).

For $-\infty < u_1 < u_2 < \infty$, $\mathscr{D}(u_1,u_2)$ denotes the convex hull of the set $\{(\sigma,w) | u_1 \leq \sigma \leq u_2,\ w \leq f(\sigma)\}$. For $\sigma \in [u_1,u_2]$, let $b(\sigma;\ u_1,u_2) = \max\{w | (\sigma,w) \in \mathscr{D}(u_1,u_2)\}$. Note that since $f(\sigma)$ is piecewise linear, $b(\sigma;\ u_1,u_2)$ is also piecewise linear and the vertices of its graph lie on the graph of f. We now define functions

$$B(u) = v_- + \int_{u_-}^{u} [b'(\sigma;\ u_-,u)]^{1/2} d\sigma, \quad u \in [u_-,\infty), \tag{3.5}$$

$$F(u) = v_+ - \int_{u_+}^{u} [b'(\sigma;\ u_+,u)]^{1/2} d\sigma, \quad u \in [u_+,\infty). \tag{3.6}$$

Leibovich [6, Proposition 2.1] shows that $B(u)$ is strictly increasing to $+\infty$ and $F(u)$ is strictly decreasing to $-\infty$. We now prove that $B(u_+) < v_+$. Note first that, since (3.3) fails, the graph of $b(\sigma;\ u_-,u_+)$ is not a straight line. Then, using Schwarz's inequality and (3.2),

$$[B(u_+) - v_-]^2 < \int_{u_-}^{u_+} b'(\sigma;\ u_-,u_+)d\sigma \int_{u_-}^{u_+} d\sigma = [f(u_+) - f(u_-)](u_+ - u_-)$$
$$= (v_+ - v_-)^2.$$

It follows that there is a unique (u_o,v_o), with $u_- < u_+ < u_o$, $v_- < v_o < v_+$, such that

$$v_o = B(u_o) = F(u_o). \tag{3.7}$$

Let $u_- = u_{-\mu} < u_{-\mu+1} < \cdots < u_{-1} < u_o$ and $u_+ = u_\nu < u_{\nu-1} < \cdots < u_1 < u_o$ be the ordinates of the vertices of the graph of $b(\sigma;\ u_-,u_o)$ and $b(\sigma;\ u_+,u_o)$, respectively. We set

$$v_\lambda = \begin{cases} B(u_\lambda) & \text{for } \lambda = -\mu,\ldots,-1 \\ F(u_\lambda) & \text{for } \lambda = 1,\ldots,\nu, \end{cases} \tag{3.8}$$

$$x_\lambda(t) = (v_\lambda - v_{\lambda+1})(u_{\lambda+1} - u_\lambda)^{-1}t, \ \lambda = -\mu,\ldots,0,\ldots,\nu - 1. \quad (3.9)$$

It can be shown that $\dot{x}_{-\mu} < \cdots < \dot{x}_{-1} < 0 < \dot{x}_o < \cdots < \dot{x}_{\nu-1}$. We define

$$(\bar{u},\bar{v})(x,t) = \begin{cases} (u,v)(x,t) & \text{for } x \leq x_{-\mu}(t), \ t \geq 0 \\ (u_\lambda, v_\lambda) & \text{for } x_{\lambda-1}(t) < x \leq x_\lambda(t), \ t > 0, \\ & \qquad \lambda = -\mu+1,\ldots,\nu-1 \\ (u,v)(x,t) & \text{for } x_{\nu-1}(t) < x, \ t \geq 0. \end{cases}$$

It is easy to prove that $(\bar{u},\bar{v})(x,t)$ is a piecewise constant local solution of (1.6) with $(\bar{u},\bar{v})(x,0) = (u,v)(x,0)$.

A tedious but straightforward estimation yields

$$\sum_{\lambda=-\mu}^{-1} \dot{x}_\lambda A(u_\lambda, u_{\lambda+1}) < \dot{x}A(u_-,u_+).$$

Moreover, it is clear that

$$\sum_{\lambda=0}^{\nu-1} \dot{x}_\lambda A(u_\lambda, u_{\lambda+1}) \leq 0.$$

Thus, by (3.4), $D_+H_{(\bar{u},\bar{v})}(0) < D_+H_{(u,v)}(0)$ which shows that the solution $(u,v)(x,t)$ does not satisfy the entropy rate criterion.

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13. ABSTRACT

The following admissibility criterion for solutions of hyperbolic conservation laws is proposed: A weak solution is admissible if the total entropy decays with the highest possible rate. The equivalence of this criterion and viscocity criterion is established for the single equation and the system of equations of one dimensional nonlinear elasticity.

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