OPTIMAL TRAJECTORY GENERATION
FOR MECHANICAL ARMS

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ABSTRACT

A general method of generating optimal trajectories between an initial and a final position of an n-degree of freedom manipulator arm with nonlinear equations of motion is proposed. The method is based on the assumption that the time history of each of the coordinates can be expanded in a series of simple time functions. By searching over the coefficients of the terms in the expansion, trajectories which minimize the value of a given cost function can be obtained.

The method has been applied to a planar three degree of freedom arm. The coordinates of the arm are the three joint angles. Two types of trajectories have been assumed. These are such that the time history for each joint angle is:

1. a series expansion of polynomials,
2. a series expansion of periodic functions.

Two integral type cost functions have been used:

1. the integral of the kinetic energy of the arm,
2. the integral of the magnitude of the joint torques.

The optimal values of the coefficients in the series expansion show a distinct pattern. For a particular combination of type of trajectory and cost function the optimal values of the coefficients have been approximated by rather simple functions. This results in suboptimal values of the coefficients, but they can be obtained without performing an on-line search. The difference between the optimal and suboptimal value of the cost function is of the order of 8%.

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CHAPTER 1

INTRODUCTION AND PROBLEM STATEMENT

The design and control of mechanical manipulators which perform functions similar to those of the human has been the subject of many recent studies. A particular area of interest in these studies is the supervisory controlled manipulator. In supervisory control the operator specifies task subgoals to a remote computer which in turn executes pieces of the task through direct command of the manipulator supported by local control loops. Visual sensors enable the operator to monitor the execution of the task. This technique is well suited for a manipulator in outer space or other remote locations where the distance between the operator and the arm causes a significant time delay in the communication. Supervisory control can be applied equally well to performance of complex non-routine manipulation tasks as the routine execution of repetitive operations. Often it is required that a task is executed optimally in the sense that a particular cost function, for instance time or the expenditure of energy is minimized.

The dynamic equations of motion of a manipulator arm are nonlinear and would require nonlinear control techniques to minimize a given cost function. These techniques may require a considerable amount of computer storage and real time computation. Townsend [1] investigated the possibility of controlling a nonlinear arm with feedback control computed for the linearized arm motion equations. The nonlinear equations of the arm are linearized about a certain desired motion.
It is assumed that the deviations of the actual motion from the desired motion are small so that linear control laws can be used to let the arm follow the desired motion. Townsend used two types of linear system controls: a regulator and a variable gain tracking technique. If the desired motion must be optimal the problem of how to generate the optimal motion strategies arises. This thesis describes a possible solution to this problem for a particular class of tasks, namely moving a manipulator arm from one position to another.

If the manipulator arm has \( n \) degrees of freedom the position of the arm with \( n \) general coordinates \( x_1, \ldots, x_n \) is described by a vector \( \mathbf{x} \) in the coordinate space. The motion of the arm between an initial position \( \mathbf{x}_i = [x_{i1}, \ldots, x_{in}] \) and a final position \( \mathbf{x}_f = [x_{f1}, \ldots, x_{fn}] \) is described by a trajectory in the coordinate space between \( \mathbf{x}_i \) and \( \mathbf{x}_f \). The motion is optimal if the value of the cost function is minimum along the trajectory.
CHAPTER 2

METHOD OF GENERATING OPTIMAL TRAJECTORIES

In this chapter a method of generating the optimal trajectory between an initial and a final position of a n degree of freedom mechanical arm is described.

2.1 General Approach

The method is based on the assumption that the time history of each of the elements of the position vector \( x_k \) \((k = 1, \ldots, n)\) between \( x_{k_1} \) and \( x_{k_f} \) can be expanded in a series of simple time functions.

\[
\begin{align*}
x_k &= A_k + a_{k0}f_0(t) + a_{k1}f_1(t) + a_{k2}f_2(t) \\
&\quad \quad + \ldots + a_{km}f_m(t)
\end{align*}
\]

(2.1)

where \( a_{k\ell} (\ell = 0, \ldots, m) \) are coefficients independent of time and \( A_k \) is constant. If \( f_0(t), \ldots, f_m(t) \) are given time functions \( x_k \) is only a function of the coefficients \( a_{k1}, \ldots, a_{km} \). The cost function \( J \) which must be minimized along the trajectory between the initial and final position of the arm is generally a function of \( x_k \), its derivatives \( \dot{x}_k \) and \( \ddot{x}_k \), and the task performance time \( T \). \( J \) takes the general form:

\[
J = \int_0^T L(x, \dot{x}, \ddot{x}) \, dt
\]

(2.2)
Substituting the functions for \( x_k, \dot{x}_k, \) and \( \ddot{x}_k \) in the expression for \( J \), \( J \) can be written as a function of the parameters \( a_{k\ell}(k = 1, ..., n; \ell = 0, ..., m) \) and the performance time \( T \). For a given \( T \), \( J \) is only a function of the \( a \)'s. So:

\[
J = J(a)
\]  

(2.3)

where \( a \) is the matrix of the parameters \( a_{k\ell}(k = 1, ..., n; \ell = 0, ..., m) \).

By this procedure the problem of finding the optimal trajectory has been reduced to a parameter optimization problem, i.e. finding the values of the parameters \( a_{k\ell} \) for which the value of the cost function \( J(a) \) is minimum.

To obtain the optimal values of the parameters, one can follow different procedures which can be divided into two main categories:

a. analytical method,

b. numerical methods.

The two methods are discussed briefly in the following sections.

2.2 Analytical Method

If there are no constraints on the possible values of the parameters \( a_{k\ell} \) and the function \( J(a) \) has first and second partial derivatives everywhere, necessary conditions for a minimum are:

\[
\frac{\partial J}{\partial a_i} = 0
\]  

(2.4)
where \( a_i \) is the \( i \)-th element of \( a \) with \( i = 2n + k \) and:

\[
\frac{\partial^2 J}{\partial a_i \partial a_j} \geq 0
\]  

(2.5)

which means that the matrix whose components are \( \frac{\partial^2 J}{\partial a_i \partial a_j} \) must be positive semidefinite. Equation (2.4) will give as many equations as there are unknown parameters. The advantage of the analytical method is that it gives all the possible solutions. However, in practice this method can present problems if the function \( J(a) \) is complicated.

2.3 Numerical Methods

There are various numerical methods available. Bryson and Ho [2], Bekey [3], Sage and Melsa [4] give a survey and a description of several of these methods. In general they are based on the following principle. Make an initial guess for the values of the parameters and supply these values as part of the input to a computer program. The program changes the values of the parameters according to a certain algorithm until it has found a set of values for the parameters which minimizes the value of the cost function. The particular numerical method one uses depends on the behavior of the function \( J(a) \) as a function of its argument \( a \). A disadvantage of these numerical methods is that, if \( J(a) \) has several local minima, only one local minimum is found, depending on the initial guess for the values of the \( a \)'s. This local minimum is not necessarily the global minimum. The numerical methods are very suitable for the cases that \( J(a) \) is a complicated function.
CHAPTER 3

GENERATION OF OPTIMAL TRAJECTORIES

FOR A PARTICULAR MANIPULATOR ARM

The generation of optimal trajectories as described in Chapter 2 has been applied to a planar three degree of freedom mechanical arm.

3.1 Description of the Manipulator Arm

The arm consists of two rigid straight links with lengths $l_1$ and $l_2$ connected to a fixed reference frame and to each other by moveable joints. As shown in Fig. 3.1 the joint with the fixed frame is considered as a double hinge with two degrees of freedom; the joint between the two links is a hinge with one degree of freedom. The mass of the arm is lumped as two point masses $m_1$ and $m_2$ at the ends of the links. The point masses have no rotational moment of inertia about the axis of the associated links. This lumping of masses simplifies the mathematics but does not affect the generality of the results.

The following joint angles have been chosen as coordinates of the arm (see also Fig. 3.2):

1. the angle $\theta_1$ between the plane through the two links and a fixed plane through axis 1,
2. the angle $\theta_2$ between link 1 and a line in the plane of the arm perpendicular to axis 1,
3. the relative angle $\theta_3$ between link 1 and link 2.
Figure 3.1. Sketch of the manipulator arm.
The arrows in Fig. 3.2 indicate the positive direction of rotation. This coordinate system is convenient both mathematically and physically for manipulators with torque drive at the joints.

The position of the manipulator arm is described by a vector $\mathbf{\theta}$ with elements $\theta_1$, $\theta_2$, and $\theta_3$. This $\mathbf{\theta}$ corresponds to $\mathbf{x}$ used in Chapter 2.

3.2 Trajectories

Trajectories can be categorized depending on the type of functions $f_{l}(t)(l = 0, \ldots, m)$ used in the expansion of the functions $\theta_k (k = 1, 2, 3)$:

$$
\theta_k = A_k + a_{k0} f_0(t) + a_{k1} f_1(t) + a_{k2} f_2(t) + \ldots + a_{km} f_m(t)
$$

(3.1)
For the purpose of this study two types of trajectories have been assumed.

Type 1:

The function for each joint angle $\theta_k$ ($k = 1, 2, 3$) between $t = 0$ and $t = T$ is a series expansion of polynomials.

\[ f_0(t) = \frac{1}{T} t \]  
(3.2a)

\[ f_1(t) = \frac{4}{T^2} t(T-t) \]  
(3.2b)

\[ f_2(t) = \frac{64}{T^3} t\left(\frac{T}{2} - t\right)(T - t) \]  
(3.2c)

Only the first three terms of the series have been taken into account, so $m = 2$.

For $t < 0$ and $t > T$ the value of $\theta_k$ is equal to the function value at $t=0$ and $t=T$ respectively. Figure 3.3 shows a plot of the functions $f_\phi(t)$ and their first and second derivatives.

From the end conditions at $t=0$ and $t=T$, i.e.

\[ \theta_k(0) = \theta_{ki} \]  
(3.3a)

\[ \theta_k(T) = \theta_{kf} \]  
(3.3b)

follows:

\[ A_k = \theta_{ki} \]  
(3.4)

and

\[ a_0 = \theta_{kf} - \theta_{ki} \]  
(3.5)
Figure 3.3. Functions $f_k(t)$ ($k=0,1,2$) and their first and second derivatives for a series expansion of polynomials.
The coefficients $a_{kl}$ and $a_{k2}$ are the ones to be chosen optimally. The expression for each $\theta_k$ becomes:

$$\theta_k = \theta_{ki} + (\theta_{kf} - \theta_{ki}) \frac{t}{T} + a_{k1} \frac{4}{T^2} t(T-t)$$

$$+ a_{k2} \frac{64}{T^3} t(\frac{T}{2} - t)(T-t)$$

(3.6)

This rather simple function does not provide a smooth start up and slow down of the manipulator arm because of the impulse singularities in the second derivatives.

Type 2:

The function for each joint angle $\theta_k$ ($k = 1, 2, 3$) is a series expansion of periodic functions of the following form:

$$f_0(t) = \frac{1}{T} (t - \frac{1}{\omega} \sin \omega t) \quad (3.7a)$$

$$f_1(t) = \begin{cases} 
\frac{2}{T} (t - \frac{1}{2\omega} \sin 2\omega t), & 0 \leq t < \frac{T}{2} \\
2 - \frac{2}{T} (t - \frac{1}{2\omega} \sin 2\omega t), & \frac{T}{2} \leq t \leq T
\end{cases} \quad (3.7b)$$

$$f_2(t) = \begin{cases} 
\frac{4}{T} (t - \frac{1}{4\omega} \sin 4\omega t), & 0 \leq t \leq \frac{T}{4} \\
2 - \frac{4}{T} (t - \frac{1}{4\omega} \sin 4\omega t), & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
-4 + \frac{4}{T} (t - \frac{1}{4\omega} \sin 4\omega t), & \frac{3T}{4} \leq t \leq T
\end{cases} \quad (3.7c)$$
Figure 3.4. Functions $f_k(t)$ ($k=0,1,2$) and their first and second derivatives for a series expansion of periodic functions.
where $\omega = 2\pi/T$. Also in this expansion only the first three terms have been taken into account.

The functions $f_k(t)$ and their derivatives are plotted in Fig. 3.4. From the conditions (3.3) follows:

$$A_k = \theta_{ki}$$

and

$$a_{k0} = \theta_{kf} - \theta_{ki}$$

As $\dot{\theta}_k$ and $\ddot{\theta}_k$ are zero at $t=0$ and $t=T$ this type of trajectory will give a smooth start up and slow down of the arm.

3.3 Cost Functions

Two cost functions have been used to optimize the trajectory between the initial and final position of the arm. Both cost functions are integral type functions.

Cost Function 1:

The integral between $t=0$ and $t=T$ of the kinetic energy of the manipulator arm:

$$J = \int_0^T KE \, dt \quad (3.8)$$

where $KE$ = kinetic energy of the arm at time $t$.

For the particular arm studied here the expression for the kinetic energy of the arm at time $t$ is:

$$KE = 0.5\{\dot{\theta}_1^2(m_1 + m_2)k_1^2 \cos^2 \theta_2 + m_2k_2^2 \cos^2(\theta_2 + \theta_3)$$

$$+ 2m_2k_1k_2 \cos \theta_2 \cos(\theta_2 + \theta_3)\} \quad (3.9)$$

CONT'D.
\[ + \dot{\theta}_2^2 (m_1 + m_2) l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 \cos \theta_3 \]
\[ + \dot{\theta}_3^2 m_2 l_2^2 \]
\[ + \ddot{\theta}_3 (2m_2 l_2^2 + 2m_2 l_1 l_2 \cos \theta_3) \]

This expression is derived in Appendix IA.

Cost Function 2:

The integral between \( t=0 \) and \( t=T \) of the sum of the magnitude of the joint torques:

\[ J = \int_0^T \left| \sum_{k=1}^{3} u_k \right| dt \]  
(3.10)

where \( u_k \) = external torque applied at the \( k \)-th axis of rotation.

This cost function is closely related to the energy consumed.

From the dynamic equations of motion:

\[ \ddot{\mathbf{T}} \theta = \mathbf{u} + \mathbf{c} \]

follows:

\[ \mathbf{u} = \mathbf{T} \ddot{\theta} - \mathbf{c} \]  
(3.12)

where \( \mathbf{u} = [u_1, u_2, u_3] \), the vector representing the external torques.

\[ \mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \]

the matrix representing the moments of inertia.
For the arm studied here

\[ T_{12} = T_{13} = T_{21} = T_{31} = 0. \]

\[ c = [c_1, c_2, c_3], \] the vector representing

the torques due to the reaction forces
to centripetal and coriolis forces.

The equations for the elements of \( \mathbf{u} \) become:

\[ u_1 = T_{11} \ddot{\theta}_1 - c_1 \quad (3.13a) \]

\[ u_2 = T_{22} \ddot{\theta}_2 + T_{23} \ddot{\theta}_3 - c_2 \quad (3.13b) \]

\[ u_3 = T_{32} \ddot{\theta}_2 + T_{33} \ddot{\theta}_3 - c_3 \quad (3.13c) \]

The expressions for the elements of \( \mathbf{T} \) and \( c \) are given in Appendix IB.

A computer program which generates the equations of motion for a manipulator arm of a given configuration was available.

In both cost functions the influence of gravity has been omitted for two reasons. First, the position of a manipulator arm with respect to gravity will differ from case to case. For any particular case it will not be difficult to incorporate the influence of gravity in the cost function. Second, if a manipulator is used in outer space the influence of gravity is absent.

3.4 Computer Programs

Using the trajectories and cost functions described in the previous sections three combinations of cost function and type of trajectory are possible.
Combination I:

Minimizing $J = \int_{0}^{T} KE \, dt$ assuming trajectories of Type 1 
($\theta_k$ is a series expansion of polynomials).

Combination II:

Minimizing $J = \int_{0}^{T} KE \, dt$ assuming trajectories of Type 2 
($\theta_k$ is a series expansion of periodic functions).

Combination III:

Minimizing $J = \int_{0}^{T} \sum_{k=1}^{3} |u_k| \, dt$ assuming trajectories of Type 2.

A combination of $J = \int_{0}^{T} \sum_{k=1}^{3} |u_k| \, dt$ with trajectories of Type 1 is not possible. The joint torques $u_k (k = 1, 2, 3)$ are functions of $\theta_k$.

For the trajectories of Type 1, $\ddot{\theta}_k$ is infinite at $t=0$ and $t=T$.

To obtain the optimal values of the parameters $a_i (i=1, \ldots, 6)$ for a particular initial and final position of the arm a fortran coded computer program has been written for each combination of trajectory and cost function. The programs consist of:

1. main program,
2. numerical search routine,
3. subroutine to compute the value of the cost function $J(a)$.

The three parts of the programs are described next.
1. Main Program

The main program reads the input data (lengths and masses of the arm, initial and final position, performance time, and initial guess for the values of the coefficients in the expansion), calls the search routine and prints out the final (optimal) values of the coefficients and the cost function. The main program is basically the same for combination I, II and III.

2. Numerical Search Routine

The search routine used in this study is called pattern search. Pattern search is a direct search routine for minimizing a function \( J(\mathbf{a}) \) of several variables \( \mathbf{a} = [a_1, a_2, \ldots] \). The argument \( \mathbf{a} \) is systematically varied until the minimum of \( J(\mathbf{a}) \) is obtained. The pattern search routine determines the sequence of values for \( \mathbf{a} \); an independent subroutine computes the functional values of \( J(\mathbf{a}) \).

A detailed description of the pattern search routine is given by Hooke and Jeeves [5].

Figure 3.5 shows a flow diagram of the search procedure as given by Hooke and Jeeves.

3. Subroutine to Compute the Value of the Cost Function

This subroutine is called by pattern search after each change in the argument \( \mathbf{a} \). It simulates the trajectory of a given type for the value of \( \mathbf{a} \) supplied by pattern search and computes the value of \( J(\mathbf{a}) \) along the trajectory. The integration is carried out with Simpson's Rule. The number of intervals between \( t=0 \) and \( t=T \) is twenty. This
25

Figure 3.5a. Descriptive flow diagram of pattern search.

Figure 3.5b. Descriptive flow diagram for exploratory moves. This routine is carried out for each coordinate separately.
subroutine is different for each of the combinations of trajectory and cost function.

The programs are listed in Appendix II.
CHAPTER 4

DESCRIPTION AND RESULTS OF THE PROGRAM RUNS

This chapter gives a description of the program runs made, possible difficulties in the use of the search routine and an interpretation of the results of the searches. All runs were made for an arm with \( \ell_1 = \ell_2 = 0.3 \, \text{m} \) and \( m_1 = m_2 = 1 \, \text{kg} \).

4.1 Initial and Final Positions

![Figure 4.1. Combinations of initial (final) angles.](image)
The initial and final positions of the arm were chosen in a certain region in θ-space at discrete points. The dots in Fig. 4.1 indicate the values of θ₂₁ and θ₃₁ or θ₂f and θ₃f. For fixed values of θ₁₁ and θ₁f this will give 256 possible combinations of initial and final position for each of the combinations I, II and III. To limit the number of runs a choice was made out of the 256 possible combinations.

For most runs θ₁₁ = -0.78 and θ₁f = 0.78 (angles in radians) were chosen. For combination III a number of runs were made with different values for θ₁₁ and θ₁f while keeping θ₂₁, θ₂f, θ₃₁ and θ₃f constant.

4.2 Results of the Searches

The results of the individual searches for θ₁₁ = -0.78 and θ₁f = 0.78 are given in Appendices III A, B and C. The following observations can be made concerning the results for combinations I, II and III.

I. For all combinations of θ₁ and θ₂ the values of a₁₂, a₂₂ and a₃₂ are zero. When both θ₂₁ = θ₂f and θ₃₁ = θ₃f the value of a₁₁ is zero too.

II. When θ₂₁ = θ₂f and θ₃₁ = θ₃f the optimal values of a₁₁, a₂₁, a₂₂ and a₃₂ are zero. The optimal value of a₁₂ lies between 0.11 and 0.18.

III. As for I the optimal values of a₁₂, a₂₂ and a₃₂ are zero or very small in all cases, and a₁₁ is zero when both θ₂₁ = θ₂f and θ₃₁ = θ₃f.
For the special cases that both $\theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$ the optimal values of the parameters $a_{21}$ and $a_{31}$ for the three combinations of trajectory and cost function are plotted as functions of $\theta_{2i} = \theta_{2f} = \theta_2$ and $\theta_{3i} = \theta_{3f} = \theta_3$ in Figs. 4.2, 4.3 and 4.4.

The results of the searches for combination III with varying $\theta_{1i}$ and $\theta_{1f}$ are listed in Appendix III D. The optimal values of $a_{21}$, $a_{22}$ and $a_{32}$ are zero in all cases. The optimal values of $a_{11}$, $a_{21}$, $a_{31}$ and $J(a)$ are plotted as function of $\theta_{1f} - \theta_{1i}$ in Figs. 4.5, 4.6 and 4.7.

Each search gives only a local optimum. Therefore one can not be sure that the optimum found is a global optimum. However by starting the search in a proper point based on physical considerations one can increase the probability that the global optimum will be found. An example of choosing a wrong starting point is given next.

For combination III with $\theta_4 = [-0.78, -0.78, 1.57]$ and $\theta_f = [0.78, -0.78, 1.57]$ the initial values of the a's were chosen all equal to zero. This resulted in a set of optimal values for the a's of $[0.0, 0.0, 0.27, 0.0, 0.0, 0.0]$ and a value of the cost function $J$ of 0.935. Starting the search at $[0.0, -0.168, 0.492, 0.0, 0.0, 0.0]$ resulted in a set of optimal values for the a's of $[0.0, -0.219, -0.626, 0.0, 0.0, 0.0]$ and a value of the cost function $J$ of 0.630 which is much less than in the other case. The new starting point was the result of an interpolation between the results for the other combinations of $\theta_2$ and $\theta_3$ (see Fig. 4.4). This example indicates that the choice of
Figure 4.2. Optimal values of coefficients $a_{21}$ and $a_{31}$ for combination I with $\theta_{21}=\theta_{2f}=\theta_2$ and $\theta_{31}=\theta_{3f}=\theta_3$.

The symbols in the figures indicate the value of $\theta_3$:
- $\times$ $\theta_3=0.0$;
- $\circ$ $\theta_3=0.78$;
- $\triangle$ $\theta_3=1.57$;
- $\square$ $\theta_3=2.36$. 
Figure 4.3. Optimal values of coefficients $a_{21}$ and $a_{31}$ for combination II with $\theta_{2i} = \theta_{2f} = \theta_2$ and $\theta_{3i} = \theta_{3f} = \theta_3$.

The symbols in the figures indicate the value of $\theta_3$:
- $\times \times \theta_3 = 0.0$;
- $\circ \circ \theta_3 = 0.78$;
- $\triangle \triangle \theta_3 = 1.57$;
- $\cdots \cdots \theta_3 = 2.36$. 
Figure 4.4. Optimal values of coefficients $a_{21}$ and $a_{31}$ for combination III with $\theta_{21}=\theta_{2f}=\theta_2$ and $\theta_{31}=\theta_{3f}=\theta_3$. 
Figure 4.5. Optimal values of coefficients $a_{21}$ and $a_{31}$ and cost function $J(a)$ as functions of $\theta_{1f} - \theta_{1i}$ for combination III with $\theta_{2i} = -0.78$ and $\theta_{2f} = 0.78$. 
Figure 4.6. Optimal values of coefficients $a_{21}$ and $a_{31}$ and cost function $J(a)$ as functions of $\theta_{1f}-\theta_{1i}$ for combination III with $\theta_{2i}=\theta_{2f}=0.0$ and $\theta_{3i}=\theta_{3f}=1.57$. 
Figure 4.7. Optimal values of coefficients $a_{11}$, $a_{21}$ and $a_{31}$ and cost function $J(a)$ as functions of $\theta_{lf}-\theta_{li}$ for combination III with $\theta_{21}=-0.78$, $\theta_{2f}=-1.57$ and $\theta_{31}=\theta_{3f}=0.78$. 
the starting point for the search can be very important. Therefore, the starting points have been chosen carefully in accordance with the physics of the problem.

4.3 Some Examples of Optimal Trajectories

In Fig. 4.8 and 4.9 two examples of how the functions for $\theta_1$, $\theta_2$ and $\theta_3$ will look like for different combinations of trajectory type and cost functions, using the optimal values of the coefficients in the series expansion. The Roman numbers in the figures indicate the combination of trajectory and cost function as mentioned in Section 3.4.
Figure 4.8. Time histories for $\theta_1$, $\theta_2$ and $\theta_3$ for
$\theta_i = [-0.78,0,0,0]$ and $\theta_f = [0.78,0,0,0]$. 
Figure 4.9. Time histories for $\theta_1$, $\theta_2$ and $\theta_3$ for

$\bar{\theta}_1=[-0.78,0.0,0.0]$ and $\bar{\theta}_f=[0.78,-0.78,1.57]$. 
CHAPTER 5

GENERALIZATION OF THE RESULTS

This chapter discusses a method of generalizing the results of the individual searches for specific combinations of trajectory and cost function. This method will lead to suboptimal values of the coefficients $a_{k\ell}$. However, these values can be obtained by on-line computation in a short time. It is not necessary anymore to perform an on-line search.

The method is worked out for combination III where

$$J = \int_0^T \sum_{k=1}^3 |u_k| \, dt$$

and the trajectories are of Type 2.

The values of $\theta_{1i}$ and $\theta_{1f}$ are -0.78 and 0.78 respectively.

For combination III the optimal values of $a_{12}$, $a_{22}$ and $a_{32}$ are zero or at least very small. Therefore the suboptimal values of these parameters have been chosen zero for all combinations of $\theta_i$ and $\theta_f$. To obtain general expressions for determining the values of $a_{11}$, $a_{21}$ and $a_{31}$ four categories of combinations of $\theta_i$ and $\theta_f$ have been considered separately. The assumptions made and the results obtained for each of the categories are described in the following sections.

5.1 $\theta_{1i} = -0.78, \theta_{1f} = 0.78, \theta_{2i} = \theta_{2f}$ and $\theta_{3i} = \theta_{3f}$

For this case the optimal values of $a_{11}$ are zero. The optimal values of $a_{21}$ and $a_{31}$ are plotted in Fig. 4.4.
It is assumed that for each value of $\theta_3$ the values of $a_{21}$ and $a_{31}$ as functions of $\theta_2$ can be approximated by parabolic functions of the form:

$$a_{21} = p(\theta_2 - a)^2 + b$$

(5.1)

$$a_{31} = q(\theta_2 - c)^2 + d$$

(5.2)

Using a linear regression technique the best fitting parabolas through the four values of $a_{21}$ and $a_{31}$ for a constant $\theta_3$ were determined. The formula used for this purpose is derived in Appendix IV. This gives four expressions for $a_{21}$ of the form (5.1) and four for $a_{31}$ of the form (5.2). Consideration of the values of $p$, $a$, $b$, $q$, $c$ and $d$ as functions of $\theta_3$ indicated that a good approximation for $a,b,q,c$ and $d$ as a function of $\theta_3$ is a straight line and for $p$ as a function of $\theta_3$ is a parabola. Fitting the curves through the points the following expressions were obtained:

$$p = 0.4936 - 0.3079 \theta_3 + 0.0504 \theta_3^2$$

(5.3)

$$a = -0.3929 - 0.2002 \theta_3$$

(5.4)

$$b = -0.6628 + 0.2619 \theta_3$$

(5.5)

$$q = -0.9985 + 0.3903 \theta_3$$

(5.6)

$$c = -0.3370 - 0.1106 \theta_3$$

(5.7)

$$d = 1.5442 - 0.5772 \theta_3$$

(5.8)
Figure 5.1. Functions for suboptimal values of coefficients $a_{21}$ and $a_{31}$ for combination III with $\theta_{2i} = \theta_{2f} = \theta_2$ and $\theta_{3i} = \theta_{3f} = \theta_3$. 
Substituting (5.3) through (5.8) in Eqs. (5.1) and (5.2) \( a_{21} \) and \( a_{31} \) can be written as functions of \( \theta_2 = \theta_{21} = \theta_{2f} \) and \( \theta_3 = \theta_{31} = \theta_{3f} \). In Figure 5.1 the functions for \( a_{21} \) and \( a_{31} \) have been plotted for four values of \( \theta_3 \). For comparison the optimal values of \( a_{21} \) and \( a_{31} \) obtained from the searches have been plotted too.

The value of the cost function \( J(a) \) has been computed for the suboptimal values of the parameters and for all parameters equal to zero. This showed that on the average the difference between the suboptimal value and the optimal value of \( J(a) \) is 7.4% of the optimal value. The difference between the value of \( J(a) \) for all \( a \)'s equal to zero and the optimal value is 27.5% of the optimal value.

5.2 \( \theta_{1i} = -0.78, \theta_{1f} = 0.78, \theta_{21} = \theta_{2f} \) and \( \theta_{31} \neq \theta_{3f} \)

For this category the expressions for \( a_{21} \) and \( a_{31} \) mentioned in 5.1 have been used to determine the suboptimal values of \( a_{21} \) and \( a_{31} \) as follows. Setting \( \theta_2 = \theta_{21} = \theta_{2f} \) and \( \theta_3 = \theta_{31} \) in Eqs. (5.1) through (5.8) gives certain values for \( a_{21} \) and \( a_{31} \), say \( a'_{21} \) and \( a'_{31} \). Setting \( \theta_2 = \theta_{21} = \theta_{2f} \) and \( \theta_3 = \theta_{3f} \) the values of \( a_{21} \) and \( a_{31} \) are \( a''_{21} \) and \( a''_{31} \). Using the following expressions a reasonable fit to the data for \( \theta_{21} = \theta_{2f} \) and \( \theta_{31} \neq \theta_{3f} \) was obtained.

\[
\begin{align*}
    a_{21} &= \min\{a'_{21}, a''_{21}\} \quad (5.9) \\
    a_{31} &= \min\{a'_{31}, a''_{31}\} + r|a'_{31} - a''_{31}| \quad (5.10)
\end{align*}
\]

where

\[
r = 1.3(\theta_2 + 0.4)^4 + 0.6 \quad (5.11)
\]
For the suboptimal value of parameter $a_{11}$ the expression

$$a_{11} = -0.20 \, \text{sgn}(2 \cos \theta_{21} + \cos(\theta_{21} + \theta_{31}) - 2 \cos \theta_{2f} - \cos(\theta_{2f} + \theta_{3f}))$$

(5.12)

gave satisfactory results.

5.3 $\theta_{11} = -0.78, \theta_{1f} = 0.78, \theta_{21} \neq \theta_{2f}$ and $\theta_{31} = \theta_{3f}$.

For this category the suboptimal values for $a_{21}$ and $a_{31}$ were obtained from (again using Eqs. (5.1) through (5.8)):

$$a_{21} = \min\{\text{all } a_{21} \text{ for } \theta_{3} = \theta_{31} = \theta_{3f} \text{ and } \theta_{2} \text{ between } \theta_{21} \text{ and } \theta_{2f}\}$$

(5.13)

$$a_{31} = \max\{\text{all } a_{31} \text{ for } \theta_{3} = \theta_{31} = \theta_{3f} \text{ and } \theta_{2} \text{ between } \theta_{21} \text{ and } \theta_{2f}\}$$

(5.14)

The value for $a_{11}$ followed from (5.12).

5.4 $\theta_{11} = -0.78, \theta_{1f} = 0.78, \theta_{21} \neq \theta_{2f}$ and $\theta_{31} \neq \theta_{3f}$.

Satisfactory values for the cost function $J(a)$ were obtained by choosing for the values of $a_{21}$ and $a_{31}$ (using (5.1) through (5.8)):

$$a_{21} = \min\{\text{all } a_{21} \text{ for } \theta_{2} \text{ between } \theta_{21} \text{ and } \theta_{2f} \text{ and } \theta_{3} \text{ between } \theta_{31} \text{ and } \theta_{3f}\}$$

(5.15)
\( a_{31} = \max\{a_{31} \text{ for } \theta_2 \text{ between } \theta_{2i} \text{ and } \theta_{2f}, \theta_3 \text{ between } \theta_{3i} \text{ and } \theta_{3f}\} \) (5.16)

The value of \( a_{11} \) was chosen from Eq. (5.12).

For the categories described in Sections 5.2, 5.3 and 5.4 the values of the cost function \( J(a) \) for all \( a \)'s equal to zero are on the average 70\% bigger than the optimal values. For the suboptimal values of the \( a \)'s the difference between the values of \( J(a) \) and the optimal values was on the average 10\%. This justifies the use of suboptimal values for the parameters \( a_{k\ell} \) very well.

5.5 Flow Diagram for Determining the Suboptimal Values of the Coefficients in the Series Expansion

The results of the previous sections can be summarized in a flow diagram as presented in Fig. 5.2. The flow diagram is for the cases that \( \theta_{11} = -0.78, \theta_{1f} = 0.78, \theta_{2i} \text{ and } \theta_{2f} \text{ between } -1.57 \text{ and } 0.78, \text{ and } \theta_{3i} \text{ and } \theta_{3f} \text{ between } 2.36 \text{ and } 0 \). The flow diagram is easy to transform into a computer program.

The generalization as described in this chapter can also be done for each of the combinations I and II. The main problem will be finding a function which fits the data for the cases that \( \theta_{2i} = \theta_{2f} \) and \( \theta_{3i} = \theta_{3f} \).
\[ \theta_{11} = -0.78 \]

\[ \theta_{1f} = 0.78 \]

\[ \theta_{2i} = \theta_{2f} \]

Yes

\[ a_{1l} = 0 \]

No

\[ E_1: a_{1l} \]

1

\[ \theta_{3i} = \theta_{3f} \]

Yes

\[ E_1: a_{1l} \]

No

\[ E_1: a_{1l} \]

2

\[ \theta_{3i} = \theta_{3f} \]

Yes

\[ E_1: a_{1l} \]

No

\[ E_1: a_{1l} \]

2

\[ \theta_{2} = \theta_{2i} = \theta_{2f} \]

\[ \theta_{3} = \theta_{3i} = \theta_{3f} \]

\[ E_2: a_{2i} \]

\[ E_3: a_{3i} \]

3

\[ \theta_{2} = \theta_{2i} = \theta_{2f} \]

\[ \theta_{3} = \theta_{3i} \]

\[ E_2: a_{2i}' \]

\[ E_3: a_{3i}' \]

4

\[ r = 1.3(\theta_2 + 0.4)^{4+0.6} \]

\[ a_{2i} = \min (a_{2i}', a_{2f}') \]

\[ a_{3i} = \min (a_{3i}', a_{3f}') + r |a_{3i}' - a_{3f}'| \]

5

\[ \theta_{3} = \theta_{3i} = \theta_{3f} \]

\[ a_{2i} = \min (all \ a_{2i} \ for \ \theta_2 \ between \ \theta_{2i} \ and \ \theta_{2f}) \]

\[ a_{3i} = \max (all \ a_{3i} \ for \ \theta_2 \ between \ \theta_{2i} \ and \ \theta_{2f}) \]

Stop

6

\[ a_{2i} = \min (all \ a_{2i} \ for \ \theta_2 \ between \ \theta_{2i} \ and \ \theta_{2f} \ and \ \theta_2' \ between \ \theta_{3i} \ and \ \theta_{3f}) \]

\[ a_{3i} = \max (all \ a_{3i} \ for \ \theta_2 \ between \ \theta_{2i} \ and \ \theta_{2f} \ and \ \theta_2' \ between \ \theta_{3i} \ and \ \theta_{3f}) \]

Stop

CONT'D
\[
E_1: a_{11} = -0.20 \text{ sgn}\{ 2 \cos \theta_{21} + \cos (\theta_{21} + \theta_{31}) \\
- 2 \cos \theta_{2f} - \cos (\theta_{2f} + \theta_{3f}) \} 
\]

\[
E_2: \begin{align*}
p &= 0.4936 - 0.3079 \theta_3 + 0.0504 \theta_3^2 \\
a &= -0.3929 - 0.2002 \theta_3 \\
b &= -0.6628 + 0.2619 \theta_3 \\
a_{21} &= p (\theta_2 - a)^2 + b
\end{align*}
\]

\[
E_3: \begin{align*}
q &= -0.9985 + 0.3903 \theta_3 \\
c &= -0.3370 - 0.1106 \theta_3 \\
d &= 1.5142 - 0.5772 \theta_3 \\
a_{31} &= q (\theta_2 - c)^2 + d
\end{align*}
\]

Figure 5.2. Flow diagram for determining the suboptimal values of the parameters \(a_{11}\), \(a_{21}\) and \(a_{31}\) for combination III with \(\theta_{11} = -0.78\) and \(\theta_{1f} = 0.78\).
The proposed method of generating optimal trajectories worked successfully for a three degree of freedom mechanical arm. However, the method is general and it can be expected that it is also applicable to more complicated arm models.

The consistency of the results with the physical understanding of the problem indicates that the pattern search routine was suitable for this problem.

The results of the searches indicate that only the first two terms of the series expansion are important, except for joint angle \( \theta_1 \) in combination II where the third term has a significant influence on the shape of the function for \( \theta_1 \).

The optimal value of the cost function \( J = \int_0^T KE \, dt \) for combination I is lower than for combination II. From this it can be concluded that in order to minimize the integral of the kinetic energy a series expansion of polynomials is more suitable than a series expansion of periodic functions. To minimize the integral of the torque magnitude only the series expansion of periodic functions is applicable.

In the special cases that \( \theta_{2f} = \theta_{2f} \) and \( \theta_{3f} = \theta_{3f} \) the optimal values of the parameters \( a_{21} \) and \( a_{31} \) showed a certain pattern, especially for combination III. For this combination the values of
a_{21} and a_{31} could be summarized by paraboloid functions with
\theta_{21} = \theta_{2f} = \theta_2 and \theta_{31} = \theta_{3f} = \theta_3 as independent variables (see
Fig. 5.1). The values of a_{21} and a_{31} obtained in this way are
suboptimal. The difference between the suboptimal and the optimal value
of the cost function is on the average 7.4%. From the suboptimal
values of a_{21} and a_{31} in the case that \theta_{21} = \theta_{2f} and \theta_{31} = \theta_{3f}
the suboptimal values of a_{21} and a_{31} in the other cases of
combination III could be derived quite easily. The suboptimal value of
the cost function in these cases is on the average 10% bigger than the
optimal value, which is satisfactory.

Using the algorithm for the suboptimal values the coefficients
in the series expansion can be obtained on-line without search. This
saves a significant amount of real time computation.
APPENDIX I

A. Derivation of the expression for the kinetic energy of the arm.

For a system of two point masses, what the arm essentially is, the kinetic energy is:

\[ KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \]  \hspace{1cm} (I.1)

where \( v_1 \) and \( v_2 \) are the velocities of the masses \( m_1 \) and \( m_2 \).

Furthermore (see figure I.1):

\[ v_1^2 = v_{1\theta_1}^2 + v_{1\theta_2}^2 \]  \hspace{1cm} (I.2)

where \( v_{1\theta_1} \) = component of \( v_1 \) due to rotation about axis 1,
\( v_{1\theta_2} \) = component of \( v_1 \) due to rotation about axis 2.

\[ v_2^2 = v_{2\theta_1}^2 + v_{2\theta_2}^2 + v_{2\theta_3}^2 + 2v_{2\theta_2}v_{2\theta_3} \cos \alpha \]  \hspace{1cm} (I.3)

where \( v_{2\theta_1} \) = component of \( v_2 \) due to rotation about axis 1,
\( v_{2\theta_2} \) = component of \( v_2 \) due to rotation about axis 2,
\( v_{2\theta_3} \) = component of \( v_2 \) due to rotation about axis 3,
\( \alpha \) = angle between \( v_{2\theta_2} \) and \( v_{2\theta_3} \).

The expressions for the components of \( v_1 \) and \( v_2 \) are:

\[ v_{1\theta_1} = l_1 \dot{\theta}_1 \cos \theta_2 \]  \hspace{1cm} (I.4)
\[ v_{1\theta_2} = l_1 \dot{\theta}_2 \]  \hspace{1cm} (I.5)
\[ v_{2\theta_1} = (l_1 \cos \theta_2 + l_2 \cos (\theta_2 + \theta_3)) \dot{\theta}_1 \]  \hspace{1cm} (I.6)
\[ v_{2\theta_2} = \sqrt{l_1^2 + l_2^2 + 2l_1 l_2 \cos \theta_3 \cos \theta_2} \]  \hspace{1cm} (I.7)
Substituting (I.2) through (I.8) in (I.1) gives the following expression for the kinetic energy:

\[
KE = \frac{1}{2}m_1 [\dot{\theta}_1^2 \cos^2 \theta_1 + \dot{\theta}_2^2]
+ \frac{1}{2}m_2 [\{\dot{\theta}_1 \cos \theta_2 + \dot{\theta}_2 \cos (\theta_2 + \theta_3)\}^2 \theta_1^2
+ \sqrt{\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos \theta_3} \dot{\theta}_2^2
+ \dot{\theta}_2^2 \theta_3
+ 2\sqrt{\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos \theta_3} \dot{\theta}_2 \dot{\theta}_3 \cos \alpha] \quad (1.9)
\]

Substituting:

\[
\cos \alpha = \frac{\dot{\theta}_2 + \dot{\theta}_1 \cos \theta_3}{\sqrt{\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1 \dot{\theta}_2 \cos \theta_3}} \quad (1.10)
\]

in Eqn. (1.9) and changing the order of the terms results in:

\[
KE = \frac{1}{2} \dot{\theta}_1^2 \{m_1 + m_2\} \dot{\theta}_1^2 \cos^2 \theta_1 + m_2 \dot{\theta}_2^2 \cos^2 (\theta_2 + \theta_3)
+ 2m_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_2 \cos (\theta_2 + \theta_3)
+ \dot{\theta}_2^2 \{m_1 + m_2\} \dot{\theta}_1^2 + m_2 \dot{\theta}_2^2 + 2m_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_3
+ \dot{\theta}_3^2 \{m_2 \dot{\theta}_2^2
+ \dot{\theta}_2 \dot{\theta}_3 \{2m_2 \dot{\theta}_2^2 + 2m_2 \dot{\theta}_1 \dot{\theta}_2 \cos \theta_3\}] \quad (3.9)
\]
B. Expressions for the elements of $T$ and $c$ in Eqn. (3.13).

\[ T_{11} = \frac{1}{2}(m_1 + m_2)k_1^2 + \frac{1}{2}m_2k_2^2 + \frac{1}{2}(m_1 + m_2)k_1^2\cos^2\theta_2 + \frac{1}{2}m_2k_2^2\cos^2(\theta_2 + \theta_3) + 2m_2k_1k_2\cos\theta_2\cos(\theta_2 + \theta_3) \]  

(I.11)

\[ T_{22} = m_1k_1^2 + m_2k_1^2 + m_2k_2^2 + 2m_2k_1k_2\cos\theta_3 \]  

(I.12)

\[ T_{23} = m_2k_2^2 + m_2k_1k_2\cos\theta_3 \]  

(I.13)

\[ T_{32} = T_{23} \]  

(I.14)

\[ T_{33} = m_2k_2^2 \]  

(I.15)

\[ c_1 = \frac{1}{\theta_1\theta_2}(m_1 + m_2)k_1^2\sin^2\theta_2 + m_2k_2^2\sin^2(\theta_2 + \theta_3) + 2m_2k_1k_2\sin(\theta_3 + \theta_2) \]  

(I.16)

\[ c_2 = -\frac{1}{\theta_1\theta_2}(m_1 + m_2)k_1^2\sin^2\theta_2 + \frac{1}{2}m_2k_2^2\sin^2(\theta_2 + \theta_3) + m_2k_1k_2\sin(\theta_3 + \theta_2) \]  

+ \theta_3^2\{m_2k_1k_2\sin\theta_3\}  

(I.17)

\[ c_3 = -\frac{1}{\theta_1\theta_2}(m_1 + m_2)k_1^2\sin^2(\theta_2 + \theta_3) + m_2k_1k_2\cos^2\theta_2\sin(\theta_2 + \theta_3) \]  

- \theta_3^2\{m_2k_1k_2\sin\theta_3\}  

(I.18)
C-----MAIN PROGRAM
C------FCRC COMPUTING THE OPTIMAL VALUES OF THE COEFFICIENTS
C------IN THE SERIES EXPANSION
C------TCRC IS THE SUBROUTINE FOR COMPUTING THE COST FUNCTION
C------TCINT IS THE VALUE OF THE COST FUNCTION

REAL LL, L2, ML, M2
EXTERNAL TORQ
DIMENSION BA(9)
COMMON TFIN, F4, FS, F6, T9, ANG1L, ANG1F, ANG2L, ANG2F,
       ANG3L, ANG3F

C------READ INPUT DATA
READ(5,2C) TFIN, LL, L2, ML, M2
2C FORMAT(F15.6/4E15.6)
READ(5,37) N
37 FORMAT(11)
READ(5,1C) DELIN, DLMIN
READ(5,5) M
5 FORMAT(12)
WRITE(6,4)
4 FORMAT(*10OPTIMAL PARAMETERS FOR MIN INT TORQUE*)
WRITE(6,25)
25 FORMAT(/, TFIN, LL, L2, ML, M2, */)
WRITE(6,26) TFIN, LL, L2, ML, M2
26 FORMAT(5E15.6)
WRITE(6,38) DELIN, DLMIN
38 FORMAT(/, DELIN = , F15.6, ' DELMIN = ', E15.6, */)
F1 = LL**2
F2 = L2**2
F3 = ML*L2
F4 = ML*F1
F5 = M2*F1
F6 = M2*F3
T9 = F2*F2
NCSET = 1

C------READ INITIAL AND FINAL POSITION OF THE ARM
6 READ(5,1C) ANG1L, ANG2L, ANG3L, ANG1F, ANG2F, ANG3F
1C FORMAT(E15.6)

C------READ INITIAL GUESS FOR THE PARAMETERS
READ(5,1C) (BA(I), I=1, 6)
WRITE(6,35)
35 FORMAT(/, *** INITIAL AND FINAL ANGLES***, */)
WRITE(6,36) ANG1L, ANG2L, ANG3L, ANG1F, ANG2F, ANG3F
36 FORMAT(3E15.6/3E15.6, */)
DEL = CELIN
CALL PATSH (BA, TOINT, N, DEL, DLMIN, TORQ)
WRITE(6,40)
4C FORMAT(/, ' FINAL RESULTS', */)
WRITE(6,50) (BA(I), I=1, 6)
5C FORMAT(6E15.6)
WRITE(6,60) TOINT
6C FORMAT(F15.6)
IF (NSET.EQ.M) GO TO 70
NCSET=NOSET+1
GOTO 6
70 CONTINUE
END
SUBROUTINE PATSH(PSI, SSI, N, DEL, DLMIN, MRIT4)

C PSI IS THE CURRENT BASEPT
C THT IS THE PREVIOUS BASEPT
C PHI IS THE TRIAL PT
C S IS THE OBJECTIVE FCT
C
DIMENSION PSI(9), PHI(9), THT(9), EPS(9)
WRITE(6, 603)
603 FORMAT(' CURRENT POINT, OBJ FCT AND STEPSIZE')
ALFA=1.02
C EVALUATE AT INIT BASEPT
10 CALL MRIT4(PSI, SSI)
C START AT BASEPT
100 S=SSI
   DC 1CI I=1,N
101 PHI(I)=PSI(I)
   ICALL=1
   WRITE(6, 599)
599 FORMAT(' ***•')
   WRITE(6, 600) (PSI(J), J=1,N)
   WRITE(6, 601) S, DEL
C MAKE EXPLORATORY MOVES
GO TO 150
C IS PRESENT VALUE ) BASEPT VALUE
160 IF(S.LT.SSI) GO TO 200
   GC TC 300
C SET NEW BASEPT
200 SSI=S
   DC 2CI I=1,N
   THT(I)=PSI(I)
   PSI(I)=PHI(I)
C MAKE PATTERN MOVE
201 PHI(I)=PHI(I)+ALFA*(PHI(I)-THT(I))
   CALL MRIT4(PHI, SPI )
   S=SPI
   WRITE(6, 599)
   WRITE(6, 599)
   WRITE(6, 600) (PHI(I), I=1,N)
600 FORMAT(8E15.6)
   WRITE(6, 601) S, DEL
601 FORMAT(2E15.6)
   ICALL=2
C MAKE EXPL MOVES
GO TO 150
C IS PRESENT VALUE ) BASEPT VALUE
260 IF(S.LT.SSI) GO TO 200
   GC TC 10C
300 IF(DEL.LT. DLMIN) RETURN
   DEL=DEL/2.
   GC TC 100
C MAKE EXPL MOVES
150 DO 180 K=1,N
   EPS(K)=.05*PHI(K)
   IF(EPS(K) .EQ. 0.) EPS(K)=.05
   PHI(K)=PHI(K)+EPS(K)*CEL
   CALL DRIT4(PHI, SPI)
155 IF(SPI.LT.S) GO TO 179
   PHI(K)=PHI(K)-.5*EPS(K)*DEL
   CALL DRIT4(PHI, SPI)
165 IF(SPI.LT.S) GO TO 179
   PHI(K)=PHI(K)+EPS(K)*CEL
   GC TC 180
179 S=SPI
180 CONTINUE
   GC TC (160, 260), ICALL
END
SUBROUTINE KNET(BA, KEINT)

C------FCR COMPUTING THE INTEGRAL OF THE KINETIC ENERGY
C------WITH TRAJECTORIES OF TYPE I
C------(SERIES EXPANSION OF POLYNOMIALS)
REAL MAL1, MAL2, MAL4, KE(21), KEINT
DIMENSION BA(9)
COMMON TFIN, F4, F5, F6, T9, ANG1I, ANG1F, ANG2I, ANG2F,
ANG3I, ANG3F
B1 = BA(1)
B2 = BA(2)
B3 = BA(3)
B4 = BA(4)
B5 = BA(5)
B6 = BA(6)
TF2 = TFIN**2

C------COMPUTE KINETIC ENERGY AT EACH INTERVAL POINT
DC 2C00 J = 1, 21
T = TFIN*(FLOAT(J)-1.)/20.
ANG2 = ANG2I+T*(ANG2F-ANG2I)/TFIN+4.*R2*T*(TFIN-T)/TF2
ANG2 = ANG2+64.*R5*T*(TFIN/2.-T)*(TFIN-T)/(3.*TF2*TFIN)
ANG3 = ANG3I+T*(ANG3F-ANG3I)/TFIN+4.*R3*T*(TFIN-T)/TF2
ANG3 = ANG3+64.*R6*T*(TFIN/2.-T)*(TFIN-T)/(3.*TF2*TFIN)
VEL1 = ANG1F-ANG1I)/TFIN+4.*B1*(TFIN-2.*T)/TF2
VEL1 = VEL1+64.*R4*(0.5*TF2-3.*TFIN*T+3.*T**2)/
(3.*TF2*TFIN)
VEL2 = (ANG2F-ANG2I)/TFIN+4.*B2*(TFIN-2.*T)/TF2
VEL2 = VEL2+64.*B5*(0.5*TF2-3.*TFIN*T+3.*T**2)/
(3.*TF2*TFIN)
VEL3 = (ANG3F-ANG3I)/TFIN+4.*B3*(TFIN-2.*T)/TF2
VEL3 = VEL3+64.*B6*(0.5*TF2-3.*TFIN*T+3.*T**2)/
(3.*TF2*TFIN)

C------SUM CF ANGLES
FAC1 = ANG2+ANG3

C------SINES ANC COSINES
CCS1 = COS(ANG2)
CCS2 = CCS(ANG3)
CCS4 = COS(FAC1)

C------COMBINE TERMS
G1 = F4+F5
G6 = F6*COS2

C------COMPLETE 2*KINETIC ENERGY
MAL1 = G1*COS1**2+T9*COS4**2+2.*F6*COS1*COS4
MAL2 = G1+T9+2.*G6
MAL4 = 2.*(T9+G6)
KE(J) = VEL1**2*MAL1+VEL2**2*MAL2+VEL3**2*T9+
VEL2*VEL3*MAL4
2C00 CONTINUE
C------COMPUTE INTEGRAL OF KINETIC ENERGy
KEINT=KE(1)+4.*KE(2)+KE(21)
DC 2050 L=2,10
2050 KEINT=KEINT+2.*KE(2*L-1)+4.*KE(2*L)
KEINT=(TFIN/60.)*KEINT/2.
RETURN
END
SUBROUTINE KINFT (BA, KEINT)

C----- FCR Computing the Integral of the Kinetic Energy
C----- With Trajectories of Type 2
C----- (Series Expansion of Periodic Functions)

REAL MAL1, MAL2, MAL4, KE(21), KEINT
DIMENSION BA(9)
COMMON TFIN, F4, F5, F6, F9, ANG1I, ANG1F, ANG2I, ANG2F,
1 ANG3I, ANG3F

TC1 = EA(1) * 2.
TC2 = EA(2) * 2.
TC3 = EA(3) * 2.
TC5 = EA(5) * 2.
TC6 = EA(6) * 2.
FC4 = EA(4) * 4.
FC5 = EA(5) * 4.
FC6 = EA(6) * 4.
OM = 6.28319 / TFIN
TCM = 2. * OM
FCM = 4. * OM

C----- Compute Kinetic Energy at Each Interval Point
DC 2600 J = 1, 21
T = TFIN * (FLOAT(J) - 1.) / 20.
OMT = CMT
TCMT = 2. * CMT
FCMT = 4. * CMT
SCMT = SIN(CMT)
CCMT = COS(CMT)
STOMT = 2. * STOMT * COMT
CTOMT = 1. - 2. * STOMT ** 2
SFOMT = 2. * STOMT * CTOMT
CFOMT = 1. - 2. * STOMT ** 2
PCNE = (T - STOMT / OM) / TFIN
QCNE = (L. - COMT) / TFIN
PTWO = (T - SFOMT / OM) / TFIN
QTWO = (1. - CFOMT) / TFIN
PFCU = (T - SFOMT / OM) / TFIN
QFCU = (1. - CFOMT) / TFIN

C----- First Approximation T Between 0 and TFIN
ANG2 = ANG2I + (ANG2F - ANG2I) * PONE
ANG3 = ANG3I + (ANG3F - ANG3I) * PONE
VEL1 = (ANG1F - ANG1I) * QONE
VEL2 = (ANG2F - ANG2I) * QONE
VEL3 = (ANG3F - ANG3I) * QONE
IF (T - TFIN / 2.) 200, 200, 300

C----- Second Approximation T Between 0 and TFIN/2
200 ANG2 = ANG2 + TC2 * PTWC
ANG3 = ANG3 + TC3 * PTWC
VEL1 = VEL1 + TC1 * QTWC
VEL2 = VEL2 + TC2 * QTWC
VEL3 = VEL3 + TC3 * QTWC
IF (T - TFIN / 4.) 600, 600, 700
C-----SECOND APPROXIMATION T BETWEEN TFIM/2 AND TFIN

300 ANG2=ANG2+TC2-TC2*PTWC
ANG3=ANG3+TC3-TC3*PTWC
VEL1=VEL1-TC1*QTWG
VEL2=VEL2-TC2*CTWG
VEL3=VEL3-TC3*QTWG
IF(T-.75*TFIN) 700,70C,800

C-----THIRD APPROXIMATION T BETWEEN 0 AND TFIN/4

600 ANG2=ANG2+FC5*PFOU
ANG3=ANG3+FC6*PFOU
VEL1=VEL1+FC4*QFUU
VEL2=VEL2+FC5*QFOU
VEL3=VEL3+FC6*QFOU
GC TC 1000

C-----THIRD APPROXIMATION T BETWEEN TFIN/4 AND 3*TFIN/4

700 ANG2=ANG2+FC5-FC5*PFOU
ANG3=ANG3+TC6-FC6*PFOU
VEL1=VEL1+FC4*CFOU
VEL2=VEL2+FC5*QFOU
VEL3=VEL3+FC6*QFOU
GC TC 1000

C-----THIRD APPROXIMATION T BETWEEN 3*TFIN/4 AND TFIN

800 ANG2=ANG2-FC5+FC5*PFOU
ANG3=ANG3-FC6+FC6*PFOU
VEL1=VEL1-FC4*CFOU
VEL2=VEL2+FC5*QFOU
VEL3=VEL3+FC6*QFOU
GC TC 1000

C-----SUM OF ANGLES
1000 FAC1=ANG2+ANG3

C-----SINES AND COSINES
COS1=COS(ANG2)
COS2=COS(ANG3)
COS3=COS(FAC1)

C-----COMBINE TERMS
G1=F4+F5
G6=F6*COS2

C-----COMPUTE 2*KINETIC ENERGY
MAL1=G1*COS1**2+T9*COS4**2+2.*F6*COS1*COS4
MAL2=G1+T9+2.*G6
MAL4=2.*(T9+G6)
KE(J)=VEL1**2*MAL1+VEL2**2*MAL2+VEL3**2*T9+
1VEL2*VEL3*MAL4

2000 CONTINUE

C-----COMPLETE INTEGRAL OF KINETIC ENERGY
KEINT=KE(1)+4.*KE(2)+KE(21)
DC 2C50 L=2,10
2C50 KEINT=KEINT+2.*KE(2*L-1)+4.*KE(2*L)
KEINT=(TFIN/60.)*KEINT/2.
RETURN
END
SUBROUTINE TORC (BA, TCINT)
C-----FCR COMPUTING THE INTEGRAL OF THE SUM OF THE
C-----ABS. VALUES OF THE JOINT TORQUES
C-----WITH TRAJECTORIES OF TYPE 2
C-----SERIES EXPANSION OF PERIODIC FUNCTIONS)
DIMENSION BA(9), AU(21)
COMMON TFIN,F4,F5,F6,T9,ANG1I, ANGI F, ANG2I, ANG2F,
LANG3I, LANG3F
TC1=BA(1)*2.
TC2=BA(2)*2.
TC3=BA(3)*2.
TC5=BA(5)*2.
TC6=BA(6)*2.
FC4=BA(4)*4.
FC5=BA(5)*4.
FC6=BA(6)*4.
OM=6.28319/TFIN

C-----COMPUTE JOINT TORQUES AT EACH INTERVAL POINT
DC 2COO J=1,21
T=TFIN*(FLOAT(J)-1.)/20.
OMT=OM*T
TCMT=2.*CMT
SCMT=SIN(CMT)
CM T=COS(CMT)
STCMT=2.*SCMT*CMT
CTCMT=1.-2.*SCMT**2
SFOMT=2.*STCMT*CTCMT
CFOMT=1.-2.*STCMT**2
QCN E=(1.-CMT)/TFIN
RCN E=CM*SCMT/TFIN
PTWC=(T-STCMT/OM)/TFIN
QTWC=(1.-CTCMT)/TFIN
RTWC=OM*STCMT/TFIN
PFCU=(T-SFOMT/OM)/TFIN
QFCU=(1.-CFOMT)/TFIN
RFCU=OM*SFOMT/TFIN
C-----FIRST APPROXIMATION T BETWEEN 0 AND TFIN
ANG2=(ANG2I+(ANG2F-ANG2I)*PONE
ANG3=(ANG3I+(ANG3F-ANG3I)*PONE
VEL1=(ANG1F-ANG1I)*QONE
VEL2=(ANG2F-ANG2I)*QONE
VEL3=(ANG3F-ANG3I)*QONE
ACC1=(ANG1F-ANG1I)*QONE
ACC2=(ANG2F-ANG2I)*QONE
ACC3=(ANG3F-ANG3I)*QONE
IF (T-TFIN/2.) 200, 200, 300
C-----SECOND APPROXIMATION T BETWEEN C AND TFIN/2
200 ANG2=ANG2+TC2*PTWC
ANG3=ANG3+TC3*PTWC
VEL1=VEL1+TC1*QTWC
VEL2=VEL2+TC2*QTWC
VEL3=VEL3+TC3*QTWC
ACC1=ACC1+TC1*RTWC
ACC2=ACC2+TC2*RTWC
ACC3=ACC3+TC3*RTWC
IF(T-TFIN/2) 600,600,700

C-----SECOND APPROXIMATION T BETWEEN TFIN/2 AND TFIN
300 ANG2=ANG2+TC2-TC2*PTWC
ANG3=ANG3+TC3-TC3*PTWC
VEL1=VEL1-TC1*QTWC
VEL2=VEL2-TC2*QTWC
VEL3=VEL3-TC3*QTWC
ACC1=ACC1-TC1*RTWC
ACC2=ACC2-TC2*RTWC
ACC3=ACC3-TC3*RTWC
IF(T-.75*TFIN) 700,700,800

C-----THIRD APPROXIMATION T BETWEEN O AND TFIN/4
600 ANG2=ANG2+FC5*PFOU
ANG3=ANG3+FC6*PFOU
VEL1=VEL1+FC4*QFOU
VEL2=VEL2+FC5*QFOU
VEL3=VEL3+FC6*QFOU
ACC1=ACC1+FC4*RFOU
ACC2=ACC2+FC5*RFOU
ACC3=ACC3+FC6*RFOU
GC TC 1000

C-----THIRD APPROXIMATION T BETWEEN TFIN/4 AND 3*TFIN/4
700 ANG2=ANG2+TC5-FC5*PFOU
ANG3=ANG3+TC6-FC6*PFOU
VEL1=VEL1-FC4*QFOU
VEL2=VEL2-FC5*QFOU
VEL3=VEL3-FC6*QFOU
ACC1=ACC1-FC4*RFOU
ACC2=ACC2-FC5*RFOU
ACC3=ACC3-FC6*RFOU
GC TC 1000

C-----THIRD APPROXIMATION T BETWEEN 3*TFIN/4 AND TFIN
800 ANG2=ANG2-FC5+FC5*PFOU
ANG3=ANG3-FC6+FC6*PFOU
VEL1=VEL1+FC4*QFOU
VEL2=VEL2+FC5*QFOU
VEL3=VEL3+FC6*QFOU
ACC1=ACC1+FC4*RFOU
ACC2=ACC2+FC5*RFOU
ACC3=ACC3+FC6*QFOU
C-----SUM OF ANGLES
1COO  FAC1=ANG2+ANG3
       FAC2=2.*FAC1
       FAC3=2.*ANG2
       FAC4=ANG3+FAC3

C-----SINES AND COSINES
       SIN2=SIN(ANG3)
       CCS1=COS(ANG2)
       CCS2=COS(ANG3)
       SIN3=SIN(FAC3)
       SIN4=SIN(FAC1)
       SIN5=SIN(FAC2)
       SIN6=SIN(FAC4)
       CCS3=CCS(FAC3)
       CCS4=CCS(FAC1)
       CCS5=CCS(FAC2)

C-----COMBINE TERMS
       G1=F4+F5
       G2=T9*SIN5
       G3=F6*SIN6
       G4=F6*COS1
       G5=F6*SIN2
       G6=F6*COS2
       G7=2.*G4*SIN4+G2
       G8=G1*SIN3+G2+2.*G3

C-----COMPUTE PRODUCTS OF VELOCITIES
       V1=.5*VEL1**2
       V2=VEL2**2
       V3=VEL3**2
       V4=VEL1*VEL2
       V5=VEL1*VEL3
       V6=VEL2*VEL3

C-----COMPUTE C-TERMS
       C1=V4*G8+V5*G7
       C2=-V1*G8+(V3+2.*V6)*G5
       C3=-V1*G7-V2*G5

C-----COMPUTE T-TERMS
       T6=T9+G6
       T5=T9+2.*G6+G1
       T8=T6
       T1=.5*(G1*(1+COS3)+T9*(1+COS5))+2.*G4*COS4

C-----COMPUTE JOINT TORQUES
       U1=T1*ACC1-C1
       U2=T5*ACC2+T6*ACC3-C2
       U3=T8*ACC2+T9*ACC3-C3

C-----COMPUTE SUM OF ABS. VALUES OF JOINT TORQUES
       AU(J)=ABS(U1)+ABS(U2)+ABS(U3)

2000 CONTINUE
C------COMPUTE INTEGRAL OF ABS. JOINT TORQUES
TOINT=AU(1)+4.*AU(2)+AU(21)
DO 2050 L=2,10
2050 TOINT=TOINT+2.*AU(2*L-1)+4.*AU(2*L)
TCINT=(TFIN/60.)*TOINT
RETURN
END
APPENDIX III

RESULTS OF THE SEARCHES

Optimal values of the coefficients in the series expansion and the cost function $J(a)$
### III A.

<table>
<thead>
<tr>
<th>$\theta_{11}$</th>
<th>$\theta_{21}$</th>
<th>$\theta_{31}$</th>
<th>$\theta_{12}$</th>
<th>$\theta_{22}$</th>
<th>$\theta_{32}$</th>
<th>$\theta_{13}$</th>
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<td>$-1.57$</td>
<td>$-1.57$</td>
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<td>$-1.57$</td>
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\[ J(a) = \int_{0}^{T} KE \, dt \] trajectories of type 1 (series expansion of polynomials).
### III A cont'd.

<table>
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<th>$\theta_3$</th>
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</tbody>
</table>

The table above represents the coefficients for the trajectories of type 1 (series expansion of polynomials).

The equation for $J(a)$ is given by:

$$J(a) = \int KE dt; \text{ trajectories of type 1 (series expansion of polynomials).}$$
<table>
<thead>
<tr>
<th>$\theta_{1f}$</th>
<th>$\theta_{2f}$</th>
<th>$\theta_{3f}$</th>
<th>$a_{11}$</th>
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<th>$J(a)$</th>
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</thead>
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\[ J(a) = \int_{0}^{T} KE \, dt \]; trajectories of type 2 (series expansion of periodic functions).
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$J(\theta) = \int_0^T \sum_{k=1}^3 u_k^2 \, dt$; trajectories of type 2 (series expansion of periodic functions).
\[
J(a) = \int_0^T \sum_{k=1}^3 |u_k| \, dt; \text{ trajectories of type 2 (series expansion of periodic functions).}
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DERIVATION OF THE LINEAR REGRESSION FORMULA

If the result of an experiment is a number of data points 
\((x_1, y_1), \ldots, (x_N, y_N)\) and the model for the experiment is assumed:

\[ y = c_0 + c_1 x + c_2 x^2 + \ldots + c_K x^K \]  \hspace{1cm} (IV.1)

one can write the following \(N\) equations for the estimates of the \(y\)'s:

\[ \hat{y}_1 = c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_K x_1^K \]
\[ \vdots \]
\[ \hat{y}_N = c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_K x_N^K \]  \hspace{1cm} (IV.2)

or in matrix form:

\[ \hat{Y} = X \hat{c} \]  \hspace{1cm} (IV.3)

where

\[ \hat{Y}^T = [y_1, y_2, \ldots, y_N] \]  
\[ X = \begin{bmatrix} 1 & x_1 & \ldots & x_1^K \\ 1 & x_2 & \ldots & x_2^K \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & \ldots & x_N^K \end{bmatrix} \]  
\[ \hat{c}^T = [c_1, c_2, \ldots, c_K] \]
The error between the actual $y$ and the estimate of it $\hat{y}$ is:

$$e = y - Xc$$  \hspace{1cm} (IV.4)

The squared error becomes:

$$e^T e = (y - Xc)^T(y - Xc)$$  \hspace{1cm} (IV.5)

The $c$ that will minimize the squared error is $\hat{c}$, the least square estimate of $c$, and follows from the necessary condition for a minimum squared error:

$$\frac{\partial e^T e}{\partial c} = 0$$  \hspace{1cm} (IV.6)

or:

$$-X^T y + X^T X c = 0$$  \hspace{1cm} (IV.7)

So that:

$$\hat{c} = [X^T X]^{-1} X^T y$$  \hspace{1cm} (IV.8)
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