THERMAL ELASTOPLASTIC STRUCTURAL ANALYSIS OF
NON-METALLIC THERMAL PROTECTION SYSTEMS

by

T. J. Chung and G. Yagawa

Final Technical Report

(NASA-CR-124005) THERMAL ELASTOPLASTIC
STRUCTURAL ANALYSIS OF NON-METALLIC
THERMAL PROTECTION SYSTEMS Final
Technical Report (Alabama Univ.,
Huntsville.) 78 p HC $6.00 CSCL 20M G3/33 16354

This research work was supported by
the National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Contract NAS8-27792

The University of Alabama in Huntsville
School of Graduate Studies and Research
Huntsville, Alabama 35807

August 1972
THERMAL ELASTOPLASTIC STRUCTURAL ANALYSIS OF
NON-METALLIC THERMAL PROTECTION SYSTEMS

by

T. J. Chung and G. Yagawa

Final Technical Report

This research work was supported by
the National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Contract NAS8-27792

The University of Alabama in Huntsville
School of Graduate Studies and Research
Huntsville, Alabama 35807

August 1972
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>PREFACE</td>
<td>i.</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ii.</td>
</tr>
<tr>
<td>NOMENCLATURE</td>
<td>iii.</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>v.</td>
</tr>
</tbody>
</table>

## SECTIONS

1. Introduction ........................................ 1
2. Thermomechanical Preliminaries ..................... 2
3. Thermoelastoplastic Behavior ....................... 3
4. Heat Conduction Equations-Finite Element Formulation ........ 8
5. Incremental Finite Element Equations of Equilibrium .......... 12
6. Solution Procedure .................................. 14
7. Applications ........................................ 15
8. Concluding Remarks ................................ 31

## REFERENCES

32

## APPENDICES

A - Capabilities and Limitation of the Program .............. 34
B - Various Integrals in Isoparametric Element ............... 35
C - Flow Chart ........................................ 38
D - Subroutine Organization Chart .......................... 40
E - Descriptions of Subroutines ............................ 41
F - Data Input Format ................................... 44
G - Program Listing ..................................... 48
PREFACE

This report presents the results of studies conducted during the period September 1, 1971 - August 31, 1972, under NASA Research Contract NAS8-27792, "Thermal Elastoplastic Structural Analysis of Non-metallic Thermal Protection Systems." This study was monitored by Mr. C. R. Zimmerman, Analytical Mechanics Division, Asttronautics Laboratory of NASA's Marshall Space Flight Center.
LIST OF FIGURES

Figure 1 - Transient thermal stress distribution in the free-free beam, uncoupled.

Figure 2 - Discretized geometry of three-dimensional solid and input data.

Figure 3 - Temperature distribution of $y = z = 0$ in Figure 2, coupled.

Figure 4 - Displacement ($w$) at $y = 100\text{mm}$, $z = 300\text{mm}$ in the x-direction.

Figure 5 - Transient displacement $w$ at point A of Figure 2.

Figure 6 - Transient temperature change at point A of Figure 2.

Figure 7 - Stress $\sigma_z$ - displacement $w$ for element B of Figure 2.

Figure 8 - Development of plastic regions, coupled and uncoupled.

Figure 9 - Comparison of coupled elastoplastic displacements ($w$) with and without surface insulation at $y = 100\text{mm}$, $z = 300\text{mm}$ in the x-direction.

Figure 10 - Discretized geometry of three-dimensional solid and input data.

Figure 11 - Temperature distribution of $y = z = 0$ in Figure 10.

Figure 12 - Displacement ($w$) at $y = 100\text{mm}$, $z = 300\text{mm}$ in the x-direction in Figure 12.

Figure 13 - Development of plastic regions in Figure 10.
NOMENCLATURE

\( B_{1j} \) = Tensor of thermoelastic constants

\( B_{1j}^* \) = Tensor of thermoplastic constants

\( c \) = Specific heat

\( d\lambda \) = Positive constant

\( D \) = Dissipation

\( E_{1jk} \) = Tensor of elastic moduli

\( \tilde{E}_{1jk} \) = Tensor of plastic moduli

\( E \) = Plastic modulus

\( F^i \) = Components of mechanical force

\( h \) = Heat supply

\( q \) = Heat flux

\( T_0, T \) = Reference temperature and temperature change

\( u^i, \dot{u}^i, \ddot{u}^i \) = Components of displacements, velocity, and acceleration

\( \varepsilon \) = Equivalent yield strain

\( \gamma_{ij} \) = Strain tensor

\( \Delta t \) = Time interval

\( e \) = Strain energy density

\( \tau \) = Entropy

\( \theta \) = Absolute temperature

\( \rho_0, \rho \) = Density at initial and deformed configuration

\( \sigma \) = Equivalent yield stress

\( \sigma^{ij} \) = Stress tensor
\( m \) = Free energy

\( \psi_{1N} \) = Normalized displacement interpolation function

\( \Omega_{r} \) = Normalized temperature interpolation function
THERMAL ELASTOPLASTIC ANALYSIS OF
NON-METALLIC THERMAL PROTECTION SYSTEMS

T. J. Chung and C. Yagawa

ABSTRACT

This report presents an incremental theory and numerical procedure to analyze a three-dimensional thermoelastoplastic structure subjected to high temperature, surface heat flux, and volume heat supply as well as mechanical loadings. Heat conduction equations and equilibrium equations are derived by assuming a specific form of incremental free energy, entropy, stresses and heat flux together with the first and second laws of thermodynamics, von Mises yield criteria and Prandtl-Reuss flow rule. The finite element discretization using the linear isotropic three-dimensional element for the space domain and a difference operator corresponding to a linear variation of temperature within a small time increment for the time domain lead to systematic solutions of temperature distribution and displacement and stress fields. Various boundary conditions such as insulated surfaces and convection through uninsulated surface can be easily treated. To demonstrate effectiveness of the present formulation a number of example problems are presented.
1. INTRODUCTION

The mechanics of thermoelastoplastic solids has attracted the attention of many investigators in recent years. The development of the theories of thermoelastoplasticity began with an attempt to consider the plastic deformation as a thermodynamic state variable and with the controversial treatment of finite deformations. Crucial difficulties lie in the proper choice of free energy functional and methods of numerical computation.

Biot [1] and Coleman and Gurtin [2] used the concept of state variable in dealing with thermodynamics of viscoelastic materials. Perzyna and Wojno [3], Kratochvil and Dillon [4], subsequently, employed the similar concept by decomposing the total strains into elastic and plastic strain components. The recent work by Valanis [5] uses also the concept of hidden variables, but introducing no yield surfaces. Finite elastoplastic deformations with thermodynamic considerations were also discussed by Green and Naghdi [7]. Schapery [8] studied a thermodynamic constitutive theory with history effects represented by single integrals. Recently, Oden and Bhandari [9] proposed a general functional theory of thermodynamics of viscoelastoplastic solids and showed that various theories of viscoelasticity, rate independent viscoplasticity and plasticity can be obtained as special cases by imposing suitable constraints on the material parameters.

All of these theories mentioned above, however, lead to considerable mathematical and computational difficulties in dealing with problems of geometrical complexity. The present paper is an attempt to establish an alternate approach. We derive incremental governing equations for three-
dimensional thermoelastoplastic solids and provide convenient solution techniques by finite elements in space domain and the difference operator in time domain. Although an extension to materials with memory presents no special difficulty, the present study is not intended for viscous behavior. In this study, the Lagrangian coordinates are used disregarding the finite strains, but large deformations are included in the formulation.

Finally, example problems are presented to demonstrate the merits of the present formulation. Effects of thermomechanical coupling, thermoplastic deformations and stress, and developments of plastic regions as a function of time are shown and evaluated. Various features not included in this study such as temperature dependent material properties or dynamically coupled inertia effects may be treated with minor modifications.

2. THERMOMECHANICAL PRELIMINARIES

It is assumed that the behavior of the body under thermomechanical environments obeys the laws of conservation of mass, balance of linear and angular momentum, conservation of energy, and the Clausius-Duhem inequality,

\[ \int_{V_0} \rho_0 \, dv_0 = \int_{V} \rho \, dv \] \hspace{1cm} (1a)

\[ \sigma_{i,j} + \rho F_{j,i} - \rho u_{,i} = 0, \quad \sigma_{i,j} = \sigma_{j,i} \] \hspace{1cm} (lb)

\[ \rho \dot{\varepsilon} = \sigma_{i,j} \gamma_{j,i} + q_{j,i} - \rho h \] \hspace{1cm} (lc)

\[ D + \frac{1}{\theta} q_{i} \theta_{,i} \geq 0 \] \hspace{1cm} (ld)
\[ D = \rho \theta \gamma - q \frac{\partial}{\partial t} - \rho h \]  

(1e)

Here \( \rho \) is the mass density with the subscript \( o \) indicating undeformed configuration. \( \sigma^{ij} \) is the second Piola-Kirchhoff stress tensor; superposed dots represent time rates; strokes and commas are covariant and ordinary differentiations; \( F \) and \( u \) are the body forces and displacements; \( \gamma_{ij} \) is the strain tensor; \( \varepsilon, h \) and \( \eta \) are the internal energy, heat supply and entropy per unit mass; \( q^j \) is the heat flux per unit area; \( \Theta \) is the absolute temperature; \( D \) is the internal dissipation. It is understood that for small strains \( \rho_o = \rho \) and for rectangular cartesian coordinates covariant and ordinary differentiations are the same.

The free energy \( \varphi \) is defined as

\[ \varphi = \varepsilon - \Theta \eta \]  

(2)

which leads to

\[ \rho \dot{\varphi} = \sigma^{ij} \gamma_{ij} - D - \rho \eta \dot{\Theta} \]  

(3)

3. THERMOELASTOPLASTIC BEHAVIOR

As discussed in Section 1, our approach is to avoid the functional theory or the state variable concept. Instead of expressing the free energy as functionals of all the histories of strains and temperature and as a function of temperature gradient, we postulate an existence of \( \varphi \) in incremental quantity so that for a small time interval \( \Delta t \)

\[ \varphi(\Delta t) = \hat{\varphi}[\gamma_{ij}(\Delta t), \gamma_{ij}'(\Delta t), \Theta(\Delta t)] \]  

(4a)

Similarly,

\[ \sigma^{ij}(\Delta t) = \hat{\sigma}[\gamma_{ij}(\Delta t), \gamma_{ij}'(\Delta t), \Theta(\Delta t)] \]  

(4b)
\[ q^i(\Delta t) = Q[\gamma_{ij}^e(\Delta t), \gamma_{ij}^p(\Delta t), \theta(\Delta t)] \] 
\[ \eta(\Delta t) = H[\gamma_{ij}^e(\Delta t), \gamma_{ij}^p(\Delta t), \theta(\Delta t)] \]

Here it is assumed that the total strain is the sum of the elastic and plastic components with \((e)\) and \((p)\) representing, respectively, "elastic" and "plastic".

The implication of (4a) through (4d) is that the free energy, stresses, heat flux, and entropy are functions of elastic strain, plastic strain and temperature only within the small time interval. It is then a simple matter to derive the governing heat conduction equations and equations of equilibrium in "incremental form". All histories may be carried over from one increment to another through the step-by-step numerical time integration.

Although the free energy can be shown in various forms depending on the material properties or the purpose of analysis, the present study is limited to a particular case including only quadratic terms as follows:

\[ \rho F(\Delta t) = \frac{1}{2} E_{ijkl}^e \gamma_{ij}^e(\Delta t) \gamma_{kl}^e(\Delta t) + \frac{1}{2} E_{ijkl}^p \gamma_{ij}^p(\Delta t) \gamma_{kl}^p(\Delta t) - B_{ijkl}^T(\Delta t) \gamma_{ij}^e(\Delta t) - B_{ijkl}^T(\Delta t) \gamma_{ij}^p(\Delta t) - \frac{c}{2} T^2(\Delta t) \]

in which \( E_{ijkl}^e \) and \( E_{ijkl}^p \) are tensors of elastic and plastic moduli; \( B_{ijkl}^T \) and \( B_{ijkl}^T \) are tensors of thermoelastic and thermoplastic moduli; \( c \) is the specific heat; \( T_0 \) and \( T \) are the reference temperature and temperature change related by \( \theta = T_0 + T \). Here \( E_{ijkl}^e \) and \( E_{ijkl}^p \) are the functions of current state of stress in the inelastic range. It should be noted that such inelastic material properties cannot be admitted had the form of free energy been expressed in entire history domain rather than in a small time interval. These arrays of plastic moduli remain constant only during the small time interval and must
be updated as the state of stress changes.

Likewise, the expression (3) in a small time increment is given by

\[
\phi(\Delta t) = \sigma_{ij}(\Delta t) \dot{\gamma}_{ij}(\Delta t) - D(\Delta t) - \rho \eta(\Delta t) \dot{\theta}(\Delta t)
\]

\[
= \sigma_{ij}(\Delta t) (\dot{\gamma}_{ij}(\Delta t) + \dot{\gamma}_{ij}(\Delta t)) - D(\Delta t) - \rho \eta(\Delta t) \dot{\theta}(\Delta t)
\] (6)

and the time rate of (5) becomes

\[
\dot{\rho} = E_{ijkl} \dot{\gamma}_{kl}(\dot{\gamma}^*(\Delta t) - \dot{\gamma}^*(\Delta t)) - B^{ijT}(\dot{\gamma}_{ij}(\Delta t)
\]

\[
- B^{ijT}(\dot{\gamma}_{ij}(\Delta t) - B^{ijT}(\dot{\gamma}_{ij}(\Delta t)
\]

\[
- B^{ijT}(\dot{\gamma}_{ij}(\Delta t) - B^{ijT}(\dot{\gamma}_{ij}(\Delta t)
\]

\[
= \frac{c T}{T_0} + \rho \eta ) T + D + E_{ijkl} \dot{\gamma}_{kl}(\dot{\gamma}^*(\Delta t) - \dot{\gamma}^*(\Delta t) - \sigma_{ij}(\dot{\gamma}^*(\Delta t)
\] (7)

In view of (6) and (7) and dropping (\Delta t) for simpler notation, we obtain

\[
( E_{ijkl} \dot{\gamma}_{kl}^{(\epsilon)} - B^{ijT} - \sigma_{ij}^{(\gamma)} + (-B^{ijT} - \dot{\gamma}_{ij}^{(\gamma)} - B^{ijT} - \dot{\gamma}_{ij}^{(\gamma)} - \sigma_{ij}^{(\gamma)} = 0
\] (8)

For all arbitrary values of \( \dot{\gamma}_{ij}^{(\epsilon)} \) and \( T \) the following relationships must be true from (8):

\[
\sigma_{ij} = E_{ijkl} \dot{\gamma}_{kl}^{(\epsilon)} - B^{ijT}
\] (9)

\[
\rho \eta = B^{ijT} + B^{ijT} + \frac{c T}{T_0}
\] (10)

\[
D = -E_{ijkl} \dot{\gamma}_{kl}^{(\gamma)} + B^{ijT} + \sigma_{ij}^{(\gamma)}
\] (11)

It is interesting to note that the internal dissipation consists of terms associated with thermoelastoplastic strains or energy dissipated by plastic deformations.

To evaluate \( E_{ijkl} \) and \( B^{ijT} \) the von Mises yield criteria and associated flow rule may be used as discussed by earlier investigators [10, 11, 12].

Writing (9) in differential form,
\[
d\sigma_{ij} = E^{ijkl}(d\gamma_{kl} - d\gamma^{(p)}_{kl}) - B^{ij}dT \tag{12}
\]

where

\[
d\gamma^{(p)}_{kl} = \frac{\partial F}{\partial \gamma_{kl}}d\lambda \tag{13}
\]

The plastic potential \(F\) is related by the equivalent yield stress \(\sigma\) and the second deviatoric stress invariant \(J\),

\[
F = \overline{\sigma}^2 = 3J \tag{14}
\]

from which we derive the incremental yield stress in the form

\[
d\sigma = \frac{3}{2\sigma} \frac{\partial F}{\partial \sigma_{ij}} d\sigma_{ij} = \lambda_{ij} d\sigma_{ij} \tag{15}
\]

and \(d\lambda\) in (13), a positive constant, can be shown to be

\[
d\lambda = d\gamma^{(p)}/2\overline{\sigma} \tag{16}
\]

with the incremental equivalent yield strain \(d\gamma^{(p)}\) related by the bilinear plastic modulus \(E\) as

\[
d\gamma^{(p)} = \frac{d\gamma^{(p)}}{E} \tag{17}
\]

Using (13) through (17) in (12) we obtain

\[
d\sigma_{ij} = E^{ijkl}(d\gamma_{kl} - Z_{kl}d\gamma^{(p)}) - B^{ij}dT \tag{18}
\]

In view of (15), (17), and (18) \(d\gamma^{(p)}\) assumes the form

\[
d\gamma^{(p)} = H^{-1}(Z_{ij}E^{ijkl}d\gamma_{kl} - Z_{ij}B^{ij}dT) \tag{19}
\]

where

\[
H^{-1} = E + \frac{1}{\tau}Z_{rs}Z_{tu}E_{rstu} \tag{20}
\]

Substituting (20) in (18) yields

\[
d\sigma_{ij} = E^{ijkl}d\gamma_{kl} + \overset{*}{E}^{ijkl}d\gamma_{kl} - B^{ij}dT - \overset{*}{B}^{ij}dT \tag{21}
\]

in which

\[
\overset{*}{E}^{ijkl} = -H^{-1}E^{ijkl}Z_{pq}Z_{mn}E^{klpq} \tag{22}
\]

\[
\overset{*}{B}^{ij} = -H^{-1}B^{mn}Z_{mn}Z_{kl}E^{ijkl} \tag{23}
\]
We now return to \( (1e) \), a representation of irreversible thermodynamic processes and express it for a small time increment in the form,

\[
\rho \dot{\theta} (\Delta t) \left[ \eta (\Delta t) \right] - q(\Delta t) \frac{\partial}{\partial t} + \phi (\Delta t) - D(\Delta t) = 0
\]

(24)

Once again dropping \( (\Delta t) \) and substituting from (10) and (11), it is possible to write (24) in the form

\[
(T_0 + T) (B^{11} \gamma_j^{(c)} + B^{11} \gamma_j^{(p)} + \frac{c^I}{T_0}) - q_j - \phi^i + E_{11}^{(p)} \gamma_{jk} \gamma_{1j}^{(p)} - B^{11} \gamma_{1j}^{(p)} - \sigma^{11} \gamma_{1j}^{(p)} = 0
\]

(25)

where

\[
\gamma_j^{(p)} = \gamma_j^{(c)} - \gamma_j^{(e)}
\]

(26)

and

\[
\gamma_j^{(c)} = \gamma_j^{(p)} = Z_{1j}^{-1} \gamma_j^{(p)} = Z_{1j}^{-1} (Z_{mn} E_{n} n_{k} \gamma_{kl} - Z_{mn} B_{n} \gamma_{n})
\]

(27)

Substituting (26) and (27) into (25) yields

\[
(T_0 + T) \left\{ (B^{11} + B^{11} + B^{11}) \gamma_{1j}^{(c)} + (B + \phi) \right\} T
\]

\[
+ \frac{c^I}{T_0} \left\{ q_j - \phi^i + E_{11}^{(p)} \gamma_{jk} \gamma_{1j}^{(p)} - E_{11}^{(p)} \gamma_{1j}^{(p)} \gamma_{1j}^{(e)}
\]

\[-B_{11} \gamma_{1j}^{(e)} + W_{1j} T - \gamma_j^{(e)} - \phi^i T - \sigma^{11} (G \gamma_{1j}^{(e)} + W_{1j} T) = 0
\]

(28)

where

\[
B^{11} = H^{-1} B^{11} Z_{p} Z_{q} E_{pq} \gamma_{kl}
\]

(29a)

\[
\tilde{B} = H^{-1} B^{11} Z_{1} Z_{p} B_{pq}
\]

(29b)

\[
\tilde{B} = H^{-1} B^{11} Z_{1} Z_{p} B_{pq}
\]

(29c)

\[
\tilde{G} = H^{-1} Z_{rs} Z_{pq} E_{rs} p_{q}
\]

(29d)

\[
W_{mn} = H^{-1} Z_{mn} Z_{pq} B_{pq}
\]

(29e)
Note that all quantities in (28) are now expressed in terms of the total strain and elastic strain but not plastic strain. The plastic behavior is exhibited by $E^{ijkl}$, $B^{ij}$, $B^{ij}$, $\tilde{B}$, $\tilde{B}$, $G$, and $W_{mn}$. It can easily be shown that for isotropic solids $B$, $\tilde{B}$, $\tilde{B}$, and $W_{mn}$ are always zero. They need be considered only for anisotropic solids. Here elastic strains follow simply from the elastic constitutive law.

The expression in (28) is the transient heat conduction equation with a complete elastoplastic coupling and internal dissipation. The formulation of incremental finite element equations from (28) will be shown in the following section.

4. HEAT CONDUCTION EQUATIONS-FINITE ELEMENT FORMULATION

To introduce the finite element application to (28) the element temperature $T$ and element displacements $u_i$ ($i = 1, 2, 3$) are replaced by a linear combination of all nodal temperatures $T^r$ and all nodal displacements $u^n$ with suitable interpolation functions [9, 16], in the form

$$T = \Omega^r u^r$$

$$u_i = \psi^i u^n$$

For the 8 node isoparametric element, we have $n = 8$ and $N = 24$. $\Omega^r$ and $\psi^i$ are the normalized interpolation functions for temperature and displacements, respectively.

The strain-displacement relationship is given by

$$\gamma_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{n',i}u_{n',j})$$

On substituting (31) into (32), we have

$$\gamma_{ij} = A_{N1} u^n + C_{NM1} u^n u^n$$
where $A_{N11}$ and $C_{NM11}$ are the strain transformation operators,

\[
A_{N11} = \frac{1}{2}(\psi_{11N} + \psi_{1N1})
\]
\[
C_{NM11} = \frac{1}{2} \psi_{kN1} \psi_{NM1}
\]

Now let it be required to solve the differential equation (28) rewritten in the form

\[
L(\gamma_{11}, \dot{\gamma}_{11}, T, \dot{T}) = 0
\]
or upon substitution of (30) into (33),

\[
L(u^{N}, \dot{u}^{N}, T^{R}, \dot{T}^{R}) = 0
\]

(34)

where $L$ is the differential operator and $L(u^{N}, \dot{u}^{N}, T^{R}, \dot{T}^{R})$ is considered as the local residual $L(u^{N}, \dot{u}^{N}, T^{R}, \dot{T}^{R})$. Requiring this local residual to be orthogonal to the subspace spanned by the functions $\Omega_{R}$ for each finite element; i.e.,

\[
\int_{V} L(u^{N}, \dot{u}^{N}, T^{R}, \dot{T}^{R}) \Omega_{R} dv = 0
\]

(35)

which is essentially the Galerkin's method, we obtain the finite element model of (28),

\[
\int_{V} \left[ \left( T_{O} + \Omega_{u} T^{u} \right) \Omega_{R} \left( \tilde{B}^{*11} + \tilde{B}^{11} + \tilde{B}^{11} \right) (A_{N11} + 2C_{MP11} u^{P}) \dot{u}^{N} \\
+ \left( \tilde{B} + \tilde{B} \right) \Omega_{S} \dot{T}^{S} + \frac{C}{T_{O}} \Omega_{S} T^{S} \right] - q^{11} \Omega_{R} - ph_{R} + \tilde{E}^{11} \Omega_{R} \left( A_{N11} u^{N} + C_{NM11} u^{M} u^{N} \right) \dot{T}
\]
\[
+ \tilde{E}^{11} \Omega_{R} \left( A_{P11} u^{P} + 2C_{P11} u^{R} \right) \dot{u}^{P} - \tilde{E}^{11} \Omega_{R} \left( A_{P11} + 2C_{P11} u^{R} \right) \dot{u}^{P} \psi^{(*)}_{R}
\]
\[
- \tilde{E}^{11} \Omega_{R} \left( A_{N11} u^{N} + C_{NM11} u^{M} u^{N} \right) \dot{T} + \tilde{E}^{11} \Omega_{R} \left( A_{N11} \dot{u}^{N} + C_{NM11} \dot{u}^{M} u^{N} \right) \dot{T}
\]
\[
+ \sigma^{11} \Omega_{R} \left( A_{N11} \dot{u}^{N} + 2C_{NM11} \dot{u}^{M} u^{N} \right) \dot{x} - \sigma^{11} \Omega_{R} \left( \dot{u}^{N} \right) \dot{T} = 0
\]

(36)

Introducing the linear Fourier law,

\[
q^{i} = \psi^{T} \psi^{i}
\]
where $\kappa^{ij}$ is the thermal conductivity, using the Green-Gauss theorem, and after some algebra in (36), we finally arrive at the finite element heat conduction equations for a kth time increment,

$$N_{Rs} \dot{T}^s_{(k)} + R_{Rs} T^s_{(k)} = P^{(q)}_{(k)} + P^{(q)}_{(k)} + p^{(c)}_{(k)} + p^{(R)}_{(k)} + p^{(T)}_{(k)}$$  \hspace{1cm} (37)

in which

heat capacity matrix

$$N_{Rs} = \int_V c_{\Omega_R \Omega_S} d\nu$$  \hspace{1cm} (38a)

conductivity matrix

$$R_{Rs} = \int_V \kappa^{ij} \Omega_R, i \Omega_S, j d\nu$$  \hspace{1cm} (38b)

volume heat supply vector

$$p^{(c)}_{(k)} = \int_V \rho h c_{\Omega_R} d\nu$$  \hspace{1cm} (38c)

surface heat flux vector

$$p^{(q)}_{(k)} = \int_A q^I n_I \Omega_R dA$$  \hspace{1cm} (38d)

Here $n_I$ is the unit normal to surface. If the surface is uninsulated and convection loss is to take place due to ambient temperature $T'$ then the following boundary condition should be met:

$$q^I n_I + q + \alpha (T - T') = 0$$

or

$$q^I n_I = -q - \alpha (T - T')$$

where $\alpha$ is the film coefficient. This requires the surface heat flux vector (38d) to be replaced by

$$p^{(q)}_{(k)} = -\int_A \Omega_R (q - \alpha T') d\nu - \int_A \Omega_R \alpha \Omega_S dA T^s_{(k)}$$  \hspace{1cm} (38d-1)
The second term of the left-hand side of (38d-1) may be added to the conductivity matrix (38b) so that

\[ R_{RS} = \int_{v} \kappa \Omega_{R}, \Omega_{S}, dV + \int_{A} \alpha \Omega_{R} \Omega_{S} dA \]  
(38b-1)

and

\[ p^{(q)}_{R(k)} = - \int_{A} \Omega_{R} (q - \alpha T') dA \]  
(38d-2)

Pseudo heat capacity vector

\[ p^{(e)}_{R(k)} = \int_{v} \left[ \frac{C}{T_{0}} \Omega_{R} \Omega_{S} T_{(k - 1)} - \Delta v \right] T_{(k - 1)} \]  
(38e)

Pseudo elastoplastic coupling vector

\[ p^{(e) p}_{R(k)} = \int_{v} \left( T_{0} + T_{(k - 1)} \right) \Omega_{R} \left( B^{11} + B^{12} + \delta^{12} \right) (A_{R1}) \]

\[ + 2 C_{M11} u_{(k - 1)}^{p} \Delta v \]  
\[ + \int_{v} \left( T_{0} + T_{(k - 1)} \right) \Omega_{R} \]  
\[ (B + \delta) \Omega_{S} \Delta v \]  
\[ \hat{T}_{S} (k - 1) \]  
(38f)

Pseudo thermoplastic dissipation vector,

\[ p^{(e)p}_{R(k)} = - \int_{v} \left[ \varepsilon_{11}^{m} c_{R}^{m} \left( A_{R} u_{(k - 1)}^{m} \right) + C_{M11} u_{(k - 1)}^{m} (k - 1) \right] \]
\[ (A_{p,m} + 2 C_{p,q} u_{(k - 1)}^{q}) \Delta v \]  
\[ + \int_{v} \varepsilon_{11}^{m} c_{R}^{m} \left( A_{p11} + 2 C_{p,q} u_{(k - 1)}^{q} \right) \Delta v \]  
\[ \gamma_{m} (k - 1) \]  
(38g)
\[
+ \int_V \sum_{i,j} \tilde{E}_{ij} \frac{\partial^2 \mathbf{w}_i}{\partial \mathbf{u}_j^2} \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \, dv \left[ (k - 1) \right] \quad (k - 1)
\]

\[
- \int_V \left[ \sum_{i,j} \mathbf{K}_{ij} \dot{\mathbf{u}}_i \mathbf{u}_j \right] \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \, dv \left[ (k - 1) \right] \quad \gamma^{(e)} \left[ i_j (k - 1) \right]
\]

\[
+ \left[ \int_V \sum_{i,j} \tilde{E}_{ij} \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \right] \left[ (k - 1) \right] + 2 \sum_{i,j} \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \left[ (k - 1) (k - 1) \right] \left[ (k - 1) \right] (k - 1)
\]

\[
+ 2 \sum_{i,j} \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \left[ (k - 1) (k - 1) \right] + \int_V \sum_{i,j} \sigma^{ij} \mathbf{C}_{ij} \mathbf{u}_i \mathbf{u}_j \left[ i_j (k - 1) \right] \left[ (k - 1) \right]
\]

Here \((k - 1)\) refers to the previous time step; and if the pseudo elastoplastic coupling vector and pseudo thermoplastic vector are removed the expression (37) is identical to the uncoupled transient heat conduction equations.

5. INCREMENTAL FINITE ELEMENT EQUATIONS OF EQUILIBRIUM

The standard finite element equations of equilibrium is given by

\[
\int_V \sigma^{ij} \frac{\partial \mathbf{w}_i}{\partial \mathbf{u}_j} \, dv = F^{(e)}
\]

in which \( F^{(e)} \) is the nodal applied load vector. The incremental form of (39) is obtained by taking a variation of (39) in the form

\[
\int_V d \sigma^{ij} \frac{\partial \mathbf{w}_i}{\partial \mathbf{u}_j} \, dv + \int_V \sigma^{ij} d \left[ \frac{\partial \mathbf{w}_i}{\partial \mathbf{u}_j} \right] \, dv = dF^{(e)}
\]
The first and second terms in the left-hand side of (40) correspond, respectively, to incremental changes of stresses and geometries. In view of (20), (21), (22), and (26) and performing appropriate differentiations, it is now possible to write (40) in incremental quantities,

\[
\int_V \left[ (E_1^{ij} + E_2^{ij}) \left( A_{NKJ} d\mathbf{u} + 2C_{NMKJ} \mathbf{u} d\mathbf{u} \right) + (B_1^{ij} + B_2^{ij}) \Omega \right] \left( A_{N1J} + 2C_{NM1J} \mathbf{u} \right) d\mathbf{v} + \int_V 2G^{ij} C_{NM1J} d\mathbf{u} d\mathbf{v} = dF^{(t)}_{N(j)}
\]

(41)

It should be noted that \( G^{ij} \) in the last term of (41) implies the initial stresses or the residual stresses in the structure just prior to a new change in geometry.

After some algebra the final form of incremental thermoelastoplastic equation of equilibrium for the \( j \)th incremental step

\[
(K^{(e)}_{NM} + K^{(G)}_{NM} + K^{(p)}_{NM}) \mathbf{d}\mathbf{u} = dF^{(e)}(j) + dF^{(t)}_{N(j)} + dF^{(N)}_{N(j)}
\]

(42)

in which \( K^{(e)}_{NM} \), \( K^{(G)}_{NM} \), and \( K^{(p)}_{NM} \) are the standard stiffness matrices representing linear elastic, geometrically nonlinear, and plastic behavior, respectively,

\[
K^{(e)}_{NM} = \int_V E_1^{ij} \mathbf{A}_{N1J} \mathbf{A}_{NKJ} d\mathbf{v}
\]

\[
K^{(G)}_{NM} = \int_V 2G^{ij} C_{NM1J} d\mathbf{v}
\]

\[
K^{(p)}_{NM} = \int_V \frac{1}{2} E_2^{ij} \mathbf{A}_{N1J} \mathbf{A}_{NKJ} d\mathbf{v}
\]

and the incremental thermoelastoplastic load vector \( dF^{(t)}_{N(j)} \) is

\[
dF^{(t)}_{N(j)} = \int_V (B_1^{ij} + B_2^{ij}) \Omega \left( A_{N1J} + 2C_{NM1J} \mathbf{u} \right) d\mathbf{v} d\mathbf{r} + \int_V (B_1^{ij} \Omega C_{NM1J} d\mathbf{v} \mathbf{u} + B_2^{ij} \Omega C_{NM1J} d\mathbf{v} \mathbf{u}) (j - 1) (j - 1)
\]

(43)

All the rest of the terms in (41) other than those mentioned above may be grouped in \( F^{(N)}_{N(j)} \) called the pseudo nonlinear load vector but may be dropped
because of their negligible effects.

6. SOLUTION PROCEDURE

We are now provided with incremental heat conduction equations and equilibrium equations. Either thermal loads or mechanical loads or both may be applied. Depending on loading and boundary conditions, we can either start from equilibrium equations or heat conduction equations, but both equations must be solved iteratively within a time increment. Any existing recursive formula for step-by-step integration or difference operator may be used to solve heat conduction equations. In the present study a difference operator for linear variation of temperature within a time increment is combined with iterative solution of plastic equilibrium equations.

The incremental temperature at any time step $k$ is given by \[14,15\],

$$
\begin{align*}
T_{(k)} &= 2 \left( \frac{\Delta t}{2} N + R \right)^{-1} \left\{ P_{(k)} - \frac{\Delta t}{2} \right\} \\
& + \frac{\Delta t}{2} R T_{(k - 1)} - T_{(k - 1)} \\
\end{align*}
$$

(44)

Here $N$, $R$, $P$, and $T$ are assembled forms of $N_{NM}$, $R_{NM}$, $p_{R}^{(e)} + p_{R}^{(p)} + p_{R}^{(e)} + p_{R}^{(p)} + p_{R}^{(p)}$, and $T^{(e)}$, respectively. The reference temperature and initial thermal input can easily be incorporated in (44) and the temperature change at the end of the first time increment calculated.

The results of this solution are used in the assembled incremental equations of equilibrium to determine the displacements and stresses. These stresses are checked, element by element, for yielding. If any element has yielded the plastic tangent stiffness matrix is constructed and the standard iterative cycles are repeated until convergence is achieved [10,11,12].
With the final values of displacements, it is possible to calculate the displacement rates by

$$\dot{u}_k = (u_k - u_{k-1})/\Delta t$$  \hspace{1cm} (45)

for use in the heat conduction equations. Returning to the heat conduction equations for the second time increment, the process is repeated as before except that the elastoplastic coupling is now to be included. The elastoplastic and thermoplastic model as determined in the converged solution of the plastic equilibrium equations of (42) will be used in the heat conduction equations. The marching with time increments, thus, continues until the desired length of time has been reached. Details of the computer program are given in Appendix A—Capabilities and Limitations of the Program, Appendix C—Flow Chart, Appendix D—Subroutines Organization Chart, Appendix E—Description of Subroutines, Appendix F—Data Input Format, and Appendix G—Program Listing.

In the present study, we use temperature and displacement approximations based on a three-dimensional linear isoparametric function and the integration is performed by an 8 point Gaussian quadrature [16] (see Appendix B).

7. APPLICATIONS

In order to verify, first of all, correctness of the present approach a comparison study was made for an uncoupled heat conduction of a beam reported by Wah [17] who used a classical series solution and substantiated his results with Boley [19]. An excellent agreement was obtained as shown in Figure 1.
Next, a series of example problems were tested to determine effects of various terms in the governing equations; namely, behavior due to coupling and non-coupling, linear elastic and elastoplastic properties. Figure 2 shows the information on geometry, boundary conditions, material constants and temperature input. Only one-quarter of the symmetrical three-dimensional solid is shown. The temperature change of 200°C is applied at nodes of the center-left end element. The material properties given in Figure 2 represent a mild steel. The transient temperature distribution in the direction of x with \( y = z = 0 \) is shown in Figure 3. Effects of elastic and elastoplastic couplings are studied in this case. It is interesting to note that there exists little difference in temperature distribution between elastic of elastoplastic coupling for the time period examined. However, the displacement \( w \) at \( y = 100 \text{ mm} \) and \( z = 300 \text{ mm} \) becomes larger for elastoplastic coupling than that for elastic coupling after approximately one hour as shown in Figure 4. The changes of \( w \) displacements vs. time at point A are plotted in Figure 5. It is seen that uncoupled displacements are larger than the coupled displacements, a trend confirmed by Oden and Poe [18] in their study of thermoelastic one-dimensional problems. It should be noted that the elastoplastic displacements are larger than the elastic displacements in both cases. Figure 6 reveals an interesting fact that temperature is also lower for a coupled case than for an uncoupled case but little difference is noted for either elastic or elastoplastic behavior. Figure 7 shows the stress in \( z \) direction \( \sigma_z \) from that for elastic coupling, a fact well known in mechanics. Finally, plastic regions developing with elapse of time are shown in Figure 8, indicating little effects of coupling.
The effects of convection through uninsulated surfaces with the film coefficient $\varphi = 1.0 \text{ Kg/mm hr } ^\circ \text{C}$, ambient temperature $T' = 1000^\circ \text{C}$ and the heat flux $q = -100 \text{ Kg/mm hr}$ at $z = 300 \text{ mm}$ on the x-y plane are investigated and the results are shown in Figure 9. Large elastoplastic displacements ($w$) result due to surface heating together with ambient conditions. Variations of material properties from element to element are accommodated in the program and an example problem for such case is described in Figure 10. Once again the transient temperature distributions are almost identical for elastic coupling and elastoplastic coupling as shown in Figure 11. Significant deviations exist, however, for displacements ($w$) (Figure 12) between elastic and elastoplastic couplings as temperature rises as noted earlier for the uniform material. Because of ambient temperature and heat flux through uninsulated surfaces and variable material properties throughout the structure, the pattern of development of plastic regions (Figure 13) differs considerably from that of the previous example of uniform material and insulated surface. In the foregoing example problems, geometric nonlinearities are excluded for the interest of computing time.

It should be noted that for the case of an isotropic solid, the tensor of thermoplastic moduli (23) and expressions of (29a), (29b), (29c), and (29e) are zero but must be updated in the case of an anisotropic solid as plastic deformation progresses [9]. Although the temperature dependent material properties, finite strains, and the dynamic-coupled inertia effects can easily be handled, such applications are not included in the present study.
Figure 1  Transient thermal stress distribution in the free-free beam, uncoupled.
DISPLACEMENT BOUNDARY CONDITIONS:

\[ u = v = w = 0 \quad \text{At} \ x = 0 \ \text{and} \ x = 600 \ \text{mm} \]

\[ v = 0 \quad \text{At} \ y = 0 \]

\[ w = 0 \quad \text{At} \ z = 0 \]

TEMPERATURE BOUNDARY CONDITIONS:

Insulated on all the surfaces except at the points \((x, y, z) = (0, 0, 0), (0, 100, 0), (0, 0, 100) \ \text{and} \ (0, 100, 100)\) which are kept at \(200^\circ C\).

CONSTANTS:

\[ E = 2.0 \times 10^4 \ \text{(kg/mm}^2\text{)}, \quad E = 2.0 \times 10^3 \ \text{(kg/mm}^2\text{)}, \quad \sigma_v = 25.0 \ \text{(kg/mm}^2\text{)} \]

\[ \nu = 0.3 \quad , \quad \alpha = 1.3 \times 10^{-6} \ \text{(°C)} \quad , \quad \kappa = 9.0 \times 10^6 \ \text{(kg/hr °C)} \]

\[ c = 0.3 \ \text{(kg/mm}^2\text{ °C)}, \quad T_0 = 27^\circ C \quad , \quad \Delta t = 0.05 \ \text{(hrs.)} \]

Figure 2  Discretized geometry of three dimensional solid and input data.
Figure 3  Temperature distribution of $y = z = 0$ in Fig. 2, coupled.
Figure 4  Displacement ($w$) at $y = 100\text{mm}$, $z = 300\text{mm}$ in the $x$ - direction.
Figure 5  Transient Displacement $w$ at point A of Fig. 2.
Figure 6  Transient temperature change at point A of Fig. 2.
Figure 7  Stress $\sigma_z$ - displacement $w$ for element B of Fig. 2.
Figure 8 Development of plastic regions, coupled and uncoupled.
Figure 9: Comparison of Coupled Elastoplastic Displacements (w) With and Without Surface Insulation at y = 100 mm, z = 300 mm in the x-Direction.
DISPLACEMENT BOUNDARY CONDITIONS:
\[ u = v = w = 0 \] at \( x = 0 \) and \( x = 600 \) mm
\[ v = 0 \] at \( y = 0 \)
\[ w = 0 \] at \( z = 0 \)

TEMPERATURE BOUNDARY CONDITIONS:
\[ T = 200 \,^\circ C \] at the points \((x,y,z) = (0,0,0), (0,100,0), (0,0,100)\) and \((0,100,100)\).
\[ \bar{g} = 1.0 (kg/mm.hr \,^\circ C), T' = 1000.0 \,^\circ C \) and \( q = -100.0 (kg/mm.hr) \) on the surface \( z = 300 \) mm.
Insulated on all other surfaces.

CONSTANTS:
\[ E = 2.0 \times 10^4 \, (kg/mm^2) \] for elements 1 - 6,
\[ E = 1.0 \times 10^4 \, (kg/mm^2) \] for elements 7 - 12,
\[ E = 0.7 \times 10^4 \, (kg/mm^2) \] for elements 13 - 18,
\[ E = 1.0 \times 10^3 \, (kg/mm^2) , \gamma = 9.5 (kg/mm^2) , \nu = 0.3 \]
\[ \alpha = 1.3 \times 10^{-5} \, (/^\circ C) , \chi = 9.0 \times 10^3 \, (kg/hr ^\circ C) \]
\[ c = 0.3 \, (kg/mm^2 ^\circ C) , T_o = 27(^\circ C) , \Delta t = 0.05 \, (hrs.) \]

Figure 10: Discretized Geometry of Three-Dimensional Solid and Input Data.
Figure 11: Temperature Distribution of $y = z = 0$ in Figure 10.
Figure 12: Displacement \( w \) at \( y = 100 \text{ mm} \), \( z = 300 \text{ mm} \) in the \( x \)-Direction in Figure 12.
Figure 13: Development of Plastic Regions in Figure 10.
8. CONCLUDING REMARKS

A three-dimensional thermoelastoplastic analysis has been carried out using the incremental theory consistent with the first and second laws of thermodynamics. Complicated mathematical operations emanating from the functional theory or state variables are replaced by proposing an incremental free energy as a function of total and inelastic strain and temperature unique only within a small time increment. The similar incremental functional dependency is then valid for stresses, entropy, and heat flux. Such treatment lends itself to numerical techniques taking advantage of the finite element method and time integration by difference operators.

For the example problems and material properties considered in this study it appears that elastoplastic coupling is significant for displacements and stresses, but that neither elastic nor elastoplastic coupling has any effect on temperature distribution. It may be argued, however, that for certain types of material and geometry these conclusions would not necessarily be true. Exhaustive study on such effects is beyond the scope of the present report.
REFERENCES


APPENDIX A
CAPABILITIES AND LIMITATION OF THE PROGRAM

General. This program analyzes a three-dimensional solid subjected to both thermal and mechanical loadings. The program takes into account the elastoplastic behavior coupled with transient heat conduction. The formulation is based on the first and second laws of thermodynamics, von Mises yield criteria, associated Prandtl-Reuss flow rule, and the linear Fourier law. The finite element discretization by means of a linear isoparametric interpolation for both temperature and displacement fields is utilized. Integration is performed by Gaussian quadrature. A step-by-step time integration assuming a linear variation of temperature within a time increment is used to solve heat conduction equations. Capabilities and limitations of this program are listed as follows:

(1) Capable of handling up to 60 nodes and 30 elements.
(2) Temperatures may be specified at nodes (50).
(3) Heat flux and heat supply may be specified on the element surface (50) and inside the entire element solid (30), respectively.
(4) Surfaces may be insulated or exposed to ambient temperatures.
(5) Capable of incorporating 100 restrained generalized coordinates.
(6) Geometric nonlinearities are not considered in the program.
(7) Capable of handling laminated structure with varying material properties from element to element.
APPENDIX B
VARIOUS INTEGRALS IN ISOPARAMETRIC ELEMENT

In the present study, linear hexahedral isoparametric elements are used to model the three dimensional solids and consequently constitute the basis for displacement and temperature fields. Although details of isoparametric elements may be found in Zienkiewicz (1972), some of the integrals essential in the program by Gaussian quadrature are shown in explicit form:

\[ \int_V \psi_N \, dV = \frac{1}{8^3} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left(1 + e_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right) |J| \, d\xi d\eta d\zeta \]

\[ \int_V \psi_N \psi_M \, dV = \frac{1}{8^3} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left(1 + e_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right)\left(1 + C_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right) |J| \, d\xi d\eta d\zeta \]

\[ \int_V \psi_N \psi_M \psi_R \, dV = \frac{1}{8^3} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left(1 + e_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right)\left(1 + C_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right)\left(1 + e_{M}\right) |J| \, d\xi d\eta d\zeta \]

\[ \times \left(1 + C_{R}\right) \left(1 + e_{M}\right) \left(1 + e_{R}\right) \left(1 + e_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right) |J| \, d\xi d\eta d\zeta \]

\[ \int_V \psi_N A_{M1} \, dV = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \left(1 + e_{S_1}\right)\left(1 + e_{S_2}\right)\left(1 + e_{S_3}\right)A_{M1} |J| \, d\xi d\eta d\zeta \]
\[
\int_{V} \psi_{N} \psi_{M} dV = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} (1 + \xi \xi_{N})(1 + \eta \eta_{N})(1 + \zeta \zeta_{N}) \\
(1 + \xi \xi_{Q})(1 + \eta \eta_{Q})(1 + \zeta \zeta_{Q}) \times A_{M_{1}N_{1}} |J| \, \delta \eta \delta \zeta
\]

\[
\int_{V} \psi_{N_{1}} \psi_{M_{1}} dV = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \psi_{N_{1}} \psi_{M_{1}} |J| \, \delta \eta \delta \zeta
\]

\[
\int_{V} A_{M_{2}N_{2}} A_{N_{1}M_{1}} dV = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} A_{M_{2}N_{2}} A_{N_{1}M_{1}} |J| \, \delta \eta \delta \zeta
\]

\[
\int_{A} \psi_{N} dA = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} (1 + \xi \xi_{N})(1 + \eta \eta_{N})(1 + \zeta \zeta_{N}) \, dA
\]

and

\[
\int_{A} \psi_{N} \psi_{M} dA = \frac{1}{8} \int_{-1}^{1} \int_{-1}^{1} (1 + \xi \xi_{N})(1 + \eta \eta_{N})(1 + \zeta \zeta_{N}) \, dA
\]

(1 + \xi \xi_{N})(1 + \eta \eta_{M})(1 + \zeta \zeta_{N})

where the differential area \( dA \) is expressed as

\[
dA = \left[ \left( \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \eta} \right)^{2} + \left( \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \eta} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} \right)^{2} + \left( \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta} \right)^{2} \right]^{\frac{1}{2}} \, d\xi d\eta \delta \zeta
\]

for \( \zeta = \pm 1 \) plane

\[
= \left[ \left( \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} \frac{\partial y}{\partial \xi} \right)^{2} + \left( \frac{\partial z}{\partial \eta} \frac{\partial x}{\partial \xi} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \xi} \right)^{2} + \left( \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \eta} \frac{\partial x}{\partial \xi} \right)^{2} \right]^{\frac{1}{2}} \, d\eta d\zeta
\]

for \( \xi = \pm 1 \) plane
Here, displacement and temperature fields are related by

\[ u = \sum_{i=1}^{8} \psi_i u_i, \text{ etc., and } T = \sum_{i=1}^{8} \psi_i T_i \]

where

\[ \psi_i = \frac{1}{8} (1 + \varepsilon \varepsilon_i) (1 + \eta_1) (1 + \zeta_1) \]

The determinant of Jacobian is given by

\[
|J| = \frac{1}{8^2} \sum_{i=1}^{8} \sum_{j=1}^{8} \sum_{k=1}^{8} \left[ \xi_i (1 + \eta_1)(1 + \zeta_1) x_1 \{ \eta_j (1 + \varepsilon \varepsilon_j)(1 + \zeta_1) \right.
\]

\[ y_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) z_k 
- \xi_i (1 + \varepsilon \varepsilon_i)(1 + \zeta_1) z_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) y_k 
\]

\[ - \xi_i (1 + \varepsilon \varepsilon_i)(1 + \zeta_1) y_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) z_k 
- \xi_i (1 + \varepsilon \varepsilon_i)(1 + \zeta_1) z_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) y_k 
\]

\[ \left. + \xi_i (1 + \varepsilon \varepsilon_i)(1 + \zeta_1) z_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) y_k 
- \xi_i (1 + \varepsilon \varepsilon_i)(1 + \zeta_1) x_j \zeta_k (1 + \varepsilon \varepsilon_k)(1 + \eta_1) x_k \right] \]

\[
\right] \right] \]
APPENDIX C: FLOW CHART

INITIALIZATION

Time step \(k = 0\), Temp \((k) = 0\), \(\sigma_{\text{max}}\) = Yield Stress \(\sigma_y\), \(u_{(k=0)} = 0\)

\[ k + 1 \]

Calculate \(T_{(k)}\) from Eq. (35)

Calculate \(u_{(k)}\) from Eq. (32)
Calculate strains and stresses
Calculate equivalent stresses, \(\bar{\sigma}_{(k)}\)

\[ \sigma_{(k)} > \sigma_{\text{max}} \]

YES

\[ d\bar{\sigma}_{(k)} = \sigma_{(k)} - \bar{\sigma}_{(k-1)} \]

Calculate \(\dot{\delta}_{(k)}\) from Eq. (27)
and pseudo plastic load \(dF_{(p)}^{(p)}\) from Eq. (33c)

Calculate elastoplastic coupling vector @ \(k - \frac{1}{2}\)

Calculate \(T_{(k)}\) from Eq. (35)

\[ dT_{(k)} = T_{(k)} - T_{(k-1)} \]

Calculate thermal load vector \(dF_{(T)}^{(T)}\) (Eq. (34))

\[ du_{(k)} = k^{-1} [dF_{(p)}^{(p)} + dF_{(T)}^{(T)}] \]
Calculate strains, and stresses $d\delta_k$.

Calculate $\sigma_{(k)} = \sigma_{(k-1)} + d\sigma_{(k)}$.

$$d\delta^{(2)}_{(k)} = z_{(k)}^{r} d\sigma_{(k)}$$

$$\tilde{\sigma}_{(k)} = \tilde{\sigma}_{(k-1)} + d\tilde{\sigma}^{(2)}_{(k)}$$

$$\text{ERR} = \left| \sqrt{\sum (d\tilde{\sigma}^{(2)}_{(k)})^2} - \sqrt{\sum (d\tilde{\sigma}^{(1)}_{(k)})^2} \right|$$

If $d\tilde{\sigma}^{(1)}_{(k)} = d\tilde{\sigma}^{(2)}_{(k)}$ NO

If $\text{ERR} < \varepsilon$ YES

$$\tilde{\sigma}_{\text{max}} = \text{AMAX}(\tilde{\sigma}_{\text{max}}, \tilde{\sigma}_{(k-1)} + d\tilde{\sigma}^{(2)}_{(k)})$$

$$u^{(k)} = u^{(k-1)} + du^{(k)}$$

$k = k + 1$
## APPENDIX E

### DESCRIPTIONS OF SUBROUTINES

<table>
<thead>
<tr>
<th>Subroutine Name</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASSIGN</td>
<td>Rearranges the nodal displacement vector</td>
</tr>
<tr>
<td>BOUND</td>
<td>Applies the boundary conditions and reorders the matrices accordingly</td>
</tr>
<tr>
<td>CONTINH</td>
<td>Defines various quantities for Gaussian Quadrature integrations</td>
</tr>
<tr>
<td>DATA</td>
<td>Reads all input data</td>
</tr>
<tr>
<td>DFVCTR</td>
<td>Calculates all pseudo coupling vectors and heat input vectors</td>
</tr>
<tr>
<td>DTHETA</td>
<td>Solves heat conduction equations</td>
</tr>
<tr>
<td>ELASTC</td>
<td>Calculates the elasticity matrix</td>
</tr>
<tr>
<td>F</td>
<td>Function subroutine for Gaussian integration</td>
</tr>
<tr>
<td>GAUSS</td>
<td>Integration by Gaussian quadrature</td>
</tr>
<tr>
<td>HSTRESS</td>
<td>Calculates the thermal load vector in equilibrium equations</td>
</tr>
<tr>
<td>MATINV</td>
<td>Matrix inversion</td>
</tr>
<tr>
<td>MATMPY</td>
<td>Matrix multiplications</td>
</tr>
<tr>
<td>Subroutine Name</td>
<td>Descriptions</td>
</tr>
<tr>
<td>----------------</td>
<td>--------------</td>
</tr>
<tr>
<td>MCODE</td>
<td>Reassigns the global node number to the local element node number</td>
</tr>
<tr>
<td>KSTACK</td>
<td>Assembles the local element matrices into a global form</td>
</tr>
<tr>
<td>PHI</td>
<td>Function subroutine involved in Gaussian integrations</td>
</tr>
<tr>
<td>PJCOB</td>
<td>Calculates Jacobian in Gaussian integration</td>
</tr>
<tr>
<td>PLAMDA</td>
<td>Function subroutine involved in Gaussian integration</td>
</tr>
<tr>
<td>PMU</td>
<td>Function subroutine involved in Gaussian integration</td>
</tr>
<tr>
<td>PQR</td>
<td>Defines constants necessary for Gaussian integration</td>
</tr>
<tr>
<td>PRINTK</td>
<td>Print stiffness matrix, heat capacity matrix, and conductivity matrix</td>
</tr>
<tr>
<td>PRINTT</td>
<td>Print nodal temperatures</td>
</tr>
<tr>
<td>PRNTDT</td>
<td>Print nodal displacement and equivalent nodal forces</td>
</tr>
<tr>
<td>PSTIFF</td>
<td>Calculates the plastic tangent stiffness matrix</td>
</tr>
<tr>
<td>REDUCE</td>
<td>Modifies the equivalent nodal vectors in correspondence to BOUND</td>
</tr>
<tr>
<td>RESTOR 1</td>
<td>Restores the equivalent nodal vectors to the form prior to REDUCE</td>
</tr>
<tr>
<td>Subroutine Name</td>
<td>Descriptions</td>
</tr>
<tr>
<td>-----------------</td>
<td>---------------------------------------------------</td>
</tr>
<tr>
<td>SKAPPA</td>
<td>Calculates the plasticity matrix</td>
</tr>
<tr>
<td>STIFF</td>
<td>Calculates the elastic stiffness matrix</td>
</tr>
<tr>
<td>STRSEL</td>
<td>Calculates stresses prior to yielding</td>
</tr>
<tr>
<td>STRSPL</td>
<td>Calculates incremental stresses after yielding</td>
</tr>
<tr>
<td>ZERO</td>
<td>Zeroes out all matrices</td>
</tr>
</tbody>
</table>
APPENDIX F

DATA INPUT FORMAT

Card 1: FORMAT (7 I5)
(1) NELEMT - Number of elements
(2) INODE - Number of nodes
(3) NB - Number of constrained displacements
(4) IPT - Number of integration points in the Gaussian quadrature
(5) NBHC - Number of prescribed nodal temperatures
(6) ITER - Number of time increments
(7) NKE - Number of type of elements

Card 2: FORMAT (3 F 10.0)
(1) RT - Reference temperature
(2) EPSS - Percent error limit, e in the elastoplastic analysis
(3) DELT - Incremental time interval, Δt

Card 3: FORMAT (8 F 10.0)
(1) E(I) - Young's modulus
(2) EP(I) - Plastic modulus
(3) SYIELD(I) - Yield stress
(4) XNU(I) - Poisson's ratio
(5) TKX(I) - Thermal conductivity
(6) SH(I) - Specific heat
(7) ALPHA(I) - Coefficient of thermal expansion
(8) DENSTY(I) - Density
Cards 4: FORMAT (16I5)

(1) NY(I) - Type of elements I

Repeat cards 4 as required to complete all elements.

Cards 5: FORMAT (8I5)

(1) to (8)

MA(I), MB(I), MS(I) - Node numbers of element I

Repeat cards 5 NELEMT times.

Cards 6: FORMAT (6F10.0)

(1) to (3)

X(I), Y(I), Z(I) - x, y and z coordinates of node number I.

Repeat cards 6 as required to complete all nodes.

Card 7: FORMAT (2I5)

(1) LSTRES = 0 if stress analysis is not desired (no elastic or elastoplastic coupling)

= 1 if stress analysis is to be included (elastic or elastoplastic coupling present)

(2) LHEAT = 0 if heat conduction analysis is not desired

= 1 otherwise

Card 8: FORMAT (2I5)

(1) KLOAD = 0 if mechanical loads are not applied

= 1 if mechanical loads are applied

(2) MLOAD = 0 if temperatures are not prescribed

= 1 if temperatures are prescribed
(An assumption is made \( LSTRES = LHEAT = MLOAD = 1 \) and \( KLOAD = 0 \) after card 9)

Cards 9: FORMAT (I5, F10.0)

(1) IIIBND(I) - Node number at which temperature is prescribed
(2) TFR(I) - Prescribed temperature

Repeat cards 9 \( NBHC \) times.

Cards 10: FORMAT (4I5, 3F10.0)

(1)* NS(L,I) - Node numbers of the uninsulated surface (I changes from 1 to 4)
(2)* SC(L,I) SC(L,1) = \( \alpha \) (film coefficient)
    SC(L,2) = \( q \) (heat flux)
    SC(L,3) = \( T' \) (ambient temperature)

*L indicates the number of uninsulated surfaces subjected to convection.

Provide a blank card to signify end of data.

Cards 11: FORMAT (4I5)

(1) NODE - Node number at which displacements are constrained.
(2) IU = 0 if U is not constrained
    = 1 if U is constrained
(3) IV = 0 if V is not constrained
    = 1 if V is constrained
(4) IW = 0 if W is not constrained
    = 1 if W is constrained

Provide a blank card to signify end of data.
Cards 12: FORMAT (8F10.0)

(1) DMV(I) - Heat supply of element I.

Repeat cards 12 as required to complete all elements.
APPENDIX C

COMPUTER PROGRAM LISTING

```
7* COMMON /BLK/MA(30), MB(30), MC(30), MD(30), MP(30), MQ(30), MR(30), HS(30),
8* COMMON /BLK/A(10), B(10), C(10), P(33), Q(33), R(33), W(10), H(10), I(10),
9* COMMON /BLK/Z(10), H(10), M(10), H(10), HS(10), MB(10), MR(10),
10* *H(10), M(10), G(10), G(10), D2(10), G2(10), G3(10), H2(10), H3(10), G1(10),
11* COMMON /BLK/TO(60), TD(60), T2(60), DT(60), DT2(60), DT3(60),
12* COMMON /BLK/NS(50, 9), SC(50, 3), AREA(90),
13* COMMON /BLK/IGAUS, ITER, EPS, TEE,
14* COMMON /BLK/DEL(10), E(10), P(10), YIELD(10), XNY(10), TX(10),
15* SH(10), ALPH(10), DEMSTY(10), NY(30), DELT, RT,
16* COMMON /BLK/S1(30), Y(30), Z(30), X(30), NHODE(8), NODE(8),
17* COMMON /BLK/UX(180), UX1(180), UX2(180), UX3(180), UX4(180), UX5(180),
18* COMMON /BLK/UX6(180), UX7(180), UX8(180), UX9(180), UX10(180), UX11(180), UX12(180),
19* COMMON /BLK/S1(30), S1(30), S1(30), S1(30), S1(30), S1(30),
20* COMMON /BLK/STRN(4, 30), STRN(4, 30), STRN(4, 30), ABC,
21* DIMENSION CI(180), BSK(180), DF(10), DBF(10), DNP(10), EDF(10),
22* DIMENSION DP(24), PR(180), PK(24, 24), AJ(4, 6), FP(180), TP(180),
23* IV(10), IV(10), IV(10), IV(10),
24* COMMON /BLK/S1(30), S1(30), S1(30), S1(30),
25* \* COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
26* \* COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
27* \* COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
28* CALL DATA(FP, BSK, CI),
29* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
30* NTMNY(10),
31* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
32* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
33* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
34* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
35* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
36* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
37* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
38* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
39* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
40* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
41* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
42* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
43* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
44* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
45* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
46* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
47* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
48* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
49* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
50* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
51* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
52* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
53* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
54* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
55* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
56* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
57* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
58* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
59* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
60* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
61* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
62* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
63* \*COMMON /BLK/DEL(10), E(10), P(10), YIELD(10),
```

```
40 CONTINUE

C**SURFACE NEAT BOUNDARY

64 CINLINE(L),E.Q.D) GO TO 92
65 A1=A1AR(A1+5C(L),2)-5C(L)+3)
66 DO 43 J=1,4
67 NINE-L
68 43 DFIN=+FIN-A1
69 L=0
70 GO TO 41
71 42 CONTINUE
73 WRITE(L) (.L1.DF1),L=1,INODE)
74 C**DTHETA VECTOR
75 CALL DTHETA(IF,DF,DFMLOAD,TO,T1,TP)
76 CALL PRINTDF(DF,DFM,T1,INODE,INPAGE,AD)
77 C**DTHETA VECTOR
78 CALL ZEO(IF,180,1)
79 IF(180,N,LT,L) NX0
80 NY0
81 ALPHAF=ALPHAIN+IF(180,L,2)*NX0(1)
82 CALL MODE(MKODE,MNODE,1,3)
83 CALL STRES(STR,FR,MMODE,MODE,NK,ALPHAF,180,1)
84 NEW
85 240 CONTINUE
86 DO 245 =1,NN
87 265 FR(IF,FUP,FUP,FUP,FUP,FUP)*FLOAT(111)
88 C**GEOGRMETRY CALLING HOMLINEAR VECTOR
89 CALL ZERO(AJ,6,6)
90 DO 264 J=1,1,NELEM
91 CALL MODE(MKODE,MNODE,1,3)
92 CALL ZEGDOP(ZK,1)
93 CALL PALIGN(PK,1,1,1)
94 CALL STRES(STR,FR,MMODE,MODE,NK,ALPHAF,180,1)
95 DO 267 J=1,29
96 NEW
97 267 DU=U(JK)
98 268 DO(IF,DFM,DFM,DFM,DFM,DFM)*DU
99 269 266 CONTINUE
100 269 CONTINUE
101 C**DTHETA VECTOR
102 CALL REQUCE(FR,180,NN,NK,180)
103 CALL MAPTF(5K,NN,FR,180)
104 CALL RSORZ(IF,FR,180,NN,180)
105 CALL RSORZ(IF,180,NN,180)
106 CALL PRDST(IF,180,180,INODE,INPAGE,180)
107 C**STRAIN & STRESS
108 WRITE(.L1,88)
109 DO 50 =1,102
100 50 L=1,NELEM
111 CALL MODE(MKODE,MNODE,1,3)
112 CALL STRES(UO,TO,TITEL,1)
113 WRITE(.L1,90) L,TIT,E(S,M,S,M,S,M),(S,M,S,M),(S,M,S,M)
114 120 CONTINUE
115 50 CONTINUE
116 IF(YIELD.EQ.1) GO TO 3000
117 99 CONTINUE
118 C**PRINT OUT
119 WRITE(6) J1,INODE
120 DO 125 J=1,3
121 IJ=1,J3
122 U1(UJ1,J1)
123 125 CONTINUE
124 DO 126 J=1,NELEM
125 SIGMA(J,1)=SIGMA(J,1)
126 DO 126 J=1,6
127 SIGMA(J,1)=SIGMA(J,1)
128 126 CONTINUE
129 GO TO 1600
130 1600 CONTINUE
131 C**FOR YIELD
132 3000 CONTINUE
133 IF(NN=6.10) STOP
134 WRITE(6) J1,1,NN
135 CALL SKAP'l
136 4000 CONTINUE
137 CALL ZER0(IF,6,1,1)
138 455 RENIND 2
139 460 CONTINUE
140 450 CONTINUE
141 465 CONTINUE
142 450 CONTINUE
143 465 CONTINUE
144 450 CONTINUE
145 465 CONTINUE
146 450 CONTINUE
147 L=1
148 NC=0
149 25 CONTINUE
150 CALL MODE(MKODE,MNODE,L,3)
151 D=DSHV(L)
152 CALL DVCSTIED(KAP,DMX,NN,NN,0)
153 MM=M(NODE)
154 35 D=DSH(V,1)(1+)OF(MNDO)
155 35 IF(1,NN,MY(L-1)) NX0
156 177
SUBROUTINE DATAFF,MSK,C1
COMMON /ALKE/NELEMT,INODE,MB,MBHC,LHEAT,LSTRES,NN,NNH,NNNH,NNNH
COMMON /ALKB/NOPT,NT,NU,NPX,NODE,KTOTAL,HLOAD,LOAD
COMMON /ALKC/SSK(2K),IK(1),LSK(8),MK(8),PT(8),PE(18),P(8)
COMMON /ALKD/SK(2Q),IK(1),LKV(2),MK(8),HP(8),HT(1),HR(8)
COMMON /ALKE/LE(3Q),ALHE(3Q),MC(3Q),M(3Q),MP(3Q),L(3Q),MS(3Q)
COMMON /ALK2/(1Q),B(1),C(1),P(33),Q(33),R(33),S(1),T(1),U(1),V(1)
COMMON /AE2/(1Q),H(1),N(1),S(1),A(1),B(1),G(1),G(8),G(8),G(8),G(8)
COMMON /AEK/(1Q),T(1),U(1),V(1),W(1),X(1),Y(1),Z(1),A(1),B(1)
COMMON /ALEK/(1Q),B(1),C(1),D(1),E(1),F(1),G(1),H(1),I(1),J(1)
COMMON /ALEN/(1Q),B(1),C(1),D(1),E(1),F(1),G(1),H(1),I(1),J(1)
DIMENSION FR(1),C(1),B(1),A(1),TR(1),J(1),I(1)
READ(1),1 1000
1 IF(IGV,EQ,1) GO TO 1000
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
READS(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
WRITE(1),A(1),B(1),C(1),D(1),E(1),F(1),G(1),H(1)
I N Y IQ4Q o9449 *o44rroroz-lh0 e*ZIrrr~rDre 0000oo oo-- _
END OF COMPILATION: NO DIAGNOSTICS.
50 SIGMA(IN) = DSIGMA(IN)
54
55
SS(1,1,1) = SIGMA(1,1,1) + SIGMA(1,2,1) + SIGMA(1,3,1) / 3.
56
57
SS(1,2,1) = SIGMA(2,1,1) + SIGMA(2,2,1) + SIGMA(2,3,1) / 3.
58
59
SS(1,3,1) = SIGMA(3,1,1) + SIGMA(3,2,1) + SIGMA(3,3,1) / 3.
60
61
\[ \text{SIGMA}(IN) = \text{DSIGMA}(IN) \]
62
63
IF(SIGMA(IN) > SIGMAX(IN)) IYIELD = 1
64
DO 70 I = 1, 6
65
70 \text{STRAIN(IN) = DEPS(IN)}
66
67
NPRINT = 0
68
NPRINT = 0
69
IF (NPRINT = EQ 0) GO TO 51
70
WRITE (6, 81)
71
WRITE (6, 80) (EDR(I,1), I = 1, 24)
72
WRITE (6, 82) (LAMDA(I), I = 1, 8)
73
WRITE (6, 83) (MU(I), I = 1, 8)
74
WRITE (6, 84) (PHI(I), I = 1, 8)
75
WRITE (6, 85) (DEPS(I), I = 1, 6)
76
WRITE (6, 86) DETJ
77
78
51 CONTINUE
79
80 FORMAT (1PE13.5)
81
82 FORMAT (1PE13.5)
83
84 FORMAT (1PE13.5)
85
86 FORMAT (1PE13.5)
87
88 FORMAT (/)
89
85 RETURN
90 END
91
END OF COMPIlATION! NO DIAGNOSTICS.
SUBROUTINE PRINT0(4, &B, &A, &AB, &ABT, &N)

C PRINTS OUT JOINT FORCES AND DISPLACEMENTS FOR 3-D STRUCTURES

DIMENSION NANN, BANN, A(11, 11), T(11, 11)

NANN = 4

LINES1 = 3

WRITE(6, 2) N

DO 5 I = 1, N

WRITE(6, 3) I

N = N + 1

WRITE(6, 4) I

5 CONTINUE

4 LINES1 = LINES1 + 1

END

END OF COMPILATION: NO DIAGNOSTICS.

SUBROUTINE PRINT0(4, &B, &A, &AB, &ABT, &N)

C PRINTS OUT NORMAL TEMPERATURES FOR 3-D STRUCTURES

DIMENSION NANN, BANN, A(11, 11), T(11, 11)

NANN = 4

LINES1 = 3

WRITE(6, 2) N

DO 5 I = 1, N

WRITE(6, 3) I

N = N + 1

WRITE(6, 4) I

5 CONTINUE

4 LINES1 = LINES1 + 1

END

END OF COMPILATION: NO DIAGNOSTICS.
END OF COMPILATION: NO DIAGNOSTICS.

14 SUBROUTINE ZEROGA(I,R,N)
15 DIMENSION A(I)
16 C-RAN
17 DO 10 I=1,K
18 RETURN
19 END

END OF COMPILATION: NO DIAGNOSTICS.

14 SUBROUTINE REDUCEF(I,R,N,NR,NM)
15 DIMENSION FNI(I),BNI(I)
16 NBN=BN
17 I=1(NBN)
18 IF(I>BN) 2,4,9
19 RETURN
20 9 CONTINUE
21 RETURN
22 END

END OF COMPILATION: NO DIAGNOSTICS.

14 SUBROUTINE RESTORIO(I,R,N,NR,NM)
15 NWR=NWR
16 I=1(NWR)
17 IF(I>NR) 1,4,6
18 RETURN
19 6 CONTINUE
20 RETURN
21 END

SUBROUTINE CONT
14 COMMON A(R),HN1(N),HN2(NR),HN3(NM),HN4(NR),HN5(NM),HN6
15 A=0.0
16 RETURN
17 END
SUBROUTINE SMATRX(A,NX,NY,NI)
DIMENSION A(1:NX,1:NY,1:NI)
IIX = 1
IY = 1
II = 1
DO 10 J = 1,NI
   IF (II.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   IF (IIX.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   CONTINUE
10   RETURN
END

SUBROUTINE BOUNDARYR(NX,NY,NI)
DIMENSION A(1:NX,1:NY,1:NI)
IIX = 1
IY = 1
II = 1
DO 10 J = 1,NI
   IF (II.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   IF (IIX.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   CONTINUE
10   RETURN
END

SUBROUTINE MINVX(A,NX,NY,NI)
DIMENSION A(1:NX,1:NY,1:NI)
IIX = 1
IY = 1
II = 1
DO 10 J = 1,NI
   IF (II.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   IF (IIX.LT.1) GO TO 4
   IF (IY.LT.1) GO TO 4
   CONTINUE
10   RETURN
END
END OF Compilation: NO DIAGNOSTICS.
SUBROUTINE MCODE(HKODE,HX,L,MH)

COMMON/KL2,HL30,H2,HL30,H30,H130,HL30,H130,H30

END
SUBROUTINE GAUSS(T,AA,M,L)

COMMON /ALK1/AA(10),R(10),C(10),A(33),R3(33),R(6),M(6),1PT
EXTERNAL F

DO 100 J=1,1PT
M(1)=C(J)
100 CONTINUE

RETURN

END

FUNCTION F(I,J,T,M,N,L,TT)

COMMON /ALK2/AA(10),B(10),C(10),P33,R33,R(6),M(6),1PT

EXTERNAL PH,PHI2,PHI3,PHI4

G(4)=G(4)+G(1)*G(2)*G(3)*G(4)+G(5)*G(6)*G(7)

RETURN

END

END OF COMPILATION: NO DIAGNOSTICS.