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DOPPLER FREQUENCY IN INTERPLANETARY RADAR AND GENERAL RELATIVITY

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16. Abstract

The change of frequency of an interplanetary radar signal sent from the Earth to another planet or to a space probe is worked out according to general relativity. The Schwarzschild space-time is employed and its null geodesics control the motion of the signals. Exact Doppler frequency formulas are derived for one-way and two-way radar in terms of an arbitrary Schwarzschild radial coordinate. A reduction to the special relativity case is used to interpret the formulas in terms of the relative radial velocity of emitter and target. The general relativity corrections are worked out approximately for each of three possible Schwarzschild radial coordinates, and a numerical example is given. The amount of the correction is different according as one or other of the Schwarzschild coordinates is identified with the radius vector deduced from classical celestial mechanics. The identification problem is discussed in the final section of the paper.

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DOPPLER FREQUENCY IN INTERPLANETARY RADAR AND GENERAL RELATIVITY

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INTRODUCTION

The object of this investigation is to present the theory of the change of frequency of signals in interplanetary radar experiments when general relativity is employed as the theory of gravitation. The basic ideas will be found in the author's *General Relativity and Cosmology* (Reference 1). No claim is made that the contents of the first three sections of this paper are entirely new and that similar results have not been obtained by others by alternative methods. The last two sections contain the author's ideas on the interpretation of the theoretical formulas. The associated problem of radar time delays is worked out in Reference 2.

It is well known that in Einstein's theory of gravitation an exact solution of the gravitational field of a set of discrete bodies is possible only when one of the bodies is of finite mass whereas the rest are of infinitesimally small mass. This is in contrast to Newton's theory of gravitation in which an exact solution for the problem of two massive bodies in motion is possible, complications arising only when three or more bodies are in question. In Einstein's theory the field of the single massive body is given by the Schwarzschild space-time. The massive body may be taken to be the Sun and the planets, their satellites, space probes, etc., must then be regarded as test particles whose own gravitational fields are to be regarded as negligibly small.

The simplest form of the metric of the Schwarzschild space-time is

$$ds^2 = \left(1 - \frac{2m}{r}\right)dt^2 - \frac{dr^2/(1 - 2m/r) + r^2\, d\Omega^2}{c^2} \tag{1}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2 ,$$

in which $r = 0$ is the center of the (spherically symmetric) massive body. The interval s and the time t are measured in time units, r is in length units, c is the local velocity of light, and the constant m is defined by

$$m = \frac{GM}{c^2} , \tag{2}$$

where G is the constant of gravitation and M is the mass of the central body. When M is the mass of the Sun,

$$m = 1.477 \times 10^5 \quad \text{cm.} \tag{3}$$

If the central body is the Earth, then

$$m = 0.443 \quad \text{cm.} \tag{4}$$

It is well known that in relativity there is no unique system of coordinates in a given space-time. The Schwarzschild space-time for a given central body is characterized by one and the same value of m, whatever coordinate system may be employed. Two of the simplest alternative systems to the one used in Equation 1 are the isotropic and the harmonic systems (Reference 3). In the isotropic system the coordinate r is replaced by $\bar{r}$ where

$$r = \bar{r} \left(1 + \frac{m}{2\bar{r}}\right)^2 , \tag{5}$$

in terms of which Equation 1 becomes

$$ds^2 = \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 dt^2 - \frac{(1 + m/2\bar{r})^4 \, (d\bar{r}^2 + \bar{r}^2 \, d\Omega^2)}{c^2} . \tag{6}$$

The analogues of rectangular coordinates may be introduced by

$$\begin{aligned} \xi &= \bar{r} \sin\theta \cos\phi , \\ \eta &= \bar{r} \sin\theta \sin\phi , \\ \zeta &= \bar{r} \cos\theta , \end{aligned} \tag{7}$$

and then the metric becomes

$$ds^2 = \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2 dt^2 - \frac{(1 + m/2\bar{r})^4 \, (d\xi^2 + d\eta^2 + d\zeta^2)}{c^2} . \tag{8}$$

The harmonic system has a radial coordinate ρ where

$$r = \rho + m \tag{9}$$

in terms of which the metric has the form

$$ds^2 = \frac{1 - m/\rho}{1 + m/\rho} \, dt^2 - \frac{[(1 + m/\rho)/(1 - m/\rho)] \, d\rho^2 + \rho^2 (1 + m/\rho)^2 \, d\Omega^2}{c^2} . \tag{10}$$

Rectangular coordinates somewhat similar to those in Equations 7 are possible with reference to this harmonic system (Reference 4).

To deal with these three radial coordinates at once, it is advantageous to write the metric of the Schwarzschild space-time as

$$ds^2 = e^{2\lambda} \, dt^2 - \frac{e^{2\mu} \, dx^2 + e^{2\nu} x^2 \, d\Omega^2}{c^2} , \tag{11}$$

where λ, μ, and ν are functions of the radial coordinate x involving the constant m. The forms of these functions are identified by comparing Equation 11 with Equations 1, 6, and 10, respectively, as x is identified with r, $\bar{r}$, or ρ. It is clear that, for any one of these identifications, the condition $m = 0$ makes λ, μ, and ν zero. Gravitational effects are then absent and the three forms of the metric reduce to

$$ds^2 = dt^2 - \frac{dx^2 + x^2\, d\Omega^2}{c^2}$$

$$= dt^2 - \frac{d\xi^2 + d\eta^2 + d\zeta^2}{c^2}\ , \tag{12}$$

with $x = r = \bar{r} = \rho$. The metric is now that of special relativity and the coordinate system (t, x, θ, ϕ) has become the inertial system in which $x = 0$ is the center of the now massless Sun. This is also an approximate description of the coordinate system (t, x, θ, ϕ) employed in Equation 11 in those regions of the Schwarzschild space-time in which x is very large compared with m.

Figure 1 shows portions ABE and FC of the orbits of E and F in space. The Earth is E and the target F is another planet or a space probe. It is assumed that the paths are coplanar and lie in the plane $\theta = \pi/2$. Figure 1 also shows the paths AMC and $CM'B$ of the outgoing and returning radar signals. They are null geodesics of the space-time of Equation 11, and it is assumed that the signal is immediately reflected at the target on its arrival there. Figure 1 refers to a configuration in which E and F are on opposite sides of the Sun. Figure 2 illustrates the configuration when E and F lie on the same side of the Sun.

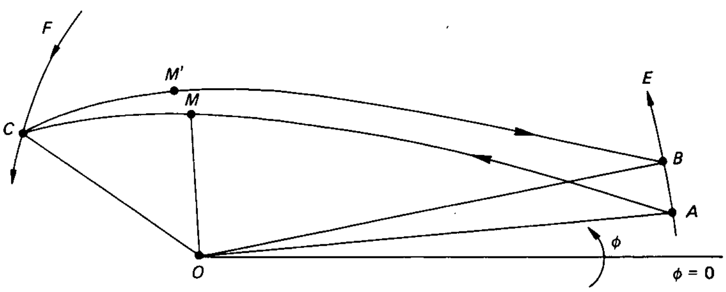

Figure 1—Diagram of the paths AMC and $CM'B$ of the radar signals sent between the Earth E and the target F. The center of the Sun is at O, and $OA = x_E$, $OB = x'_E$, $OC = x_F$, and $OM = X$.

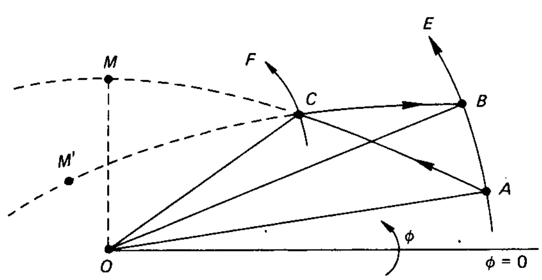

Figure 2—Diagram for the paths AC and CB of the radar signals sent between E and F. The paths have been extended to M and M', respectively. Other definitions are as in Figure 1.

Because E and F are "material particles," their invariant proper times are nonzero. Consider E which is at $(x_E, \pi/2, \phi_E)$ at time t_E and at $(x_E + dx_E, \pi/2, \phi_E + d\phi_E)$ at time $t_E + dt_E$. By applying to Equation 11 the argument that produced Equation 1.8 of Reference 2, it follows that the invariant velocity of E is V_E, where

$$V_E^2 = [e^{-2\lambda}(e^{2\mu}\, \dot{x}^2 + e^{2\nu}x^2\, \dot{\phi}^2)]_E\ . \tag{13}$$

The dot denotes d/dt and the subscript E means that every term within the bracket must be evaluated at $x = x_E$, $\phi = \phi_E$. Also by Equation 11 the two events in question are separated by the orbital proper time interval

$$ds_E = \left[e^{\lambda}\left(1 - \frac{V^2}{c^2}\right)^{1/2}\right]_E dt_E\ . \tag{14}$$

As in Reference 2, this terrestrial proper time will be identified with the time kept by atomic clocks on Earth. Corresponding formulas for F at $C(x_F, \pi/2, \phi_F)$ at time t_F are

$$\begin{aligned} V_F^2 &= [e^{-2\lambda}(e^{2\mu}\, \dot{x}^2 + e^{2\nu}x^2\, \dot{\phi}^2)]_F \\ ds_F &= \left[e^{\lambda}\left(1 - \frac{V^2}{c^2}\right)^{1/2}\right]_F dt_F\ . \end{aligned} \tag{15}$$

DOPPLER FREQUENCY FORMULAS

It will be assumed that the radar signal consists of a pulse of electromagnetic waves all of the same frequency ν. It will be necessary to have a prescription for the frequency relative to clocks that read proper times proceeding at different rates. That is, if s and s' are two of these proper times, then $ds/ds' \neq 1$. The prescription to be employed is that of Kruger.[1] He argues that, in radar experiments, it is the phase of the signal which is measured. Let Φ be the phase of the wave, and let ds and ds' be the proper time intervals during which the same increment of phase $d\Phi$ is measured. Then the frequencies are to be defined as the rates of change of phase with respect to proper time. Thus,

$$\nu = \frac{d\Phi}{ds}\ ,$$

$$\nu' = \frac{d\Phi}{ds'}\ ,$$

and therefore

$$\frac{\nu'}{\nu} = \frac{ds}{ds'}\ . \tag{16}$$

To express this result in terms of the (t, x, θ, ϕ) coordinates of the Earth and the target, the method described in Section 5.4 of Reference 1 will be employed. This method consists of defining the emission and the receipt of a wave by pairs of events as follows. In Figure 1, let the Earth E be at

[1] B. Kruger: *The Doppler Equation in Range and Range Rate Measurement.* NASA GSFC Report X-507-65-385, 1965.

$A(x_E, \pi/2, \phi_E)$ at time t_E. Then the emission of an electromagnetic wave of frequency ν_E will be described by the two events $E_1(t_E, x_E, \pi/2, \phi_E)$ and $E_2(t_E + dt_E, x_E + dx_E, \pi/2, \phi_E + d\phi_E)$, where dt_E may be regarded as the period of the wave in coordinate time. Similarly the receipt of the wave of frequency ν_F by the target F at C is defined by the pair of events $F_1(t_F, x_F, \pi/2, \phi_F)$ and $F_2(t_F + dt_F, x_F + dx_F, \pi/2, \phi_F + d\phi_F)$. The events (E_1, F_1) lie on a null geodesic in the plane $\theta = \pi/2$ of the space-time of Equation 11, whose differential equations may be written (by Equations 2.308 and 2.309 of Reference 1) as

$$0 = e^{2\lambda}\left(\frac{dt}{dp}\right)^2 - \frac{1}{c^2}\left[e^{2\mu}\left(\frac{dx}{dp}\right)^2 + e^{2\nu}x^2\left(\frac{d\phi}{dp}\right)^2\right],$$

$$e^{2\nu}x^2\frac{d\phi}{dp} = D\beta c,$$

$$\frac{dt}{dp} = \beta e^{-2\lambda},$$

where p is a nonzero parameter varying continuously along the null geodesic and D and β are constants of integration. These three equations reduce after some calculation to

$$\frac{d\phi}{dt} = \frac{e^{2(\lambda-\nu)}Dc}{x^2} \tag{17}$$

$$\left(\frac{dx}{dt}\right)^2 = c^2 e^{2(\lambda-\mu)}\left(1 - \frac{e^{2(\lambda-\nu)}D^2}{x^2}\right). \tag{18}$$

It is convenient to introduce here two functions of x:

$$Q^2(x, D) = 1 - \frac{e^{2(\lambda-\nu)}D^2}{x^2} \tag{19}$$

$$P(x, D) = e^{\mu-\lambda}Q(x, D). \tag{20}$$

The point M on a null geodesic such as AMC of Figure 1 is defined to be the point where x has its minimum value, so that $dx/dt = 0$ at M. Thus if $x = X$ at M, Equation 18 shows that

$$D = Xe^{\nu(X)-\lambda(X)}. \tag{21}$$

Along the portion A to M of the null geodesic, x decreases as t increases, whereas x and t increase together from M to C. Thus the negative square root of Equation 18 applies from A to M and the positive one from M to C. Hence integration of Equation 18 gives for the moments of emission at A and receipt at C (t_E and t_F, respectively) the relation

$$c(t_F - t_E) = \left(\int_X^{x_E} + \int_X^{x_F}\right)\frac{e^{\mu-\lambda}\,dx}{Q(x, D)}. \tag{22}$$

Similar arguments apply to the result of dividing Equation 17 by the appropriate square root of Equation 18. Integration then yields

$$\phi_F - \phi_E = D\left(\int_X^{x_E} + \int_X^{x_F}\right) \frac{e^{\lambda+\mu-2\nu}}{x^2 Q(x,D)}\, dx\,. \tag{23}$$

Again, the events E_2 and F_2 also lie on a null geodesic whose integrated equations may be obtained from Equations 22 and 23 by writing $t_E + dt_E$, $t_F + dt_F$, $x_E + dx_E$, $x_F + dx_F$, $\phi_E + d\phi_E$, and $\phi_F + d\phi_F$ for t_E, t_F, x_E, x_F, ϕ_E, and ϕ_F, respectively, and also by altering the constant of integration D to $D + dD$. It is also easy to show that

$$Q^{-1}(x, D + dD) = Q^{-1}(x,D)\left(1 + \frac{D\, dD\, e^{2(\lambda-\nu)}}{x^2 Q^2(x,D)}\right). \tag{24}$$

The differences $dt_F - dt_E$ and $d\phi_F - d\phi_E$ are now required; and, since the second difference probably involves the more complicated calculation, it is worked out in detail.

The null geodesic equation (Equation 23) for E_1 and F_1 and its analogue for E_2 and F_2 yield

$$d\phi_F - d\phi_E = (D + dD)\left(\int_{X+dX}^{x_E+dx_E} + \int_{X+dX}^{x_F+dx_F}\right) \frac{e^{\lambda+\mu-2\nu}}{x^2 Q(x, D+dD)}\, dx$$

$$-D\left(\int_X^{x_E} + \int_X^{x_F}\right) \frac{e^{\lambda+\mu-2\nu}}{x^2 Q(x,D)}\, dx\,,$$

where dX is obtained in terms of dD by taking differentials of both sides of Equation 21. If the right-hand side of the last equation is evaluated to the first order in the differentials, it is found after some calculation and the use of Equation 24 that

$$\frac{D}{c}(d\phi_F - d\phi_E) = \frac{D\, dD}{c}\left(\int_X^{x_E} + \int_X^{x_F}\right) \frac{e^{\lambda+\mu-2\nu}}{x^2 Q^3(x,D)}$$

$$+ \frac{D^2}{c}\left[J(x_E)\, dx_E + J(x_F)\, dx_F - 2J(X)\, dX\right]\,, \tag{25}$$

where, in the last three terms,

$$J(x) = \frac{e^{\lambda+\mu-2\nu}}{x^2 Q(x,D)}\,.$$

A corresponding calculation applied to Equation 22 and its analogue for E_2 and F_2 yields

$$dt_F - dt_E = \frac{D\, dD}{c}\left(\int_X^{x_E} + \int_X^{x_F}\right) \frac{e^{\lambda+\mu-2\nu}}{x^2 Q^3(x,D)}\, dx$$

$$+ \frac{1}{c}\left[I(x_E)\, dx_E + I(x_F)\, dx_F - 2I(X)\, dX\right]\,, \tag{26}$$

where

$$I(x) = \frac{e^{\mu - \lambda}}{Q(x, D)} .$$

If now Equation 25 is subtracted from 26 and Equations 19 to 21 are also used, it follows that

$$dt_F - \frac{D}{c}\,d\phi_F - P(x_F, D)\frac{dx_F}{c} = dt_E - \frac{D}{c}\,d\phi_E + P(x_E, D)\frac{dx_E}{c} , \tag{27}$$

which is the final result for the outgoing signal A to C. It has been established without the necessity of finding dD or dX. Presumably Equation 27 is still valid when $x_F = X$ so that C and M coincide, in spite of the fact that in these circumstances $Q(X, D) = 0$ and therefore also $P(X, D) = 0$. The result is also expressible in terms of the t-time rates of change $(dx/dt, d\phi/dt \equiv \dot{x}, \dot{\phi})$ of the coordinates of E and F. In fact, Equation 27 is also

$$\frac{dt_E}{dt_F} = \frac{1 - D\dot{\phi}_F/c - P(x_F, D)\dot{x}_F/c}{1 - D\dot{\phi}_E/c + P(x_E, D)\dot{x}_E/c} . \tag{28}$$

The returning signal is immediately reflected at C so that none of the symbols with subscript F are altered. But the point of closest approach is now M' in Figure 1, for which the constant is D'. The t-time period is dt'_E at B whose coordinates and their rates of change are $x'_E, \phi'_E, \dot{x}'_E$, and $\dot{\phi}'_E$. The signal is also now traveling from C to B. When all these changes are taken into account, and the preceding calculations are repeated, it turns out that

$$\frac{dt_F}{dt'_E} = \frac{1 + D'\dot{\phi}'_E/c - P(x'_E, D')\dot{x}'_E/c}{1 + D'\dot{\phi}_F/c + P(x_F, D')\dot{x}_F/c} . \tag{29}$$

Formulas 28 and 29 refer to the configuration of Figure 1. When C, A, and B all lie "on the same side" of the Sun, as in Figure 2, the calculations are simplified because, in the integrals corresponding to Equations 22 and 23, the limit X does not occur. The limits in the integrals are simply x_E and x_F. The path AC of the outgoing signal now yields

$$\frac{dt_E}{dt_F} = \frac{1 - D\dot{\phi}_F/c + P(x_F, D)\dot{x}_F/c}{1 - D\dot{\phi}_E/c + P(x_E, D)\dot{x}_E/c} , \tag{30}$$

while, for the returning signal along CB,

$$\frac{dt_F}{dt'_E} = \frac{1 + D'\dot{\phi}'_E/c - P(x'_E, D')\dot{x}'_E/c}{1 + D'\dot{\phi}_F/c - P(x_F, D')\dot{x}_F/c} . \tag{31}$$

The final step consists in transforming from coordinate-time intervals to proper time intervals and thence to frequencies through Equation 16 and results such as Equations 28 to 31. A frequency ν_E emitted from A is received as ν_F at C, reemitted as ν_F at C and received as ν'_E at B. The two-way radar Doppler frequency formula is then

$$\frac{\nu'_E}{\nu_E} = \frac{ds_E}{ds'_E} = \frac{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_E}{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_{E'}}\,\frac{dt_E}{dt'_E} , \tag{32}$$

and the one-way radar formulas are

$$\frac{\nu_F}{\nu_E} = \frac{ds_E}{ds_F} = \frac{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_E}{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_F} \frac{dt_E}{dt_F} \tag{33}$$

$$\frac{\nu'_E}{\nu_F} = \frac{ds_F}{ds'_E} = \frac{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_F}{[e^{\lambda}(1 - V^2/c^2)^{1/2}]_{E'}} \frac{dt_F}{dt'_E} \ . \tag{34}$$

In these formulas the ratios of the coordinate-time differentials are obtained from Equations 28 and 29 for the Figure 1 configuration and from Equations 30 and 31 for that of Figure 2.

Though Formulas 33 and 34 have been called Doppler frequency formulas, they nevertheless incorporate the gravitational red-shift effect. Consider the case for which E and F are at rest, so that in Figure 1, for example, A and B are coincident and $x'_E \equiv x_E$. Though now $\dot{x}$, $\dot{\phi}$, and V are all zero at C and A, yet ν_F/ν_E and ν'_E/ν_F are in general different from unity because $\lambda(x_F) \neq \lambda(x_E)$. However, in the two-way radar formula (Equation 32) the gravitational red-shift effect has been eliminated and $\nu'_E = \nu_E$.

SPECIAL RELATIVITY

The results of the preceding section may be elucidated by considering their meanings in the degenerate case of special relativity. In this theory, λ, μ, and ν are all zero, the geometry is Euclidean, and the paths of the signals are Euclidean straight lines. Hence by Equation 13 the velocity of either E or F is given by

$$\begin{aligned} V^2 &= \dot{x}^2 + x^2\dot{\phi}^2 \\ &\equiv u^2 + v^2 \ , \end{aligned} \tag{35}$$

and, by Equation 21, $D = X$ and $D' = X'$. The angle ψ may be defined by

$$\frac{D}{x} = \cos\psi \ , \tag{36}$$

and then Equations 19 and 20 show that

$$P(x, D) = Q(x, D) = \left(1 - \frac{D^2}{x^2}\right)^{1/2} = \sin\psi \ . \tag{37}$$

Replacement of D by D' in the last two formulas gives corresponding results for an angle ψ'. The velocity diagram for the outgoing signal, A to C, of Figure 1 is shown in Figure 3. The one-way Doppler frequency formula (Equation 33) may be converted by Equations 28 and 35 to

$$\frac{\nu_F}{\nu_E} = \left(\frac{1 - V_E^2/c^2}{1 - V_F^2/c^2}\right)^{1/2} \frac{1 - (D/x_F)(v_F/c) - P(x_F, D)u_F/c}{1 - (D/x_E)(v_E/c) + P(x_E, D)u_E/c} \ .$$

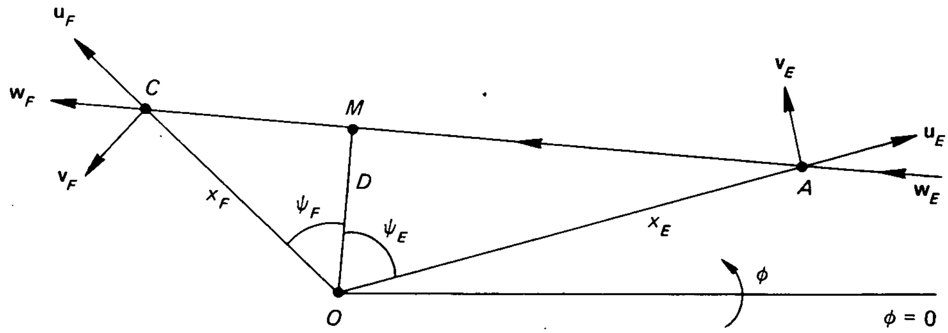

Figure 3—Velocity components of E at A and F at C.

But

$$\frac{D}{x_F} v_F + P(x_F, D)u_F = v_F \cos \psi_F + u_F \sin \psi_F$$

$$= w_F$$

$$\frac{D}{x_E} v_E - P(x_E, D)u_E = v_E \cos \psi_E - u_E \sin \psi_E$$

$$= w_E ,$$

where the velocity components w_F and w_E, of F and E along the (straight) line of sight, are reckoned to be positive in the direction A to C. Thus

$$\frac{\nu_F}{\nu_E} = \frac{(1 - V_E^2/c^2)^{1/2}}{(1 - V_F^2/c^2)^{1/2}} \; \frac{1 - w_F/c}{1 - w_E/c} . \tag{38}$$

If F is identified as an "observer" who is at rest at C so that $w_F = 0$ and $V_F = 0$, then the emission frequency in the rest frame of E at A is ν_E and the frequency received by the observer at C is ν_F. In terms of wavelengths, $\lambda_E = c/\nu_E$ and $\lambda_F = c/\nu_F$; thus the last equation is

$$\frac{\lambda_F}{\lambda_E} = \frac{1 + (-w_E/c)}{(1 - V_E^2/c^2)^{1/2}} , \tag{39}$$

where the case of a negative value of w_E is presupposed so that E has a velocity of recession along the line of sight as viewed from C. Equation 39 is the well-known Doppler formula of special relativity (Reference 5) including the transverse Doppler effect represented by the term in V_E^2/c^2.

The velocity diagram for the returning signal C to B is shown in Figure 4. The factors linear in the velocity components are those shown in Equation 29. The w components are again taken in the

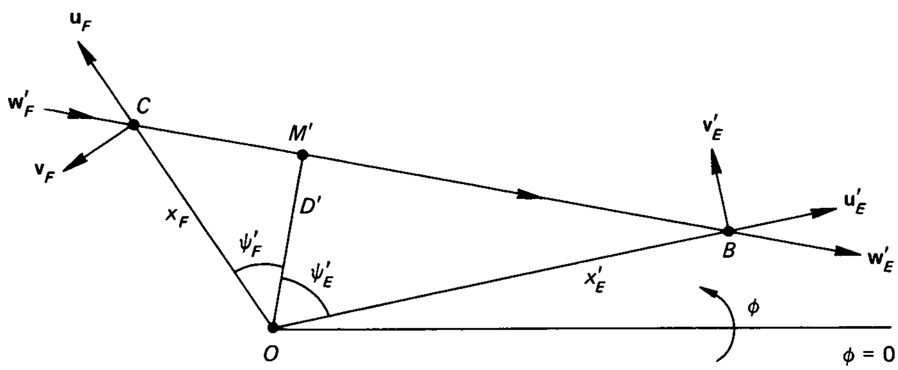

Figure 4—Velocity components of E at B and F at C.

direction of motion of the signal. Hence

$$\frac{D'}{x_F} v_F + P(x_F, D')u_F = v_F \cos \psi'_F + u_F \sin \psi'_F$$

$$= - w'_F$$

$$\frac{D'}{x'_E} v'_E - P(x'_E, D')u'_E = v'_E \cos \psi'_E - u'_E \sin \psi'_E$$

$$= - w'_E \; .$$

Thus the special relativity version of Equation 34 becomes

$$\frac{\nu'_E}{\nu_F} = \left(\frac{1 - V_F^2/c^2}{1 - V_E'^2/c^2}\right)^{1/2} \frac{1 - w'_E/c}{1 - w_F/c} , \tag{40}$$

which is essentially the same formula as Equation 38 when it is remembered that ν_F now plays the part of the emitted frequency. The reader may draw his own velocity diagrams for the Figure 2 configuration. It is to be borne in mind that the v components are drawn in the direction of ϕ increasing and the w components are always in the direction of motion of the signal. Then Equation 38 again follows from the special relativity versions of Equations 30 and 33, and Equation 39 follows from those of Equations 31 and 34.

The main point of this discussion is that the ratios of the t-time differentials in Equations 32 to 34 give the effects of the line-of-sight velocities of E at A and B and F at C and this interpretation may be expected to hold for the non-Euclidean geometry of the Schwarzschild space-time. In this case the principal complication arises through the definitions of the angles ψ and ψ'. An illustration, based on the use of the metric of Equation 1, will be found in Section 3 of Reference 2.

NUMERICAL VALUES

To obtain an estimate of the orders of magnitude involved in the terms of the general relativity formulas, the target will be assumed to move in the solar system under the gravitational attraction of the Sun alone, which is in accordance with the definition of the Schwarzschild space-time. Inspection of formulas 28 to 34 shows that the arithmetic will be simplified if the Earth is regarded as at rest during the radar experiment. Under these conditions, the Figure 1 configuration has A and B coincident, so that $D = D'$. Then Equation 32, with the aid of Equations 28 and 29, reduces to

$$\frac{\nu_E'}{\nu_E} = \frac{1 - D\dot{\phi}/c - P(x, D)\dot{x}/c}{1 + D\dot{\phi}/c + P(x, D)\dot{x}/c} \, , \tag{41}$$

where the subscript F has been omitted on the right-hand side. The expression to be evaluated may then be written

$$\eta(x) = \frac{D}{x}\frac{(x\dot{\phi})}{c} + P(x, D)\frac{\dot{x}}{c} \, . \tag{42}$$

Suppose that it is decided to use the coordinate r of Equation 1 in the first instance. Hence $x \equiv r$ in Equation 11 and

$$\begin{aligned} e^{2\lambda} &= 1 - \frac{2m}{r} \, , \\ e^{2\mu} &= \left(1 - \frac{2m}{r}\right)^{-1} \, , \\ e^{2\nu} &= 1 \, , \end{aligned} \tag{43}$$

so that Equation 21 with $X \equiv R$ is

$$D = R\left(1 - \frac{2m}{R}\right)^{-1/2} \, , \tag{44}$$

and by Equations 19 and 20

$$P(r, D) = \left(1 - \frac{2m}{r}\right)^{-1} \left[1 - \frac{R^2}{r^2}\frac{(1 - 2m/r)}{(1 - 2m/R)}\right]^{1/2} \, . \tag{45}$$

These are exact formulas that would have to be used if r were approximately equal to R; i.e., for a configuration in which C lay close to M. In a large number of configurations, however, the ratio R^2/r^2 is substantially less than unity. Thus it is possible to use expressions for D and P that are of the first order in m/r and m/R, namely,

$$\begin{aligned} D &= R\left(1 + \frac{m}{R}\right) \\ P(r, D) &= \left(1 - \frac{R^2}{r^2}\right)^{1/2} \left[1 + \frac{2m/r - (m/r + m/R)(R^2/r^2)}{1 - R^2/r^2}\right] \, . \end{aligned} \tag{46}$$

The exact expression for $\eta(r)$ is obtained by substituting $x = r$ and using Equations 44 and 45 in Equation 42. The approximate expression derived from Equation 46 is

$$\eta(r) = \frac{w(r)}{c}\left\{1 + \left[\frac{m}{r}\frac{r\dot{\phi}}{w(r)} + \frac{2m/r - (m/r + m/R)(R^2/r^2)}{(1 - R^2/r^2)^{1/2}}\frac{\dot{r}}{w(r)}\right]_m\right\} \tag{47}$$

where

$$w(r) = \frac{R}{r}(r\dot{\phi}) + \left(1 - \frac{R^2}{r^2}\right)^{1/2}\dot{r}. \tag{48}$$

In Formula 47 and in all subsequent equations for η, the contributions of the terms involving m (the m terms) are enclosed in square brackets with subscript m. It was shown that when $m = 0$, and special relativity applies, η reduces to w, which is the radial velocity of F at C relative to the stationary Earth at A.

General relativity, however, contains no prescription that compels the use of r in preference to $\bar{r}$, to ρ, or to any other radial coordinate that may legitimately be used in the Schwarzschild space-time. Suppose, therefore, that x is next identified with $\bar{r}$ so that now

$$e^{2\lambda} = \left(\frac{1 - m/2\bar{r}}{1 + m/2\bar{r}}\right)^2$$

$$e^{2\mu} = e^{2\nu} = \left(1 + \frac{m}{2\bar{r}}\right)^4 .$$

An alteration in the value of m is, of course, forbidden because the same central body is in question. Then Equation 21 with $X \equiv \bar{R}$ gives

$$\bar{D} = \bar{R}\frac{(1 + m/2\bar{R})^3}{1 - m/2\bar{R}} \tag{49}$$

and Equations 19 and 20 yield

$$P(\bar{r}, \bar{D}) = \frac{(1 + m/2\bar{r})^3}{1 - m/2\bar{r}}\left[1 - \frac{\bar{R}^2(1 + m/2\bar{R})^6}{\bar{r}^2(1 - m/2\bar{R})^2}\frac{(1 - m/2\bar{r})^2}{(1 + m/2\bar{r})^6}\right]^{1/2} . \tag{50}$$

The approximate form of $\eta(\bar{r})$ derived from these expressions and from Equation 42 is then

$$\eta(\bar{r}) = \frac{w(\bar{r})}{c}\left\{1 + \left[\frac{2m}{\bar{r}}\frac{\bar{r}\dot{\phi}}{w(\bar{r})} + \frac{2[m/\bar{r} - (m/\bar{R})(\bar{R}^2/\bar{r}^2)]}{(1 - \bar{R}^2/\bar{r}^2)^{1/2}}\frac{\dot{\bar{r}}}{w(\bar{r})}\right]_m\right\}, \tag{51}$$

where $w(\bar{r})$ is obtained from Equation 48 by replacing R and r by $\bar{R}$ and $\bar{r}$, respectively. Lastly the coordinate ρ of Equation 10 could be employed so that $x = \rho$ and

$$e^{2\lambda} = \frac{1 - m/\rho}{1 + m/\rho},$$

$$e^{2\mu} = \frac{1 + m/\rho}{1 - m/\rho},$$

$$e^{2\nu} = \left(1 + \frac{m}{\rho}\right)^2 .$$

Then, if $X \equiv \mathcal{R}$ and $D \equiv \Delta$, Equations 19 to 21 yield

$$\Delta = \frac{(1 + m/\mathcal{R})^{3/2}}{(1 - m/\mathcal{R})^{1/2}} \mathcal{R} \tag{52}$$

$$P(\rho, \Delta) = \frac{1 + m/\rho}{1 - m/\rho} \left[1 - \frac{\mathcal{R}^2}{\rho^2} \frac{(1 + m/\mathcal{R})^3}{1 - m/\mathcal{R}} \frac{1 - m/\rho}{(1 + m/\rho)^3} \right]^{1/2} . \tag{53}$$

It is noticed, by comparing Equation 49 with 52 and Equation 50 with 53, that the exact dependence of $\overline{D}$ on $\overline{R}$ is different from that of Δ on $\mathcal{R}$ and similarly for $P(\overline{r}, \overline{D})$ as compared with $P(\rho, \Delta)$. But to the first order for the m terms

$$\eta(\rho) = \frac{w(\rho)}{c} \left\{ 1 + \left[\frac{2m}{\rho} \frac{\rho\dot{\phi}}{w(\rho)} + \frac{2[m/\rho - (m/\mathcal{R})(\mathcal{R}^2/\rho^2)]}{(1 - \mathcal{R}^2/\rho^2)^{1/2}} \frac{\dot{\rho}}{w(\rho)} \right]_m \right\} , \tag{54}$$

where $w(\rho)$ is obtained from Equation 48 by writing $\mathcal{R}$ and ρ for R and r, respectively. Comparison of Equations 51 and 54 shows that $\eta(\overline{r})$ depends on $\overline{r}$ and $\overline{R}$ in the same way as $\eta(\rho)$ does on ρ and $\mathcal{R}$. To this order of approximation therefore, it is immaterial whether the isotropic coordinate $\overline{r}$ or the harmonic coordinate ρ is employed. But this would not be true for configurations in which $\overline{r}$ were nearly equal to $\overline{R}$ and ρ to $\mathcal{R}$.

The Newtonian analogue for the motion of F is given by the elementary two-body problem in which the mass of the Sun $M_\odot$ is regarded as infinitely great compared with the mass of any planet or space probe. As an example, let F be a space probe whose orbit, calculated by Newtonian theory, is an ellipse with the Sun at one focus, of perihelion distance $a_n/5$ and eccentricity $e_n = 2/3$, where a_n is the astronomical unit of Newtonian orbit theory. The semimajor axis of the orbit of F is then $3a_n/5$ and the semilatus rectum is $a_n/3$. At any instant of absolute Newtonian time t_n, the radial and cross-radial components of the velocity of F are

$$u_n = \frac{dr_n}{dt_n}$$

$$v_n = r_n \frac{d\phi_n}{dt_n} ,$$

where (r_n, ϕ_n) are the coordinates of F computed by Newtonian gravitational theory. They are coordinates in Euclidean space; such Euclidean coordinates, which arise from Newtonian theory, will be called N coordinates. Now let Figure 1 refer to a configuration in which F is at that end of the semilatus rectum of its orbit at which r_n (equal to $a_n/3$) and ϕ_n are increasing together. The choice is dictated by the well-known Newtonian theorem which states that the radial component of velocity is greatest at an end of the latus rectum. At such a point

$$\frac{u_n}{c} = \left(\frac{GM_\odot}{c^2} \frac{e_n^2}{1 - e_n^2} \frac{5}{3a_n} \right)^{1/2} . \tag{55a}$$

$$\frac{v_n}{c} = \left(\frac{GM_\odot}{c^2} \frac{1}{1 - e_n^2} \frac{5}{3a_n}\right)^{1/2} , \tag{55b}$$

Incidentally, reference to Equation 2 shows that both of these are of order $(m/a)^{1/2}$, a consideration that might be employed in what follows. However, it is probably easier to go directly to the numerical values, which are given in Equations 56.

In addition to placing F at one end of the latus rectum of its orbit, it will be assumed that F is seen from E at grazing incidence to the solar surface. In the sequel this will result in the m terms of Equations 47, 51, and 54 being as large as possible. The r_n coordinate of the point M of closest approach to the Sun's center will be identified with the solar radius $R_\odot$.

It is assumed that the quantities in the right-hand sides of Equations 47, 51, and 54 may be identified with the corresponding Newtonian values. Thus there are three alternative identifications according to whether x is chosen to be r, $\bar{r}$, or ρ, and X to be R, $\bar{R}$, or $\mathcal{R}$. The three alternatives can be represented by (Reference 6).

$$\begin{aligned}
t &= t_n , \\
x &= \frac{a_n}{3} \\
&= \frac{1}{3}(1.496 \times 10^{13}) \quad \text{cm}, \\
X &= R_\odot \\
&= 6.960 \times 10^{10} \quad \text{cm}, \\
\frac{\dot{x}}{c} &= \frac{u_n}{c} \\
&= 1.147 \times 10^{-4} , \\
\frac{x\dot{\phi}}{c} &= \frac{v_n}{c} \\
&= 1.720 \times 10^{-4} , \\
m &= \frac{GM_\odot}{c^2} \\
&= 1.477 \times 10^{5} \quad \text{cm}.
\end{aligned} \tag{56}$$

The values of u_n and v_n are calculated from Equations 55 with $e_n = 2/3$. It then follows from

Equation 48 and its analogues for the coordinates $\bar{r}$ and ρ that

$$
\begin{aligned}
\frac{w(r)}{c} &= \frac{w(\bar{r})}{c} \\
&= \frac{w(\rho)}{c} \\
&= 1.171 \times 10^{-4} [1 - 9.539 \times 10^{-5} + O(10^{-9})] ,
\end{aligned}
\tag{57}
$$

and that the m terms, if η is to be correct to the order 10^{-12} in Equations 47, 51, and 54, are, respectively,

$$
\begin{aligned}
[r]_m &= 1.011 \times 10^{-7} , \\
[\bar{r}]_m &= [\rho]_m \\
&= 1.443 \times 10^{-7} .
\end{aligned}
\tag{58}
$$

Thus, for the particular configuration in which F is at one end of the latus rectum of its orbit and is seen from E at grazing incidence to the solar surface, the m terms differ in value by as much as 42 percent as r, $\bar{r}$, or ρ is identified with the N coordinate r_n. This is not a property peculiar to this configuration. Examination of the m terms in Equations 47, 51, and 54 shows that differences would be found in other configurations also.

A two-way radar experiment would, of course, give ν'_E/ν_E rather than η. However, Equation 41 to a sufficient accuracy is

$$
\begin{aligned}
\frac{\nu'_E}{\nu_E} &= \frac{1 - \eta}{1 + \eta} \\
&= 1 - 2\eta + 2\eta^2 .
\end{aligned}
$$

Thus for the r coordinate, Equations 57 and 58 give

$$
\frac{\nu'_E}{\nu_E} = 1 - 2.342 \times 10^{-4} + 4.978 \times 10^{-8} + O(10^{-12}) - [2.368 \times 10^{-11} + \cdots]_m . \tag{59}
$$

If either $\bar{r}$ or ρ had been employed instead, the amount of the leading m term would have been $[3.379 \times 10^{-11}]_m$. This example shows that an accuracy of one part in 10^{11} would be required in the measurement of ν'_E/ν_E if the values of the m terms were to be deducible from two-way radar experiments.

The motion of the Earth produces additional mathematical complications in the evaluation of ν'_E/ν_E from Equation 32 and either Equations 28 and 29 or 30 and 31. One complication arises because the difference between D and D' must now be calculated. Another is that the values of these constants are not known directly, as has been assumed in the foregoing example. In fact only the coordinates of A, C, and B can be regarded as calculable from the orbits of E and F. The quantities D and D' must therefore be expressed in terms of the coordinates of these three points. An approximate method of dealing with these matters, in terms of the r coordinate, is given in Reference 2.

COORDINATE IDENTIFICATIONS

The smallness of the m terms and the considerations mentioned at the end of the last section indicate that orbits of high accuracy for E and F are needed so that the coordinates of A, C, and B and the velocity components at those points can be predicted as exactly as possible. To obtain such orbits, it is necessary to allow for the attraction of the planets on each other or on a space probe. This is done for the *American Ephemeris* by a method which is "in accordance with the Newtonian law of gravitation, modified by the theory of general relativity" (Reference 7). This means that the Euclidean space and absolute time of Newtonian theory are employed in writing down the equations of motion of planets and space probes. These equations contain terms that allow for the motion of the Sun and for the mutual attractions of the planets. But in addition, the equations contain terms, essentially ad hoc, that produce a small rotation of the line of apsides of a planetary orbit. This small rotation has the same value as that predicted for the orbit of a test particle in the Schwarzschild space-time, where, as already stated, the planetary perturbations are by definition zero. A similar procedure appears to be employed at the Lincoln Laboratory (Reference 4). A relativity perturbation is calculated for a test particle in the Schwarzschild space-time in terms of harmonic coordinates. It is multiplied by an arbitrary constant factor (to be found eventually from observation) and is then inserted into the complete Newtonian equations of motion to produce an additional perturbing force. In either case, it can be argued that N coordinates, and their associated Euclidean velocity components, are obtained. Both methods will be called "classical celestial mechanics."

The problem of identifying the N coordinates with some set of coordinates in the non-Euclidean space of the Schwarzschild field has still to be solved. One way of proceeding is by definition, as was done explicitly in the preceding section. Or the definition may be made implicitly as in the Lincoln Laboratory method. The insertion of the relativity perturbation into the Newtonian equations of motion carries with it the implications that the t time of the Schwarzschild field is identical with the Newtonian absolute time t_n and that the harmonic coordinates are identical with the N coordinates. It is true that Fock (Reference 3) has attempted to prove theoretically that harmonic coordinates are in some sense superior to others, a contention that contradicts Einstein's aim of putting all coordinate systems on a par with one another. Fock's idea might carry weight if it could be shown that the m terms in the $\eta(x)$ of Equation 42 always had their smallest numerical values when harmonic coordinates were employed. It might then be argued that these coordinates most closely approximated the N coordinates. But it has been shown that harmonic coordinates do not always produce the smallest values of the m terms.

When radar time delays were considered (Reference 2), the m terms were found to be larger, of the order of 10^{-7}, than those shown in the Doppler frequency formula (Equation 59). They are also different in value according to the identification employed. Thus the same problem of coordinate identification arises.

An alternative solution of the problem is possible in principle. Orbits of E and F would be calculated by classical celestial mechanics combined with observations of E and F for those configurations in which the m terms could be expected to be negligibly small. The N coordinates thus produced would then be identified in turn with the coordinates used in Equations 1, 6, and 10 to predict the values of the m terms for those configurations in which these terms were expected to be as large as possible. Simultaneously, measurements would be made of time delays and Doppler frequency ratios in these configurations with large m terms. In this way, not only would the predictions of general relativity be checked, but the Schwarzschild coordinate system that most closely corresponds to the N coordinates would be identified.

In conclusion, reference may be made to a point that is more easily discussed with respect to Doppler frequency formulas than to time delays. The frequency ratios, in Equation 16 or 32, for example, are obtained as the ratios of two invariant intervals ds. It might therefore be argued that the particular coordinate system used in the calculations could not affect the result. This is true so long as coordinate transformations such as those in Equations 5 and 9 are in question. In these transformations, to every r there corresponds a different $\bar{r}$ or ρ calculated from the two formulas, and vice versa. But this is not the operation envisaged in the coordinate-identification procedure. In the previous discussion, for example, the question was whether r, $\bar{r}$, or ρ should be identified with $a_n/3$; this is not a transformation of the kind envisaged in Equations 5 and 9. Thus, invariance of the Doppler frequency ratio is not to be expected.

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