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ABSTRACT

Wave propagation and refraction of "Type I" irregularities in the equatorial electrojet are investigated. Quantitative calculation of wave refraction in a model electrojet shows that the direction of wave refraction must change sign at one altitude. Waves propagating with the electrons rotate their wave vectors upwards in the upper electrojet and downwards in the lower electrojet during the day, and vice versa at night. Furthermore, the altitude region of largest linear growth rate is also the one with the weakest refraction rate. Consequently computations of the ray-path integrated wave growth shows that this region would dominate the backscatter spectrum from the electrojet if linear theory were valid, and it is further noted that the maximum amplitude wave should have phase velocities exceeding the ion acoustic speed. We therefore conclude that propagation alone, without inclusion of nonlinear effects, cannot explain backscatter observations of a constant Doppler frequency shift given by the ion acoustic speed.

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Field-aligned irregularities in the equatorial electrojet have been extensively investigated by radar backscatter (Bowles et al., 1960; Cohen and Bowles, 1967; Balsley, 1969; Balsley and Farley, 1971; Farley and Balsley, 1972). Balsley (1969) classified the irregularities into types I and II. Type II irregularities, which will not be considered here, are thought to be generated by a plasma density gradient in crossed electric and magnetic fields (Hoh, 1963; Maeda et al., 1963, Simon, 1963; Knox, 1964; Reid, 1968; Register and D'Angelo, 1970; Whitehead, 1971). The instability driven by electrons drifting through ions across the magnetic field proposed by Farley (1963a,b) and Buneman (1963) is generally accepted as the generation mechanism for Type I irregularities. The kinetic theory of the Type I instability has been extended by Lee et al. (1971) to include the high frequency and short wavelength modes which are generated when the electron drift exceeds twice the ion acoustic speed.

While the above theories account for the thresholds and wavelengths of the Type I irregularities, at least the most straightforward application of linear instability theory fails to account for one mystery. The backscatter radar detects Type I irregularities over a wide range of elevation angles looking both east and west, with a constant Doppler frequency shift, given roughly by $\pm 2k_T c_T$, where $k_T$ is the wave number of the transmitted wave and $c_T$ is the ion acoustic speed. In order for the irregularities observed by backscatter at oblique incidence
to be unstable, the electron polarization drift $V_p$ must exceed $c_T$. However, the linear theory indicates that the maximally growing waves should have phase velocities near $V_p$, so that Doppler frequency shifts in excess of $\pm 2k_Tc_T$ would be expected. It is equally puzzling on this basis that the same Doppler shift is found at many backscatter radar elevation angles.

Recently Kaw (1972) suggested a way out of this dilemma. By including propagation and refraction in an inhomogeneous electron drift velocity altitude profile, Kaw argued that at any altitude the largest amplitude waves in the convectively unstable case would always be at local marginal stability. Since the marginally stable waves always have phase velocities equal to the ion acoustic speed, the observations of constant Doppler shift could be accounted for. In this paper, we submit Kaw's suggestion to quantitative test. We compute the linear amplification of convectively unstable waves in a simple vertically stratified electrojet model appropriate to the geomagnetic equator. We find that linear theory alone does not account for the observations. This is because the fastest growing waves occur where $V_p$ has its maximum; it is precisely there that the waves do not refract and Kaw's argument fails. Yet on the basis of linear theory, these would have the largest amplification, and should therefore dominate the cross-section for backscatter observations without height resolution. We believe, however, that propagation arguments, coupled with the appropriate nonlinear saturation theory, still promises to account for the constant Doppler shifts. Recently, a nonlinear theory involving convective amplification
in an electrojet profile made irregular by longwavelength Type II instabilities has been proposed by Sudan et al. (1972).

In Section 2.1 we develop a simple model for the vertical profile of the electron drift in the equatorial electrojet. In 2.2 we review the linear theory of the Type I electrojet and in 2.3 we review the predictions of the linear theory neglecting propagation concerning backscatter observations. In Section 3.2 we discuss the propagation of waves in the electrojet profile derived in 2.1, and in 3.3 we derive the net amplification of unstable waves along a ray path. We close with a brief discussion in Section 4.

2. ELECTROJET MODEL AND LINEAR INSTABILITY THEORY

2.1 Electrojet Model

We choose a Cartesian coordinate appropriate to the geomagnetic equator where \( \mathbf{e}_z \) is vertically upward, \( \mathbf{e}_x \) is eastward, and \( \mathbf{e}_y \) is northward, in the direction of the geomagnetic field \( \mathbf{B}_0 \). We will use a simplification of the electrojet model of Sugiura and Cain (1966), wherein the vertical current driven by atmospheric tidal motions is entirely inhibited by a vertical polarization field \( \mathbf{E}_p = E_p \mathbf{e}_z \), where \( E_p \) at the magnetic dip equator is given by
\[
V_{Dj} = \frac{e_j \omega_j r_j}{1 + \Omega_j \tau_j^2} \frac{c_e e_j}{B_0} + \frac{\omega_j^2 \tau_j}{1 + \Omega_j \tau_j^2} \frac{c_p e_j \times e_z}{B_0}
\]

\[
= e_j F_i \frac{c_e e_j}{B_0} + e_j \frac{c_p e_j \times e_z}{B_0}
\]

where \( j = i, e \) denotes ions or electrons, \( e_i = +1, e_e = -1 \), \( \omega_i \) the cyclotron frequency, \( \tau_j = 1/v_j \) the collision time, \( v_j \) the collision frequency, and \( \tau_j \) the sum of the electric field \( E \) and the ambipolar electric field arising from plasma gradients

\[
E_j = E_j - e_j \frac{\nu P_j}{N_j e}
\]

where \( P_j \) and \( N_i = N_e = N \) are the pressure and density, and \( e \) is the electronic charge. Henceforth we will assume isothermal electrons and ions, so that \( P_j = N T_j \) and \( T_i = T_e = T \) where \( T \) is the temperature in energy units. The condition that the vertical components of the electron and ion drifts be equal leads to an expression for \( E_p \)

\[
\frac{c E_p}{B_0} = \frac{T}{e} \frac{d \ln N}{dz} \frac{(F_i - F_e)}{F_i + F_e} + \frac{G_e - G_i}{F_i + F_e} \frac{c E_D}{B_0}
\]

where the plasma density gradient was assumed vertical.

Assuming \( \nu_i / \omega_i \gg 1 \) and \( \nu_e / \omega_e \ll 1 \), and expanding (3) to first order in \( \omega_i / \nu_i \) and \( \nu_e / \omega_e \), we arrive at a simplified expression for \( E_p \) appropriate to the electrojet
\[
\frac{cE_p}{B_0} = I \frac{d \ln N}{dz} \frac{(1-\alpha)}{1+\alpha} + \frac{v_i}{\Omega_i(1+\alpha)} \frac{cE_D}{B_0}
\]

(4)

Since \(E_D \approx 1 \text{ mV/m}\), and \(T/e \approx \frac{1}{20} V\), \(\frac{d \ln N}{dz} \approx (6 \text{ km})^{-1}\), the ambipolar field may be neglected. Henceforth, we will denote the polarization factor \(v_i \Omega_i(1+\alpha)\) by \(P(z)\), the parameter \(\alpha = \frac{v_e v_i}{\Omega_e \Omega_i}\), the \(x\)-component of the current drift \(V_{ex} - V_{ix} = -cE_p B_0 \equiv V_p\) to order \((\Omega_i/v_i)^2\), when equation (4) is valid.

2.2 Linear Theory of Electrojet Instability

The frequency and growth rate \(\gamma\) of electrostatic irregularities propagating perpendicular to the magnetic field has been derived (Lee, 1972) using the two-fluid equations originally introduced into this problem by Buneman (1963)

\[
\omega = \frac{k \cdot V_D}{1+\alpha} + \frac{ak \cdot U_D}{(1+\alpha)}
\]

(5)

\[
\gamma = \frac{\nu_e}{\Omega e \Omega_i(1+\alpha)} \sqrt{\frac{(k \cdot V_D - k \cdot U_D)^2}{(1+\alpha)^2} - k^2 c_T^2}
\]

(6)

where \(c_T^2 = \frac{T_e + T_i}{M_i} = 2T/M\) and \(V_{Di} = \frac{eE_p}{M_i v_i} e_z\) is a small vertical ion drift which can ordinarily be neglected except in wave propagation calculations. Equations (5) and (6) are valid for sufficiently long wavelengths, \(k < \nu_i / c_T\), where the fluid approximation applies. However, \(\gamma\), from equation (6) increases as \(k^2\) without limit, so that to find the maximum growth rate at short wavelengths, kinetic theory must be used. Farley (1963a,b)
has studied marginal stability conditions from kinetic theory, and Lee et al. (1971) have calculated the growth rates using kinetic theory. While the knowledge of the maximally unstable wavenumber \( k \) is essentially for nonlinear saturation calculations, fluid equations do adequately describe the range of wavenumbers typically sampled by present backscatter measurements. Since the collision frequencies increase with decreasing altitude, the fluid theory is valid for increasingly short wavelengths with decreasing altitude.

We further reduce (5) and (6) by substituting the electron and ion drifts derivable from our model, equations (1) and (2), whereupon

\[
\omega = V_p \left\{ -\frac{k_x}{(1+\alpha)} + \frac{\Omega_i}{V_i} k_z \right\} \tag{7a}
\]

\[
\gamma = \frac{v_e k_x^2}{\Omega_e (1+\alpha)} \left\{ \frac{V_p^2}{(1+\alpha)^2} - k^2 c_T^2 \right\} \tag{7b}
\]

where the \( z \)-component of the electron drift and the ion Pedersen drift are mutually cancelled in (7b) due to the assumption of zero vertical current. If, further, \( k_x = k \cos \phi \), where \( \phi \) shown in Figure (1) is the angle between the wavevector and the horizontal direction, we may rewrite (7a) and (7b)

\[
\omega = k c_T \left\{ \frac{\cos \phi}{\cos \phi_c} + \frac{\Omega_i}{V_i} \frac{\sin \phi (1+\alpha)}{\cos \phi_c} \right\} \tag{8a}
\]

\[
\gamma = \frac{v_e k_x^2 c_T^2}{\Omega_e (1+\alpha)} \left\{ \cos \frac{2 \phi}{\cos \phi_c} - 1 \right\} \tag{8b}
\]
where we have defined the critical wave normal angle $\phi_c$ as the angle to the horizontal for which waves are marginally stable

$$\cos \phi_c = \frac{(1+\alpha)c_T}{V_p} \frac{V_c}{V_p}$$

(8c)

a formulation which is valid only for marginally stable or unstable waves. From (7) or (8), we conclude that whether or not waves are unstable depends only upon the magnitude of the horizontal polarization drift $cE_p/B_0$. As $E_p$ increases, it eventually reaches the threshold for instability, defined by $\cos \phi_c = 1$ or $V_p = cE_p/B_0 = c_T(1+\alpha)$, at which point only horizontal waves are excited. As $E_p$ increases still further, two symmetrical wedges of propagation angles $\phi$ are unstable, with their axes parallel and antiparallel to the polarization with the half-angle $\phi_c$. Thus, the angular width of the unstable spectrum increases with increasing $V_p/V_c$. The most unstable waves continue to propagate horizontally, parallel and antiparallel to the polarization drift. Equation (8c) indicates that the strength of an unstable electrojet current may be labelled by $V_p/V_c$ or $\phi_c$ interchangeably. We will often use $\phi_c$ for this purpose, since the angular width of the unstable spectrum can be measured by backscatter techniques simply by varying the angle of incidence of the probing radar beam.

The ratio of the critical drift $V_c$ to the thermal speed $c_T$ for the present electrojet model is plotted in the left hand side of Figure (3), as the solid line. Above 102.5 km, the parameter $\alpha$ is small, and $V_c \approx c_T$ independent of height. The corresponding height profiles of $V_p/V_c$, or alternately $\phi_c$, are displayed on the right of Figure (2). Below 102.5 km, $\alpha > 1$ and
the critical drift increases sharply. At 98 km, $V_c = 2c_T$. For driving fields $E_D = 3/4$ mV/m and $E_D = 1$ mV/m, the electron polarization drifts $V_p$, normalized to $c_T$, are also plotted as curves as a function of height on the left of Figure (2). The values of $E_D$ were chosen to yield maximum values of $V_p/V_c = 1.5, 2$. Instability is possible between 98.5 and 108 km for $E_D = 3/4$ mV/m, and between 97 and 110 km for $E_D = 1$ mV/m. A very important feature for propagation considerations is that $V_p$ has a local maximum within the unstable region at 100 km, and that the ratio $V_p/V_c$, or equivalently $\varphi_c$, maximizes at $\sim 102.5$ km. Simple differentiation indicates that $V_p/V_c$ maximizes at the altitude where $\alpha = 1/3$, or $z = 102.5$ km. Thus, since $E_p$ scales with $E_D$, as $E_D$ first exceeds $\sim 1/2$ mV/m, waves near 102.5 km will first become unstable; as $E_D$ further increases, the altitude range for linear instability increases with increasing $E_D$. The kinetic theory of the Type I instability (Lee et al., 1971) indicates that the wavelength of the fastest growing mode decreases with increasing $V_p/V_c$. Therefore, if the linear theory adequately described the instability when $V_p$ exceeds $V_c$, the broadest angular spectrum and the shortest wavelengths would be expected near 102.5 km. This conclusion is subject to modification by propagation and nonlinear effects. One further point of importance for our subsequent discussion is that since equation (3) indicates that the growth rate $\gamma$ scales as the electron collision frequency $v_e$, for a given $k$ and $V_p/V_c$ the waves at the bottom of the electrojet will grow five times as fast as those at the top.

Schieldge et al. (1972a,b,c) have kindly provided us with
a plot of the diurnal variation of the driving field $E_D$ derived from atmospheric tidal motions, which is displayed as Figure (4). From this figure we conclude that at times, both during the day and night, $E_D$ can exceed 1.5 mV/m, which for this model implies $V_D/V_C > 3$ at $z = 102.5$ km. Furthermore, $E_D$ exceeds 1.2 mV/m for considerable portions of the day. The electrojet reverses direction at night. Henceforth, we will direct our discussion to the daytime electrojet. However, our arguments can be carried through to the nighttime electrojet by reversing the sign of $E_D$ wherever it appears in the discussion to follow.

While our electrojet and instability model is subject to some uncertainties due to our neglect of vertical currents, its qualitative features will remain intact in any internal variation of its basic parameters because the electron drift $V_p$ and critical drift $V_D$ depend only on the collision frequencies normalized to the cyclotron frequencies. Varying the neutral density at 100 km will raise or lower the profiles of Figure (2) in altitude, while changing the scale height $H$ will mainly change the vertical scale of the unstable region. Thus, the basic features of importance in the discussion to come, local maxima in $V_p/c_T$ and $V_p/V_C$, and large growth rates at the bottom of the electrojet, will be preserved qualitatively and only modified quantitatively. Finally, it is our contention that the great variations in plasma and instability parameters over the unstable region of the electrojet represented by Figures (1) and (2) must never be forgotten in any discussion of nonlinear saturation.
2.3 Backscatter Observations

A probing plane electromagnetic wave in scattering from ionospheric irregularities must satisfy the following matching conditions

\[ \omega_s = \omega_T + \omega \]

\[ k_s = k_T + k \]

where \((\omega_s, k_s)\) and \((\omega_T, k_T)\) are the frequency and wave number of the scattered (s) and transmitted (T) waves, and \((\omega, k)\) are the frequency and wavenumber of the ionospheric irregularities. Thus an ideal backscatter system emitting a plane wave \((\omega, k_T)\) will sense excited fluctuations with a single \(k\). If the system also had perfect altitude resolution, the scattered wave would have a unique frequency shift, corresponding to the solution of the dispersion relation (4) for \(k = -2k_T\) at a given altitude. Most backscatter experiments thus far have not had good altitude resolution; consequently, to a first approximation, the experiments sense all those excited waves with \(k = -2k_T\) in the unstable region of the electrojet. In this case, we must expect that the scattered wave will have a continuous spectrum of frequency shifts. The scatter cross-section should be dominated by that altitude where the waves of a given \(k = -2k_T\) have the largest amplitude.

At a given altitude, and for a given \(V_p/V_e\), linear theory
leads us to expect waves with wave vectors in the angular ranges $(-\phi_C, +\phi_C), (\pi-\phi_C, \pi+\phi_C)$. A probing radar can detect only propagating waves with $k_z < 0$. An ideal plane wave backscatter radar with perfect height resolution located at the magnetic dip equator would detect waves, looking both eastward and westward, for all elevations of the radar dish less than $\phi_C$ with respect to the horizontal. Since typical backscatter experiments sample all altitudes, scattered signals would be expected for all elevations less than $\phi_C_{\text{max}}$, which corresponds to the maximum of the electrojet. Thus, the elevations at which enhanced scattered from Type I irregularities is observed is directly related in linear theory to the degree to which $V_p/V_C$ exceeds 1. Since scatter returns are observed at large elevations, $V_p/V_C$ must typically exceed 1 at at least one altitude in the unstable electrojet.

Examination of the spectral content of the scattered signal has proven extremely illuminating. Since $\Omega/V_i << 1$, equation (8a) reduces approximately to

$$\omega/\kappa_T = \frac{\cos \omega}{\cos \phi_C}$$

Thus, marginally stable waves always have phase velocities equal in magnitude to the acoustic speed $c_T$, and unstable wave phase velocities exceed $c_T$ regardless of altitude and driving field. An ideal backscatter radar with perfect height resolution would therefore detect Doppler frequency shifts of $\pm 2\kappa_T c_T$ at the maximum
elevation where enhanced scatter is observed, and larger frequency shifts at all lower elevations but at the same altitude. A backscatter radar without altitude resolution would, from Figure (4), sample marginally stable waves only from two altitudes; the waves sampled from altitudes in between would be unstable, and produce Doppler frequency shifts exceeding $2k_Tc_T$. The linear theory developed thus far would suggest that the unstable waves should have large amplitudes. Therefore, we would expect to find the Doppler shift frequency at the peak of the scattering cross-section to exceed $2k_Tc_T$, in general. Since reasonable driving fields lead to $V_p/V_c \lesssim 3$, the Doppler shifts should range between 2 and $6k_Tc_T$, though it must be remembered that the waves with the largest phase velocities have $\cos \phi \approx 1$ and would be detectable only at very small radar elevation angles. The above phenomenology derived from the simplest application of linear theory conflicts with observation. The Doppler shift of the spectral peak is very often observed to lie near $\pm 2k_Tc_T$ for many elevation angles looking both east and west (Balsley and Farley, 1971).

3. LINEAR THEORY OF CONVECTIVE HALL CURRENT INSTABILITY

3.1 Introduction

An ingenious attempt to explain the constant Doppler shift has been proposed by Kaw (1972), who included propagation in the linear instability theory. He noticed that in an inhomogeneous electrojet profile the wave group velocity has a positive
vertical component and that the angle \( \varphi \) changes monotonically as the wave propagates. Suppose \( \varphi \) always decreases. Then at a given altitude \( z \), the largest amplitude wave at a given \( k \) would be the one generated by fluctuations at the angle \( \varphi_c(z') \) at \( z < z' \) which would then propagate upwards and arrive at \( z \) with the angle \( -\varphi_c(z') \). Waves at \( z \) with \( \varphi > \varphi_c \) have not amplified as much while those with \( \varphi < \varphi_c \) at \( z \) are already damping. Thus, height-resolved backscatter measurements would detect the maximum scatter at the elevation corresponding to \( \varphi_c(z') \) with Doppler shift of \( |2kT_c| \). On the other hand, since by this argument the maximally amplified waves at any altitude should be locally marginally stable, backscatter measurements without height resolution should find the Doppler spectral peak near \( |2kT_c| \) at a variety of radar elevation angles, provided the electrojet has a significant range of \( V_p/V_c \), or \( \varphi_c \), as the linear theory suggests.

In this chapter we propose to investigate this mechanism in detail. For simplicity, Kaw (1972) assumed an altitude profile for the electron polarization drift and had not considered the altitude variation of the collision frequencies, whereas in fact the altitude profile of \( V_p \) is due directly to this variation. The existence of a maximum in the \( V_p/V_c \) profile implies that there will be three characteristic regions of wave propagation; below, near, and above the maximum \( V_p/V_c \). Secondly, it was implicitly assumed that linear theory was adequate, i.e., that the integrated wave growth \( \Gamma \) would not be too large. In view of the observations of backscatter at large elevations which suggest large \( \gamma \), this assumption must be evaluated. Kaw also assumed that marginally stable waves dominate the spectrum at each and every altitude. However,
for backscatter without height resolution, it is unclear, on the basis of linear theory, whether the still-amplifying waves, say near the altitude maximum of $V_p/V_c$, actually should have smaller amplitudes than the marginally stable waves elsewhere, or whether they should dominate, producing Doppler frequency shifts exceeding $|2k_Tc_T|$. 

3.2 Propagation

In this section we shall assume that the polarization field is unmodified by the presence of the turbulent fluctuations and consequently is related by equation (1) to the horizontally uniform driving field only through the collision frequencies, whereupon (7a) reduces to

$$\omega = \frac{cE_D}{B_0} \left( \frac{-k_x v_i/\Omega_i}{(1+\alpha)^2} + \frac{k_z}{(1+\alpha)} \right)$$

As a consequence of (11), we infer that all waves have the same vertical component of group velocity, $\frac{\partial \omega}{\partial k_z} = \frac{cE_D/B_0}{(1+\alpha)}$, which is upward during the day and downward at night. If the electrojet is homogeneous in the east-west direction, $k_x$ will be conserved as the waves propagate. From the condition that in WKB approximation $\omega$ is also conserved, we may find the rate of change of $k_z$ with $z$

$$\frac{dk_z}{dz} = \frac{1}{(1+\alpha)\Omega_i} \left( \frac{v_i}{k_x} \frac{(3\alpha-1)}{1+\alpha} - 2ak_z \right)$$
where we have used \( \left( \frac{v_i}{v_0} \right) = \left( \frac{v_i}{v_0} \right) e^\frac{-z-z_0}{H}, \alpha = \alpha_0 e^\frac{-2(z-z_0)}{H} \) where 
\( z_0 = 100 \text{ km} \). Since \( \nu_i/\Omega_i > 1 \), the first term of (11) dominates except in a narrow altitude region near \( z = 102.5 \text{ km} \) where \( \alpha = 1/3 \), the same point where \( \nu_p/\nu_c \) maximizes. Thus, there are three distinct regions of differing wave refraction properties: where 
\( \alpha > 1/3, (z < 102.5 \text{ km}) \) waves antiparallel to the electron drift, 
\( k_x > 0, \) increase \( k_z, \) and waves parallel to the electron drift decrease \( k \). Conversely, where \( \alpha < 1/3, (z > 102.5 \text{ km}) \), the refraction properties are reversed. Finally, in a narrow altitude region near \( z = 102.5 \text{ km} \), refraction is very weak, but is always towards the horizontal. Since the group velocity reverses at night, the above refraction regions also reverse roles at night.

Equation (12) may be rewritten in terms of the propagation angle \( \varphi \) by defining \( \tan \varphi = -\frac{k_z}{k_x} \), whereupon

\[
\frac{d\varphi}{dz} = -\frac{\cos^2 \varphi \left[ 2\alpha \tan \varphi + \frac{v_i(3\alpha - 1)}{\Omega_i (1+\alpha)} \right]}{(1+\alpha)H (2\alpha \tan \varphi + \frac{v_i(3\alpha - 1)}{\Omega_i (1+\alpha)})} \tag{13}
\]

In Figure (5) we have plotted the daytime \( \frac{d\varphi}{dz} \) against \( \varphi \) at \( z = 102.8 \text{ km} \), where waves parallel to the electron drift increase \( \varphi \); at \( z = 102.5 \text{ km} \), where refraction is towards the horizontal; and at \( z = 102.2 \text{ km} \), where waves rapidly decrease \( \varphi \). Equation (13) indicates that the maximum angular refraction rate occurs for horizontally propagating waves \( \varphi = 0 \) except near \( z = 102.5 \text{ km} \). This daytime maximum rate is plotted as a function of altitude in Figure (6); the complicated structure near \( z = 102.5 \text{ km} \) has been omitted from this figure.
3.3 Amplification

The amplitude of a given wave mode at a given altitude \( z \) is determined by its net amplification after generation at a lower altitude \( z' \)

\[
N_k(z) = N_k(z') e^{\int_{z}^{z'} \Gamma_k(z) \, dz}
\]

where

\[
\Gamma_k = \int_{z}^{z'} \frac{\gamma_k}{\omega/\partial k_z} \, dz
\]

Equation (15) can be integrated analytically since the altitude dependence of \( k_z \) may be found from (11), assuming \( \omega \) and \( k_x \) constant, \( \partial \omega/\partial k_z \) may also be derived from (11), and \( \gamma_k \) is given in terms of \( k_z \) and the collision frequencies, which are simple exponential functions of altitude, from (7b) together with (1).

We start our computations with a wave of a given \( \phi \) at altitude \( z \). As we follow the wave backwards over its propagation path, its wave vector will rotate monotonically until it arrives back at marginal stability at altitude \( z' \) where \( \phi = \phi_c \). We assume the wave to be generated by thermal fluctuations at that point, and plot the net gain \( \Gamma_k(z, z') \) between these two points. The angular dependence of \( e^{\Gamma_k} \) would then give a rough indication of the angular spectrum predicted by linear theory. Figure (7) shows computations of the angular dependence of \( \Gamma_k \) for a variety
of altitudes in the unstable electrojet assuming $E_D = 3/4 \text{ mV/m}$. For $z < 102.5 \text{ km}$, $\Gamma_k$ maximizes for negative $\varphi$ near $\varphi_c$, corresponding to the fact that waves in the direction of the electron drift rotate their wave vectors downward in this region. Above 102.5 km, the reverse is true. Near 102.5, the spectrum maximizes at $\varphi = 0^\circ$, corresponding to the weak refraction expected in this region. The amplifications for waves antiparallel to the electron drift are entirely analogous.

Several points must be made concerning Figure (7). First, we computed the amplification only for $z - z' < 1 \text{ km}$. Second, some waves at large $\varphi$ had rotated their wave vector into a damping region of $k$ space before arriving at observation point $z$. Those that in so doing suffered a net damping greater than 15 are indicated by dashed curves. Third, $\Gamma_k$ was computed for a wavelength of three meters at the observation point $z$; the results may easily be rescaled to other wavelengths by noting that $\gamma - k^2$.

Figure (8) investigates the degree to which the angular spectrum becomes more sharply peaked near $\varphi = \varphi_c$ with increasing amplitude. Shown are the relative growths at $z = 100$ and 106 km, integrated over successively longer propagation paths. As expected, with increasing propagation path, refraction is more pronounced, and the angular spectrum is more sharply peaked.

Above and below the 102 - 103 km altitude region, the wave vectors refract rapidly and so rotate through the unstable angular cone as they propagate. For example, a wave and $\varphi = \varphi_c$ at 106 km cannot have been unstable at 102.5 km,
linear growth at 102.5 km. On the other hand, waves at 102.5 km refract very slowly and receive very large growths. The termination of our calculations at $\Delta z = 1$ km was most stringent for these modes. However, nonlinear effects are likely to invalidate any conclusions we might draw from the larger growths obtained by extending $\Delta z$. In conclusion, we believe the termination by refraction of an already smaller amplification to be the reason why the peaked spectra for $z \neq 102 - 103$ km have smaller amplitudes than the more isotropic spectrum at the amplification maximum and refraction minimum at $z = 102 - 103$ km.

3.4 Discussion and Summary

From Sections 3.1-3.3, we may abstract the following conclusions:

(1) In altitude regions where $\frac{d\phi}{dz}$ is large, the maximum amplification is for waves near $\varphi = \varphi_c$, as proposed by Kaw (1972). Waves with $\varphi > \varphi_c$ must suffer significant damping, and so the spectrum peaks at $\varphi = \varphi_c$.

(2) From this argument and Figure (6) we may conclude that a backscatter radar looking eastward will detect from these altitudes during the day waves from $z < 102.5$, whereas if it is pointed westward it will detect waves from $z > 102.5$, provided that the altitude profile of the polarization electric field is given by linear theory.

(3) However, the largest amplification of all occurs for the symmetric spectrum near 102 - 103 km altitude, where refraction
is very weak. The net amplification at \( z = 102 - 103 \) km for all propagation angles there exceeds the maximum amplification at any angle at essentially all other altitudes. Thus, if this model were correct, waves from 102.5 km would dominate the backscatter return for experiments without altitude resolution. Since \( \cos \theta / \cos \phi_c \) is largest at this altitude, Doppler shifts in excess of \( 2k_T c_T \) would be observed. Thus, we conclude that linear theory alone does not support Kaw's (1972) hypothesis.

The magnitude of the driving field \( E_D \) indicated by Figure (3) indicates that the net linear amplifications are so large, \( \Gamma_k = 10^2 - 10^3 \), that nonlinear effects must certainly be operative. It is interesting to note that extremely large amplifications take place over small altitude intervals, \( \Delta z < 1 \) km, which implies that any saturation mechanism which reduces \( \Gamma_k \) may be considered as acting locally. This paper suggests that we must look to the combination of nonlinear saturation and propagation for an explanation of the constant Doppler shift.

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Figure 1. Coordinate System

$E_S$ is the total static electric field. $E_S = E_p e_z + E_D e_x$.

$E_D$ is the horizontal driving electric field due to atmospheric tidal motions. $E_p$ is the polarization electric field given by equation (1). $k_\parallel$ is the propagation vector. Since $k_{\parallel}$ is assumed zero $k = k_{\perp}$.

Figure 2. Collision Frequencies

Shown here is the altitude profile of the ion-neutral and electron-neutral collision frequencies, $\nu_i$ and $\nu_e$ respectively, for the electrojet model used in this paper. We also plot the parameter $\alpha$, which appears in the linear instability theory.

Figure 3. Electron Drift Altitude Profiles

On the left is plotted the ratio of the electron polarization drift to the ion acoustic speed $c_T$ for two driving fields, $V'_p$ corresponding to $E_D = 0.75 \text{ mV/m}$ and $V'_p$ to 1 mV/m. The solid line is the ratio of the critical drift $V_c = c_T(1+\alpha)$ to $c_T$. The same information is plotted on the right but normalized to the critical drift $c_T(1+\alpha)$. The maximum ratio of electron to critical drift occurs at $z = 102.5 \text{ km}$ in this model. At this altitude, the wave should have the largest growth rates, the widest unstable angular spectrum, and the shortest wavelengths. Furthermore, since the vertical gradients of the electron drift divided by $1+\alpha$ are smallest there, wave refractions should be the weakest.

Figure 4. Diurnal Variation of Driving Field

Shown here is the diurnal variation, for a given day, of the equivalent driving field $E_D$ derived from electrojet magnetic field observations by Schieldge et al. (1972c). We see that for a good portion of the day and night $|E_D| > 1 \text{ mV/m}$, which according to Figure 3, implies $|V'_D V_c| > 1.5$ and $\gamma_e > 45^\circ$ over a considerable altitude region.
Figure 5. Angular Refraction Rates Near the Critical Altitude, z = 102.5 km

Shown here are the angular refraction rates for waves propagating with the electron drift near z = 102.5 km, plotted from equation (12) and Figure (2). Precisely at z = 102.5 km the refraction rates are small. Furthermore, waves propagating upwards through z = 102.5 near φ = 0 will be refracted towards the horizontal, since dφ/dz > 0 for φ < 0 and dφ/dz < 0 for φ > 0. The character of the region changes within ±0.3 km. At z = 102.8 km the wave normal vectors are rapidly rotated upwards, and at z = 102.2 km, they are rotated downwards. The waves propagating antiparallel to the electron drift have the sense of their wave normal vector rotations precisely reversed.

Figure 6. Altitude Dependence of Wave Vector Rotation Rate

Shown here is the maximum refraction rate at φ = 0 for waves propagating along the electron drift as a function of altitude. The complicated dependence near z = 102.5 km has been omitted. Above z = 102.5 km the maximum refraction rate is roughly constant, +10°/km. Below 102.5 km the refraction rate is larger, altitude dependent, and negative. The refraction rate for all other φ at a given altitude scales as cos φ. The refraction rate changes sign for waves propagating opposite to the electron drift.

Figure 7. Integrated Convective Growth

Shown here is the integrated growth of 3 meter waves propagating with the electrons at various altitudes between 99 and 107 km for $E_0 = 3/4$ mV/m. The computations were started at a
given \( \varphi \) and altitude \( z \), and were carried backwards in time until marginal stability was reached, or \( \Delta z = 1 \) km, whichever happened first. \( e^{-r(\varphi)} \) is some measure of the angular spectrum predicted by linear theory. Below 102 km, where the wave normal vector is rotated downward, the amplitude maximum is displaced to negative angles. The angular displacement is larger at 101 km than at 99 km, corresponding to the fact that \( |\varphi_c| \) is larger at 101 km than at 99 km. Precisely the reverse situation holds above 103 km, as expected. At \( z = 102 - 103 \) km, where refraction is weak and toward the horizontal, the spectrum is symmetrical and peaked near \( \varphi = 0 \). And furthermore, its amplitude at nearly all angles exceeds the peak amplitudes elsewhere. Consequently this region should dominate backscatter measurements without height resolution, and a constant Doppler shift would not be expected.

The dotted lines indicate that these waves were damped with \( |\Gamma| > 15 \) before reaching the computation altitude.

**Figure 8.** Amplification over Successively Longer Propagation Paths

Here the integrations were cut off at \( \Delta z = 1 \) km and 0.25 km, at \( z = 100 \) km, left inset, and at \( \Delta z = 1 \) km, 0.5 km, and 0.25 km at \( z = 106 \) km, right inset. As expected, the angular spectrum becomes increasingly skewed with increasing propagation path, and consequently refraction. The net amplifications of Figures 6 and 7 are so large that nonlinear effects must be expected.
Figure 1
Figure 2
Figure 3

\[ \phi_c = \cos^{-1} \frac{V_c}{V_p} \]
Rate of angular change

$\frac{\partial \phi}{\partial z}$ in deg/km

Figure 5
Figure 6
Figure 7
Figure 8
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