NASA CR-114524
ASRL TR 166-2
Available to the Public.

THE NONLINEAR INSTABILITY IN FLAP-LAG OF ROTOR BLADES IN FORWARD FLIGHT

Pin Tong

AEROELASTIC AND STRUCTURES RESEARCH LABORATORY
DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139

October 1971

Prepared for

AMES DIRECTORATE
U.S. ARMY AIR MOBILITY RESEARCH AND DEVELOPMENT LABORATORY
AMES RESEARCH CENTER
MOFFETT FIELD, CALIFORNIA

Contract No. NAS2-6175

67

1. Report No. NASA CR -114524	2. Government Accession No.	3. Recipient's Catalog No.
4. Title and Subtitle THE NONLINEAR INSTABILITY IN FLAP-LAG OF ROTOR BLADES IN FORWARD FLIGHT		5. Report Date October 1971 6. Performing Organization Code
7. Author(s) Pin Tong		8. Performing Organization Report No. ASRL TR 166-2
9. Performing Organization Name and Address Aeroelastic and Structures Research Laboratory Department of Aeronautics and Astronautics Massachusetts Institute of Technology Cambridge, Massachusetts 02139		10. Work Unit No. 11. Contract or Grant No. NAS2-6175
12. Sponsoring Agency Name and Address Ames Directorate, U.S. Army Air Mobility Research and Development Laboratory Ames Research Center, Moffett Field, California 94035		13. Type of Report and Period Covered Contractor Report 14. Sponsoring Agency Code

15. Supplementary Notes
Technical Monitor: Dr. R.A. Ormiston,
Ames Directorate, U.S. Army Air Mobility Research and Development Laboratory,
Ames Research Center, Moffett Field, California 94035

16. Abstract
The nonlinear flap-lag coupled oscillation of torsionally rigid rotor blades in forward flight is examined using a set of consistently derived equations by the asymptotic expansion procedure of multiple time scales. The regions of stability and limit cycle oscillation are presented. The roles of parametric excitation, nonlinear oscillation, and forced excitation played in the response of the blade are determined.

17. Key Words (Suggested by Author(s))		18. Distribution Statement
Helicopter Aeroelasticity Rotor Blade Dynamics Nonlinear Structural Dynamics Nonlinear, Blade Stability Rotor Blade Flutter Nonlinear Flutter	Dynamic Stability Perturbation Methods Multiple Time Scaling	Unclassified, Unlimited

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price*
Unclassified	Unclassified	66	~~$3.00~~

* For sale by the National Technical Information Service, Springfield, Virginia 22151

NASA-C-168 (Rev. 6-71)

FOREWORD

This research was conducted by the Aeroelastic and Structures Research Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts under Contract No. NAS2-6175, supervised by Dr. R.A. Ormiston of the Ames Directorate, U.S. Army Air Mobility Research and Development Laboratory, Ames Research Center, Moffett Field, California.

The author wishes to express his appreciation to Dr. R.A. Ormiston for his useful comments and suggestions.

The author is deeply indebted to Professors R.H. Miller, N.D. Ham, T.H.H. Pian, and E.A. Witmer, and to Dr. Peretz Friedmann for their helpful advice and constructive criticism in the course of this study.

The digital computations were carried out at the MIT Information Processing Center.

ABSTRACT

The nonlinear flap-lag coupled oscillation of torsionally rigid rotor blades in forward flight is examined using a set of consistently derived equations by the asymptotic expansion procedure of multiple time scales. The regions of stability and limit cycle oscillation are presented. The roles of parametric excitation, nonlinear oscillation, and forced excitation played in the response of the blade are determined.

CONTENTS

LIST OF ILLUSTRATIONS

I. Introduction

The problem of the coupling between nonlinear oscillation, parametric excitation, and forced oscillation is of great interest in rotor dynamics. The present paper will restrict consideration to the flap-lag motion. Few authors have considered the nonlinear oscillation of a torsionally rigid blade. Young (Ref. 1) has used rough approximations to derive the solution of a set of nonlinear equations. Hohenemser and Heaton (Ref.2) derived the nonlinear equations of motion in forward flight approximately and solved the equations by numerical integration. The present author (Ref. 3) used the equations derived by Hohenemser and applied the asymptotic expansion technique to establish the limit cycle oscillation amplitude and clarified the role played by nonlinear oscillation, parametric excitation, and forced oscillation of the rotor blades. In the present paper, the nonlinear motion of flap-lag in forward flight are carefully rederived, which is extended from the linear flap-lag equations in hover derived by Ormiston and Hodge (Ref. 4). A technique similar to that of Ref. 3 is applied to treat these new equations. Extensive results on the forced response, stability boundary, and limit cycle amplitude will be presented.

II. Formulation

In deriving the equation of motion, we shall make the following assumptions: (1) the pitch angle, θ, the static equilibrium coning angle β_o, and the static equilibrium lag angle ζ_o are small, so that θ^2, β_o^2 and ζ_o^2 can be neglected as compared to one; (2) the effective angle of attack is small and the two-dimensional quasi-

*All of the symbols used are defined in Appendices A and B.

steady aerodynamic theory can be used; (3) the ratio of the profile drag coefficient to the lift slope is small and can be neglected compared with unity; and (4) nonlinear terms up to the third order will be retained. However, since the aerodynamic damping is much higher in flap than that in lag, the perturbed motion in lag is much larger than that in flap; thus, it is expected that the nonlinear terms of the perturbed lag motion ζ will be more important than the perturbed flap motion β. It is also expected that the nonlinear terms are more important in the lag equation than in the flap equation.* Thus, the nonlinear terms such as $\beta\zeta$, $\beta\zeta^2$, $\theta\zeta^2$ and their time derivatives will be retained in the flap equation while β^2, $\beta^2\zeta$, $\theta\beta\zeta$ etc. will be retained in the lag equations.

From the above assumptions, it has been shown (Appendix A) that the equations of motion in flap and in lag are, respectively:

$$
\begin{aligned}
&\ddot{\beta} + g_\beta \dot{\beta} + (1+\Omega_\beta^2)\beta + \Omega_{\beta\zeta}\zeta - \underline{X}\dot{\zeta} \\
&= -2\beta\dot{\zeta} - (\beta_0+\beta-\tfrac{\gamma}{8}\theta)\dot{\zeta}^2 - \tfrac{\gamma}{8}\dot{\beta}\dot{\zeta} \\
&\quad + \tfrac{\gamma}{8}\mu\left[(\tfrac{8}{3}\theta - c)\sin\psi - \tfrac{4}{3}\beta_0\cos\psi\right] \\
&\quad + \tfrac{\gamma}{8}\mu\left\{(\tfrac{8}{3}\theta\dot{\zeta} + \tfrac{4}{3}\beta_0\zeta - \tfrac{4}{3}\dot{\beta})\sin\psi + \left[(\tfrac{8}{3}\theta - c)\zeta - \tfrac{4}{3}\beta - \tfrac{4}{3}\beta_0\dot{\zeta}\right]\cos\psi\right\} \\
&\quad + \tfrac{\gamma}{8}\mu\left[\tfrac{4}{3}\beta\zeta\sin\psi + (\tfrac{8}{3}\theta\dot{\zeta}\zeta - \tfrac{4}{3}\beta\dot{\zeta} - \tfrac{4}{3}\dot{\beta}\zeta)\cos\psi\right] \\
&\quad + \tfrac{\gamma}{8}\mu^2(2\theta\sin^2\psi - \beta_0\sin 2\psi) \\
&\quad + \tfrac{\gamma}{8}\mu^2(2\theta\zeta - \beta)\sin 2\psi
\end{aligned}
\qquad (1)
$$

*This is indicated by the results of ASRL 166-1 (Ref. 3).

$$\ddot{\zeta} + g_\zeta \dot{\zeta} + \Omega_\zeta^2 \zeta + \Omega_{\beta\zeta} \beta - Y \dot{\beta}$$

$$= 2\beta\dot{\beta} + \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\dot{\beta}\dot{\zeta} + \frac{\gamma}{8}\dot{\beta}^2 + 2\beta\dot{\beta}\dot{\zeta}$$

$$+\frac{\gamma}{8}\mu\left[-\left(\frac{8}{3}\frac{C_{d_0}}{a} + C\theta\right)\sin\psi - \left(\frac{4}{3}\beta_0\theta - 2\beta_0 C\right)\cos\psi\right] + \frac{\gamma}{8}\mu\left\{\left[\left(\frac{4}{3}\theta - 2C\right)\beta_0\zeta\right.\right.$$

$$\left.\left. - \frac{8}{3}\frac{C_{d_0}}{a}\dot{\zeta} - \frac{4}{3}\theta\dot{\beta}\right]\sin\psi + \left[\left(2C - \frac{4}{3}\theta\right)\beta + \frac{4}{3}\beta_0\left(2\dot{\beta} - \theta\dot{\zeta}\right) - C\theta\zeta\right]\cos\psi\right\}$$

$$+ \frac{\gamma}{8}\mu\left\{\left[\left(\frac{4}{3}\theta - 2C\right)\beta\zeta + \frac{4}{3}\beta_0\zeta\left(2\dot{\beta} + \theta\dot{\zeta}\right)\right]\sin\psi\right.$$

$$\left. + \left[\frac{8}{3}\beta\dot{\beta} - \frac{4}{3}\theta\left(\beta\dot{\zeta} + \dot{\beta}\zeta\right)\right]\cos\psi\right\} \qquad (2)$$

$$+ \frac{\gamma}{8}\mu^2\left[2\beta_0^2\cos^2\psi - \beta_0\theta\sin 2\psi - 2\frac{C_{d_0}}{a}\sin^2\psi\right]$$

$$+ \frac{\gamma}{8}\mu^2\left\{4\beta_0\beta\cos^2\psi - \left[2\left(\beta_0^2 + \frac{C_{d_0}}{a}\right)\zeta + \theta\beta\right]\sin 2\psi\right\}$$

where $(\dot{\ }) = \frac{d}{d\psi}$, $\psi = \Omega t$, γ is the Lock number, μ is the advance ratio and the rest of the parameters are defined in Eqs. (A.9), (A.22), (A.23), and (A.24).

The critical condition of the system is determined by the existence of a small amplitude undamped oscillation for the linearized version of Eqs. (1) and (2). Namely,

$$\begin{aligned} &\ddot{\beta} + g_\beta \dot{\beta} + (1+\Omega_\beta^2)\beta + \Omega_{\beta\zeta}\zeta - Z\dot{\zeta} = 0 \\ &\ddot{\zeta} + g_\zeta \dot{\zeta} + \Omega_\zeta^2 \zeta + \Omega_{\beta\zeta}\beta - Y\dot{\beta} = 0 \end{aligned} \qquad (3)$$

Let

$$\beta, \zeta \sim e^{ip\psi}$$

in (3), we have the characteristic equation for p

$$\left(1+\Omega_\beta^2 - p^2 + i\,p\,g_\beta\right)\left(\Omega_\zeta^2 - p^2 + i p g_\zeta\right) - \left(\Omega_{\beta\zeta} - i p X\right)\left(\Omega_{\beta\zeta} - i p Y\right) = 0 \qquad (4)$$

where $i = \sqrt{-1}$. The critical condition corresponds to $p = \Omega_F$ which is called the flutter frequency. Substituting into (4) and using subscript "c" to denote the critical condition for the parameter, one obtains

$$\Omega_F^2 = \left[g_{\beta c}\Omega_\zeta^2 + g_{\zeta c}(1+\Omega_\beta^2) + \Omega_{\beta\zeta c}(X_c + Y_c)\right] / \left(g_{\beta c} + g_{\zeta c}\right) \qquad (5)$$

$$\left(X_c Y_c - g_{\beta c} g_{\zeta c}\right)\Omega_F^2 = \Omega_{\beta\zeta c}^2 + \frac{\left[g_{\beta c}(1+\Omega_\beta^2 - \Omega_\zeta^2) - (X_c+Y_c)\Omega_{\beta\zeta c}\right]\left[g_{\zeta c}(1+\Omega_\beta^2 - \Omega_\zeta^2) + (X_c+Y_c)\Omega_{\beta\zeta c}\right]}{\left[g_{\beta c} + g_{\zeta c}\right]^2} \qquad (6)$$

If all the physical parameters such as Ω_β, Ω_ζ, $\Omega_{\beta\zeta}^{*}$, η_β, and η_ζ are given, one can determine the critical pitch angle, say θ_c from (5) and (6). It can be shown that for $\theta<\theta_c$ all the small amplitude motion will be damped and for $\theta>\theta_c$ there exists a branch of solution of (3), which will grow exponentially.

Ormiston and Hodge, Ref. 4, have shown that when η_β, η_ζ, e, and $\Omega_{\zeta\beta} = 0$, it is necessary to have $0<\Omega_\beta^2<1$ for the existence of θ_c. Indeed in this case, it can be shown that

$$(\theta_c - A_c)^2 \geq \frac{1+\Omega_\beta^2}{2\,\Omega_\beta^2\,(1-\Omega_\beta^2)} \qquad (7)$$

---

* $\Omega_{\beta\zeta}$ is the elastic coupling effect, no expression for it is derived at present. It will be treated in a forthcoming report. It was shown in Ref. 4 to have an important influence on flap-lag instability.

We define the differential operators L and M, the functions f's and h's, and various quantities α's, β_{ij}, ζ_{ij}, $\underset{\sim}{N}$, P's and κ's in Appendix B. All these quantities are to be used in the following sections for the perturbation solutions.

III. Perturbation Solution below the Critical Condition

If the angle of the pitch is less than θ_c, for a given set of physical parameters, all the solution of (3) are damped. A perturbation solution of (1) and (2) can be established by assuming

$$\begin{aligned} \beta &= \mu\beta_1 + \mu^2\beta_2 + \cdots \\ \zeta &= \mu\zeta_1 + \mu^2\zeta_2 + \cdots \end{aligned} \tag{8}$$

Substituting (8) into (1) and (2), requiring all terms of the same order in μ satisfying (1) and (2) separately, we obtain

$O(\mu)$

$$\begin{aligned} L(\beta_1,\zeta_1) &= f_1(\psi,\theta) \\ M(\beta_1,\zeta_1) &= h_1(\psi,\theta) \end{aligned} \tag{9}$$

$O(\mu^2)$

$$\begin{aligned} L(\beta_1,\zeta_1) &= f_2(\psi,\theta) + f_3(\psi,\theta;\beta_1,\zeta_1) + f_4(\beta_1,\zeta_1) \\ M(\beta_1,\zeta_1) &= h_2(\psi,\theta) + h_3(\psi,\theta;\beta_1,\zeta_1) + h_4(\beta_1,\zeta_1) \end{aligned} \tag{10}$$

- - - - - - - -

where L, M, f's and h's are defined by (B.1) through (B.17) in Appendix B.

Since the homogeneous solutions in (9), (10), etc. are damped (below critical condition), we only have to consider the particular solutions. Using (B.4) and (B.5), we may write (9) in the form

$$
\begin{aligned}
L(\beta_1, \zeta_1) &= P_7 e^{i\psi} + P_7^* e^{-i\psi} \\
M(\beta_1, \zeta_1) &= P_8 e^{i\psi} + P_8^* e^{-i\psi},
\end{aligned}
\tag{11}
$$

where P_7 and P_8 are defined in (B.34) and (B.35). $(\ )^*$ denotes the complex conjugate of the corresponding quantities. The particular solution for β_1 and ζ_1 is simply

$$
\begin{pmatrix} \beta_1 \\ \zeta_1 \end{pmatrix} = \begin{pmatrix} \beta_{12} \\ \zeta_{12} \end{pmatrix} e^{i\psi} + \begin{pmatrix} \beta_{12}^* \\ \zeta_{12}^* \end{pmatrix} e^{-i\psi}
\tag{12}
$$

where β_{12} and ζ_{12} are defined in (B.36). Substituting (12) into (10) and using (B.5) through (B.10), we have

$$
\begin{aligned}
L(\beta_2, \zeta_2) &= P_{13} e^{2i\psi} + P_{13}^* e^{-2i\psi} + P_{15} \\
M(\beta_2, \zeta_2) &= P_{14} e^{2i\psi} + P_{14}^* e^{-2i\psi} + P_{16}
\end{aligned}
\tag{13}
$$

and the particular solution for β_2 and ζ_2 is

$$
\begin{pmatrix} \beta_2 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \beta_{25} \\ \zeta_{25} \end{pmatrix} e^{2i\psi} + \begin{pmatrix} \beta_{25}^* \\ \zeta_{25}^* \end{pmatrix} e^{-2i\psi} + \begin{pmatrix} \beta_{26} \\ \zeta_{26} \end{pmatrix}
\tag{14}
$$

where $P_{13}, \ldots P_{16}$, β_{25}, ζ_{25}, β_{26}, and ζ_{26} are defined in (B.41) through (B.48) in which $\mathcal{X}_0, \wp_0, \mathcal{X}$ and $\wp$ are set to 1.

By a similar procedure one can obtain the solutions for higher order expansions.

IV. Perturbation Solution in the Neighborhood of the Critical Condition

In the previous section, we only seek the particular solution of β and ζ. If the pitch angle is larger than θ_c, at least one branch of the homogeneous solution of

$$L(\beta,\zeta)=0$$
$$M(\beta,\zeta)=0$$

is undamped. This undamped solution must be included in the solution of β_1, ζ_1 etc. In the neighborhood of θ_c, the value of β_{12}, ζ_{12}, β_{25}, etc. becomes very large, i.e., the solution in (12) or (14) etc. are large (see B.36 and B.45 through B.48), the expansion in (8) will break down. Under these circumstances, we shall establish a different expansion by perturbing the system near the critical condition. Introduce a small parameter ε and let

$$\begin{aligned} \beta &= \varepsilon^{\frac{1}{2}}\beta_1 + \varepsilon\beta_2 + \varepsilon^{3/2}\beta_3 + \cdots \\ \zeta &= \varepsilon^{\frac{1}{2}}\zeta_1 + \varepsilon\zeta_2 + \varepsilon^{3/2}\zeta_3 + \cdots \end{aligned} \tag{15}$$

for the dependent variables and let

$$\begin{aligned} \theta &= \theta_c + \varepsilon\theta_2 + \cdots \\ g_\beta &= g_{\beta c} + \frac{\partial g_\beta}{\partial\theta}\varepsilon\theta_2 + \cdots = g_{\beta c} + \varepsilon\theta_2 g_{\beta 2} + \cdots \\ g_\zeta &= g_{\zeta c} + \varepsilon\theta_2\frac{\partial g_\zeta}{\partial\theta} + \cdots = g_{\zeta c} + \varepsilon\theta_2 g_{\zeta 2} + \cdots \\ \Omega_{\beta\zeta} &= \Omega_{\beta\zeta c} + \varepsilon\theta_2\frac{\partial\Omega_{\beta\zeta}}{\partial\theta} + \cdots = \Omega_{\beta\zeta c} + \varepsilon\theta_2\,\Omega_{\beta\zeta 2} + \cdots \end{aligned} \tag{16}$$

The subscript "c" refers to the value at the critical condition. We also introduce different time scales.

$$\tau_m = \varepsilon^m \psi \quad ; \qquad m = 0, 1, 2 \cdots \tag{17}$$

then

$$\frac{\partial}{\partial \psi} = \frac{\partial}{\partial \tau_0} + \varepsilon \frac{\partial}{\partial \tau_2} + \cdots$$

$$\frac{\partial^2}{\partial \psi^2} = \frac{\partial^2}{\partial \tau_0^2} + 2\varepsilon \frac{\partial^2}{\partial \tau_2 \partial \tau_0} + \cdots$$

For different values of Ω_F, different order of magnitude of parametric excitations and forcing functions, i.e., different order of μ, μf_1, μh_1, $\mu^2 h_2$ and $\mu^2 h_2$, are permitted in the expansion series of (15). We write

$$\begin{aligned}
\mu f_1(\tau_0,\theta) &= \mu \mathcal{X} f_1'(\tau_0,\theta) \\
\mu h_1(\tau_0,\theta) &= \mu \mathcal{X} h_1'(\tau_0,\theta) \\
\mu^2 f_2(\tau_0,\theta) &= \mu^2 \varphi f_2'(\tau_0,\theta) \\
\mu^2 h_2(\tau_0,\theta) &= \mu^2 \varphi h_2'(\tau_0,\theta)
\end{aligned} \tag{18}$$

Assuming f_1', h_1', f_2' and h_2' are functions at most of order one, we expand μ, $\mathcal{X}$ and φ in the following forms:

$$\begin{aligned}
\mu &= \mu_1 \varepsilon^{\frac{1}{2}} + \mu_2 \varepsilon + \cdots \\
\mathcal{X} &= \mathcal{X}_0 + \mathcal{X}_1 \varepsilon^{1/2} + \mathcal{X}_2 \varepsilon + \cdots \\
\varphi &= \varphi_0 + \varphi_1 \varepsilon^{1/2} + \varphi_2 \varepsilon + \cdots
\end{aligned} \tag{19}$$

The constraint in the choice of the value of the coefficients in (19) is to make the expansion series (15) a periodic solution in the fast time variable τ_0. Since we shall only consider the solution up to order $\varepsilon^{3/2}$, we may, in general, have only one of the coefficients as nonzero. In particular, within the above mentioned constraint, only the leading coefficient is chosen to be nonzero.

Substituting (15) through (19) into (1) and (2), requiring terms of the same order in ε satisfying (1) and (2) separately, we will obtain the differential equations for the determination of β_1, ζ_1, etc. The equations can be expressed in the following form:

$O(\varepsilon^{1/2})$

$$\begin{aligned} L(\beta_1,\zeta_1) &= \mu_1 \varkappa_0 f_1'(\tau_0,\theta) \\ M(\beta_1,\zeta_1) &= \mu_1 \varkappa_0 h_1'(\tau_0,\theta) \end{aligned} \tag{20}$$

$O(\varepsilon)$

$$\begin{aligned} L(\beta_2,\zeta_2) &= (\mu_1\varkappa_1+\mu_2\varkappa_0) f_1'(\tau_0,\theta) + \mu_1^2 \varphi_0 f_2'(\tau_0,\theta) + \mu_1 f_3(\tau_0,\theta;\beta_1,\zeta_1) + f_4(\beta_1,\zeta_1) \\ M(\beta_2,\zeta_2) &= (\mu_1\varkappa_1+\mu_2\varkappa_0) h_1'(\tau_0,\theta) + \mu_1^2 \varphi_0 h_2'(\tau_0,\theta) + \mu_1 h_3(\tau_0,\theta;\beta_1,\zeta_1) \\ &\quad + h_4(\beta_1,\zeta_1) \end{aligned} \tag{21}$$

$O(\varepsilon^{3/2})$

$$\begin{aligned} L(\beta_3,\zeta_3) &= \mu_1\varkappa_2 f_1'(\tau_0,\theta) + \mu_1^2 \varphi_1 f_2'(\tau_0,\theta) + \mu_2 f_3(\tau_0,\theta;\beta_1,\zeta_1) + \mu_1 f_3(\tau_0,\theta;\beta_2,\zeta_2) \\ &\quad + \mu_1 f_5(\tau_0,\theta;\beta_1,\zeta_1) + \mu_1^2 f_6(\tau_0,\theta;\beta_1,\zeta_1) + f_7(\beta_1,\zeta_1;\beta_2,\zeta_2) + \cdots \\ M(\beta_3,\zeta_3) &= \mu_1\varkappa_2 h_1'(\tau_0,\theta) + \mu_1^2 \varphi_1 h_2'(\tau_0,\theta) + \mu_2 h_3(\tau_0,\theta;\beta_1,\zeta_1) + \mu_1 h_3(\tau_0,\theta;\beta_2,\zeta_2) \\ &\quad + \mu_1 h_5(\tau_0,\theta;\beta_1,\zeta_1) + \mu_1^2 h_6(\tau_0,\theta;\beta_1,\zeta_1) + h_7(\beta_1,\zeta_1;\beta_2,\zeta_2) + \cdots \end{aligned} \tag{22}$$

- - - - - - -

where L, M, f's and h's are defined in (B.1) through (B.17). In this section, the parameters used in L, M and $\underset{\sim}{N}$ [(B.1), (B.2) and (B.18)] are referred to the value at the critical conditions.

The homogeneous solution of (20) is the same as (3). At the critical condition, there is only one branch of the solution which is undamped. Therefore, we may just include that particular branch in the solution of β_1 and ζ_1 and

$$\begin{pmatrix}\beta_1\\ \zeta_1\end{pmatrix} = 2\,\mathrm{Re}\left[A\begin{pmatrix}u\\ 1\end{pmatrix}e^{i\Omega_F\tau_0} + \mu_1\frac{\varkappa_0}{\varkappa}\begin{pmatrix}\beta_{12}\\ \zeta_{12}\end{pmatrix}e^{i\tau_0}\right]$$

$$\begin{aligned}\begin{pmatrix}\beta_2\\ \zeta_2\end{pmatrix} = 2\,\mathrm{Re}\Bigg[&A^2\begin{pmatrix}\beta_{21}\\ \zeta_{21}\end{pmatrix}e^{2i\Omega_F\tau_0} + \frac{AA^*}{2}\begin{pmatrix}\beta_{22}\\ \zeta_{22}\end{pmatrix} + \mu_1 A^*\begin{pmatrix}\beta_{23}\\ \zeta_{23}\end{pmatrix}e^{i(1-\Omega_F)\tau_0}\\ &+ \mu_1 A\begin{pmatrix}\beta_{24}\\ \zeta_{24}\end{pmatrix}e^{i(1+\Omega_F)\tau_0} + \mu_1^2\begin{pmatrix}\beta_{25}\\ \zeta_{25}\end{pmatrix}e^{2i\tau_0}\\ &+ \frac{\mu_1^2}{2}\begin{pmatrix}\beta_{26}\\ \zeta_{26}\end{pmatrix} + \frac{\mu_1\varkappa_1+\mu_2\varkappa_0}{\varkappa}\begin{pmatrix}\beta_{12}\\ \zeta_{12}\end{pmatrix}e^{i\tau_0}\Bigg]\end{aligned} \tag{23}$$

where β_{ij} and ζ_{ij} are defined in (B.30), (B.31), (B.36), (B.47) and (B.48). In (23), the μ's, $\varkappa$'s and ϱ's are so chosen such that there are no secular terms in β's and ζ's, i.e., the solutions are periodic in the variable τ_0. A substitution of (23) into (22) yields

$$
\begin{aligned}
L(\beta_3,\zeta_3) = &\left[P_1\frac{\partial A}{\partial\tau_2} + (\theta_2 P_3 + \mu_1^2 P_{17})A + P_5 A^2 A^*\right] e^{i\Omega_F\tau_0} \\
&+\mu_1\left[\frac{\varkappa_2}{\varkappa} P_7 + P_{25}\, AA^* + \mu_1^2 P_{27}\right] e^{i\tau_0} \\
&+\mu_2 P_9 A^* e^{i(1-\Omega_F)\tau_0} + \mu_1^2 P_{23} A^* e^{i(2-\Omega_F)\tau_0} \\
&+\mu_1 P_{19} A^{*2} e^{i(1-2\Omega_F)\tau_0} + \mu_1 P_{19}^* A^2 e^{i(2\Omega_F-1)\tau_0} \\
&+\mu_1^3 P_{21} e^{3i\tau_0} + \mu_1^2\frac{\varphi_1}{\varphi} P_{29} e^{2i\tau_0} + \cdots
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
M(\beta_3,\zeta_3) = &\left[P_2\frac{\partial A}{\partial\tau_2} + (\theta_2 P_4 + \mu_1^2 P_{18})A + P_6 A^2 A^*\right] e^{i\Omega_F\tau_0} \\
&+\mu_1\left[\frac{\varkappa_2}{\varkappa} P_8 + P_{26} AA^* + \mu_1 P_{28}\right] e^{i\tau_0} \\
&+\mu_2 P_{10} A^* e^{i(1-\Omega_F)\tau_0} + \mu_1^2 P_{24} A^* e^{i(2-\Omega_F)\tau_0} \\
&+\mu_1 P_{20} A^{*2} e^{i(1-2\Omega_F)\tau_0} + \mu_1 P_{20}^* A^2 e^{i(2\Omega_F-1)\tau_0} \\
&+\mu_1^3 P_{22} e^{3i\tau_0} + \mu_1^2\frac{\varphi_1}{\varphi} P_{30} e^{2i\tau_0} + \cdots
\end{aligned}
$$

In the right hand side of (24), many terms have the same frequencies as Ω_F, i.e., the solution of β_3, ζ_3 will have secular terms, i.e., β_3, $\zeta_3 \sim \tau_0 e^{i\Omega_F\tau_0}$, unless the right hand side is orthogonal to the solution of the adjoint operator

$$\begin{pmatrix} v \\ 1 \end{pmatrix} e^{-i\Omega_F\tau_0}$$

of L and M (Ref. 3), i.e.

$$\int\left[v\,L(\beta_3,\zeta_3) + M(\beta_3,\zeta_3)\right] e^{-i\Omega_F\tau_0}\, d\tau_0 = 0 \tag{25}$$

in which v is defined in (B.63). Equation (25) will provide us the equation for A which can be functions of τ_2, τ_4,...etc.

Consider the following cases:

(1) <u>Hovering $\mu(=\mu_1=\mu_2)=0$.</u> In this case, all the forcing functions and the parametric excitations (terms with coefficient

μ_1 or μ_2 in (20) - (25)] are zero. To suppress the secular terms of β_3 and ζ_3, (25) is equivalent to requiring

$$v\left(P_1\frac{\partial A}{\partial \tau_2} + \theta_2 P_3 A + P_5 A^2 A^*\right) + P_2\frac{\partial A}{\partial \tau_1} + \theta_2 P_4 A + P_6 A^2 A^* = 0$$

or

$$\frac{\partial A}{\partial \tau_2} = \theta_2 \kappa_2 A + \kappa_3 A^2 A^* \tag{26}$$

where v, κ_2, κ_3 are defined in (B.63) through (B.65). Equation (26) is in the same form as that of Eq. (49) in Reference 3. It can be solved easily. The solution is

$$A = \rho e^{i\phi} \tag{27}$$

where $(\ )_R$ and $(\ )_I$ are used to denote the real and the imaginary part,

$$\rho = \rho_0 e^{\kappa_{2R}\theta_2\tau_2} \Big/ \left[1 + \frac{\kappa_{3R}}{\kappa_{2R}\theta_2}\rho_0^2\left(1 - e^{2\kappa_{2R}\theta_2\tau_2}\right)\right]^{\frac{1}{2}}$$
$$\phi = \phi_0 + \kappa_{2I}\theta_2\tau_2 - \frac{\kappa_{3I}}{2\kappa_{3R}} \ln\left[1 + \frac{\kappa_{3R}}{\kappa_{2R}\theta_2}\rho_0^2\left(1 - e^{2\kappa_{2R}\theta_2\tau_2}\right)\right] \tag{28}$$

and ρ_0 and ϕ_0 are the values of ρ and ϕ at $\tau_2=0$. The conclusions concerning the stability of the solution is the same as that in Sec. IVa of Ref. 3; namely, (a) if $\theta_2\kappa_{2R}>0$ (above critical condition) and $\kappa_{3R}>0$, the blade is unstable; (b) if $\theta_2\kappa_{2R}<0$ (below critical condition) and $\kappa_{3R}<0$ the blade is stable; (c) if $\theta_2\kappa_{2R}>0$ and $\kappa_{3R}<0$, there exists a limit cycle oscillation with amplitude being

$$|A|_{\tau_2\to\infty} = \left(-\frac{\kappa_{2R}\theta_2}{\kappa_{3R}}\right)^{\frac{1}{2}} \tag{29}$$

(d) if $\theta_2\kappa_{2R}<0$ and $\kappa_{3R}>0$, there exists an unstable limit cycle. The blade is stable if the disturbance in $|A|$ is smaller than that in (29), and the blade becomes unstable if the disturbance in $|A|$ is larger than that in (29). Let $\theta_2=1$, by (15), (16) and (22), the amplitude of the blade motion is $(\theta-\theta_c)^{1/2}A$.

It should be noted that κ_{2R} is positive for the present problem. Therefore, the origin ($A\tilde{=}0$) is unstable above the critical condition ($\theta_2>0$), and is stable below the critical conditions ($\theta_2<0$).

(2) $\mu\neq 0$, $\Omega_F=1/2+\epsilon\nu$, where ν is of order one or smaller.

In this case β_{23} and ζ_{23} in the second equation of (23) can be very large or even singular, i.e., the solution of β_2 and ζ_2 will have secular terms. This is because the nonlinear coupling of the first order homogeneous solution, β_1 and ζ_1, with the forced response due to f_1 and h_1 and its coupling with the parametric excitation are in resonance with the system. The difficulty can be avoided by requiring

$$\mu_1=0 \qquad (30a)$$

and the other coefficients in (19) may be taken as

$$\begin{aligned} \mathcal{X}_0 &= \varphi_0 = 1 \\ \mathcal{X}_1 &= \mathcal{X}_2 = \varphi_1 = \varphi_2 = 0 \end{aligned} \qquad (30b)$$

i.e.

$$\begin{aligned} \mu &= \mu_2\varepsilon \\ \mathcal{X} &= \varphi = 1 \end{aligned}$$

In (24), terms proportional to $e^{i\Omega_F\tau_0}$ and $e^{i(1-\Omega_F)\tau_0}$ will cause the secular terms in β_3 and τ_3. To suppress it, it is required that (25) be satisfied which is equivalent to requiring

$$\nu\left[P_1\frac{\partial A}{\partial\tau_2}+\theta_2P_3A+\mu_2P_9A^*e^{-2i\nu\tau_2}+P_5A^2A^*\right]$$
$$+P_2\frac{\partial A}{\partial\tau_2}+\theta_2P_4A+\mu_2P_{10}A^*e^{-2i\nu\tau_2}+P_6A^2A^*=0$$

or

$$\frac{\partial A}{\partial\tau_2}=\theta_2\kappa_2A+\mu_2\kappa_5A^*e^{-2i\nu\tau_2}+\kappa_3A^2A^* \tag{31'}$$

changing variable by $\tilde{A}=Ae^{i\nu\tau_2}$, it becomes

$$\frac{\partial\tilde{A}}{\partial\tau_2}=(\theta_2\kappa_2+i\nu)\tilde{A}+\mu_2\kappa_5\tilde{A}^*+\kappa_3\tilde{A}^2\tilde{A}^* \tag{31}$$

where κ's are defined in (B.65). Equation (31) is in the same form as (10) of Ref. 3. Thus its behavior can be investigated in the same manner, that is, if

$$\mu_2<|\kappa_{2I}\theta_2+\nu|/|\kappa_5| \tag{32}$$

the origin, $|A|=0$ is a focal point; if

$$\frac{|\kappa_{2I}\theta_2+\nu|}{|\kappa_5|}<\mu_2<\frac{|\kappa_2\theta_2+i\nu|}{|\kappa_5|} \tag{33}$$

the origin is a node; if

$$\frac{|\kappa_2\theta_2+i\nu|}{|\kappa_5|}<\mu_2 \tag{34}$$

the origin is a saddle point. Since (31) is homogeneous, if

$$\mathrm{Re}\left[\kappa_{2R}\theta_2+\sqrt{\mu_2^2|\kappa_5|^2-(\kappa_{2I}\theta_2+\nu)^2}\right]<0$$

the solution near the origin (A=0) is stable. In this case, if $\kappa_{3R}<0$, there will be no large amplitude flutter response, i.e. the blade will simply respond to the forcing function (no flutter mode) due to advancing flight. If $\theta_2\kappa_{2R}\kappa_{3R} > 0$ no close trajectory for A can be formed on the phase plane. In this case, if $\kappa_{3R}<0$ $(\theta_2\kappa_{2R}<0)$, the solution is expected to converge to one of the singular points of (31); if $\kappa_{3R}>0$ $(\theta_2\kappa_{2R}>0)$, the solution is unstable. If $\theta_2\kappa_{2R}\kappa_{3R}<0$, a closed trajectory for A can exist on the phase plane. The maximum distance of the trajectory from the origin is bounded by

$$\max\left(-\frac{\kappa_{2R}\theta_2}{2\kappa_{3R}}, -\frac{\kappa_{2R}\theta_2}{\kappa_{3R}}-\frac{\mu_2|\kappa_5|}{|\kappa_{3R}|}\right)<|A|^2_{max}<-\frac{\kappa_{2R}\theta_2}{\kappa_{3R}}+\frac{\mu_2|\kappa_5|}{|\kappa_{3R}|} \tag{35}$$

In this case, it is expected that if $\kappa_{3R}<0$ $(\theta_2\kappa_{2R}>0)$, the closed trajectory is stable and if $\kappa_{3R}>0$ $(\theta_2\kappa_{2R}<0)$, it is unstable.

Let $\mu_2=1$, i.e. $\varepsilon=\mu$, $\theta_2=(\theta-\theta_c)/\mu$ and $\tau_2=\mu\psi$. By (15), (23) and (31), the amplitude of the blade motion will be $\mu^{1/2}A(\theta-\theta_c/\mu, \mu\psi)$. For fixed $(\theta-\theta_c)/\mu$ and as μ increases, the amplitude will grow as $\mu^{1/2}$ and will vary more rapidly with time scale $\sim 1/\mu$. A will tend to $\left[-\frac{\theta-\theta_c}{\mu}\frac{\kappa_{2R}}{\kappa_{3R}}\right]^{1/2}$ as $(\theta-\theta_c)/\mu\to\infty$, if $\theta_2\kappa_{2R}>0$ and $\kappa_{3R}<0$.

<u>(3) $\mu\neq 0$, $\Omega_F=1+\varepsilon\nu$, where ν is of order one or smaller.</u> In this case, β_{12} and ζ_{12} in the first equation (23) can be very large or even singular, i.e., the solution of β_1 and ζ_1 has secular terms in τ_0. This is because the forcing function f_1 and h_1 have almost the same frequency as the flutter frequency. This difficulty can be avoided by requiring both f_1 and h_1 to be small; namely, $\mathcal{L}=\varepsilon$ or

$$\mathcal{X}_0 = \mathcal{X}_1 = 0$$
$$\mathcal{X}_2 = 1 \tag{36a}$$

in (19). This is a reasonable assumption because both f_1 and h_1 are proportional to the pitch angle which are indeed relatively small. We may take the other coefficients in (19) as

$$\varphi_0 = 1$$
$$\varphi_1 = \varphi_2 = \mu_2 = 0 \tag{36b}$$

i.e.

$$\mu = \mu_1 \varepsilon^{1/2}$$
$$\mathcal{X} = \varepsilon$$
$$\varphi = 1$$

(In reference 3, μ was taken to be $\mu_3 \varepsilon^{3/2}$, $\mathcal{X}$ to be one. The present expansion is valid for a larger range of μ.)

In (24), the terms proportional to $e^{\pm i\Omega_F \tau_0}$, $e^{\pm i\tau_0}$, $e^{\pm i(2-\Omega_F)\tau_0}$ and $e^{\pm i(2\Omega_F-1)\tau_0}$ will cause the secular solution for β_3 and ζ_3. Equation (25), in this case, is equivalent to requiring

$$\nu\Big[P_1\frac{\partial A}{\partial \tau_2} + (\theta_2 P_3 + \mu_1^2 P_{17})A + P_5 A^2 A^* + \mu_1 P_{19}^* A^2 e^{i\nu\tau_2}$$
$$+ \mu_1\Big(P_{25} AA^* + \frac{P_7}{\varepsilon} + \mu_1^2 P_{27}\Big)e^{-i\nu\tau_2} + \mu_1^2 P_{23} A^* e^{-2i\nu\tau_2}\Big]$$
$$+ P_2\frac{\partial A}{\partial \tau_2} + (\theta_2 P_4 + \mu_1^2 P_{18})A + P_6 A^2 A^* + \mu_1 P_{20}^* A^2 e^{i\nu\tau_2}$$
$$+ \mu_1\Big(P_{26} AA^* + \frac{P_8}{\varepsilon} + \mu_1^2 P_{28}\Big)e^{-i\nu\tau_2} + \mu_1^2 P_{24} A^* e^{-2i\nu\tau_2} = 0$$

or

$$\frac{\partial A}{\partial \tau_2} = (K_2\theta_2 + \mu_1^2 K_6)A + \mu_1^2 K_9 A^* + \mu_1\Big(K_{10} AA^* + \mu_1^2 K_{11} + \frac{K_9}{\varepsilon}\Big)e^{-i\nu\tau_2}$$
$$+ K_3 A^2 A^* + \mu_1 K_{13} A^2 e^{i\nu\tau_2} \tag{37}$$

Equation (37) is quite complicated. The complex amplitude "A" depends on μ_1, θ_2, and ν which are coefficients in the expansion of the parameters. Their physical meaning is not quite obvious because ε itself is just a conveniently chosen small parameter. To see the effects of pitch angle, advance ratio and the value of the flutter frequency, it is better to rewrite (37) in terms of physically more meaningful quantities by defining:

$$\bar{A} = \varepsilon^{\frac{1}{2}} A e^{i\nu\tau_2} = \varepsilon^{\frac{1}{2}} A e^{i(\Omega_F - 1)\psi} \tag{38}$$

where $2|\bar{A}|$ is the amplitude of the first order solution of the lag motion in radians. Substituting (38) into (37), and using (16) - (19), we have

$$\frac{d\bar{A}}{d\psi} = [\kappa_2(\theta-\theta_c) + i(\Omega_F - 1)]\bar{A} + \mu^2[\kappa_6\bar{A} + \kappa_9\bar{A}^*] + \kappa_3\bar{A}^2\bar{A}^* + \mu[\kappa_{13}\bar{A} + \kappa_{10}\bar{A}^*]\bar{A} + \mu\kappa_4 + \mu^3\kappa_{11} \tag{39}$$

The third bracket is the contribution due to the nonlinear coupling of the flutter mode and the parametric excitation and the term $\kappa_3\bar{A}^2\bar{A}^*$ is from the nonlinear coupling of the flutter mode. If the nonlinear terms are neglected, we obtain the result for the linearized equations of (1) and (2), i.e.

$$\frac{d\bar{A}}{d\psi} = [\kappa_2(\theta-\theta_c) + i(\Omega_F - 1) + \mu^2\kappa_6]\bar{A} + \mu^2\kappa_9\bar{A}^* + \mu\kappa_4 + \mu^3\kappa_{11} \tag{40}$$

The homogeneous solution of (40) is in the form $\bar{A}_o e^{(\lambda_{1,2})\psi}$

$$\lambda_{1,2} = K_{2R}(\theta-\theta_c)+\mu^2 K_{6R} \pm \sqrt{\mu^4 |K_4|^2-\left[K_{2I}(\theta-\theta_c)+\Omega_F-1+\mu^2 K_{6I}\right]^2}$$

Therefore, the linearized solution, i.e., the origin ($\bar{A}=0$), is unstable if $Re(\lambda_1)$ or $Re(\lambda_2)>0$. Its stability boundary is given by

$$Re(\lambda_1) = 0 \tag{41}$$

Since (39) is inhomogeneous, regardless of the range of the parameters, the solution for A will not be zero, i.e., the flutter mode is always excited.

To examine the behavior of a nonlinear solution, one must use the full equation (39) or (37) to investigate the characteristic of its singular points. One can then estimate its behavior by the method used in Ref. 3. It can be shown that if μ is sufficiently small and $(\theta-\theta_c)/\mu^{2/3}$ is of order one or larger, it reduces to (73) of reference 3, i.e., $|\bar{A}|\sim\mu^{1/3}$ for small value of μ and the time scale for the growth is proportional to $1/\mu^{2/3}$. For a larger value of μ, i.e., $(\theta-\theta_c)/\mu^{1/2}$ is of order one, $|\bar{A}|$ will be of order μ and the time scale for growth is proportional to $1/\mu$.

<u>(4) $\mu\neq 0$, $\Omega_F=2+\varepsilon\nu$.</u> In this case, the forcing functions f_2 and h_2 and the nonlinear coupling of the forced response due to f_1 and h_1, and its coupling with the parametric excitation will cause secular terms for β_2 and ζ_2, i.e. β_{25} and ζ_{25} can be large or even singular in the second equation of (23). The difficulty

is avoided by setting

$$\varkappa_0 = \varphi_0 = 0$$
$$\varkappa_1 = \varphi_1 = 1 \tag{42a}$$

The other parameters may be taken as

$$\mu_2 = 0$$
$$\varkappa_2 = \varphi_2 = 0 \tag{42b}$$

We have

$$\mu = \mu_1 \varepsilon^{1/2}$$
$$\varkappa = \varphi = \varepsilon^{1/2} \tag{43}$$

In the right hand side of (24), only terms proportional to $e^{i\Omega_F\tau_0}$ and $e^{2i\tau_0}$ will cause secular solution of β_3 and ζ_3. To suppress the secular terms, (25) is equivalent to requiring

$$v\left[P_1\frac{\partial A}{\partial\tau_2} + (\theta_2 P_3 + \mu_1^2 P_{17})A + P_5 A^2 A^* + \mu_1^2 \frac{P_{29}}{\varepsilon^{1/2}} e^{-i\nu\tau_2}\right.$$
$$\left. + P_2\frac{\partial A}{\partial\tau_2} + (\theta_2 P_4 + \mu_1^2 P_{18})A + P_6 A^2 A^* + \mu_1^2 \frac{P_{30}}{\varepsilon^{1/2}} e^{-i\nu\tau_2}\right] = 0$$

or

$$\frac{\partial A}{\partial\tau_2} = (K_2\theta_2 + \mu^2 K_6)A + K_3 A^2 A^* + \mu_1^2 \frac{K_{12}}{\varepsilon^{1/2}} e^{-i\nu\tau_2} \tag{44}$$

Transforming back to the physically meaningful quantities by

$$\bar{A} = \varepsilon^{\frac{1}{2}} A e^{i\nu\tau_2} = \varepsilon^{\frac{1}{2}} A e^{i(\Omega_F - 2)\psi}$$

we have

$$\frac{d\bar{A}}{d\psi} = \left[K_2(\theta - \theta_c) + \mu^2 K_6 + i(\Omega_F - 2)\right]\bar{A} + K_3 A^2 A^* + \mu^2 K_{12} \tag{45}$$

The behavior of $\bar{A}$ near the origin ($\bar{A}=0$) depends only on the sign of $\mathrm{Re}[\kappa_2(\theta-\theta_c)+\mu^2\kappa_6]$, i.e., stable if it is negative and unstable if it is positive. One can examine the general behavior of $\bar{A}$ by the method used in Sec. IV of Ref. 3. The results are the following: Let

$$\delta_2 = \kappa_2(\theta-\theta_c)+\mu^2\kappa_6+i(\Omega_f-2)$$

If $\delta_{2R}\kappa_{3R}>0$, there exists no closed trajectory for $\bar{A}$ in the phase plane. Since the sign of $\partial|\bar{A}|/\partial\tau_2$ is the same as κ_{3R} for large values of $|\bar{A}|$. If $\kappa_{3R}<0$ ($\delta_{2R}<0$), the solution will converge to the singular points of (44). If $\kappa_{3R}>0$ ($\delta_{2R}>0$), the solution is unstable. If $\delta_{2R}\kappa_{3R}<0$, there exists no trajectory for $\bar{A}$ on the phase plane in the region $|\bar{A}|^2<-\delta_{2R}/2\kappa_{3R}$. However, in this case, it can be shown that the maximum value of the closed trajectory from the origin is bounded by

$$\mathrm{Max}\left(-\frac{\delta_{2R}}{2\kappa_{3R}},\ \rho_1^2\right)\le|A|^2_{max}\le\rho_0^2 \tag{46}$$

where $\rho_0(>0)$ and $\rho_1(<0)$ are the largest and the smallest real roots of

$$|\kappa_{3R}|\left(-\frac{\delta_{2R}}{\kappa_{3R}}-\rho^2\right)\rho+\mu^2|\kappa_4|=0 \tag{47}$$

It is expected that if $\kappa_{3R}<0$ ($\delta_{2R}>0$), the solution converges to the closed trajectory and vice versa.

For fixed $\theta-\theta_c$ and sufficiently small μ, say $\frac{\theta-\theta_c}{\mu^{1/2}}\sim 0(1)$, or $\frac{\theta-\theta_c}{\mu}>>1$, equation (45) can be solved similar to (26), with terms proportional to μ^2 being neglected. In this case, $\bar{A}$ is of order $(\theta-\theta_c)^{\frac{1}{2}}$ or $\mu^{1/2}$, which is the same result as obtained in Ref. 3. For larger values of μ, say $(\theta-\theta_c)/\mu \sim 0(1)$ the full equation (45), must be considered. The solution of $\bar{A}$ is of order μ.

Since (45) is inhomogeneous, $\bar{A}$ will not be zero, i.e., the flutter mode will always be excited.

(5) $\mu \neq 0$, $\Omega_F=1/2+\varepsilon\nu, 1+\varepsilon\nu$ or $2+\varepsilon\nu$. In this case, there is no secular term for the solution of (β_1, ζ_1) and (β_2, ζ_2). Therefore, we may take

$$\begin{aligned} \varkappa_0 &= \varphi_0 = 1 \\ \varkappa_1 &= \varkappa_2 = \varphi_1 = \varphi_2 = 0 \end{aligned} \tag{48}$$

i.e.

$$\begin{aligned} \mu &= \mu_1 \varepsilon^{\frac{1}{2}} \\ \varkappa &= \varphi = 1 \end{aligned} \tag{49}$$

To suppress the secular terms of β_3 and ζ_3, there are three cases

(a) $\Omega_F=1/3+\varepsilon\nu$. In (24), the terms proportional to $e^{\pm i\Omega_F\tau_0}$ and $e^{\pm i(1-2\Omega_F)\tau_0}$ will cause secular terms. Therefore, (25) is equivalent to requiring

$$\nu\left[P_1\frac{\partial A}{\partial\tau_2}+(\theta_2P_3+\mu_1^2P_{17})A+P_5A^2A^*+\mu_1P_{19}A^{*2}e^{-3i\nu\tau_2}\right]$$
$$+P_2\frac{\partial A}{\partial\tau_2}+(\theta_2P_4+\mu_1^2P_{18})A+P_6A^2A^*+\mu_1P_{20}A^{*2}e^{-3i\nu\tau_2}=0$$

or

$$\frac{\partial A}{\partial\tau_2}=(K_2\theta_2+\mu_1^2K_6)A+K_3A^2A^*+\mu_1K_7A^{*2}e^{-3i\nu\tau_2} \tag{50}$$

Near the origin (A=0), the solution is

$$K_{2R}\theta_2 + \mu_1^2 K_{6R} \gtrless 0 \quad \begin{matrix}\text{unstable}\\ \text{stable}\end{matrix} \tag{51}$$

The general behavior can be examined by the method used in Ref. 3. Let

$$\delta = K_2\theta_2 + \mu_1^2 K_6 + i(\Omega_F - \tfrac{1}{3}) \tag{52}$$

If $\delta_R \kappa_{3R} > 0$, there exists no closed trajectory for A in the phase plane. If $\delta_R \kappa_{3R} < 0$, the maximum distance of the closed trajectory from the origin is bounded by

$$\begin{aligned} \max\left[\sqrt{-\frac{\delta_R}{2K_{3R}}},\ \sqrt{\frac{\mu_1^2|K_7|^2}{4|K_{3R}|^2} - \frac{\delta_R}{K_{3R}}} - \frac{\mu_1^2|K_7|}{|K_{3R}|}\right] &\le |A|_{max} \\ &\le \sqrt{\frac{\mu_1^2|K_7|^2}{4|K_{3R}|^2} - \frac{\delta_R}{K_{3R}}} + \frac{\mu_1^2|K_7|}{|K_{3R}|} \end{aligned} \tag{53}$$

(b) $\underline{\Omega_F = 3 + \varepsilon\nu}$. In (24), the terms proportional to $e^{\pm i\Omega_F\tau_0}$ and $e^{3i\tau_0}$ will cause secular terms. Therefore (25) is equivalent to requiring

$$\begin{aligned} &\nu\left[P_1\frac{\partial A}{\partial\tau_2} + (\theta_2 P_3 + \mu_1^2 P_{17})A + P_5 A^2 A^* + \mu_1^3 P_{21} e^{-i\nu\tau_2}\right] \\ &+ P_2\frac{\partial A}{\partial\tau_2} + (\theta_2 P_4 + \mu_1^2 P_{18})A + P_6 A^2 A^* + \mu_1^3 P_{22} e^{-i\nu\tau_2} = 0 \end{aligned}$$

or

$$\frac{\partial A}{\partial\tau_2} = (K_2\theta_2 + \mu_1^2 K_6)A + K_3 A^2 A^* + \mu_1^3 K_8 e^{-i\nu\tau_2} \tag{54}$$

Equation (54) is similar to (45), it can be examined accordingly.

(c) Others: In (24), only the terms proportional to $e^{i\Omega_F \tau_0}$ will cause secular terms. Therefore, (25) is equivalent to requiring

$$\frac{\partial A}{\partial \tau_2} = (\kappa_2 \theta_2 + \mu_1^2 \kappa_6) A + \kappa_3 A^2 A^* \tag{55}$$

Equation (48) can be solved in a manner similar to that used for (26); therefore, the conclusions in Sec. IV.1 hold for (55) with $\kappa_2\theta$ being replaced by $\kappa_2\theta_2 + \mu^2\kappa_6$.

V. Uniformly Valid Expansion

The solutions given in Secs. III and IV can be combined to establish a single expansion which is valid for small advance ratios μ and all range of pitch angles θ up to a small neighborhood above the critical conditions. We shall first show that there exists a range of θ where the expansion given in both Secs. III and IV are valid, i.e., to establish the matching of the solutions. Then using a technique similar to the one used in singular perturbation theory (Refs. 5,6), a uniformly valid expansion can be constructed.

Let us use subscripts "b" and "n" for the solution in Secs. III and IV to denote respectively the asymptotic expansion for θ below and near the critical pitch angle θ_c, i.e.

$$\begin{aligned} \begin{pmatrix} \beta \\ \zeta \end{pmatrix}_b &= \mu \begin{pmatrix} \beta_1 \\ \zeta_1 \end{pmatrix}_b + \mu^2 \begin{pmatrix} \beta_2 \\ \zeta_2 \end{pmatrix}_b + \cdots \\ \begin{pmatrix} \beta \\ \zeta \end{pmatrix}_n &= \varepsilon^{\frac{1}{2}} \begin{pmatrix} \beta_1 \\ \zeta_1 \end{pmatrix}_n + \varepsilon \begin{pmatrix} \beta_2 \\ \zeta_2 \end{pmatrix}_n + \cdots \end{aligned} \tag{56}$$

the relation between ε and μ will depend on the cases (see Sec. IV) considered. Let $\tilde{\varepsilon}$ be a small parameter such that

$$0<\tilde{\varepsilon}\,, \quad \frac{\varepsilon}{\tilde{\varepsilon}} << 1 \tag{57}$$

We shall show that for $\theta<\theta_c$ and

$$\theta-\theta_c = -\frac{\varepsilon}{\tilde{\varepsilon}} \tag{58}$$

both expansions in (56) match, that is,

$$\frac{\begin{pmatrix}\beta\\ \zeta\end{pmatrix}_n - \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_b}{\tilde{\varepsilon}^{\alpha}} \longrightarrow 0 \tag{59}$$

as $\varepsilon,\tilde{\varepsilon}\to 0$ with τ_2 being fixed and $\alpha=1/2,\ 1,\ldots$ Then the uniformly valid expansion is simply

$$\begin{pmatrix}\beta\\ \zeta\end{pmatrix} \simeq \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_n + \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_b - \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_c \tag{60}$$

The last term of (60) is the common part in the expansions $\binom{\beta}{\zeta}_n$ and $\binom{\beta}{\zeta}_b$.

We shall consider the four cases of Sec. IV separately.

(1) Hovering, $\mu=0$. In this case

$$\begin{pmatrix}\beta\\ \zeta\end{pmatrix}_b = 0$$

and $\binom{\beta}{\zeta}_n$ is defined in Sec. IV.1, and is proportional to A which is the solution of (26). Since $\kappa_2\theta_2$ is linearly proportional to θ_2 $\left(=\frac{\theta-\theta_c}{\varepsilon} = -\frac{1}{\tilde{\varepsilon}}\right)$ (see B.59, B.29, B.39, 16 and A.24), its real part is a large negative quantity. The solution of (26) can be approximately expressed in the form

$$A \doteq A_0 e^{\kappa_2 \theta_2 \tau_2} = A_0 e^{\kappa_2 \frac{\theta - \theta_c}{\tilde{\varepsilon}} \tau_2} \tag{61}$$

where A_0 is a constant.* A is exponentially decaying, so is $\binom{\beta}{\zeta}_n$ i.e., (59) is automatically satisfied. We should point out that for the conditionally unstable situation of Sec.IV.1.d the approximate solution (61) may be invalid for a sufficiently large initial disturbance. However, this situation may be avoided by having sufficiently small ε, thus seriously limiting the range of validity of the uniformly valid expansion.

(2) $\Omega_F = 1+\varepsilon\nu$, (Sec. IV.2). The expansion $\binom{\beta}{\zeta}_n$ is in the form similar (up to the order considered) to that of the previous case except that A is now governed by (37). In (37), within the range given by (58) and μ_1 of order one or smaller, the real part of $\kappa_2\theta_2(\sim -1/\tilde{\varepsilon})$ is a large negative quantity; the solution for A can be approximately expressed in the form

$$A = A_0 e^{\kappa_2 \theta_2 \tau_2} - \frac{\mu_1 \kappa_4}{\theta_2 \kappa_2 + i\nu} e^{-i\nu\tau_2} \tag{62}$$

[This is valid only if $\dfrac{\mu^2 |\kappa_3 \kappa_4^2|}{|i(\Omega_F - 1) + \kappa_2(\theta - \theta_c)|^3} << 1.$]

where A_0 is a constant to be determined from initial conditions. The expansion near the critical value θ_c can be written as

$$\begin{aligned} \binom{\beta}{\zeta}_n = 2\,\mathrm{Re}\Big\{ & -\frac{\mu_1 \kappa_4}{\kappa_2(\theta-\theta_c) + i(\Omega_F - 1)} \binom{u}{1} e^{i(\Omega_F - \varepsilon\nu)\psi} + \frac{\mu^2 \kappa_4^2}{[\kappa_2(\theta-\theta_c) + i(\Omega_F-1)]^2} \binom{\beta_{21}}{\zeta_{2b}}_n e^{2i(\Omega_F - \varepsilon\nu)\phi} \\ & + \frac{\mu^2}{2} \left| \frac{\kappa_4}{\kappa_2(\theta-\theta_c) + i(\Omega_F - 1)} \right|^2 \binom{\beta_{22}}{\zeta_{22}}_n + (\;---\;) e^{\kappa_2 \theta_2 \tau_2} \\ & + O(\tilde{\varepsilon}^{\frac{3}{2}}) \Big\} \end{aligned} \tag{63}$$

---

*A_0 depends on the initial conditions.

We shall examine the expression $\binom{\beta}{\zeta}_b$ by expressing

$$1 = \Omega_F - \varepsilon\nu$$

where ν is a constant of order 1. Expanding the parameter in the form of (16), we have, (B.18, B.19 and B.36)

$$D(1) = -\varepsilon\left(\Omega_\beta^2+1-\Omega_F^2+i\Omega_F g_{\beta c}\right)\left[-i\nu(P_1v+P_2)+P_3v+P_4\right]+O(\varepsilon^2) \quad (64)$$

$$\begin{pmatrix}\beta_{12}\\ \zeta_{12}\end{pmatrix}_b = \underset{\sim}{N}(1)\begin{pmatrix}P_7\\ P_8\end{pmatrix} = \frac{\Omega_\beta^2+1-\Omega_F^2+ig_{\beta c}\Omega_F}{D(1)}\begin{pmatrix}uv+O(\varepsilon) & u+O(\varepsilon)\\ v+O(\varepsilon) & 1\end{pmatrix}\begin{pmatrix}P_7\\ P_8\end{pmatrix}$$

$$= -\frac{P_7v+P_8}{\varepsilon\left[-i\nu(P_1v+P_2)+(P_3v+P_4)\right]}\left[\begin{pmatrix}u\\1\end{pmatrix}+O(\varepsilon)\right] = -\frac{K_4}{\varepsilon(K_2\theta_2+i\nu)}\left[\begin{pmatrix}u\\1\end{pmatrix}+O(\varepsilon)\right] \quad (65)$$

From (14), (B.47), (B.48)

$$\begin{pmatrix}\beta_{25}\\ \zeta_{25}\end{pmatrix}_b = \underset{\sim}{N}(2\Omega_F)\begin{pmatrix}-2i\beta_{12}\zeta_{12}+(\beta_0-\frac{\Gamma}{8}\theta)\zeta_{12}^2+\frac{\Gamma}{8}\beta_{12}\zeta_{12}\\ 2\beta_{12}^2 i-(2\beta_0-\frac{\Gamma}{8}\theta)\beta_{12}\zeta_{12}-\frac{\Gamma}{8}\beta_{12}^2\end{pmatrix}+O\left(\frac{\tilde{\varepsilon}}{\varepsilon}\right) \quad (66)$$

$$= \left[\frac{K_4}{i(\Omega_F-1)+K_2(\theta-\theta_c)}\right]^2\begin{pmatrix}\beta_{21}\\ \zeta_{21}\end{pmatrix}_n + O\left(\frac{\tilde{\varepsilon}}{\varepsilon}\right)$$

$$\begin{pmatrix}\beta_{26}\\ \zeta_{26}\end{pmatrix} = \underset{\sim}{N}(0)\begin{pmatrix}-2i(\beta_{12}^*\zeta_{12}-\beta_{12}\zeta_{12}^*)-2(\beta_0-\frac{\Gamma}{8}\theta)\zeta_{12}\zeta_{12}^*-\frac{\Gamma}{8}(\beta_{12}^*\zeta_{12}+\beta_{12}\zeta_{12}^*)\\ (2\beta_0-\frac{\Gamma}{8}\theta)(\beta_{12}^*\zeta_{12}+\beta_{12}\zeta_{12}^*)+\frac{\Gamma}{8}2\beta_{12}\beta_{12}^*\end{pmatrix}+O\left(\frac{\tilde{\varepsilon}}{\varepsilon}\right)$$

$$= \left|\frac{K_4}{i(\Omega_F-1)+K_2(\theta-\theta_c)}\right|^2\begin{pmatrix}\beta_{22}\\ \zeta_{22}\end{pmatrix}_n + O\left(\frac{\tilde{\varepsilon}}{\varepsilon}\right) \quad (67)$$

Then the expansion $\binom{\beta}{\zeta}_b$ can be written as

$$\begin{pmatrix}\beta\\ \zeta\end{pmatrix}_b = 2\,\mathrm{Re}\left\{\frac{-\mu K_4}{i(\Omega_F-1)+K_2(\theta-\theta_c)}\begin{pmatrix}u\\ 1\end{pmatrix}e^{i\psi} + \frac{\mu^2 K_4^2}{[i(\Omega_F-1)+K_2(\theta-\theta_c)]^2}\begin{pmatrix}\beta_{21}\\ \zeta_{21}\end{pmatrix}_n e^{2i\psi}\right.$$

$$\left. + \frac{\mu^2}{2}\left|\frac{K_4}{i(\Omega_F-1)+K_2(\theta-\theta_c)}\right|^2\begin{pmatrix}\beta_{22}\\ \zeta_{22}\end{pmatrix}_n + O(\tilde{\varepsilon}^{\frac{3}{2}})\right\} \tag{68}$$

Comparing (63) and (68), evidently the matching requirement (59) is satisfied. The common parts of the two expansions within the order of approximation in the expansion are the same as in (68). Therefore, we have the uniformly valid expansion

$$\begin{pmatrix}\beta\\ \zeta\end{pmatrix} = \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_n + \begin{pmatrix}\bar{\beta}\\ \bar{\zeta}\end{pmatrix} \tag{69}$$

where

$$\begin{pmatrix}\bar{\beta}\\ \bar{\zeta}\end{pmatrix} = \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_b - \begin{pmatrix}\beta\\ \zeta\end{pmatrix}_c = 2\,\mathrm{Re}\left\{\mu\begin{pmatrix}\bar{\beta}_{12}\\ \bar{\zeta}_{12}\end{pmatrix}e^{i\psi} + \mu^2\begin{pmatrix}\bar{\beta}_{25}\\ \bar{\zeta}_{25}\end{pmatrix}e^{2i\psi} + \frac{\mu^2}{2}\begin{pmatrix}\bar{\beta}_{26}\\ \bar{\zeta}_{26}\end{pmatrix} + \cdots\right\} \tag{70}$$

$$\begin{pmatrix}\bar{\beta}_{12}\\ \bar{\zeta}_{12}\end{pmatrix} = \left[\underset{\sim}{N}(1,\theta) - \underset{\sim}{N}(1,\theta_c)\right]\begin{pmatrix}P_7\\ P_8\end{pmatrix} \tag{71}$$

and $\bar{\beta}_{25}$, $\bar{\zeta}_{25}$, $\bar{\beta}_{26}$ and $\bar{\zeta}_{26}$ are defined in the same way as $\beta_{25}\cdots$ etc. in (B.47) and (B.48), except that β_{12} and ζ_{12} is replaced by $\bar{\beta}_{12}$ and $\bar{\zeta}_{12}$ and $\varkappa_0 = \varkappa = \wp_0 = \wp = 1$.

<u>(3) $\Omega_F=2+\varepsilon\nu$.</u> Similar to the previous subsection, it can be shown that β and ζ can be expressed in the same form as (69) in which

$$\begin{pmatrix}\bar{\beta}\\ \bar{\zeta}\end{pmatrix} = 2\,\mathrm{Re}\left[\mu\begin{pmatrix}\beta_{12}\\ \zeta_{12}\end{pmatrix}_b e^{i\psi} + \mu^2\begin{pmatrix}\bar{\beta}_{25}\\ \bar{\zeta}_{25}\end{pmatrix}e^{2i\psi} + \frac{\mu^2}{2}\begin{pmatrix}\bar{\beta}_{26}\\ \bar{\zeta}_{26}\end{pmatrix} + \cdots\right] \tag{72}$$

where

$$\begin{pmatrix} \bar{\beta}_{25} \\ \bar{\zeta}_{25} \end{pmatrix} = \left[\underset{\sim}{N}(2,\theta) - \underset{\sim}{N}(2,\theta_c) \right] \begin{pmatrix} P_{13} \\ P_{14} \end{pmatrix}$$

$$\begin{pmatrix} \bar{\beta}_{26} \\ \bar{\zeta}_{26} \end{pmatrix} = \left[\underset{\sim}{N}(0,\theta) - \underset{\sim}{N}(0,\theta_c) \right] \begin{pmatrix} P_{15} \\ P_{16} \end{pmatrix} \tag{73}$$

with $\mathscr{X}_0=\mathscr{Y}_0=\mathscr{X}=\mathscr{Y}=1$ in (B.41)-(B.44).

For the other cases, the expansion near the critical conditions matches that below the critical condition, i.e., the uniformly valid expansion is of the same form as that in Sec. IV except that when evaluating β_{12}, ζ_{12}, β_{25}, ζ_{25}, β_{26} and ζ_{26}, the parameters in the matrix $\underset{\sim}{N}$ (B.18) are referred to their value at the corresponding pitch angle θ, rather than using the value at θ_c as done in Sec. IV.

VI. Conclusions and Numerical Results

To summarize the results for the pitch angle θ in the neighborhood of θ_c:

(1) The value of κ_{3R}, defined in (B.65), characterizes the behavior of the blade at large amplitude motion because the sign of $\frac{\partial|A|}{\partial\tau_2}$ is the same as that of κ_{3R} when $|A|$ is sufficiently large. Therefore, if κ_{3R} has large positive value, the nonlinear effect is strongly destabilizing and vice versa. It is a very undesirable situation to have the blade with κ_{3R} as positive, because instability can always be excited if the disturbance is sufficiently large. In practice, large disturbances are likely to occur due to gust or maneuver. The value of κ_{3R} is independent

of the advance ratio; however, it depends on θ, Ω_β, Ω_ζ, g_β, g_ζ $\beta_{p.c.}$ in a complicated way. In the two cases considered in Figs. (2) and (3), when $\Omega_\zeta^2>1+\Omega_\beta^2$, κ_{3R} is positive or has a very small negative value, i.e., nonlinear effect is destabilizing or is very weak. In the region $\Omega_\zeta^2<1+\Omega_\beta^2$, κ_{3R} is generally negative, and has a larger value than that for $\Omega_\zeta^2 > 1 + \Omega_\beta^2$, i.e., the nonlinear effect is stabilizing and is stronger.

(2) When flutter occurs, the amplitude of lag motion is much larger than that of flap. This is due to the fact that the total damping in lag is much smaller than that in flap. The ratio of the amplitudes is

$$\frac{|\beta|}{|\zeta|} = |u|$$

where u is defined in (B.20) and usually has a value of 0.1 to 0.2.

(3) In hovering, if $\theta>\theta_c$ and $\kappa_{3R}<0$, the limit cycle amplitude is

$$2\left[-\frac{\kappa_{2R}(\theta-\theta_c)}{\kappa_{3R}}\right]^{1/2}$$

for lag motion and

$$2|u|\left[-\frac{\kappa_{2R}(\theta-\theta_c)}{\kappa_{3R}}\right]^{1/2}$$

for flap motion.

The following are the cases for θ near θ_c in forward flight. The value of Ω_F has strong effect on the behavior of blade due to periodic forcing function and the periodic parametric excitation.

(4) If $\Omega_F \doteq 1/2$, the amplitude of the flutter mode is governed by (31). By taking $\mu_2=1$, the amplitude of the lag motion is $\mu^{1/2}A(\tau_2)$. In other words, when flutter occurs, the lag motion grows as $\mu^{1/2}$. The time scale τ_2 for A is $\mu\psi$, i.e., A varies more rapidly in the time scale ψ ($\sim 1/\mu$) for higher μ. The stability near the origin [A=0] is governed by the sign of

$$\kappa = Re\left[\kappa_{2R}(\theta-\theta_c)+\sqrt{\mu^2|\kappa_5|^2-[\kappa_{2I}(\theta-\theta_c)+\Omega_F-\tfrac{1}{2}]^2}\right] \quad (74a)$$

or in general, the stability of the singular points of (31) is governed by the sign of

$$\kappa = Re\left(\delta_{2R}+\sqrt{|\delta_5|^2-\delta_{2I}^2}\right) \quad (74b)$$

where

$$\delta_2 = \kappa_2(\theta-\theta_c)+2\varepsilon\,\kappa_3\tilde{A}_0\tilde{A}_0^* + i\left(\Omega_F-\tfrac{1}{2}\right)$$

$$\delta_5 = \mu\,\kappa_5 + \varepsilon\,\kappa_3\tilde{A}_0^2 \quad (75a)$$

in which $\tilde{A}=\tilde{A}_o$ is a singular point of (31). (Note $\tilde{A}_o=0$ is a singular point). If κ of (74a) is less than zero, the origin is stable and if κ_{3R} is also less than zero, since (31) is homogeneous, the flutter mode is unlikely to be excited.

(5) If $\Omega_F \doteq 1$, the flutter amplitude A is governed by (37) or (39). For small value of μ, (small as compared to $\theta-\theta_c$), i.e. $\mu_1=\varepsilon$ or $\mu=\varepsilon^{3/2}$ with θ_2 being of order one, (if $\mu<0(\varepsilon^{3/2})$ for $\theta-\theta_c=0(\varepsilon)$ forward flight will have little effect on the flutter of the blade), the lag amplitude is

$$\varepsilon^{\frac{1}{2}}A(\tau_2) = \mu^{\frac{1}{3}}A\left(\mu^{\frac{2}{3}}\psi\right)$$

i.e., the lag motion grows as $\mu^{1/3}$ and the length of time for A to grow is proportional to $1/\mu^{2/3}$. For larger values of μ (large as compared to $\theta-\theta_c$), i.e., $\mu_1=1$ or $\mu=\varepsilon^{1/2}$ with θ_2 being of order one or smaller, the lag amplitude becomes

$$\varepsilon^{\frac{1}{2}} A(\tau_2) = \mu A(\mu^2\psi)$$

i.e., the lag motion grows as μ and A varies more rapidly in the time ψ ($\sim 1/\mu^2$) for higher μ. The origin ($\bar{A}=0$) on the phase plane is no longer a singular point since (39) is inhomogeneous. Therefore, depending on the distance from the singular point to the origin, the stability prediction by the linearized theory such as (41) may not be meaningful. Let $\bar{A}=\bar{A}_o$ be a singular point of (39), then the stability of the singular depends on the sign of κ [defined in (74b)] where

$$\begin{aligned} \delta_2 &= K_2(\theta-\theta_c)+i(\Omega_F-1)+\mu^2K_6+2K_3\bar{A}_o\bar{A}_o^* + \mu(2K_{13}\bar{A}_o+K_{10}\bar{A}_o^*) \\ \delta_5 &= \mu^2 K_9 + \mu K_{10}\bar{A}_o + K_3\bar{A}_o^2 \end{aligned} \qquad (75b)$$

Since (39) is inhomogeneous, $\bar{A}_o \neq 0$, i.e., the flutter mode is always excited.

(6) If $\Omega_F \doteq 2$, the flutter amplitude is governed by (44). For small values of μ, i.e., $\mu_1=\varepsilon^{1/2}$ or $\mu=\varepsilon$ with θ_2 being of order one, the lag amplitude is

$$\varepsilon^{\frac{1}{2}} A(\tau_2) = \mu^{\frac{1}{2}} A(\mu\psi)$$

For large μ, i.e., $\mu_1=1$ or $\mu=\varepsilon^{1/2}$ with θ_2 being of order one, the lag amplitude is

$$\varepsilon^{\frac{1}{2}} A(\tau_2) = \mu A(\mu^2\psi)$$

The stability of the singular point depends on the sign of κ [defined in (74b)] in which

$$\delta_2 = \kappa_2(\theta-\theta_c)+\mu^2\kappa_6+i(\Omega_F-2)+\kappa_3\bar{A}_0\bar{A}_0^*$$

$$\delta_5 = \kappa_3\bar{A}_0^2$$

where $\bar{A}=\bar{A}_0$ is a singular point of (45). Since $\bar{A}_0=0$ because (45) is homogeneous, the flutter mode is always excited.

(7) If $\Omega_F \doteq 1/3$ or 3, the flutter amplitude is governed by (50), or (54). Taking $\mu_1=1$, i.e., $\mu=\varepsilon^{1/2}$, the lag amplitude is

$$\varepsilon^{\frac{1}{2}}A(\tau_2) = \mu^{\frac{1}{2}}A(\mu\psi)$$

The stability equation for the singular point is also (74b) in which

$$\begin{aligned} \delta_2 &= \kappa_2(\theta-\theta_c)+\mu^2\kappa_6+i(\Omega_F-\tfrac{1}{3})+2\varepsilon\kappa_3\tilde{A}_0\tilde{A}_0^* \\ \delta_5 &= \varepsilon\kappa_3\tilde{A}_0^2+2\mu\kappa_7\tilde{A}_0 \end{aligned} \qquad (75c)$$

for $\Omega_F-1/3=0(\varepsilon)$ and $\tilde{A}_0e^{-i\nu\tau_2}$ is a singular point of (50),

$$\begin{aligned} \delta_2 &= \kappa_2(\theta-\theta_c)+\mu^2\kappa_6+i(\Omega_F-3)+2\varepsilon\kappa_3\tilde{A}_0\tilde{A}_0^* \\ \delta_5 &= \varepsilon\kappa_3\tilde{A}_0 \end{aligned} \qquad (75d)$$

for $\Omega_F=3=0(\varepsilon)$ and $\tilde{A}_0e^{-i\nu\tau_2}$ is a singular point of (54). In the latter case, (54) is inhomogeneous, the flutter mode will be excited.

(8) Others. The amplitude equation (55) is similar to that of hovering. Therefore, the conclusion (3) also holds in this case if $\kappa_{2R}(\theta-\theta_c)$ is replaced by $\kappa_{2R}(\theta-\theta_c)+\mu^2\kappa_{6R}$.

(9) When evaluating κ_6 according to (B.65), if the quadratic terms in β_{12} and ζ_{12} are neglected, and in (75) $\bar{A}_o$ (or $\tilde{A}_o$) are set to zero, $\kappa=0$ of (74) gives the stability boundary of the linearized solution of (1) and (2).

(10) If $\kappa_{3R}<0$, Ω_F is not near one, two, or three and if the blade is operated under the condition $\kappa<0$, the flutter mode will not be excited. Therefore, in practical design, $\Omega_F \doteq 1$, 2, or 3 should be avoided.

(11) The amplitude of the blade motion is of order $\varepsilon|A|$, how ε relates to the advance ratio μ depends on the value of Ω_F. In other words, for the same value of μ, the magnitude of the response can be of different order. In particular, if $\Omega_F=1$, 2, or 3, the flutter mode is always excited; one cannot just from the observation of the motion of the blade conclude whether the blade is in the region of stability or instability. Because the motion cannot simply be the forced response, for different value Ω_F, the response is magnified to different magnitudes.

Numerical results are presented in the figures. The induced flow is approximated by

$$\lambda = \frac{\pi\sigma}{8}\left(\sqrt{1+\frac{12\theta}{\pi\sigma}} - 1\right)$$

where σ is the solidity ratio (Ref. 4). Throughout the computation, the following parameters are used:

$$\sigma = 0.05$$

$$\eta_\beta = \eta_\zeta = \Omega_{\beta\zeta} = 0$$

$$C_{d_0} = 0.01$$

$$a = 2\pi$$

For practical interest, we shall only compute the cases $1+\Omega_\beta^2 \simeq 1.1$ and $\Omega_\zeta^2 \simeq 1$.

In Figures 2 and 3, the locus of the critical conditions for various θ_c are presented; $\gamma=5$ in Fig. 2 and $\gamma=10$ in Fig. 3. These loci are similar to that of Fig. 4 of reference 4. The region within the locus is unstable according to linear aeroelasticity theory. For $\gamma=10$, such regions are much larger than that for $\gamma = 5$ for the same θ_c. The numbers given on this locus are the values $|\kappa_{2R}/\kappa_{3R}|^{1/2}$ for the corresponding parameters at $\theta=\theta_c$. The solid line is a portion of the locus where $\kappa_{3R}<0$ and the dotted line is $\kappa_{3R}>0$. That is, over the solid line portion of the locus, there exists a stable limit cycle of amplitude $2(\theta-\theta_c)^{1/2}|\kappa_{2R}/\kappa_{3R}|^{1/2}$ for lag motion and $2(\theta-\theta_c)^{1/2}|\kappa_{2R}/\kappa_{3R}|^{1/2}|u|$ for the flap when $\theta>\theta_c$. $|u|$ is usually $0.1 \sim 0.2$ because of low aerodynamic damping in lag. The blade is unconditionally stable if $\theta<\theta_c$. Over the dotted portion of the locus, the blade is unconditionally unstable if $\theta>\theta_c$ and conditionally stable if $\theta<\theta_c$, i.e., if the disturbance in $|A|$ is smaller than $|\kappa_{2R}/\kappa_{3R}|^{1/2}$, it will die out and if it is larger than $|\kappa_{2R}/\kappa_{3R}|^{1/2}$, it will grow. In these two cases, the limit cycle amplitude can be quite large for $\theta>\theta_c$, because $|\kappa_{2R}/\kappa_{3R}|a$ is quite large, i.e., the nonlinear effect is weak. For such a blade, it is not desirable to operate above the critical condition.

Figures 4 and 5 are the lag response of the blade in hovering for a given initial disturbance, obtained by method of numerical integration. In both cases, $\sqrt{1+\Omega_\beta^2}=1.09544$, $\gamma=5$, $\theta=0.3$ and $\theta_c=0.25$. $\Omega_\zeta=1.0017$ for Fig. 4 and $\Omega_\zeta=1.19747$ for

Fig. 5, while the numerical values for κ_{3R} are -0.0074 and 0.001 respectively for both cases. It is clear that the motion in Fig. 4 approaches a limit cycle oscillation while the amplitude in Fig. 5 grows steadily.

The case considered in Fig. 4 has a flutter frequency Ω_F=1.00418. It is expected that A is to be governed by (37). In this case,

$$\kappa_2 = 0.01178 + 0.017006\,i$$
$$\kappa_3 = -0.007424 + 0.0107\,i$$
$$\kappa_4 = -0.00963 - 0.00051\,i$$
$$\kappa_6 = -0.01154 - 0.007753\,i$$
$$\kappa_9 = -0.006385 - 0.01480\,i$$
$$\kappa_{10} = -0.02840 + 0.000624\,i$$
$$\kappa_{11} = 0.001121 + 0.002754\,i$$
$$\kappa_{13} = 0.01167 + 0.02472\,i$$
$$\Omega_F - 1 = 0.00418$$

The time variation of $|A|$ are plotted in Fig. 6. As predicted in Sec. IV.2, when $\Omega_F \simeq 1$, the amplitude of the response is of $\mu^{1/3}$ for $(\theta-\theta_c)/\mu^{2/3}$ of order 1 and the time scale is proportional to $1/\mu^{2/3}$. Figures 7 and 8 are the lag responses of the blade obtained by numerical integration having the same physical parameters as that of Fig. 6. μ=0.03 for Fig. 7 and μ=0.04 for Fig. 8. $(\theta-\theta_c)/\mu^{2/3}$ are held to be one. The lag motion is normalized by dividing by $\mu^{1/3}$, and ψ is normalized by multiplying by $\mu^{2/3}$. It can be seen that the amplitudes of the two curves are very similar in the time scale $\tau_2=\mu^{2/3}\psi$.

REFERENCES

1. Young, M.J., "A Theory of Rotor Blade Motion Stability in Flight," J.A.H.S., Vol. 9, No. 3, July 1964, pp. 12-15.
2. Hohenemser, K.H. and Heaton, P.W., Jr., "Aeroelastic Instability of Torsionally Rigid Helicopter Blades," J.A.H.S., Vol. 12, No. 2, April 1967, pp. 1-13.
3. Tong, P., "Nonlinear Instability of Rotor Blades in Flap-Lag Using Hohenemser's Equations," Unpublished Report, MIT Aeroelastic and Structures Research Laboratory, May 1971.
4. Ormiston, R.A. and Hodges, D.H., "Linear Flap-Lag Dynamics of Hingeless Helicopter Rotor Blades in Hover," J.A.H.S. Vol. 17, No. 2, April 1972, pp. 2-14.
5. Cole, J., "Perturbation Method in Applied Mathematics," Blaisdell Publishing Company, Waltham, Massachusetts.
6. Kaplun, S., "Fluid Mechanics and Singular Perturbations," edited by P.A. Lagerstrom, L.N. Howard, and C.-S. Liu, New York: Academic Press, 1967.

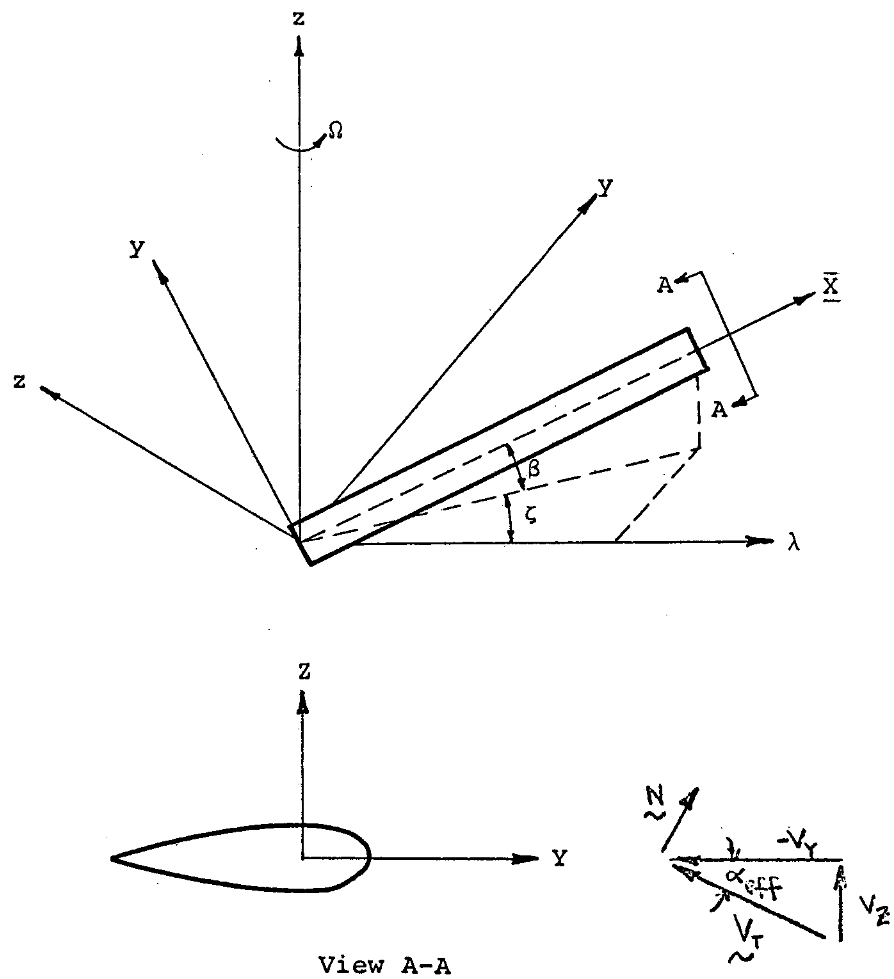

FIG. 1 GEOMETRY AND COORDINATES

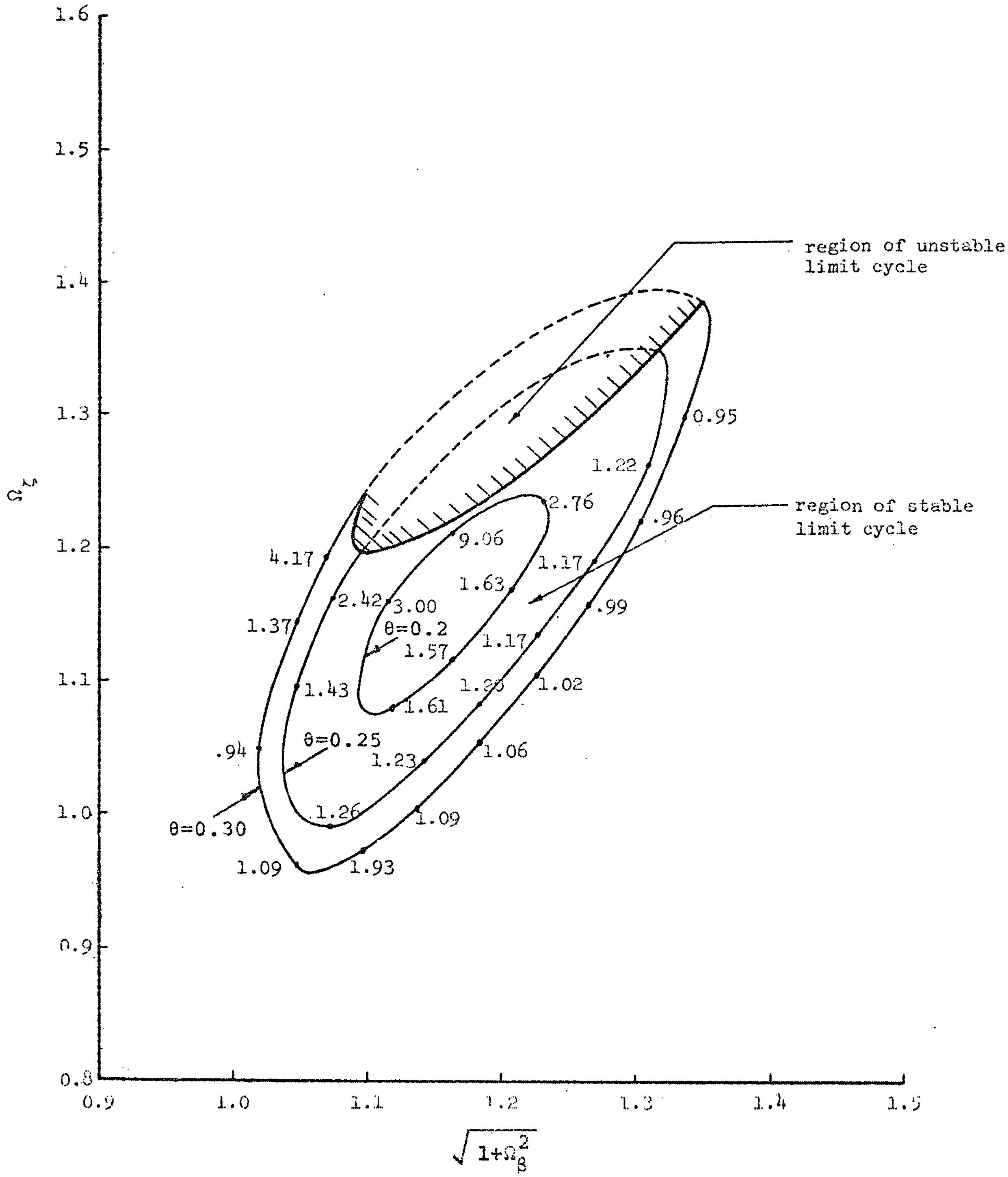

FIG. 2 LOCUS OF CRITICAL CONDITIONS FOR VARIOUS PITCH ANGLES θ_c FOR $\gamma=5$, $\sigma=0.05$, $C_{d_o}=0.01$, $a=2\pi$, $\eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=0$

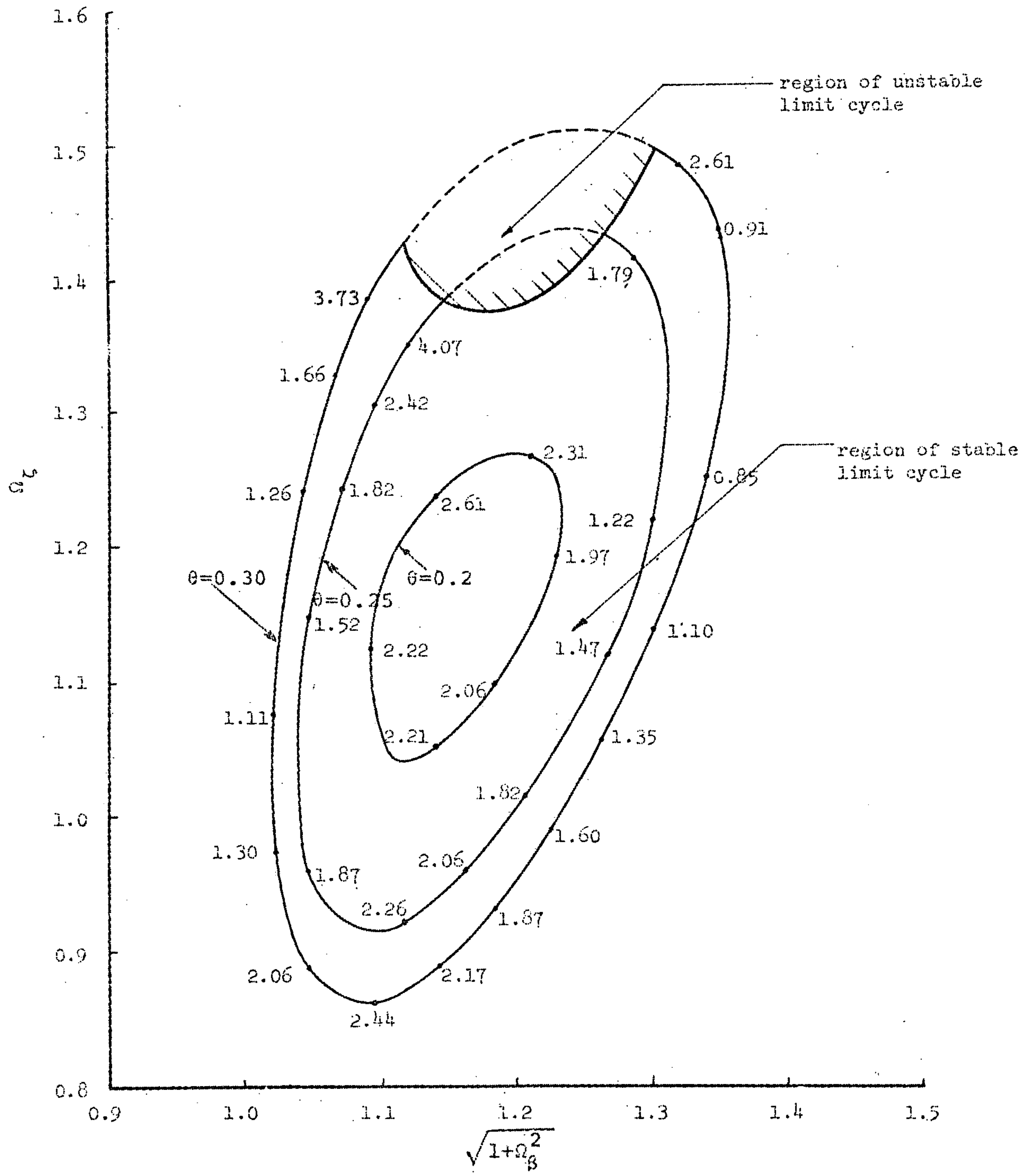

FIG. 3 LOCUS OF CRITICAL CONDITIONS FOR VARIOUS PITCH ANGLES θ_c FOR $\gamma=10$, $\sigma=0.05$, $C_{d_o}=0.01$, $a=2\pi$, $\eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=0$

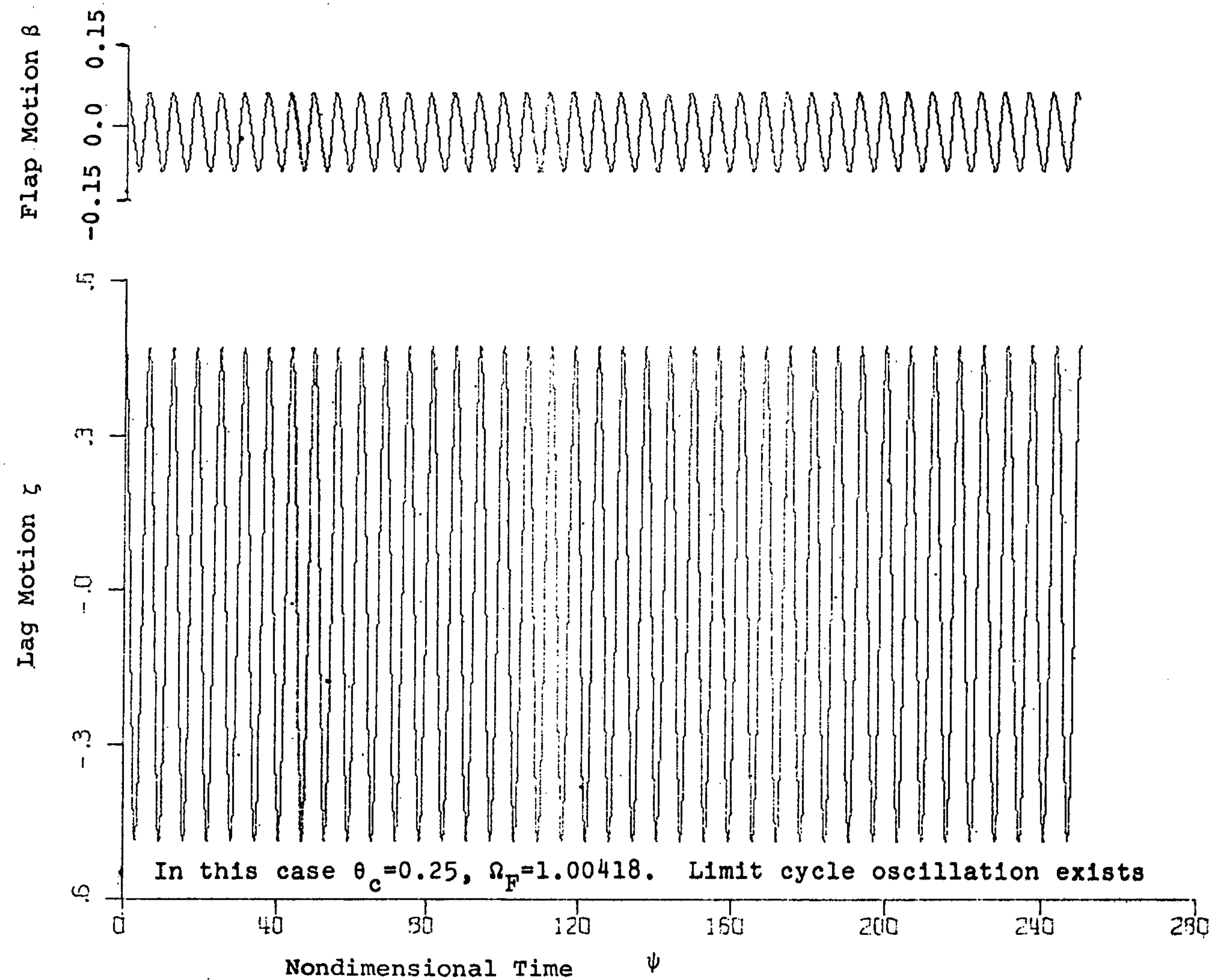

FIG. 4 FLAP AND LAG MOTION IN HOVERING FOR PITCH ANGLE ABOVE CRITICAL CONDITION FOR θ=0.29, γ=5, σ=0.05, C_{d_o}=0.01, $a=2\pi$, $\eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=0$, $\sqrt{1+\Omega_\beta^2}$=1.09544, Ω_ζ=1.0017

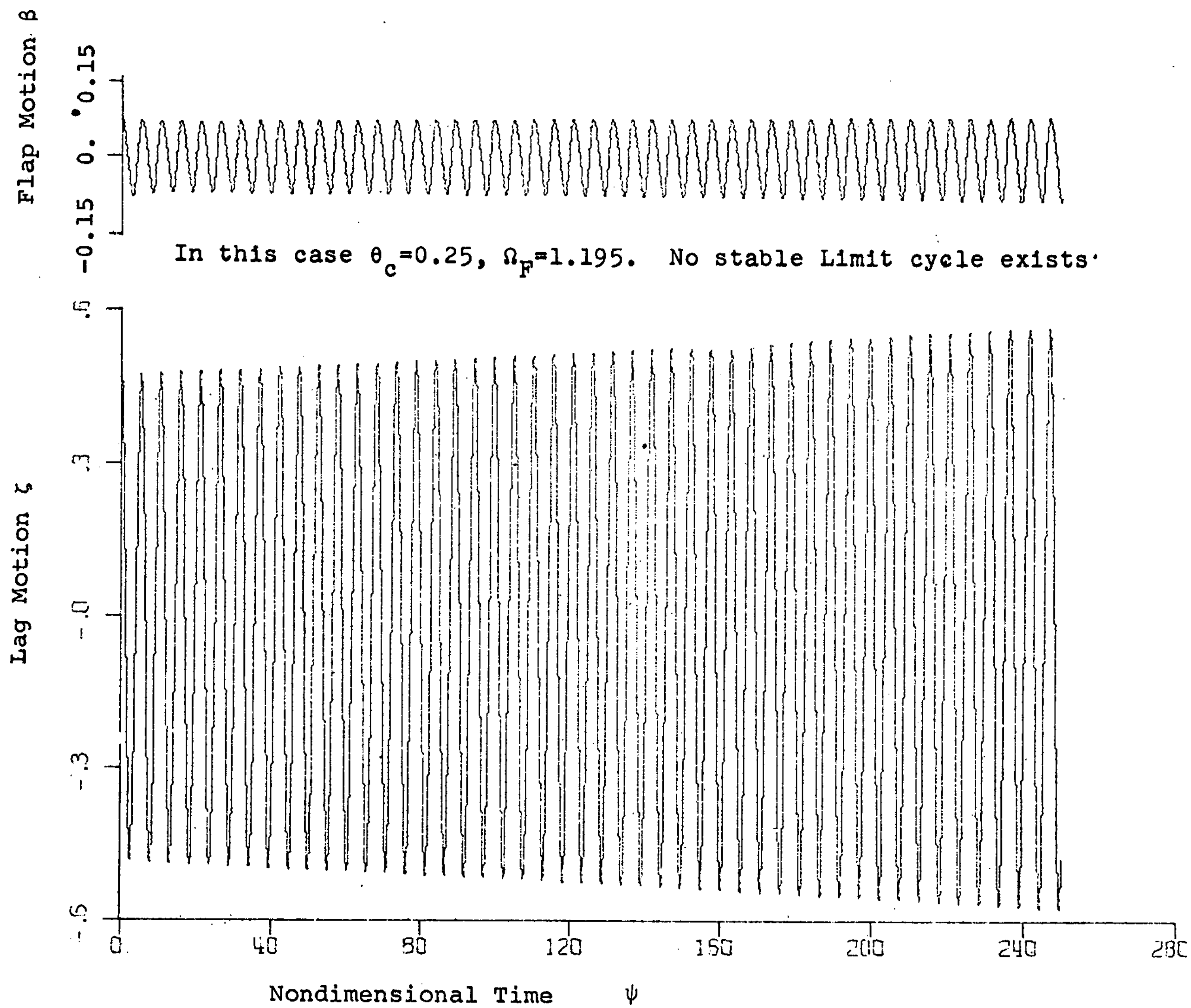

FIG. 5 FLAP AND LAG MOTION IN HOVERING FOR PITCH ANGLE ABOVE CRITICAL CONDITION FOR θ=0.25, γ=5, σ=0.05, C_{d_o}=0.01, $a=2\pi$, $\eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=0$, $\sqrt{1+\Omega_\beta^2}$ =1.09544, Ω_ζ=1.19747.

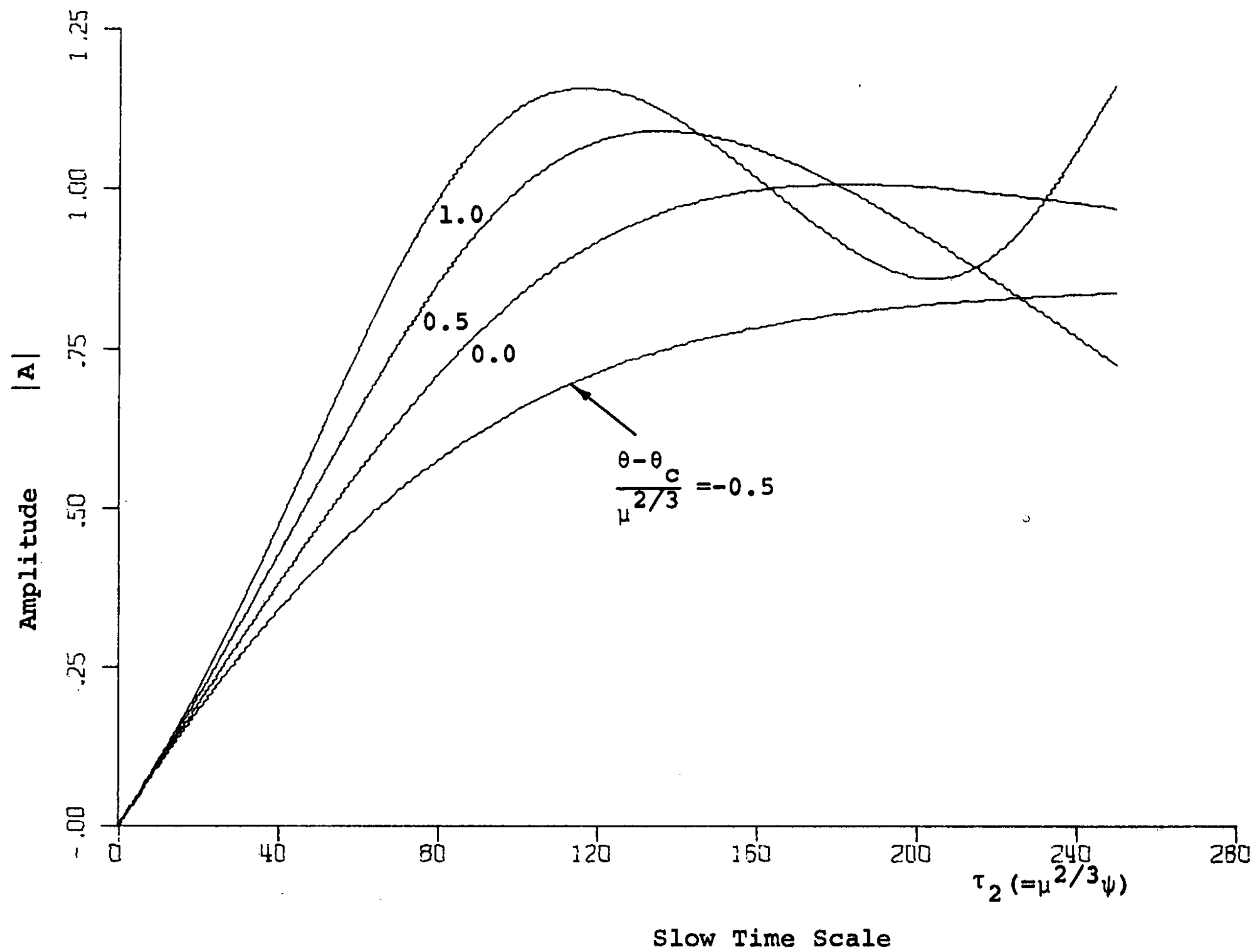

FIG. 6 TIME VARIATION OF $|A|$ FOR VARIOUS $(\theta-\theta_c)/\mu^{2/3}$ AND $\gamma=5$ FOR $\sigma=0.05$, $C_{d_o} = 0.01$, $a=2\pi$, $\eta_\beta=\eta_\zeta=\Omega_{\beta\zeta}=0$, $\sqrt{1+\Omega_\beta^2}=1.09544$, AND $\Omega_\zeta=1.0017$.

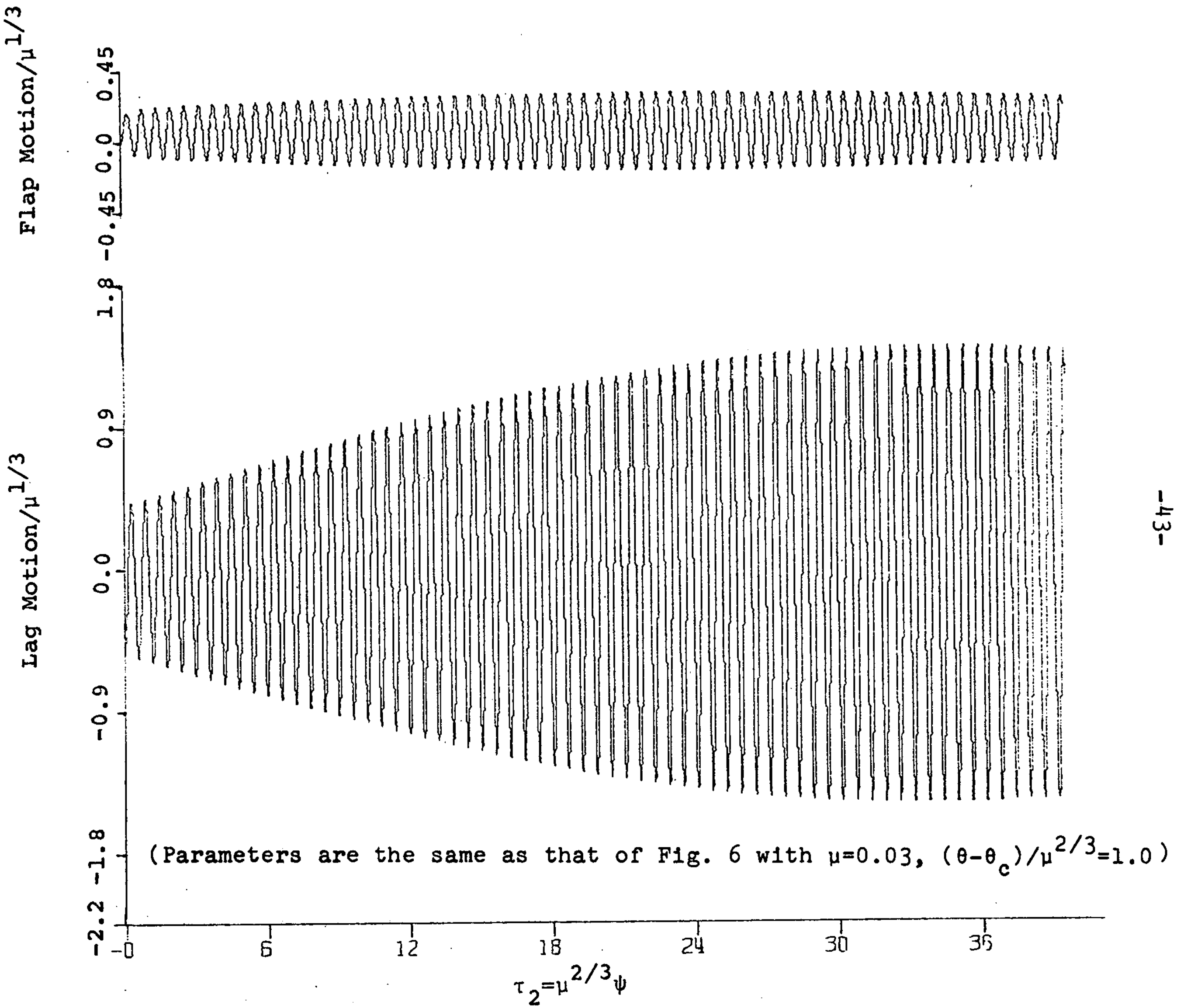

FIG. 7 NORMALIZED FLAP AND LAG MOTIONS VS. SLOW TIME SCALE $\tau_2(=\mu^{2/3}\psi)$ BY DIRECT NUMERICAL INTEGRATION OF EQS. (1) and (2) FOR μ=0.03.

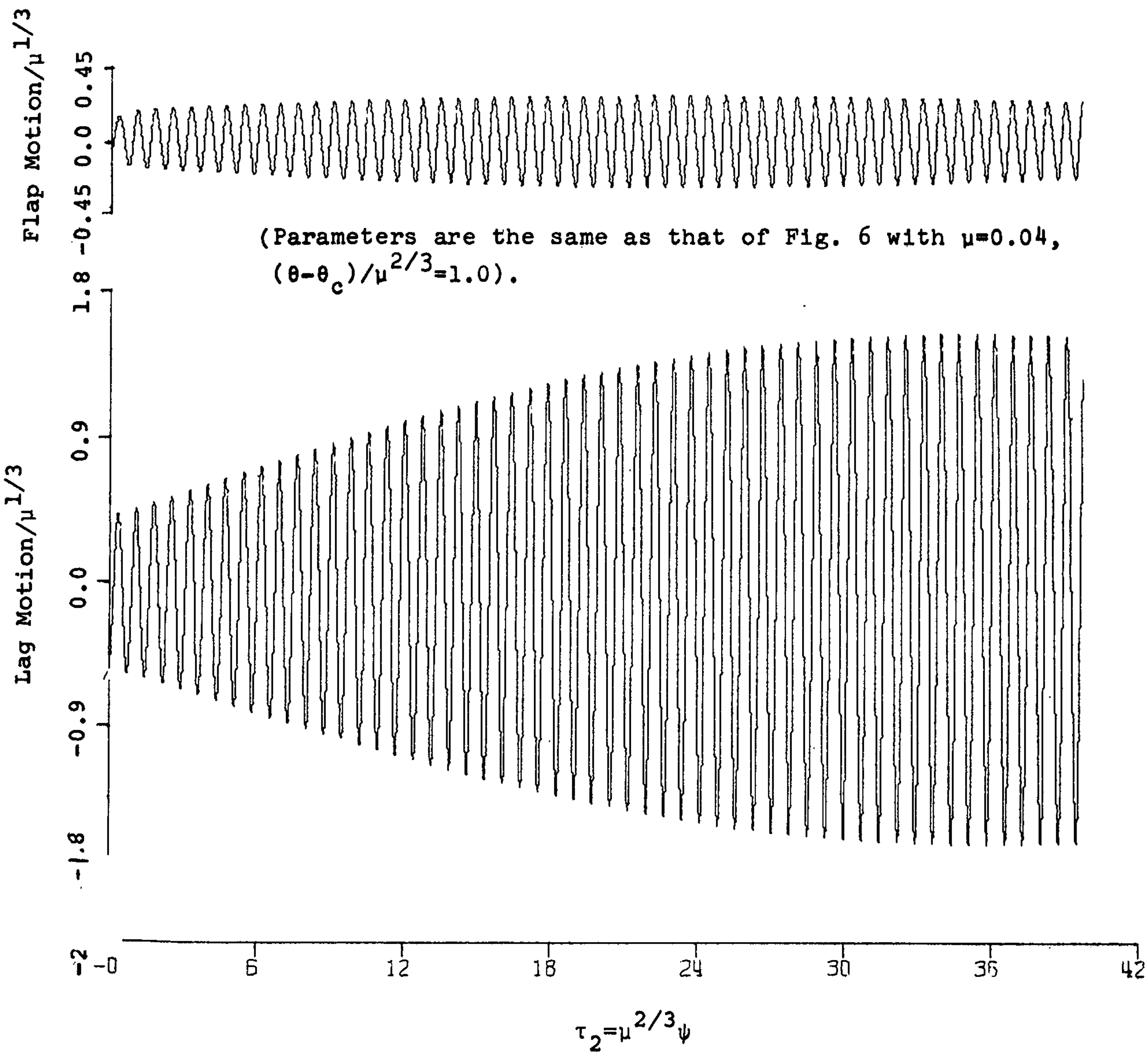

FIG. 8 NORMALIZED FLAP AND LAG MOTIONS VS. SLOW TIME SCALE $\tau_2(=\mu^{2/3}\psi)$ BY DIRECT NUMERICAL INTEGRATION OF EQS. (1) AND (2) FOR μ=0.04.

Appendix A. Derivation of Flap-Lag Equations

In deriving the equations of motion of the blade in flap-lag, β, ζ, and θ are used for flap, lag, and pitch angle respectively. $(\dot{\ })$ denotes differentiation with respect to ψ and $\psi=\Omega t$ with Ω being the rotating speed of the shaft itself. Two coordinate systems are used, namely, (x,y,z) rotating with the blade about the shaft and (X,Y,Z) rigidly attached to the blade. The base vectors of the two coordinate systems are related by (Fig. 1):

$$
\begin{aligned}
\underset{\sim}{i} &= \cos\beta\,\underset{\sim}{I} - \sin\beta\sin\theta\,\underset{\sim}{J} - \sin\beta\cos\theta\,\underset{\sim}{K} \\
\underset{\sim}{j} &= \cos\theta\,\underset{\sim}{J} - \sin\theta\,\underset{\sim}{K} \\
\underset{\sim}{k} &= \sin\beta\,\underset{\sim}{I} + \cos\beta\sin\theta\,\underset{\sim}{J} + \cos\beta\cos\theta\,\underset{\sim}{K}
\end{aligned}
\qquad \text{(A-1)}
$$

Let the distance of a point on the blade from its support be x, or $R\xi$ where R is the length of the blade. The position vector, the velocity, and acceleration of the point can be written, as

$$\underset{\sim}{r} = R\xi\left(\cos\beta\,\underset{\sim}{i} + \sin\beta\,\underset{\sim}{k}\right) \qquad \text{(A-2)}$$

$$\underset{\sim}{v} = \Omega R\xi\left[-\dot{\beta}\sin\beta\,\underset{\sim}{i} + (1+\dot{\zeta})\cos\beta\,\underset{\sim}{j} + \dot{\beta}\cos\beta\,\underset{\sim}{k}\right] \qquad \text{(A-3)}$$

$$
\begin{aligned}
\underset{\sim}{a} = \Omega^2 R\xi\Big\{&-\left[\ddot{\beta}\sin\beta + \dot{\beta}^2\cos\beta + (1+\dot{\zeta})^2\cos\beta\right]\underset{\sim}{i} \\
&+\left[\ddot{\zeta}\cos\beta - 2(1+\dot{\zeta})\dot{\beta}\sin\beta\right]\underset{\sim}{j} \\
&+\left[\ddot{\beta}\cos\beta - \dot{\beta}^2\sin\beta\right]\underset{\sim}{k}\Big\}
\end{aligned}
\qquad \text{(A-4)}
$$

Let m be the mass per unit length, then the inertia moment about the z-axis (the hub) is

$$\underset{\sim}{M}_I = -\int_0^1 m\underset{\sim}{r} \times \underset{\sim}{a}\, R\, d\xi$$

$$= I\Omega^2 \left[\ddot{\zeta}\cos\beta - 2(1+\dot{\zeta})\dot{\beta}\sin\beta\right]\left(\sin\beta\,\underset{\sim}{i} - \cos\beta\,\underset{\sim}{k}\right) \qquad \text{(A-5)}$$

$$+ I\Omega^2\left[\ddot{\beta} + (1+\dot{\zeta})^2\sin\beta\cos\beta\right]\underset{\sim}{j}$$

where

$$I = \int_0^1 R^3 \xi^2 m\, d\xi \qquad \text{(A-6)}$$

The flow velocity relative to the blade is

$$\underset{\sim}{V}_A = \mu R\Omega\left[\cos(\psi+\zeta)\underset{\sim}{i} - \sin(\psi+\zeta)\underset{\sim}{j}\right]$$

$$+ \underset{\sim}{V}_{Ind} - \Omega R\xi\left[-\dot{\beta}\sin\beta\,\underset{\sim}{i} + (1+\dot{\zeta})\cos\beta\,\underset{\sim}{j} + \dot{\beta}\cos\beta\,\underset{\sim}{k}\right] \qquad \text{(A-7)}$$

where $\underset{\sim}{V}_{IND}$ is the induced velocity. To evaluate lift and drag, it is more convenient to express $\underset{\sim}{V}_A$ in terms of (X,Y,Z)-coordinates, which move together with the blades, i.e.

$$\underset{\sim}{V}_A = V_X\underset{\sim}{I} + V_Y\underset{\sim}{J} + V_Z\underset{\sim}{K} \qquad \text{(A-8)}$$

It is clear that V_x is the velocity component in the axial direction along the blade, while V_y and V_z are in the cross section of the blade.

$$\underset{\sim}{V}_{Ind} = -\Omega R\lambda\,\underset{\sim}{K} \qquad \text{(A-9)}$$

using (A-1) we find

$$V_Y = -\Omega R\left\{\mu\left[\cos(\psi+\zeta)\sin\beta\sin\theta + \sin(\psi+\zeta)\cos\theta\right] + \xi\left[\cos\beta(1+\dot{\zeta})\cos\theta + \dot{\beta}\sin\theta\cos^2\beta\right]\right\} \qquad \text{(A-10)}$$

$$V_Z = -\Omega R\left\{\lambda + \mu\left[\cos(\psi+\zeta)\sin\beta\cos\theta - \sin(\psi+\zeta)\sin\theta\right] + \xi\left[-\cos\beta(1+\dot{\zeta})\sin\theta + \dot{\beta}\cos\theta\right]\right\}$$

We shall assume that quasi-steady two-dimensional aerodynamic theory holds for each cross section, i.e., the aerodynamic load vector can be written as

$$\underset{\sim}{L}_A = \frac{1}{2}\rho_A a c\,|V_T|^2\left(\alpha_{eff}\,\underset{\sim}{N} + \frac{C_{d_o}}{a}\,\frac{\underset{\sim}{V_T}}{|\underset{\sim}{V_T}|}\right) \tag{A-11}$$

where ρ_A is the air density, "a" lift slope, c chord length, C_{d_o} profile drag coefficient

$$\underset{\sim}{V_T} = V_Y\,\underset{\sim}{J} + V_Z\,\underset{\sim}{K} \tag{A-12}$$

α_{eff} is the effective angle of attack, i.e., (Fig. 1.C):

$$\alpha_{eff} = \tan^{-1}\frac{V_Z}{V_Y} \cong \frac{V_Z}{V_Y} \tag{A-13}$$

and $\underset{\sim}{N}$ is a unit vector normal to $\underset{\sim}{V_T}$ and in the plane of the blade cross section defined by

$$\underset{\sim}{N} = \frac{\underset{\sim}{I}\times\underset{\sim}{V_T}}{|\underset{\sim}{V_T}|}\;\frac{V_Y}{|V_Y|} \tag{A-15}$$

If $V_y>0$, i.e., the flow is from the trailing edge to the leading edge, this is called reversed flow. In our present consideration, we assume that $V_y<0$, and $\alpha^2_{eff}<<1$, $\frac{C_{d_o}}{a}<<1$. A substitution of (A-12) through (A-15) into (A-11) yields

$$\underset{\sim}{L}_A = (L_x)_A\,\underset{\sim}{i} + (L_y)_A\,\underset{\sim}{j} + (L_z)_A\,\underset{\sim}{k} \tag{A-16}$$

where

$$(L_x)_A = -\frac{1}{2}\rho_A ac\left\{\left[V_Z^2 + \frac{C_{d_o}}{a}|V_Y|V_Y\right]\sin\theta + \left[\frac{C_{d_o}}{a}|V_Y|V_Z - V_Z V_Y\right]\cos\theta\right\}\sin\beta$$

$$(L_y)_A = \frac{1}{2}\rho_A ac\left\{\left[V_Z^2 + \frac{C_{d_o}}{a}|V_Y|V_Y\right]\cos\theta - \left[\frac{C_{d_o}}{a}|V_Y|V_Z - V_Z V_Y\right]\sin\theta\right\} \tag{A-17}$$

$$(L_z)_A = -(L_x)_A\frac{\cos\beta}{\sin\beta}$$

The aerodynamic moment at the hub (z-axis) is

$$\underset{\sim}{M}_A = \int_0^1 \underset{\sim}{r} \times \underset{\sim}{L}_A R\, d\xi \tag{A-18}$$

The equations of motion of the blade are

$$(\underset{\sim}{M}_I + \underset{\sim}{M}_A)\cdot\underset{\sim}{j} + I\omega_\beta^2(\beta - \beta_{pc}) + I\omega_{\beta\zeta}\zeta = 0 \tag{A-19}$$

in flap and

$$(\underset{\sim}{M}_I + \underset{\sim}{M}_A)\cdot\underset{\sim}{K} + I\omega_\zeta^2\zeta + I\omega_{\beta\zeta}(\beta - \beta_{pc}) = 0 \tag{A-20}$$

in lag, where β_{pc} is the preconing angle, and ω_β , ω_ζ are the bending frequencies in the flap and in the lag direction, $\omega_{\beta\zeta}$ is due to the elastic coupling. (Note that, depending on the construction of the blade, ω_β , ω_ζ , $\omega_{\beta\zeta}$ can be a function of the pitch angle). In (A-19) and (A-20), it can be shown that by neglecting small terms, such as α_{eff}^2, $\frac{C_{d_o}}{a}$ compared to one,

$$
\begin{aligned}
(\underset{\sim}{M}_I + \underset{\sim}{M}_A)\cdot\underset{\sim}{j} &= I\Omega^2\left[\ddot\beta + (1+\dot\zeta)^2 \sin\beta\cos\beta\right] \\
&+ \frac{\gamma}{8} I\Omega^2 \Big\{ B\sin\theta + \left[A\dot\beta - \mu\cos(\psi+\zeta)\sin\beta\, C\right]\sin\theta\cos\theta \\
&- \left[A\cos\beta(1+\dot\zeta) + \mu\sin(\psi+\zeta)\,C\right](1+\sin^2\theta) \\
&+ \left[\cos\beta(1+\dot\zeta)^2 + \frac{8}{3}\mu(1+\dot\zeta)\sin(\psi+\zeta) + 2\mu^2\sin^2(\psi+\zeta)\right]\sin\theta \\
&- \Big[\cos\beta\,\dot\beta(1+\dot\zeta) + \frac{4}{3}\mu\sin\beta\cos\beta\cos(\psi+\zeta)(1+\dot\zeta) \\
&+ \frac{4}{3}\mu\dot\beta\sin(\psi+\zeta) + 2\mu^2\cos(\psi+\zeta)\sin(\psi+\zeta)\sin\beta\Big]\cos\theta\Big\}
\end{aligned}
\tag{A-21}
$$

$$
\begin{aligned}
(\underset{\sim}{M}_I + \underset{\sim}{M}_A)\cdot\underset{\sim}{K} &= I\Omega^2\left[\ddot\zeta\cos\beta - 2(1+\dot\zeta)\dot\beta\sin\beta\right] \\
&+ \frac{\gamma}{8} I\Omega^2 \Big\{ B\cos\theta + \left[A\dot\beta + \mu\cos(\psi+\zeta)\sin\beta\, C\right](1+\cos^2\theta) \\
&- \left[A\cos\beta(1+\dot\zeta) + \mu C\sin(\psi+\zeta)\right]\sin\theta\cos\theta \\
&+ \left[\dot\beta^2 + \frac{8}{3}\mu\dot\beta\cos(\psi+\zeta)\sin\beta + 2\mu^2\cos^2(\psi+\zeta)\sin^2\beta\right]\cos\theta \\
&- \Big[\cos\beta\,\dot\beta(1+\dot\zeta) + \frac{4}{3}\mu\sin\beta\cos\beta\cos(\psi+\zeta)(1+\dot\zeta) + \frac{4}{3}\mu\dot\beta\sin(\psi+\zeta) \\
&+ 2\mu^2\cos(\psi+\zeta)\sin(\psi+\zeta)\sin\beta\Big]\sin\theta \\
&- \frac{C_{d_0}}{a}\left[\cos^2\beta(1+\dot\zeta)^2 + \frac{8}{3}\mu\cos\beta(1+\dot\zeta)\sin(\psi+\zeta) + 2\mu^2\sin^2(\psi+\zeta)\right]\cos\theta\Big\}
\end{aligned}
$$

where

$$\gamma = \frac{\rho_a\, a\, c\, R^4}{I}$$

is the Lock number

$$A = 4\int_0^1 \lambda \xi^2 d\xi$$
$$B = 4\int_0^1 \lambda^2 \xi \, d\xi \qquad \text{(A-22)}$$
$$C = 4\int_0^1 \lambda \xi \, d\xi$$

To further simplify (A-19) and (A-20), we shall assume that elastic coupling $\omega_{\beta\zeta} \leq 0(\theta)$, the pitch angle θ, the static equilibrium flap angle β_0, and the static equilibrium lag angle ζ_0 are small, so that θ^2, β_0^2 and ζ_0 can be neglected as compared to 1. The quantities β_0 and ζ_0 satisfy

$$\Omega^2\beta_0 + \omega_\beta^2(\beta_0 - \beta_{pc}) + \omega_{\beta\zeta}\zeta_0 = \frac{\gamma}{8}(\theta - A)\Omega^2$$
$$\omega_{\beta\zeta}(\beta_0 - \beta_{pc}) + \omega_\zeta^2 \zeta_0 = -\frac{\gamma}{8}\left[A\theta + \frac{C_{d_0}}{a} - B\right]\Omega^2 \qquad \text{(A-23)}$$

We shall define the following non-dimensional parameters

$$\Omega_\beta^2 = \omega_\beta^2/\Omega^2$$
$$\Omega_\zeta^2 = \omega_\zeta^2/\Omega^2$$
$$\Omega_{\beta\zeta} = \omega_{\beta\zeta}/\Omega^2$$
$$g_\beta = 2\Omega_\beta \eta_\beta + \frac{\gamma}{8}$$
$$g_\zeta = 2\Omega_\zeta \eta_\zeta + \frac{\gamma}{8}\left(A\theta + 2\frac{C_{d_0}}{a}\right) \qquad \text{(A-24)}$$
$$X = \frac{\gamma}{8}(2\theta - A) - 2\beta_0$$
$$Y = 2\beta_0 - \frac{\gamma}{8}(\theta - 2A)$$

Substituting (A-23) into (A-19) and (A-20), replacing β by $\beta+\beta_0$ and ζ by $\zeta+\zeta_0$, i.e. using β, ζ to denote the perturbed motion flap and lag, respectively, and retaining only proper nonlinear terms we obtain Eqs. (1) and (2).

Appendix B. Definition of Various Functions and Quantities

$$L(\beta,\zeta) = \ddot{\beta} + g_\beta \dot{\beta} + (1+\Omega_\beta^2)\beta + \Omega_{\beta\zeta}\zeta - X\dot{\zeta} \quad \text{(B-1)}$$

$$M(\beta,\zeta) = \ddot{\zeta} + g_\zeta \dot{\zeta} + \Omega_\zeta^2 \zeta + \Omega_{\beta\zeta}\beta - Y\dot{\beta} \quad \text{(B-2)}$$

$$f_1(\psi,\theta) = \frac{\gamma}{8}\left[\left(\frac{8}{3}\theta - c\right)\sin\psi - \frac{4}{3}\beta_0\cos\psi\right] \quad \text{(B-3)}$$

$$f_1'(\psi,\theta) = f_1(\psi,\theta)/\varkappa \quad \text{(B-3)'}$$

$$h_1(\psi,\theta) = -\frac{\gamma}{8}\left[\left(\frac{8}{3}\frac{C_{d_0}}{a} + c\theta\right)\sin\psi + \left(\frac{4}{3}\theta - 2c\right)\beta_0\cos\psi\right] \quad \text{(B-4)}$$

$$h_1'(\psi,\theta) = h_1(\psi,\theta)/\varkappa \quad \text{(B-4)'}$$

$$f_2(\psi,\theta) = \frac{\gamma}{8}(\theta - \beta_0\sin 2\psi - \theta\cos 2\psi) \quad \text{(B-5)}$$

$$f_2'(\psi,\theta) = f_2(\psi,\theta)/\varphi \quad \text{(B-5)'}$$

$$h_2(\psi,\theta) = \frac{\gamma}{8}\left[\beta_0^2 - \frac{C_{d_0}}{a} - \beta_0\theta\sin 2\psi + \left(\beta_0^2 + \frac{C_{d_0}}{a}\right)\cos 2\psi\right] \quad \text{(B-6)}$$

$$h_2'(\psi,\theta) = h_2(\psi,\theta)/\varphi \quad \text{(B-6)'}$$

$$f_3(\psi,\theta;\beta,\zeta) = \frac{\gamma}{8}\left\{\left(\frac{8}{3}\theta\dot{\zeta} - \frac{4}{3}\dot{\beta} + \beta_0\dot{\zeta}\right)\sin\psi + \left[\left(\frac{8}{3}\theta - c\right)\zeta - \frac{4}{3}\beta - \frac{4}{3}\beta_0\dot{\zeta}\right]\cos\psi\right\} \quad \text{(B-7)}$$

$$h_3(\psi,\theta;\beta,\zeta) = \frac{\gamma}{8}\left\{-\left(\frac{8}{3}\frac{C_{d_0}}{a}\dot{\zeta} + \frac{4}{3}\theta\dot{\beta}\right)\sin\psi + \left[\left(2c - \frac{4}{3}\theta\right)\beta - \frac{4}{3}\beta_0\theta\dot{\zeta} + \frac{2}{3}\beta_0\dot{\beta} - c\theta\zeta\right]\cos\psi\right\} \quad \text{(B-8)}$$

$$f_4(\beta,\zeta) = -2\beta\dot{\zeta} - \left(\beta_0 - \frac{\gamma}{8}\theta\right)\dot{\zeta}^2 - \frac{\gamma}{8}\dot{\beta}\dot{\zeta} \quad \text{(B-9)}$$

$$h_4(\beta,\zeta) = 2\beta\dot{\beta} + \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\dot{\beta}\dot{\zeta} + \frac{\gamma}{8}\dot{\beta}^2 \quad \text{(B-10)}$$

$$f_5(\psi,\theta;\beta,\zeta)=\frac{r}{8}\left[\frac{4}{3}\beta\zeta\sin\psi+\left(\frac{8}{3}\theta\dot{\zeta}\zeta-\frac{4}{3}\beta\dot{\zeta}-\frac{4}{3}\dot{\beta}\zeta\right)\cos\psi\right] \tag{B-12}$$

$$\begin{aligned}h_5(\psi,\theta;\beta,\zeta)=\frac{r}{8}\Big\{&\left[\left(\frac{4}{3}\theta-2c\right)\beta\zeta+\frac{8}{3}\beta_0\dot{\beta}\zeta+\frac{4}{3}\beta_0\theta\zeta\dot{\zeta}\right]\sin\psi\\&+\left[\frac{8}{3}\beta\dot{\beta}-\frac{4}{3}\theta\dot{\zeta}\beta-\frac{4}{3}\theta\zeta\dot{\beta}\right]\cos\psi\Big\}\end{aligned} \tag{B-13}$$

$$f_6(\psi,\theta;\beta,\zeta)=\frac{r}{8}(2\theta\zeta-\beta)\sin 2\psi \tag{B-14}$$

$$h_6(\psi,\theta;\beta,\zeta)=\frac{r}{8}\left\{2\beta_0\beta-\left[2\left(\beta_0^2+\frac{C_{d_0}}{a}\right)\zeta+\theta\beta\right]\sin 2\psi+2\beta_0\beta\cos 2\psi\right\} \tag{B-15}$$

$$\begin{aligned}f_7(\beta_1,\zeta_1;\beta_2,\zeta_2)=&-2\beta_1\dot{\zeta}_2-2\beta_2\dot{\zeta}_1-2\left(\beta_0-\frac{r}{8}\theta\right)\dot{\zeta}_1\dot{\zeta}_2-\beta_1\dot{\zeta}_1^2\\&-\frac{r}{8}(\dot{\beta}_1\dot{\zeta}_2+\dot{\beta}_2\dot{\zeta}_1)-2\frac{\partial^2\beta_1}{\partial\tau_2\partial\tau_0}-g_{\beta c}\frac{\partial\beta_1}{\partial\tau_2}+X_0\frac{\partial\zeta_1}{\partial\tau_2}\\&-g_{\beta 2}\frac{\partial\beta_1}{\partial\tau_0}+X_2\frac{\partial\zeta_1}{\partial\tau_0}-\Omega_{\beta\zeta 2}\zeta_1\end{aligned} \tag{B-16}$$

$$\begin{aligned}h_7(\beta_1,\zeta_1;\beta_2,\zeta_2)=&\,2\beta_1\dot{\beta}_2+2\dot{\beta}_1\beta_2+\left(2\beta_0-\frac{r}{8}\theta\right)(\dot{\beta}_1\dot{\zeta}_2+\dot{\beta}_2\dot{\zeta}_1)\\&+\frac{r}{4}\dot{\beta}_1\dot{\beta}_2+2\beta_1\dot{\beta}_1\dot{\zeta}_1-2\frac{\partial^2\zeta_1}{\partial\tau_2\partial\tau_0}-g_{\zeta c}\frac{\partial\zeta_1}{\partial\tau_2}+Y_0\frac{\partial\beta_1}{\partial\tau_2}\\&-g_{\zeta 2}\frac{\partial\zeta_1}{\partial\tau_0}+Y_2\frac{\partial\beta_1}{\partial\tau_0}-\Omega_{\beta\zeta 2}\beta_1\end{aligned} \tag{B-17}$$

$$\underset{\sim}{N}(\alpha,\theta)=\begin{pmatrix}\Omega_\zeta^2-\alpha^2+i\alpha g_\zeta & i\alpha X-\Omega_{\beta\zeta}\\ i\alpha Y-\Omega_{\beta\zeta} & 1+\Omega_\beta^2-\alpha^2+i\alpha g_\beta\end{pmatrix}\frac{1}{D(\alpha,\theta)} \tag{B-18}$$

$$D(x) = (\Omega_\beta^2 + 1 - x^2 + i g_\beta x)(\Omega_\zeta^2 - x^2 + i g_\zeta x) - (\Omega_{\beta\zeta} - i Z x)(\Omega_{\beta\zeta} - i Y x) \quad \text{(B-19)}$$

$$u = \frac{i \Omega_F Z_0 - \Omega_{\beta\zeta c}}{1 + \Omega_{\beta c}^2 - \Omega_F^2 + i g_{\beta c} \Omega_F} \quad \text{(B-20)}$$

u^* = complex conjugate of u (B-21)

$$\alpha_1 = -2 i \Omega_F u + \left(\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F^2 + \frac{\gamma}{8}\Omega_F^2 u \quad \text{(B-22)}$$

$$\alpha_2 = 2 i \Omega_F u^2 - \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F^2 u - \frac{\gamma}{8}\Omega_F^2 u^2 \quad \text{(B-23)}$$

$$\alpha_3 = -2 i \Omega_F (u^* - u) - 2\left(\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F^2 - \frac{\gamma}{8}\Omega_F^2 (u^* + u) \quad \text{(B-24)}$$

$$\alpha_4 = \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F^2 (u^* + u) + \frac{\gamma}{8} \cdot 2 \Omega_F^2 u^* u \quad \text{(B-25)}$$

$$P_1 = -(2 i \Omega_F + g_{\beta 0}) u + X_0 \quad \text{(B-26)}$$

$$P_2 = -(2 i \Omega_F + g_{\zeta 0}) + Y_0 u \quad \text{(B-27)}$$

$$P_3 = -i g_{\beta 2} \Omega_F u + i \Omega_F Z_2 - \Omega_{\beta\zeta 2} \quad \text{(B-28)}$$

$$P_4 = -i \Omega_F g_{\zeta 2} + (i \Omega_F Y_2 - \Omega_{\beta\zeta 2}) u \quad \text{(B-29)}$$

$$\begin{pmatrix}\beta_{21}\\ \zeta_{21}\end{pmatrix} = \underset{\sim}{N}(2\Omega_F,\theta_c)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \tag{B-30}$$

$$\begin{pmatrix}\beta_{22}\\ \zeta_{22}\end{pmatrix} = \underset{\sim}{N}(0,\theta_c)\begin{pmatrix}\alpha_3\\ \alpha_4\end{pmatrix} \tag{B-31}$$

$$\begin{aligned} P_5 = &-2u^*(2i\Omega_F)\zeta_{21} - 2i\Omega_F(\beta_{22}-\beta_{21}) + \Omega_F^2(u^*-2u) \\ &-(\beta_0-\tfrac{\gamma}{8}\theta)4\Omega_F^2\zeta_{21} - \tfrac{\gamma}{4}u^*\Omega_F^2\zeta_{21} - \tfrac{\gamma}{4}\Omega_F^2\beta_{21} \end{aligned} \tag{B-32}$$

$$\begin{aligned} P_6 = &\,2i\Omega_F u^*\beta_{21} + 2ui\Omega_F\beta_{22} + 2(2\beta_0-\tfrac{\gamma}{8}\theta)u^*\Omega_F^2\zeta_{21} \\ &+(2\beta_0-\tfrac{\gamma}{8}\theta)2\Omega_F^2\beta_{21} + \tfrac{\gamma}{2}\Omega_F^2 u^*\beta_{21} + 2u^2\Omega_F^2 \end{aligned} \tag{B-33}$$

$$P_7 = \frac{\gamma}{8}\left[-\left(\frac{8}{3}\theta - c\right)\frac{i}{2} - \frac{2}{3}\beta_0\right] \tag{B-34}$$

$$P_8 = \frac{\gamma}{8}\left[\left(\frac{8}{3}\frac{C_{d_0}}{a} + C\theta\right)\frac{i}{2} + \left(c - \frac{2}{3}\theta\right)\beta_0\right] \tag{B-35}$$

$$\begin{pmatrix}\beta_{12}\\ \zeta_{12}\end{pmatrix} = \underset{\sim}{N}(1,\theta)\begin{pmatrix}P_7\\ P_8\end{pmatrix} \tag{B-36}$$

$$P_9 = \frac{\gamma}{8}\left\{\left[\frac{8}{3}\theta(-i\Omega_F) + \frac{4}{3}u^* i\Omega_F + \frac{4}{3}\beta_0\right]\left(-\frac{i}{2}\right) + \frac{1}{2}\left(\frac{8}{3}\theta - c - \frac{4}{3}u^* + \frac{4}{3}\beta_0\Omega_F i\right)\right\}$$

$$+\left[-2u^*\zeta_{12} i + 2\beta_{12}\Omega_F i - 2\left(\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F\zeta_{12} - \frac{\gamma}{8}\Omega_F\beta_{12}\right]\varkappa_0 \qquad \text{(B-37)}$$

$$P_{10} = \frac{\gamma}{8}\left[\frac{4}{3}\frac{C_{d_0}}{a}\Omega_F + \frac{2}{3}\theta u^*\Omega_F + \left(c - \frac{2}{3}\theta\right)u^* + \frac{4}{3}\beta_0(-i\Omega_F)u^* + \frac{2}{3}\theta\beta_0\Omega_F i - \frac{c\theta}{2}\right]$$

$$+\left[2u^*\beta_{12} i - 2u^*\beta_{12}\Omega_F i + \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\left(u^*\Omega_F\zeta_{12} + \Omega_F\beta_{12}\right)\right. \qquad \text{(B-38)}$$

$$\left. + \frac{\gamma}{4}\Omega_F u^*\beta_{12}\right]\varkappa_0$$

$$P_{11} = \frac{\gamma}{8}\left(\frac{4}{3}\theta\Omega_F - \frac{2}{3}u\Omega_F - \frac{2}{3}\beta_0 i + \frac{4}{3}\theta - \frac{c}{2} - \frac{2}{3}u - \frac{2}{3}\beta_0 i\Omega_F\right) + \left[-2u\zeta_{12} i\right.$$

$$\left. - 2\beta_{12}\Omega_F i + 2\left(\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F\zeta_{12} + \frac{\gamma}{8}\Omega_F\left(u\zeta_{12} + \beta_{12}\right)\right]\varkappa_0 \qquad \text{(B-39)}$$

$$P_{12} = \frac{\gamma}{8}\left[-\frac{4}{3}\frac{C_{d_0}}{a}\Omega_F - \frac{2}{3}\theta u\Omega_F + \left(c - \frac{2}{3}\theta\right)u + \frac{4}{3}\beta_0 i\Omega_F u - \frac{2}{3}\theta\beta_0 i\Omega_F - \frac{c\theta}{2}\right]$$

$$+\left[2u\beta_{12} i(1+\Omega_F) - \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\Omega_F\left(u\zeta_{12} + \beta_{12}\right) - \frac{\gamma}{4}\Omega_F u\beta_{12}\right]\varkappa_0 \qquad \text{(B-40)}$$

$$P_{13} = \frac{\gamma}{8}\left[\frac{4}{3}\theta\zeta_{12} - \frac{2}{3}\beta_0\zeta_{12} i - \frac{2}{3}\beta_{12} + \left(\frac{4}{3}\theta - \frac{c}{2}\right)\zeta_{12} - \frac{2}{3}\beta_{12} - \frac{2}{3}\beta_0\zeta_{12} i\right]\varkappa_0$$

$$+\frac{\gamma}{8}\left(\frac{\beta_0 i}{2} - \frac{\theta}{2}\right)\frac{\varkappa}{\varphi}\varphi_0 + \left[\cancel{2\beta_{12}}\zeta_{12} i + \left(\beta_0 - \frac{\gamma}{8}\theta\right)\zeta_{12}^2 + \frac{\gamma}{8}\beta_{12}\zeta_{12}\right]\varkappa_0^2 \qquad \text{(B-41)}$$

$$P_{14} = \frac{\gamma}{8}\left[-\frac{4}{3}\frac{C_{d_0}}{a}\zeta_{12} - \frac{2}{3}\theta\beta_{12} + \left(c - \frac{2}{3}\theta\right)\beta_{12} + \frac{4}{3}\theta_0\beta_{12} i - \frac{2}{3}\theta\beta_0\zeta_{12} i - \frac{1}{2}c\theta\zeta_{12}\right]\varkappa_0$$

$$+\frac{\gamma}{16}\left(\beta_0\theta i + \beta_0^2 + \frac{C_{d_0}}{a}\right)\frac{\varkappa}{\varphi}\varphi_0 + \left[2\beta_{12}^2 i - \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\beta_{12}\zeta_{12} - \frac{\gamma}{8}\beta_{12}^2\right]\varkappa_0^2 \qquad \text{(B-42)}$$

$$P_{15} = \frac{\gamma}{8}\left[-\frac{c}{2}(\zeta_{12}^{*}+\zeta_{12})\varkappa_0+\frac{\theta\varkappa\varphi_0}{\varphi}\right]+\left[2i(\beta_{12}^{*}\zeta_{12}+\beta_{12}\zeta_{12}^{*})-2\left(\beta_0-\frac{\gamma}{8}\theta\right)\zeta_{12}\zeta_{12}^{*}\right.$$
$$\left.-\frac{\gamma}{8}(\beta_{12}^{*}\zeta_{12}+\beta_{12}\zeta_{12}^{*})\right]\varkappa_0^2 \qquad \text{(B-43)}$$

$$P_{16} = \frac{\gamma}{8}\left[\frac{4}{3}\frac{C_{d_0}}{a}(\zeta_{12}^{*}+\zeta_{12})+C(\beta_{12}^{*}+\beta_{12})+\frac{4}{3}\beta_0 i(\beta_{12}-\beta_{12}^{*})+\frac{2}{3}\theta i(\zeta_{12}^{*}-\zeta_{12})\right.$$
$$\left.-\frac{C\theta}{2}(\zeta_{12}^{*}+\zeta_{12})\right]\varkappa_0+\frac{\gamma}{8}\left(\beta_0^2-\frac{C_{d_0}}{a}\right)\frac{\varkappa}{\varphi}\varphi_0+\left(2\beta_0-\frac{\gamma}{8}\theta\right)(\beta_{12}^{*}\zeta_{12}+\beta_{12}\zeta_{12}^{*})\varkappa_0 \qquad \text{(B-44)}$$
$$+\frac{\gamma}{4}\beta_{12}\beta_{12}^{*}\varkappa_0^2$$

$$\begin{pmatrix}\beta_{23}\\ \zeta_{23}\end{pmatrix} = \underset{\sim}{N}(1-\Omega_F,\theta_c)\begin{pmatrix}P_9\\ P_{10}\end{pmatrix} \qquad \text{(B-45)}$$

$$\begin{pmatrix}\beta_{24}\\ \zeta_{24}\end{pmatrix} = \underset{\sim}{N}(1+\Omega_F,\theta_c)\begin{pmatrix}P_{11}\\ P_{12}\end{pmatrix} \qquad \text{(B-46)}$$

$$\begin{pmatrix}\beta_{25}\\ \zeta_{25}\end{pmatrix} = \underset{\sim}{N}(2,\theta)\begin{pmatrix}P_{13}\\ P_{14}\end{pmatrix} \qquad \text{(B-47)}$$

$$\begin{pmatrix}\beta_{26}\\ \zeta_{26}\end{pmatrix} = \underset{\sim}{N}(0,\theta)\begin{pmatrix}P_{15}\\ P_{16}\end{pmatrix} \qquad \text{(B-48)}$$

$$
\begin{aligned}
P_{17} = \frac{\gamma}{8}\Big\{ & \frac{4}{3}\theta(\Omega_F-1)\zeta^*_{23} - \frac{2}{3}(\Omega_F-1)\beta^*_{23} - \frac{2}{3}\beta_0\zeta^*_{23} i + \left(\frac{4}{3}\theta - \frac{c}{2}\right)\zeta^*_{23} - \frac{2}{3}\beta^*_{23} \\
& - \frac{2}{3}\beta_0(\Omega_F-1)\zeta_{23} i - \frac{4}{3}\theta(\Omega_F+1)\zeta_{24} + \frac{2}{3}(\Omega_F-1)\beta_{24} + \frac{2}{3}\beta_0\zeta_{24} \\
& + \left(\frac{4}{3}\theta - \frac{c}{2}\right)\zeta_{24} - \frac{2}{3}\beta_{24} - \frac{2}{3}\beta_0(\Omega_F+1) i\zeta_{24} + \Big[\frac{2i}{3}(u\zeta_{12} - u\zeta^*_{12} + \beta_{12} - \beta^*_{12}) \\
& + \left(\frac{4}{3}\theta - \frac{2}{3}u\right) i\Omega_F(\zeta^*_{12} + \zeta_{12}) + \left(\frac{4}{3}\theta - \frac{2}{3}u\right)(\zeta_{12} - \zeta^*_{12}) i - \frac{2}{3} i\Omega_F(\beta^*_{12} + \zeta^*_{12}) \\
& - \frac{2}{3} i(\beta_{12} - \beta^*_{12})\Big]\varkappa_0\Big\} - 2\Big[(\Omega_F-1) i\beta_{12}\zeta^*_{23} + (\Omega_F+1) i\beta^*_{12}\zeta_{24} \qquad \text{(B-49)} \\
& \qquad + i\zeta_{12}\beta^*_{23} - i\zeta^*_{12}\beta_{24}\Big]\varkappa_0 - (\beta_{12}\zeta^*_{12} - \beta^*_{12}\zeta_{12})\Omega_F\varkappa_0 \\
& - 2\left(\beta_0 - \frac{\gamma}{8}\theta\right)\Big[(\Omega_F+1)\zeta^*_{12}\zeta_{24} - (\Omega_F-1)\zeta_{12}\zeta^*_{23}\Big]\varkappa_0 - 2i\Omega_F\beta_{26} \\
& + \frac{\gamma}{8}\Big[(\Omega_F+1)(\beta^*_{12}\zeta_{24} + \zeta^*_{12}\beta_{24}) - (\Omega_F-1)(\beta_{12}\zeta^*_{23} + \zeta_{12}\beta^*_{23})\Big]\varkappa_0
\end{aligned}
$$

$$
\begin{aligned}
P_{18} = \frac{\gamma}{8}\Big\{ & (\Omega_F+1)\left(\frac{4}{3}\frac{ca_0}{a}\zeta_{24} + \frac{2}{3}\theta\beta_{24}\right) - (\Omega_F-1)\left(\frac{4}{3}\frac{ca_0}{a}\zeta^*_{23} + \frac{2}{3}\theta\beta^*_{23}\right) + \left(c - \frac{2}{3}\theta\right)(\beta^*_{23} + \beta_{24}) \\
& + (\Omega_F+1) i\beta_0\left(\frac{4}{3}\beta_{24} - \frac{2}{3}\theta\zeta_{24}\right) + (\Omega_F-1) i\beta_0\left(\frac{4}{3}\beta^*_{23} - \frac{2}{3}\theta\zeta^*_{23}\right) - \frac{c\theta}{2}(\zeta^*_{23} + \zeta_{24}) + 2\beta_0 u \\
& + \Big(\left(c - \frac{2}{3}\theta\right)\left[u(\zeta^*_{12} - \zeta_{12}) + \beta^*_{12} - \beta_{12}\right] i + \frac{4}{3}\beta_0\left[u\Omega_F(\zeta^*_{12} - \zeta_{12}) - \beta^*_{12} - \beta_{12}\right] \\
& + \frac{2}{3}\theta\beta_0\left[\Omega_F(\zeta^*_{12} - \zeta_{12}) - \zeta^*_{12} - \zeta_{12}\right] + \frac{4}{3} u i\left[\beta_{12} - \beta^*_{12} + \Omega_F(\beta_{12} + \beta^*_{12})\right] \qquad \text{(B-50)} \\
& - \frac{2}{3}\theta i\left[u\Omega_F(\zeta^*_{12} + \zeta_{12}) - \beta^*_{12} + \beta_{12}\right] - \frac{2}{3}\theta i\left[u(\zeta_{12} - \zeta^*_{12}) + \Omega_F(\beta^*_{12} + \beta_{12})\right]\Big)\varkappa_0\Big\} \\
& + 2i\Omega_F(\beta_{12}\beta^*_{23} + \beta^*_{12}\beta_{24})\varkappa_0 + 2i\Omega_F u\beta_{26} \\
& + \left(2\beta_0 - \frac{\gamma}{8}\theta\right)\Big[(\Omega_F+1)(\beta^*_{12}\zeta_{24} + \beta_{24}\zeta^*_{12}) - (\Omega_F-1)(\beta_{12}\zeta^*_{23} + \zeta_{12}\beta^*_{23})\Big]\varkappa_0 \\
& + \frac{\gamma}{4}\Big[(\Omega_F+1)\beta^*_{12}\beta_{24} - (\Omega_F-1)\beta_{12}\beta^*_{23}\Big] - \Omega_F u(\beta_{12}\zeta^*_{12} - \beta^*_{12}\zeta_{12})\varkappa_0^2 \\
& + u(\beta_{12}\zeta^*_{12} + \beta^*_{12}\zeta_{12})\varkappa_0^2
\end{aligned}
$$

$$P_{19} = \frac{\gamma}{8}\Big\{-\frac{8}{3}\theta\Omega_F\zeta_{21}^* + \frac{4}{3}\Omega_F\beta_{21}^* - \frac{2i}{3}\beta_0\zeta_{21}^* + \left(\frac{4}{3}\theta - \frac{c}{2} + \frac{4}{3}\beta_0\Omega_F i\right)\zeta_{21}^* - \frac{2}{3}\beta_{21}^*$$

$$-\frac{2}{3}u^* i + \Omega_F i(-\theta + u^*)\frac{4}{3}\Big\} - 2i\left[u^*\zeta_{23}(1-\Omega_F) - 2\beta_{12}\zeta_{21}^*\Omega_F \varkappa_0\right]$$

$$- i(\zeta_{12}\beta_{21}^* - \Omega_F\beta_{23}\varkappa_0) - 2\Omega_F\left(\beta_0 - \frac{\gamma}{8}\theta\right)\left[\zeta_{23}(1-\Omega_F) + 2\zeta_{12}\zeta_{21}^*\varkappa_0\right] \quad \text{(B-51)}$$

$$-\frac{\gamma}{8}\Omega_F\left[u^*\zeta_{23}(1-\Omega_F) + 2\beta_{12}\zeta_{21}^*\varkappa_0\right] - \frac{\gamma}{8}\Omega_F\left[\beta_{23}(1-\Omega_F) + 4\zeta_{12}\beta_{21}^*\varkappa_0\right]$$

$$+\Omega_F\left[-2\zeta_{12}u^* + \Omega_F\beta_{12}\right]\varkappa_0$$

$$P_{20} = \frac{\gamma}{8}\Big\{\frac{8}{3}\frac{c d_0}{a}\Omega_F\zeta_{21}^* + \frac{4}{3}\theta\Omega_F\beta_{21}^* + \left(c - \frac{2}{3}\theta - \frac{8}{3}\beta_0\Omega_F i\right)\beta_{21}^* + \left(\frac{4}{3}\theta\beta_0 i\Omega_F - \frac{c\theta}{2}\right)\zeta_{21}^*$$

$$-\left(\frac{2}{3}\theta - c\right)u^* i - \frac{4}{3}\beta_0 u^*\Omega_F - \frac{2}{3}\beta_0\theta\Omega_F - \frac{4}{3}u^{*2}\Omega_F + \frac{4}{3}\theta u^* i\Omega_F$$

$$+2i\left[u^*\beta_{23}(1-\Omega_F) - \beta_{12}\beta_{21}^* 2\Omega_F\varkappa_0\right] + 2i\left[-\Omega_F u^*\beta_{23} + \beta_{12}\beta_{21}^*\varkappa_0\right]$$

$$+\left(2\beta_0 - \frac{\gamma}{8}\theta\right)\left[(1-\Omega_F)(u^*\zeta_{23} + \beta_{23}) + 2(\beta_{12}\zeta_{21}^* + \zeta_{12}\beta_{21}^*)\varkappa_0\right]\Omega_F \quad \text{(B-52)}$$

$$+\frac{\gamma}{4}\Omega_F\left[u^*\beta_{23}(1-\Omega_F) + 2\beta_{12}\beta_{21}^*\varkappa_0\right]$$

$$+\left[2\Omega_F\beta_{12}u^* + 2\Omega_F u^*\left(u^*\zeta_{12} - \Omega_F\beta_{12}\right)\right]\varkappa_0$$

$$P_{21} = \frac{\gamma}{8}\Big\{\frac{8}{3}\theta\zeta_{25} - 2i\beta_0\zeta_{25} - 2\beta_{25} + \left(\frac{4}{3}\theta - \frac{c}{2}\right)\zeta_{25} - \frac{2}{3}\beta_{12}\zeta_{12} i\varkappa_0^2$$

$$+\varkappa_0^2\left[\frac{4}{3}\theta i\zeta_{12}^2 - \frac{4}{3}i\zeta_{12}\beta_{12}\right] + (\beta_{12} - 2\theta\zeta_{12})\frac{i}{2}\varkappa_0\Big\}$$

$$+\varkappa_0\Big[4\beta_{12}\zeta_{25} i - 2i\zeta_{12}\beta_{25} + \left(\beta_0 - \frac{\gamma}{8}\theta\right)4\zeta_{12}\zeta_{25} \quad \text{(B-53)}$$

$$+\frac{\gamma}{8}(\beta_{12}\zeta_{25} + \beta_{25}\zeta_{12})\Big] + \beta_{12}\zeta_{12}^2\varkappa_0^3$$

$$P_{22} = \frac{\gamma}{8}\Big\{-\frac{8}{3}\frac{Cd_0}{a}\zeta_{25} - \frac{4}{3}\theta\beta_{25} + \left(c - \frac{2}{3}\theta + \frac{4}{3}\beta_0 i\right)\beta_{25} - \left(\frac{4}{3}\theta\beta_0 i + \frac{C\theta}{2}\right)\zeta_{25}$$

$$+\left(\left[(c-2\theta)i + \frac{4}{3}\beta_0\right]\beta_{12}\zeta_{12} + \frac{2}{3}\beta_0\theta\zeta_{12}^2 + \frac{4i}{3}\beta_{12}^2\right)\varkappa_0^2 + \beta_0\beta_{12}\varkappa_0 \tag{B-54}$$

$$+\frac{i}{2}\theta\beta_{12}\varkappa_0 + \left(\beta_0^2 + \frac{Cd_0}{a}\right)\zeta_{12} i\varkappa_0\Big\} + \Big[b\beta_{12}\beta_{25} i$$

$$-\left(2\beta_0 - \frac{\gamma}{8}\theta\right)\left(\beta_{12}\zeta_{25} + \beta_{25}\zeta_{12}\right) - \frac{\gamma}{2}\beta_{12}\beta_{25} - 2\zeta_{12}\beta_{12}^2\Big]\varkappa_0$$

$$P_{23} = \frac{\gamma}{8}\left[(2-\Omega_F)\left(\frac{4}{3}\zeta_{23} - \frac{2}{3}\beta_{23} - \frac{2i}{3}\beta_0\zeta_{23}\right) - \frac{c}{2}\zeta_{23} - \left(\theta - \frac{u^*}{2}\right)i\right]$$

$$+2i\left[\zeta_{25}\left(-2u^* + 2\beta_0 i\Omega_F - \frac{\gamma}{4}i\Omega_F\right) - \Omega_F\beta_{25}\right] \tag{B-55}$$

$$-\frac{\gamma}{4}\Omega_F\left[u^*\zeta_{25} + \beta_{25}\right]$$

$$P_{24} = \frac{\gamma}{8}\Big\{\zeta_{23}\left[-\frac{4}{3}\frac{Cd_0}{a}(1-\Omega_F) + C\beta_0 i - \frac{C\theta}{2} - \frac{2}{3}\theta\beta_0 i(2-\Omega_F)\right]$$

$$+\beta_{23}\left[-\frac{2}{3}\theta(2-\Omega_F) + C + \frac{4}{3}\beta_0 i(1-\Omega_F) + \left(\beta_0^2 + \frac{Cd_0}{a}\right)i\right. \tag{B-56}$$

$$\left.+\frac{\theta u^* i}{2} + \beta_0 u^* + 4u^*\Omega_F\beta_{25}\right\} + 2i\beta_{25}u^*(2-\Omega_F)$$

$$+ 2\Omega_F\left(2\beta_0 - \frac{\gamma}{8}\theta\right)\left(u^*\zeta_{25} + \beta_{25}\right)$$

$$P_{25} = -2i\left[u(1-\Omega_F)\zeta_{23} + u^*(1+\Omega_F)\zeta_{24} + \Omega_F\beta_{23} - \Omega_F\beta_{24}\right]$$

$$-2\Omega_F\left(\beta_0 - \frac{\gamma}{8}\right)\left[(1+\Omega_F)\zeta_{24} - (1-\Omega_F)\zeta_{23}\right] - \frac{\gamma}{8}\,\frac{2}{3}(u+u^*)i \tag{B-57}$$

$$- \frac{\gamma}{8}\Omega_F\left[u^*(1+\Omega_F)\zeta_{24} - u(1-\Omega_F)\zeta_{23} + (1+\Omega_F)\beta_{24} - (1-\Omega_F)\beta_{23}\right]$$

$$P_{26} = 2i\left[u\beta_{23} + u^*\beta_{24}\right] + \Omega_F\left(2\beta_0 - \frac{\gamma}{8}\theta\right)\left[(1+\Omega_F)u^*\zeta_{24} - (1-\Omega_F)u\zeta_{23}\right.$$

$$\left. + (1+\Omega_F)\beta_{24} - (1-\Omega_F)\beta_{23}\right] + \frac{\gamma}{4}\Omega_F\left[u^*(1+\Omega_F)\beta_{24} - u(1-\Omega_F)\beta_{23}\right] \quad \text{(B-58)}$$

$$+ \frac{\gamma}{8}\left[\left(c - \frac{2}{3}\theta\right)(u+u^*)i + \frac{4}{3}\Omega_F\beta_0(u-u^*)\right]$$

$$P_{27} = \frac{\gamma}{8}\left[\left(\frac{4}{3}\theta - \frac{2}{3}\beta_0 i - \frac{c}{2}\right)\zeta_{26} - \left(\frac{4}{3}\theta + \frac{c}{2} + \frac{2}{3}\beta_0 i\right)\zeta_{25} - \frac{2}{3}\beta_{26} + \frac{2}{3}\beta_{25}\right] \quad \text{(B-59)}$$

$$P_{28} = \frac{\gamma}{8}\left\{\left(c - \frac{2}{3}\theta\right)i\beta_0\zeta_{26} + \frac{8}{3}\frac{C_{d_0}}{a}\zeta_{25} - \left(\frac{2}{3}\theta + c\right)\beta_0\zeta_{25}i - \frac{c\theta}{2}\zeta_{26}\right.$$

$$\left. + \left(c - \frac{2}{3}\theta\right)\beta_{26} + \left(c + \frac{2}{3}\theta\right)\beta_{25} + \frac{8}{3}\beta_0\beta_{25}i - \frac{c\theta}{2}\zeta_{25}\right\} \quad \text{(B-60)}$$

$$P_{29} = \frac{\gamma}{16}(\beta_0 - \theta) \quad \text{(B-61)}$$

$$P_{30} = \frac{\gamma}{16}\left(\beta_0\theta i + \beta_0^2 + \frac{C_{d_0}}{a}\right) \quad \text{(B-62)}$$

$$v = \frac{i\Omega_F Y_0 - \Omega_{\beta sc}}{1 + \Omega_{\beta c}^2 - \Omega_F^2 + i\Omega_F g_{\beta c}} \quad \text{(B-63)}$$

$$K_1 = v P_1 + P_2 \quad \text{(B-64)}$$

$$K_j = -(v P_{2j-1} + P_{2j})/K_1 \qquad j = 2, 3, 4, 5 \quad \text{(B-65)}$$

$$K_j = -(v P_{2j+5} + P_{2j+6})/K_1 \qquad j = 6, 7, \cdots, 12 \quad \text{(B-66)}$$

$$K_{13} = -(v P_{19}^* + P_{20}^*)/K_1 \quad \text{(B-67)}$$