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RESEARCH ON UNSTEADY TRANSONIC FLOW THEORY

BY James D. Revell

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SUMMARY

The problem considered is a two-dimensional theory for the unsteady flow disturbances caused by aeroelastic deformations of a thick wing at high subsonic freestream Mach numbers, having a single, internally embedded supercritical (locally supersonic) steady flow region adjacent to the low pressure side of the wing. The theory develops a matrix of unsteady aerodynamic influence coefficients (AICs) suitable as a strip theory for aeroelastic analysis of large aspect ratio thick wings of moderate sweep, typical of a wide class of current and future aircraft.

The theory derives the linearized unsteady flow solutions separately for both the subcritical and supercritical regions. These solutions are coupled together to give the requisite (wing pressure/downwash) AICs by the intermediate step of defining flow disturbances on the sonic line, and at the shock wave; these intermediate quantities are then algebraically eliminated by expressing them in terms of the wing surface downwash.

A unique feature of the present theory is the idealization of the non-uniform, supercritical, steady flow field as a layered medium where regions of uniform flow are separated by vortex sheets where the acoustic impedance changes discontinuously.

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INTRODUCTION

There has been a recognized need for realistic steady and unsteady transonic airload prediction methods suitable for flutter, gust load, and static aeroelastic analysis of flight structures (particularly wings and control surfaces) since the first advent of high speed, compressibility-induced control problems of World War II aircraft (References 1-3). Although the level flight speeds at that time rarely exceeded 400 mph, the contemporary thick, unswept wings occasionally encountered serious supercritical flow problems in high speed dives (Reference 1, Chapter 9).

The advent of the turboprop and turbojet engine led to an entirely new generation of fighters, bombers, ASW aircraft, and both military and civilian transport aircraft operating at high subsonic Mach numbers. Transport aircraft, in particular, are characterized by thick, moderately swept wings of large aspect ratio. These characteristics are required for efficient, long range cruise. Also, adequate high-lift performance is needed to provide efficient airport performance within noise limits and with minimum wing structural weight to obtain a maximum ratio of useful load to gross weight.

The first generation of jet transport aircraft, in fact, have been operating with supercritical wing flow at the high speed cruise point. Typically, one finds local steady flow Mach numbers, referred to the swept chord, ranging from 1.2 to as high as 1.6 at high angles of attack. This is the typical situation for a 35° swept wing, for example, at a flight Mach number of .85, where the freestream Mach number component normal to the quarter chord line is only .7. Thus, one often sees a two-fold increase of local Mach number in the "supercritical" (locally supersonic) flow region.

The current generation of turbofan-powered wide-body jets operate in essentially the same speed and supercritical flow regime. Primarily, the new turbofan powered aircraft provide improved fuel economy and important reductions in airport jet noise, rather than increased speed. It is

predicted that future technology advances in steady flow aerodynamics may be used to allow: (1) increases of wing thickness, (2) reductions of sweep, (3) increases of cruise speeds, or (4) STOL (powered high lift) capability.

In view of the current stress on economy and noise reduction, the first two possibilities may gain greater emphasis than speed increases for CTOL, and especially for STOL aircraft; however, such trade-offs are a proper function of mission analysis. Suffice it to say that the aircraft evolved from any of the above described advances in technology, including STOL aircraft, will yield designs whose wings will operate well into the supercritical flow regime. Furthermore, military aircraft must be flutter certified to $V = 1.15\ V_D$ (V_D is design or structural limit speed), and commercial aircraft must certify to $1.2\ V_D$, which further extends the transonic domain of interest to the aeroelastician. In military aircraft the current emphasis is upon transonic maneuvering at high C_L, to allow higher load factors under buffet-free conditions.

For the purposes of the present study, the extensive body of experimental data and recent calculation procedures for steady flow can be regarded as available for defining a non-uniform steady flow field "environment" (with variable thermodynamic state variables) into which small amplitude, unsteady aerodynamic flow perturbations must propagate. The aeroelastically induced perturbations are small compared to the steady flow variables; therefore, it is appropriate to <u>linearize</u> the equations of motion with respect to the <u>unsteady</u> perturbations. Since the steady flow field varies, this approach is often referred to as "local linearization" in the literature (References 4 - 7). While this is an appropriate description of the mathematical process, the term "local linearization" is sometimes identified with methods employing additional simplifying assumptions with respect to the boundary conditions, especially neglect of sonic line reflections in the supercritical region. The removal of this deficiency, which is prevalent in all of the current literature on unsteady transonic flow, will be a primary feature of the present analysis.

Also, incorporated in the present method is a series of suggestions for possible simplifying key assumptions for idealizing the steady flow, based on extensive steady flow research within the aerospace industry with regard to transonic airfoils and wings, including the results of airfoil research, and experience derived from aerodynamic wing design and development of transport aircraft.

The above mentioned key assumptions will facilitate the application of other previously well developed pieces of technology in: (1) unsteady aerodynamics for purely subsonic and purely supersonic flow, and (2) relevant technology in acoustic propagation through non-uniform (layered) moving media.

In summary, the analytical approach outlined below and in various Appendices is believed to be one which is physically well grounded, and which will appeal to practical flutter analysts because it will:

- Automatically reduce to standard results under subcritical flow or purely supersonic flow conditions.
- Should cause only a nominal increase in computation time when efficiently implemented on the digital computer.
- Will require only a few readily available input parameters to characterize the essential features of the supercritical steady flow environment.
- Is capable of continuous refinement, as improved steady flow field data description becomes more routinely available.

SYMBOLS

a	speed of sound
A	transfer matrix, Eq (E-4); also coefficients in Eqs (E-94,E-95)
b	wing reference length (semichord), Eq (D-3)
B	$= \sqrt{M^2-1}$, Fig A1, Eqs (A-35, C-3); also, transfer matrix, Eq (E-4); coefficient in Eqs (E-94, E-95)
c	chord (Fig 3); section coefficient (lift, drag or pitching moment) (Fig 3)
C,D	coefficients in velocity potential solution, Eqs (A-15, E-2)
d	differential operator (d/dz)
e	base of natural logarithm
E	~~transfer function or coefficient, with different meanings,~~ depending on special subscripts or supercripts. See Eqs (A-20 to A-32, E-24, E-26, E-30-31)
f	frequency , Hz
F	particular integral forcing functions, Eq (E-5)
G	Green's function transfer matrix for shock induced downwash at wing surface, Eqs (B-20, B-21); also Green's function transfer matrix for shock impingement disturbances due to W_A, Eqs B-27 to B-29; also other meanings, depending on subscripts and superscripts, Eqs (B-38, B-42, B-46)
i	$\sqrt{-1}$ complex number operator
$\vec{i},\vec{j},\vec{k}$	unit vectors
I	integral, Eq (C-19)
I_A	inverse of matrix A, Eq (E-25)

k	unit vector (see above); also reduced frequency, Eq D-3
K	aerodynamic downwash induction matrix, Eqs (B-3, B-49); also, wave parameter, Eqs (A-10, A-11, C-2)
L	Laplace transform operator Eq (E-79)
M	Mach number
$\vec{n}$	unit normal vector
p	lifting pressure; absolute static pressure (Fig 2)
P	interference downwash induction matrices, Eqs (B-58 - B-71)
Q_p	pressure/downwash aerodynamic influence coefficient (AIC) matrix
s	Laplace transform complex variable
t	time, sec.
T	transfer matrix across shock wave, Eq (B-26, D-15); transfer matrices between surface downwash and sonic line disturbances, Eq (B-33); transfer matrix between disturbances entering shock wave and sonic line disturbances (Eq (B-34))
$\vec{u}$	absolute velocity vector
U	local freestream steady flow velocity; also, streamwise velocity perturbation (backwash) in shock wave and sonic line transfer functions (Eqs B-12 - B-39)
V	perturbation velocity normal to shock wave, Eq (B-19)
W	vertical velocity perturbation (downwash)
X	streamwise coordinate with origin at wing leading edge
x	streamwise coordinate with origin at intersection of sonic line and the wing surface
Z	vertical coordinate with origin at wing chordplane (measured perpendicular to wing chord plane)

$z = (Z - Z_{k-1})$ relative vertical coordinate for the kth layer of the local supercritical region, with origin at the interface between the kth and (k-1) layers. (For k = 1, the origin is at the wing surface). See Eqs (A-15 - A-18, C-7)

α reduced frequency parameter in shock wave transfer matrices, Eq (D-2), also coefficient in constraint equation , Eq (E-13)

β $=\sqrt{1-M^2}$, Prandtl-Glauert factor, Eq (A-35); also constraint matrix between freestream and outermost supersonic layer Eqs (E-7, E-12)

γ see Eqs (E-16, E-17)

δ interface streamline slope, Eq (E-19)

Δ difference operator; also $\Delta E_1^{(1,2)}$ has a special meaning, Eqs (E-24, E-31b); also determinant, Eqs (E-49, E-50)

ζ dummy variable of integration, Eq (C-13)

Λ matrices defined by Eqs (B-53 - B-57)

ξ dummy variable of integration (Eqs A-22, E-85)

π 3.14159 . . .

ρ mass density of fluid

Σ summation operator

σ state vectors, Eqs (B-33 - B-41)

ϕ perturbation velocity potential

ω $=2\pi f$, circular frequency, rad./sec

Special Matrix Notation

$[\ ]$ rectangular or square matrix, See Appendix B

$\{\ \}$ column vector matrix (one column only), See Appendix B

$\begin{bmatrix} 1 \end{bmatrix}$ unit diagonal matrix (off diagonal elements are zero) see Appendix B.

SPECIAL PARTIAL DERIVATIVE NOTATION

$(\ )_x$	$= \partial(\ )/\partial x$, etc
$(\ )zz$	$= \partial^2(\ )/\partial z^2$, etc

OTHER SPECIAL NOTATION

$\tilde{(\ )}$	Laplace transformed quantity, Eq (A-5)
$\bar{(\ )}$	complex amplitude for case of harmonic time dependence, Eq (A-1 - A-4); also in Appendix B, used to define modified matrices, Eqs (B-53 - B-57, B-60a, B-68 - B-70, B-76 - B-78)
$\hat{(\ )}$	used to define modified matrix, see Eq B-60b
$\vec{(\ )}$	vector quantity

SUPERSCRIPTS AND SUBSCRIPTS

a	airfoil or wing surface
A	ahead of shock wave
B	aft of shock wave
H	homogeneous (solution of a differential equation), Eqs (B-1, C-13)
I	interference, Eqs (B-58 - B-65); also, imaginary part, Fig. (A-2).
j	quantity evaluated at a particular layer interface, Eq (E-5)
k	superscript denoting a certain interior layer and its associated values of velocity, density, speed of sound and disturbance variables, Eqs (A-19, C-1, E-5)
ℓ	section lift (coefficient), Fig 3
L	local (freestream) Eq (1) of text; also lower wing surface, Fig (B-1), Eqs (B-7, B-13); also refers to layer in multilayer Eq (B-12).
M	refers to multilayer, Eq (B-12)
m	section moment coefficient (Fig 3)

N	refers to interface between outermost supersonic layer and external freestream
n	normal component
p	refers to wing surface pressure, Eq (B-42)
P	refers to particular integral (of a non-homogeneous differential equation), Eqs (A-15 - A-17, A-19 - A-22, B-27 - B-36, B-42, etc.)
R	real part (of s), Fig A-2.
s	refers to wing surface, Eq (B-42)
U	refers to backwash velocity, Eq (B-32); also refers to upper wing surface, or flow field above the wing
W	refers to downwash velocity, Eq B-32
X	refers to streamwise direction or backwash velocity component, Eqs (B-16 - B-18)
Z	refers to vertical direction or downwash velocity component Eqs (B-16 - B-18)
ϕ	refers to dependence on perturbation velocity potential (Eqs B-16 - B-18)
∞	refers to external freestream

NUMERICAL SUPERSCRIPTS AND SUBSCRIPTS

1,2	superscripts refer to interior and exterior side of a certain layer interface, Eqs (E-1 - E-7); also subscripts refer to elements of 2 x 2 transfer matrices, Eqs (E-4, E-7).
o	refers to quantities evaluated from purely subcritical flow theory, Eqs (B-3 - B-6)

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MULTIPLE SUBSCRIPTS AND SUPERSCRIPTS
HAVING UNIQUE MEANINGS

AA,AB,BA,BB	partitions of aerodynamic induction and influence coefficient matrices Eqs (B-3, B-10)
AB	shockwave transfer relationship Eqs (B-26, D-11)
APSH	disturbances impinging on upstream side of shock wave caused by particular integral terms includes (ϕPSH, UPSH, WPSH), Eqs (B-32, B-34)
BSH	conditions on downstream side of shock wave
HML	homogeneous, multilayer (Eq B-12)
PML	particular integral contribution to multilayer solution, Eq (B-12)
$p_s A_U$	refers to wing surface pressure beneath supercritical region Eq (B-42)
SH	shock wave
SL	sonic line
WA	caused by wing surface downwash ahead of shock
WA_L	interference downwash induced at lower wing surface ahead of shock wave Eq B-45
WB_L	interference downwash induced at wing surface aft of the shock wave
WAP	downwash contribution at wing surface ahead of shock wave due to particular integral term, Eqs (B-32, B-33)
WASH	refers to downwash induced by shock wave disturbance at the wing surface ahead of shock wave Eq (B-49)
WBSH	refers to downwash induced by shock wave disturbance at the wing surface aft of the shock wave

BACKGROUND AND PREVIOUS RESEARCH

Review of Existing Unsteady Transonic Flow Methods and Exploratory Research

Older work and its limitations.- It is well known (References 8,9) that the equations of unsteady transonic flow may be linearized for high reduced frequencies; however, it is not clear even in this case that one can neglect the sonic line reflections, associated with a supersonic (supercritical) region embedded in an exterior subsonic flow, and bounded by the sonic line and terminal shock (see Figure 1).

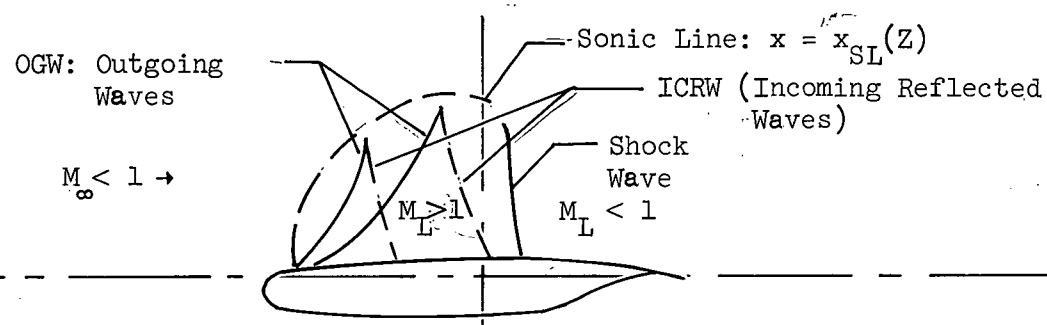

Figure 1 Schematic of Supercritical Flow Embedded in a Subsonic External Flow

Notwithstanding neglect of sonic line reflections, Landahl's book (Reference 9) is devoted almost entirely to the linearized theory of transonic unsteady flow, except for Chapter 10, dealing with aileron buzz, which contains some usable results, particularly the "shock-compatibility" relation between unsteady perturbations on the upstream and downstream side of a shock.

It is recognized (References 8 and 9) that for thin wings, especially low aspect ratio wings, linearization at higher reduced frequencies is reasonably valid. Explicit calculation of non-linear thickness effects for unsteady subsonic and supersonic flow past slender bodies by Revell (Reference 10 and 11) indicate that for slender bodies, non-linear thickness effects on dynamic derivatives are rather weak. These results are to be expected qualitatively because three dimensional relief effects via the continuity equation, require smaller excess flow-field velocities to circumvent a body

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of given lateral width/length ratio. Thus, one does not find large local variations of Mach numbers. In fact, the first order approximation is the well known slender wing or slender body theory due to Max Munk and R. T. Jones (see References 12-14). In this case the lateral lift distribution for steady or unsteady flow is independent of Mach number if $\beta AR \ll 1$; $\beta = \sqrt{1 - M_\infty^2}$ and AR is the aspect ratio of a slender wing. The application of slender wing theory to unsteady AICs in the form of Equation (5) has been explicitly given by Rodden and Revell (Reference 15).

In connection with unsteady linear theories at $M_\infty = 1$, several schemes have been devised for computation, using the sonic limit for the kernel function in the well known procedure of Watkins, Runyan, Woolston, and Cunningham, (References 16, 17) and also in "transonic box method" procedures (References 9, 15, 18, 19).

While the essential importance of the non-linear terms in transonic small disturbance flow theories has long been recognized in steady flow circles (References 4-9,13,14) little has been done for the unsteady problem. For steady flow, Spreiter and Alksne (Reference 5) introduced a notion called local linearization in which the non-linear transonic term estimated by successive approximation, yielding equations having the same form as for purely subsonic or supersonic flow, but with the local Mach number varying with space. These methods have met with some success in problems of slender airfoils and axisymmetric bodies at zero lift near Mach 1. However, the method is a "simple wave" theory considering only outgoing waves and neglecting incoming wave reflections from the sonic line and from the intermediate acoustic impedance changes associated with the gross variation of Mach number, density and acoustic speed which occurs between the wing surface and the undisturbed flow.

The work of Rubbert and Landahl (Reference 20) also considers non-linear steady effects on thick airfoils using the "method of parametric differentiation", but also fails to consider incoming wave reflections from the sonic line.

Andrew and Stenton (Reference 21) have attempted to apply local linearization to unsteady transonic flow, and have done some interesting exploratory studies on acoustic ray tracing; however, they confined their studies to two dimensional propagation in the plane of the wing, whereas, supercritical flow effects are most serious for thick large aspect ratio wings at moderate to high lift coefficients.

In this case the ray tracing(aside from geometric acoustic limitations) would be more appropriate in the vertical plane, suggesting a "strip theory" as a start on the general problem. This idea is related to the proposed approach which regards the supercritical region as a layered, moving, medium into which unsteady disturbances (sound waves) propagate. Acoustic impedance changes for such media interfaces are given, for example, by Morse, Miles and Ribner (References 22 - 24). These impedance changes cause reflections from the outer flow to airfoil surface. These reflections occur even in the locally subsonic regime; there, however, the impedance changes are quantitatively weaker, and cruder approximations may suffice. Before pursuing this approach further, some other recent developments in transonic flow theory must be discussed.

Recent research. - Several symposia (References 6, 7) and a recent bibliography (Reference 25) are available describing transonic research since 1950. Most of this theoretical and experimental research is directed towards the steady flow problems of; (1) predicting pressures and flow fields past given bodies, (2) designing delayed drag rise and "shock free" airfoils. In the case of airfoil design there has been a renewal of interest in the hodograph method (References 4, 12, 7, 26, 27), an indirect transformation method, using velocity components as independent variables, and yielding a linear problem for steady flow, at the expense of boundary condition complications. The hodograph method appears to be applicable only to steady, two-dimensional flow; therefore, it will not be considered further, except it is noted as a possible tool for description of the steady flow past a special airfoil shape whose unsteady airloads may be sought.

There are two other threads in recent transonic research which have led to improved steady flow analytical methods.

- o Lax-Wendroff type methods (References 7, 25, 28, 29)
- o Mixed finite difference methods (References 30 - 33)

The Lax-Wendroff (forward marching in time) (References 7, 25, 28, 29) type of method is actually a transient approach to solving steady flow problems wherein an initial change (in the surface boundary condition, for example) is introduced, and the asymptotic (in time) limit of the solution is sought. At each time step, the spatial derivatives are calculated by finite difference methods using data for the previous time step. One knows, at the previous instant, whether the flow is locally subsonic or supersonic and can accordingly adjust the finite difference procedures (e.g., central vs. backwards differences). This method may ultimately be applicable to oscillatory aerodynamics; however, it is plagued by several problems which would appear to render it impractical for flutter application at the present time for the following reasons:

(1) Excessive computation time. Reference 28 presents one of the most realistic and careful of the Lax-Wendroff schemes for steady flow. It is stated by the authors of Reference 28 to require about 2 hours on a large scientific computer for a steady 2D problem. Considering that in flutter analysis a large number of reduced frequencies, Mach numbers, angles of attack and wing section shapes, may have to be analyzed, such a method would have to be sparingly applied, perhaps only as a check on simpler methods.

(2) The method has not been demonstrated on oscillatory problems.

(3) The method shows extreme sensitivity to boundary conditions and a tendency towards numerical instability.

(4) The physical significance of the "artificial viscosity" (used to stabilize the calculations) is, as yet, controversial.

With regard to mixed, steady flow finite difference methods (References 30-33) these appear faster and more stable for computations than the Lax-Wendroff(transient) approach. The results of Steger and Lomax (Reference 32) and Bailey (Reference 33), based on pioneer work by J. D. Cole and, Krupp and Murman, (Reference 30,31) are impressive. However, no unsteady applications have yet been developed, though such a scheme might be devised in principle. Again, the prospect for unsteady flow would be computationally tedious, requiring many flow quantities to be defined, at each frequency, and at a large mesh of points, both on and off the body. Past subsonic and supersonic flutter methods have all used surface aerodynamic singularity distributions, without having to explicitly calculate off-body flow field data (excluding component interference problems). Therefore, any finite difference methods(steady or unsteady) such as References 28 - 33, would represent a tremendous escalation of computer time, even if their extension to oscillatory flow had already been completed with demonstrated reliability (which is not likely in the near future).

The above remarks are not intended to discourage development of unsteady, finite difference methods, which might well be feasible after a few new generations of digital computers. On the other hand, it would appear that in the immediate future, some simpler, useful methods are urgently needed which could shed light on the essential features of transonic unsteady flow for flutter analysis purposes, and this is the thrust of the present theory.

Experimental data. - Ultimately it is desired to compare any theoretical result with experimental data. It is noted that some oscillatory 2D section data, notably for trailing edge control surfaces,is available in References 37 and 38, using a pressure measurement system described in Reference 36. Reference 35 is noted to contain some oscillatory delta wing pressure data. It is contemplated that the emphasis of the present approach will be upon large aspect ratio thick wings, , where supercritical flow effects govern a more significant range of Mach number and lift coefficient, than for low aspect ratio thin wings.

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Therefore, 2D data correlations would be a first objective as a building block in a 2D strip theory as a replacement for present methods for local sections, along the span of a finite wing of moderate to large aspect ratio (AR 4 to 8, say), which are adjacent to regions of locally supercritical flow.

Physical Foundation for the Present Approach

Layered medium idealization of the steady supercritical flow past a 2D section. - We shall outline, in the next section a two-dimensional strip theory for evaluation of unsteady airload on wing sections at local spanwise stations where the steady flow field has chordwise regions of locally supercritical flow, such as is shown in Figure 2 . The steady flow field will be approximated as a layered moving medium into which disturbances propagate (see Figure 2).

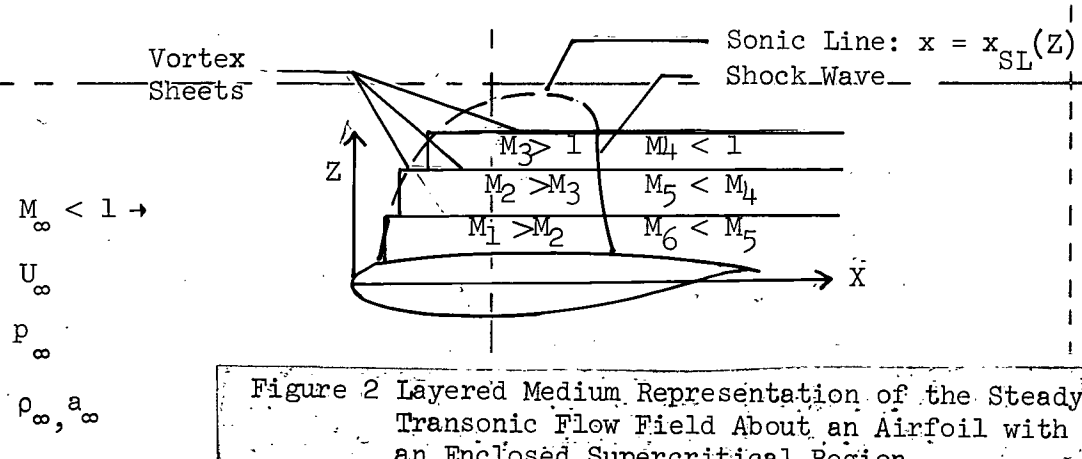

Figure 2 Layered Medium Representation of the Steady Transonic Flow Field About an Airfoil with an Enclosed Supercritical Region

Empirical substantiation of strip theory in steady flow past large aspect ratio thick wings. - Before discussing the 2D strip theory approach, some brief justification will be offered, including an approximate method for representing finite span effects, which is consistent with present state-of-the-art (3D subsonic and supersonic lifting surface theory). First we consider some empirical observations concerning steady transonic flow past wings and airfoils with local supercritical enclosures which help substantiate the usefulness of strip theory on finite wings which are thick enough to have significant supercritical flow effects.

During the past several years, a considerable amount of experimental and analytical research on wings and airfoils has been conducted throughout the world (References 6, 7, 25), along with extensive wing design and development studies in the aerospace industry for the wide-body jets. One interesting and useful fact emerges which is described as follows: If one calculates the surface pressure on an airfoil by methods which do not account for supercritical regions on the airfoil (any of several will suffice) then, in the regions where local Mach numbers are predicted to be subsonic, it is found that the theoretical and experimental pressures agree rather well despite the neglect of supercritical effects. Also it is found that some empirical methods (somewhat like Sinnott's method, Reference 40) will fairly adequately describe surface pressures in the supercritical region. The present theory is an advance, since it also accounts for subsonic interference effects caused by the presence of local supercritical flow regions.

The first point above is illustrated in Figure 3 for an airfoil having local supercritical flow. Similar results have been found in wing development studies in which subsonic lifting surface theory and 3D non-lifting potential flow methods have been applied. This indicates that a method based on local supercritical flow corrections to a 3D subsonic prediction method has had some empirical foundation in steady flow. Also, many studies have been made to validate simplified wing design procedures based on application of 2D airfoil data. These studies have repeatedly substantiated the nearly correct prediction of local chordwise pressures on 3D wings using 2D airfoil data with appropriate sweep corrections, except in the wing root and tip regions.

As a final point, it is noted that aerodynamic wing development generally leads, by deliberate design, to achieving chordwise pressure distributions which are nearly the same along the span, except for practical limitations caused by necessary spanwise variations of wing thickness.

On the basis of the above considerations, it is suggested that a 2D strip

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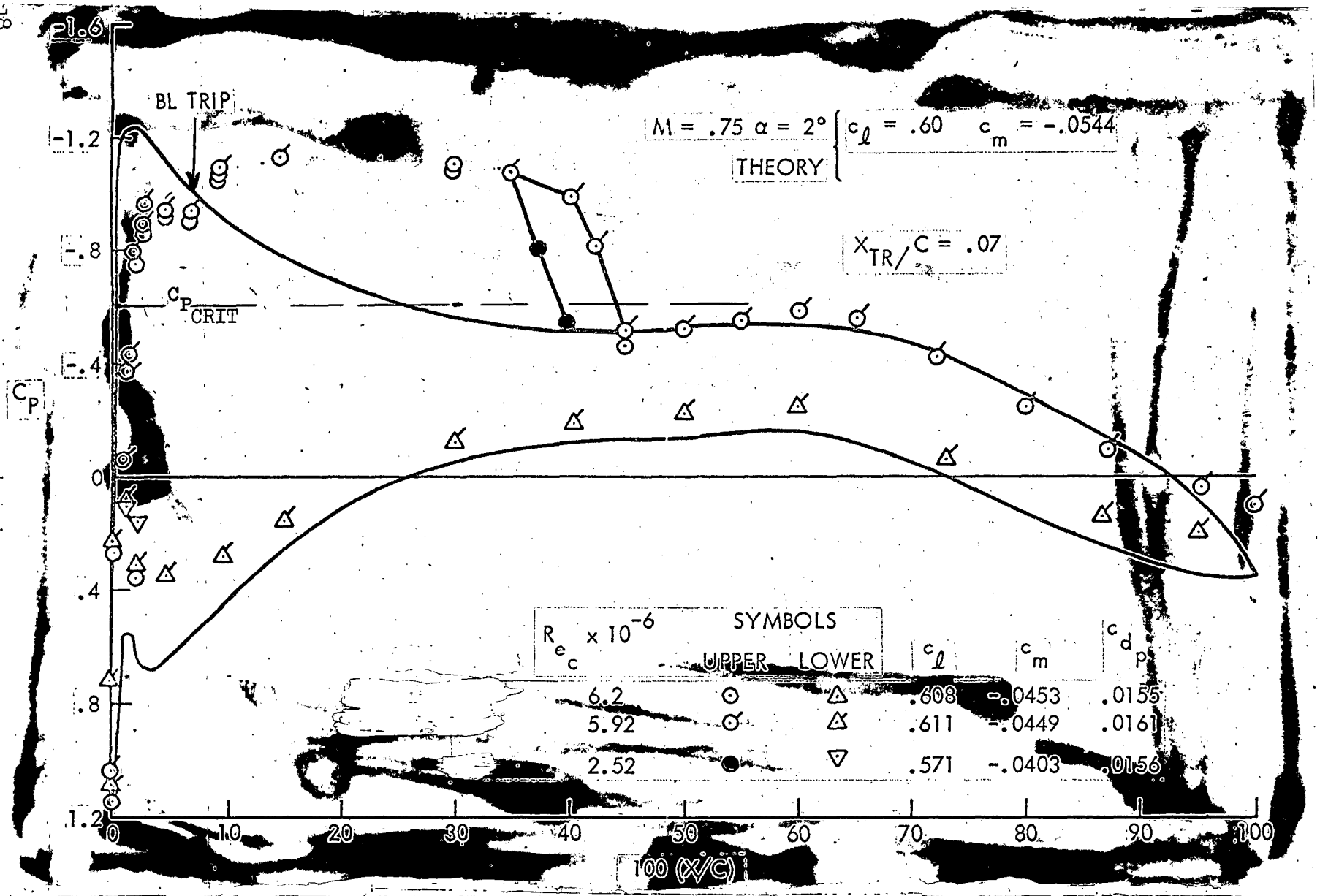

Figure 3. Comparison of Subsonic Theoretical and Experimental Pressure Data for an Airfoil with Adjacent Locally Supercritical Regions (Data taken from an unpublished Lockheed study.)

theory for unsteady supercritical flow would be a useful tool for aeroelastic analysis.

Empirical evaluation of 3D flow effects.

Prediction of local steady flow environment. - The review of analytical methods given in the previous paragraphs indicates that reliable steady supercritical flow prediction has only recently been achieved for 2D lifting airfoils (References 28 - 32) and non-lifting axisymmetric bodies (Reference 33). Even in subsonic steady flows, though methods exist, they often give poor results on practical 3D wing body configurations due to the extreme complexities of the geometries of wing root fillets, pylons, nacelles, etc., which all cause substantial interference. Also viscous and flow separation effects are especially difficult to assess for complex geometries.

Despite these 3D analysis difficulties, wing pressure data is generally available from routine aerodynamic wing development, which must be accomplished prior to commencement of serious flutter and gust analysis; those data are also needed to substantiate structural loads before even the strength requirements for member sizing can be finalized. Therefore, it is reasonable to assume that one has steady flow wing pressure data available to aid in constructing an unsteady supercritical flow theory in the following ways:

- Locally supercritical regions are mapped, both chordwise and spanwise.
- The correct steady flow section lift coefficients and surface pressures are available to "tune" a local 2D description of the steady flow above and below the wing.

Empirical evaluation of 3D effects in unsteady flow. - The following approach is suggested:

- It is proposed that unsteady airloads on those wing sections having local supercritical regions be evaluated by the theory described analytically in the next section. This method includes

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some evaluation of subsonic interference effects aft of the shock wave as discussed in Appendix B.

- In spanwise regions where the flow is locally subsonic, apply the usual subsonic unsteady lifting surface theory for unsteady airloads. This is suggested because the methods are approximately valid in steady flow as discussed above.

OUTLINE OF ANALYTICAL THEORY

Analytical Assumptions for Layered Medium Analysis of Unsteady Flow Past 2D Sections Having Local Supercritical Regions

Preliminary remarks.- In the following sections the basic assumptions are stated for the analysis of the locally supersonic and locally subsonic regions. This will be followed in the section, "Analysis Outline", by an outline of the analytical theory which will be a guide to mathematical details contained in the several Appendices.

Supercritical (locally supersonic) region.- The following is a list of highlighted assumptions, key mathematical expressions, and pertinent observations:

- The steady flow field is known, by independent calculation, by methods such as Reference 32, by semi-empirical methods, or from experimental data.
- The local supersonic supercritical flow regime can be adequately approximated in the (Z) direction (perpendicular to the chord) by a finite number of "Z" layers, each having local Mach number and thermodynamic state variables which are, at most, a function of x, and for a given layer ($Z_k \leq Z \leq Z_{k+1}$) are a suitable average of the known continuous distribution in the direction perpendicular to the surface.

- The "Z" layers are separated by vortex sheets (surfaces of tangential velocity discontinuity) across which occur finite changes of steady flow properties (and, hence, changes in acoustic impedance).

- The acoustic impedance changes and reflection and transmission coefficients across discontinuities are governed by pressure and flow direction continuity conditions (Miles, Reference 23) and with some modification allow use of previously developed results of Reference 23 for reflection and transmission coefficients at the discontinuities. Numerical evaluation of the theory of Reference 23 has provided layering criteria for the purposes of the present study. The methodology of References 23 and 24 is equivalent and is incorporated in the present theory by a simpler approach which employs the Laplace transformation with respect to the streamwise variable, x.

- Each "Z" layer in the supercritical region may, in principle, be further subdivided into chordwise sub-intervals of constant Mach number and fluid state properties separated by discontinuous impedance changes. The present theory is further idealized by assuming a single average Mach number and thermodynamic state per "Z" layer. As justification, it is noted that typical steady state pressure data in the supercritical region for many airfoils shows a tendency towards a nearly constant local Mach number at a short distance downstream from the sonic line (see Figure 3, and consult References 6, 7, 28 and 32 for examples of the near constancy of local Mach number in the supercritical region).

- In any of the supersonic subregions of constant Mach number (defined by the above mentioned layering assumptions) the unsteady velocity potential, and all perturbation quantities will be governed by the locally linearized 2D unsteady supersonic theory:

$$\vec{u} = \vec{U}_L + \nabla\phi$$

$$\Delta p = -\rho_L\left(U_L \frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial t}\right) \tag{1}$$

where, if U_L, a_L, ρ_L, $M_L = U_L/a_L$ are local fluid properties then:

$$\nabla^2\phi = \frac{1}{a_L^2}\left(U_L \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2 \phi \tag{2}$$

subject to impedance change boundary conditions at each "z" layer interface and matching the known airfoil surface normal velocity (prescribed downwash) condition.

- Only harmonic motion is considered, therefore any quantity Q is of the form

$$Q(t) = \bar{Q}(\omega)e^{i\omega t} \tag{3}$$

- For the local supersonic regions, disturbances on the sonic line $x_{SL}(z)$ are regarded as known initial values, which suggests the use of a Laplace transformation on x for solving the local supersonic flow field.

Subcritical (locally subsonic) region. -

- The layered medium concept applies in principle to the subsonic region; however, the range of variation of local flow properties is less than in the supersonic region. Therefore, subsonic induction effects will be calculated from standard subsonic theory as perturbations to the freestream steady flow, regarded as uniform.

- The principal effects on the subsonic flow field due to the presence of the embedded supercritical region are described in detail in Appendix B. Briefly they are assumed to be as follows:

(1) The subsonic doublet strength in the region forward of the shock wave is reduced to one-half its usual subsonic theoretical value because of the loss of the contribution ordinarily made by the supercritical region above the upper wing surface. This doublet strength pertains to the lift contribution from the subsonic region on the pressure side of the wing, opposite the supercritical region.

(2) The subsonic doublet strength is changed, from its usual subsonic value, both forward and aft of the shock wave position due to two effects:

(a) The above mentioned reduction forward of the shock also changes the downwash induced at the airfoil surface aft of the shock wave (compared to usual subsonic theory).

(b) Additional interference downwash at the airfoil surface, caused by disturbances convected through the shock wave from the supersonic region. These disturbances can be represented by a source distribution spread vertically across the shock wave front which terminates the supercritical region.

(3) Because of the presence of the supercritical region above the upper wing forward of the shock, the usual anti-symmetry of lifting pressure above and below the wing is destroyed forward of the shock. Therefore, the upper surface lift is calculated directly from the local supersonic surface pressure, while the lower surface lift is calculated as one-half of the subsonic doublet loading (including the interference effects described in (1) and (2), above).

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(4) It is assumed (and proven in Appendix B) that the modified subsonic doublet loading can be expressed as a linear combination of airfoil surface downwash due to prescribed motion. This provides the requisite "aerodynamic influence coefficients" (lifting pressure increments/unit downwash) in the manner required for aeroelastic analysis applications.

Outline of Analytical Results

Preliminary remarks. - Because of the length of the mathematical derivations the reader is referred to the Appendices for all of the mathematical details. Only key equations will be displayed in the following text to emphasize the important results. The basic equations and assumptions have been described in the preceding paragraphs, or in the Appendices.

Formulation of the aerodynamic influence coefficients. - The chosen form for the aerodynamic influence coefficients (AICs) is of the class (pressure/unit downwash) in matrix algebraic format. The matrix formulation of these results is contained in Appendix B which is the most important body of analytical results, since it describes the interactions between the subcritical and supercritical regions via shock wave and sonic line disturbances. The reader is urged to study Appendix B closely to obtain the primary thrust of the present analytical formulation. Appendix A contains the details of the local supersonic solution; however, it can be seen from Appendix B that the local supersonic solution is just one of the pieces in the overall analysis, and that Appendix B is the "big picture" framework whose details are further delineated by the other Appendices. It is now of interest to highlight the AIC matrix formulation of Appendix B.

The desired end result is given by Eqs (B-27 - B74) and (B-79 - B-81). In these equations as shown in Figure B-1,p.53, the superscript, A_U, refers to the upper wing local supercritical flow regime which is terminated by a

shock wave whose locus is $x_{SH}(Z)$. The superscript, A_L, refers to the subsonic region on the underside of the wing forward of the chordwise position of the upper surface shock wave. The superscripts, B_U, and, B_L, refer to the upper and lower wing subsonic regions aft of the upper wing shock wave position. The downwash values due to surface motion are called simply W_A and W_B, forward and aft of the shock wave, respectively. The present analysis is obviously restricted to a single supercritical region; however, the terms "upper side" and "lower side" could refer to "suction side" and "pressure side", and apply equally well to the case of a single supercritical region on the lower side of the wing (at a negative steady state angle of attack, for example).

From Appendix B, Eqs (B-72- B-74, B-79 - B-81) the desired (pressure/downwash) AICs are given in the following partitioned form, which emphasizes the difference in the nature of the aerodynamic coupling between wing downwash control points, depending upon whether they are forward or aft of the shock wave. The lifting pressure forward of the shock wave is

$$\left\{p^{A}\right\} = \left\{p^{A_L} - p^{A_U}\right\} \tag{4}$$

$$= \left[Q_p^{AA}\right]\left\{W_A\right\} + \left[Q_p^{AB}\right]\left\{W_B\right\} \tag{5}$$

The lifting pressure aft of the shock wave is

$$\left\{p^{B}\right\} = \left\{p^{B_L} - p^{B_U}\right\} \tag{6}$$

$$= \left[Q_p^{BA}\right]\left\{W_A\right\} + \left[Q_p^{BB}\right]\left\{W_B\right\} \tag{7}$$

The matrices $\left[Q_p\right]$, are the partitions of the pressure/downwash AICs which are required for aeroelastic analysis. Either the lumped aerodynamic forces at structural analysis grid points or the generalized aerodynamic forces, can be obtained from these lifting pressures by suitable numerical integration schemes (see References 2, 4, 10 or 14, for examples); therefore,

the pressure AICs provide the necessary data for either collocation or modal flutter analyses. Because the downwashes, W_A and W_B, are allowed arbitrary chordwise distribution, the analysis is applicable to any chordwise deformation pattern, including chordwise camber changes due to arbitrary static aeroelastic deformations.

In deriving the above equations several building block matrices are employed. The first is the basic subsonic lifting surface solution which is assumed available from any of several "kernel function" or "vortex lattice" type solutions (see References 4, 15-18, 39). These results are expressed in the following form (see Eqs B-3 to B-6)

$$\begin{Bmatrix} W_A \\ W_B \end{Bmatrix} = \begin{bmatrix} K_o^{AA} & K_o^{AB} \\ K_o^{BA} & K_o^{BB} \end{bmatrix} \begin{Bmatrix} p_o^{A} \\ p_o^{B} \end{Bmatrix} \tag{8}$$

The subscript o implies purely subsonic flow. In the subsonic case, the distinction between the A and B regions is dropped and Eq 8 is written as

$$\{W\} = [K_o]\{p_o\} \tag{9}$$

The desired pressure/downwash AICs are then

$$\{p_o\} = [Q_{p_o}]\{W\} \tag{10}$$

where the inverse is

$$[Q_{p_o}] = [K_o]^{-1} \tag{11}$$

A partitioned form is useful in the transonic solution and is written as

$$[Q_{p_o}] = \begin{bmatrix} Q_{p_o}^{AA} & Q_{p_o}^{AB} \\ Q_{p_o}^{BA} & Q_{p_o}^{BB} \end{bmatrix} \tag{12}$$

This notation is frequently employed in Appendix B.

In the case of purely subcritical flow, the matrices $[Q_p]$ and $[Q_{p_o}]$ are

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identical, by definition. In the case of transonic flow, these matrices differ by an amount which is roughly proportional to the chordwise extent of the steady supercritical flow region (presumed known) times the difference between upper surface lift for purely supersonic flow and for purely subsonic flow. As will be discussed in Appendix A, and below, the local supersonic solution is proportional to the Garrick and Rubinow solution (References 41, 2, 3, 8, 9) times a layered medium factor.

Lift asymmetry. - One of the basic distinctions between the lifting surface solution for a subsonic flow, vs. a transonic flow with an embedded supercritical region, is that the lifting pressure forward of the shock is no longer antisymmetric above and below the wing for the transonic case. This is explained in Appendix B, where a technique is presented for separately calculating the upper and lower surface contributions to the lifting pressure. The upper surface lifting pressure is derived from the local supersonic solution (discussed more fully in Appendices A, C, and E). This includes the effects of sonic line disturbances which are regarded as initial conditions in the local supersonic solution. These initial conditions are in turn coupled to the subsonic lift distribution by means of subsonic flow field induction matrices which are assumed to be available (at least in principle) from standard subsonic theory.

Sonic line disturbances. - The sonic line disturbances due to the subsonic wing loading in the regions A_L, B_U, and B_L are described in Appendix B in terms of a discrete set of the unsteady perturbation values for the velocity potential, the streamwise velocity (backwash), and the downwash. These discrete initial values are assumed to be calculated from subsonic theory at the sonic line locus $x_{SL}(Z_k)$, where Z_k define vertical coordinates of the layer interfaces used in Appendices A, C, and E to describe the local supersonic region, A_U, as a layered medium. From Appendix B, Eqs (B-13 to B-15), the sonic line disturbances are represented as follows for K interface layers:

$$\left\{U_{SL}\right\} = \left[K_{o,A}^{XSL}\right]\left\{\frac{1}{2}(p_o^{\ A} + \Delta p^{A_L})\right\} + \left[K_{o,B}^{XSL}\right]\left\{p_o^{\ B} + \Delta p^{B}\right\} \quad (13)$$

$$\left\{W_{SL}\right\} = \left[K_{o,A}^{ZSL}\right]\left\{\frac{1}{2}(p_o^{\ A} + \Delta p^{A_L})\right\} + \left[K_{o,B}^{ZSL}\right]\left\{p_o^{\ B} + \Delta p^{B}\right\} \quad (14)$$

$$\left\{\phi_{SL}\right\} = \left[K_{o,A}^{\phi SL}\right]\left\{\frac{1}{2}(p_o^{\ A} + \Delta p^{A_L})\right\} + \left[K_{o,B}^{\phi SL}\right]\left\{p_o^{\ B} + \Delta p^{B}\right\} \quad (15)$$

These matrices have K rows and N_{DW} columns, if N_{DW} is the number of chordwise downwash points

The above equations are summarized more compactly in Eq (B-35) by defining a "sonic line state vector"

$$\left\{\sigma_{SL}\right\} = \begin{Bmatrix} \phi_{SL} \\ U_{SL} \\ W_{SL} \end{Bmatrix} \quad (16)$$

The sonic line state vector defined by Eq (16) possesses 3K rows and one column. Eqs (13 to 15) are then summarized in Eq (B-4) by using the sonic line state vector and regarding the matrices $\left[K_{o,A}^{XSL}, \text{etc.}\right]$ as partitions of what will be called sonic line induction matrices $\left[K_{o,A}^{SL}\right]$ and $\left[K_{o,B}^{SL}\right]$ defined by

$$\left\{\sigma_{SL}\right\} = \left[K_{o,A}^{SL}\right]\left\{\frac{1}{2}(p_o^{\ A} + \Delta p^{A_L})\right\} + \left[K_{o,B}^{SL}\right]\left\{p_o^{\ B} + \Delta p^{B}\right\} \quad (17)$$

These sonic line induction matrices $\left[K_{o,A}^{SL}\right]$ and $\left[K_{o,B}^{SL}\right]$ are of rank $3K \times N_{DW}$. It is noted that $\left\{p_o^{\ A}\right\}$ and $\left\{p_o^{\ B}\right\}$ are known from the subsonic solution, Eqs (8 to 12), and have the ranks $N_{DW_A} \times 1$ and $N_{DW_B} \times 1$, where N_{DW_A} and N_{DW_B} are respectively the number of downwash control points forward, and aft of the shock wave, for the supercritical case. The "sonic line induction matrices" defined by Eqs (13 to 17) are regarded as the second set of known building blocks available for manipulation in the theoretical development of Appendix B.

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The terms designated as $\left\{\frac{1}{2}\Delta p^{A_L}\right\}$ and $\left\{\Delta p^{B}\right\}$ are the interference lift distribution in the subsonic regions in A_L (below the wing ahead of the shock and aft of the shock, respectively). These terms arise because of the presence of supercritical flow in A_U. The factor 1/2 denotes the loss of the upper wing contribution to doublet strength, compared with the usual subsonic theory, as explained in Appendix B. The remainder of Appendix B describes how these subsonic interference load increments are derived explicitly in terms of the wing surface downwashes $\left\{W_A\right\}$ and $\left\{W_B\right\}$ to obtain the desired (pressure/downwash) AICs described in the preceding paragraphs.

Subsonic interference downwash due to shock wave. - In Appendix B, Eqs (B-22 and B-23) there is defined a shock wave induced interference downwash at the airfoil surface, which is assumed as a reasonable approximation, to satisfy the classical "subsonic kernel function" type of lifting surface relationship

$$\left\{\Delta W_{A_L}^{SH}\right\} = \left[K_o^{AA}\right]\left\{\frac{1}{2}\left(\Delta p^{A_L} - p_o^{A}\right)\right\} + \left[K_o^{AB}\right]\left\{\Delta p^{B}\right\} \tag{18}$$

$$\left\{\Delta W_{B_L}^{SH}\right\} = \left[K_o^{BA}\right]\left\{\frac{1}{2}\left(\Delta p^{A_L} - p_o^{A}\right)\right\} + \left[K_o^{BB}\right]\left\{\Delta p^{B}\right\} \tag{19}$$

Eqs (18 and 19) take into account the basic subsonic equations (8 to 12) which relate the kinematic (motional) downwash at the wing surface to the subsonic theoretical lifting pressures $\left\{p_o^{A}\right\}$ and $\left\{p_o^{B}\right\}$.

It is now necessary to relate the interference downwash on the left hand side of Eqs (18 and 19) to the disturbances impinging on the shock wave from the supersonic side. This relationship is expressed in Appendix B, Eqs (B-20, B-21, and B-46):

$$\begin{Bmatrix} \Delta W_{A_L}^{SH} \\ \Delta W_{B_L}^{SH} \end{Bmatrix} = \begin{bmatrix} G_{SL}^{WA_L} \\ G_{SL}^{WB_L} \end{bmatrix} \left\{\sigma_{SL}\right\} + \begin{bmatrix} G_{WA}^{WASH} \\ G_{WA}^{WBSH} \end{bmatrix} \left\{W_A\right\} \tag{20}$$

Using Eqs (17 to 20) eliminates the sonic line static vector yielding, as in Eq (B-49).

$$\begin{Bmatrix} \Delta W_{A_L}^{SH} \\ \Delta W_{B_L}^{SH} \end{Bmatrix} = \begin{bmatrix} K_{AA}^{WASH} & K_{AB}^{WASH} \\ K_{BA}^{WBSH} & K_{BB}^{WBSH} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}(p_o^A + \Delta p^{A_L}) \\ (p_o^B + \Delta p^B) \end{Bmatrix} + \begin{bmatrix} G_{WA}^{WASH} \\ G_{WA}^{WBSH} \end{bmatrix} \{ \dot{W}_A \} \quad (21)$$

The system of Eq (16 to 21) is solved simultaneously in Appendix B (Eqs B-58 and B-65) to yield the interference loadings in the subsonic region in the following form

$$\begin{Bmatrix} \frac{1}{2} \Delta p^{A_L} \\ \Delta p^B \end{Bmatrix} = \begin{bmatrix} P_I^{AA} & P_I^{AB} \\ P_I^{BA} & P_I^{BB} \end{bmatrix} \begin{Bmatrix} \frac{1}{2} p_o^A \\ p_o^B \end{Bmatrix} + \begin{bmatrix} P_{I,WA}^{A} \\ P_{I,WA}^{B} \end{bmatrix} \{ W_A \} \quad (22)$$

This gives the interference loading as the sum of two quantities; (1) a term proportional to the classical subsonic doublet loads, via sonic line disturbances, and (2) a term caused by disturbances generated directly by the wing motion, W_A, in the locally supersonic region, which then impinge upon the shock wave and are transmitted through the shock wave to cause a source distribution on the subsonic side of the shock. The interference loads defined by Eq (22) are expressed in terms of wing motion downwash by eliminating of $\left\{ \frac{1}{2} p_o^A, \; p_o^B \right\}$ using Eq (10) to obtain the desired AIC format given by Eqs (5 and 7).

One of the key relations in the above sequence is Eq (20), which is Eq (B-46b) of Appendix B. The first term multiplying the sonic line state vector, is the product of three factors; (1) a subsonic source factor, $\left[G_{BSH}^{W_{A,B}} \right]$ defining subsonic interference downwash in terms of disturbances on the downstream side of the shock wave (see Eq B-45); (2) a shock wave transfer factor $\left[T_{SH}^{BA} \right]$ relating shock wave disturbance state vectors on either side of the shock (Eq B-40) and, (3) a factor, $\left[T_{SL}^{APSH} \right]$, which accounts for shock wave disturbances transmitted through the supercritical region from the sonic line.

Of the three factors in the first term of Eq(B46a), the first subsonic source factor has not been written explicitly, but is considered a trivial exercise in subsonic theory, and is regarded as definable, in principle, along with all of the other "subsonic induction" matrices. The second factor (the shock wave transfer factor) is developed in Appendix D, for normal shock waves; this is merely a convenient statement of results given by Landahl (Reference 9, pp. 110-113). The third factor, relating to transmission of sonic line disturbances through the supercritical region, is a matrix statement of results which are developed in Appendices A, C, and E, in terms of a solution, by Laplace transformation on x, in the layered supercritical region. The matrix statement of these results (given in Appendix B in Eqs (B-20, B-21, B-27 - B-34, B-38, B-42, - B-46) implies the inverse Laplace transformation of quantities defined in Appendices A, C, and E, by a convolution integral method (Reference 42). Evaluation of the convolution integrals by numerical integration then lends itself directly to a matrix formulation.

The second term of Eq (20) (or Eq B-46) likewise consists of three factors, the first two factors being the same ones discussed above. The third factor in this case defines the propagation, to the shock wave, through the layered supercritical region, of disturbances generated directly by the upper wing surface motion forward of the shock wave. This term is like the Garrick and Rubinow solution with an impedance factor. The matrix statement of this term in Appendix B implies the inversion of Laplace transformed quantities derived in Appendices A and E.

Criteria for layering of the local supersonic region. First some comments will be made concerning the nature of the required layering; then, there will follow an outline of the results presented in Appendices A, C, and E.

Prior to commencement of the Laplace transform solution presented in this report, an assessment was made of the feasibility of using a multiple reflected wave approach, utilizing previous acoustic theory results (References 22 to 24). These results calculate coefficients of reflection

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and transmission for plane acoustic waves, of varying incidence angle, impinging upon a layer interface, across which there is a discontinuous change of the quantities, local stream velocity, local speed of sound and density. Computations were made, using the theory of References 23 and 24 to evaluate the magnitude of the reflection coefficient for single reflection of a plane sound wave, with an arbitrary angle of incidence,(defined as the angle between the wave front and the interface) and striking the interface across which a discontinuous change in local freestream Mach number is assumed to occur. It is also assumed that the local density and sound speed are related to the local Mach number by isentropic flow equations throughout the supersonic region, which is a good approximation.

The results of the above described calculation show,for a single reflection, that the reflection coefficient is equal to about .5 times the change in local Mach number across the layer. This would suggest that the supercritical region might be approximated by layers of sufficient vertical depth to allow a change of local Mach number of the order of .2 across each layer. In such a layer model, one would find reflected waves whose strengths are ten percent of the incident wave strength for each reflection. It can be seen from the analyses in Appendices A, C, and E, that many reflections and refractions of waves occur; however, they are systematically accounted for by the Laplace transform analysis which has been employed in these Appendices.

There are two basic reasons for not employing the reflected wave approach directly; (1) the disturbances arising from various points generate cylindrical waves which can be described as a "bundle of plane waves" of varying incidence angle; (2) for an arbitrary steady flow field, dependent on varying values of freestream Mach number and angle of attack, the width of the layers will vary, as will the sonic line locus and shock wave positions; therefore, it would be logically complicated to describe how many wave reflections have contributed to the pressure at any fixed point on the airfoil.

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For the first of the above reasons, the reflection coefficient at an interface, due to even a single point source (leading to a Garrick and Rubinow type of disturbance) must be obtained by integrating the reflection coefficient for all angles of incidence of the plane wave bundle which represents the point source (cylindrical wave) disturbance. On the other hand, the layer interface impedance ratios are shown in Appendices A and E, to be point functions of the complex Laplace transform variable which simplifies the handling of layer interface boundary conditions. The inversion of the Laplace transform of the impedance leads to a convolution integral for the reflection, from the interface, of a point source disturbance; therefore, this method is, in principle, equivalent to the integration over a bundle of plane waves described above.

One further subtle point must be made to justify the physical basis for the use of the layered medium model for the supercritical region. In the analyses of References 23 and 24, the impedance ratio across the layer interface is proportional to the ratio of ρa^2 values where, ρ, is density and, a, is the speed of sound. For a perfect gas, ρa^2 is proportional to static pressure, which must be continuous throughout the fluid. However, because of the steady flow field streamline curvature, there is a gradient to the static pressure in the direction perpendicular to the wing surface. The present model replaces the continuous static pressure variation by stepwise changes at the layer interfaces which can be regarded as similar to membranes capable of supporting static pressure differences due to the steady flow curvature. However, to the incoming incremental aeroelastic disturbances, the membrane appears as a porous wall, having continuity of streamline slope and equality of incremental pressure on either side of the membrane. Thus, the finite layering method chosen here is similar to an approach sometimes used for example, in evaluating the effects of atmospheric pressure variation upon the propagation of a sonic boom from high altitude towards the ground.

Layered medium analysis of the local supercritical region. - The following is primarily a qualitative discussion of Appendices A, C, and E. These Appendices contain the detailed mathematical development.

Appendix A contains a formulation of the boundary value problem for the small amplitude, steady or oscillatory disturbances to the local supercritical region, having a non-uniform steady flow as shown in Figure 1, which is further idealized as a layered medium as shown in Figure 2. The physical basis of this idealization has already been discussed. Eqs A-1 to A-4 describe, respectively, the convected wave equation governing the small disturbances, the wing surface downwash boundary condition, and the first layer interface pressure and streamline slope continuity conditions. Eqs A-5 to A-14 show these same equations after Laplace transformation on the streamwise variable x, and taking into account initial values of disturbance quantities on the sonic line, which are temporarily regarded as known quantities. (Appendix B, in fact, shows how they are coupled to the subsonic loading to close the analytical feedback loop!) Eqs A-15 to A-18 show the analytical solutions within each of two interior supersonic layers, bounded by an external subsonic freestream. These solutions indicate the presence of particular integrals required to satisfy the non-homogeneous "forcing functions" appearing on the right hand side of the Laplace transformed version of the convected wave equation (Eqs A-6, A-7 and A-19). Eqs A-20 to A-23 show the local supersonic surface pressure as a convolution integral of the classical Garrick and Rubinow solution for unsteady disturbances to a steady supersonic stream (References 41, 2, 3, 8) plus a particular integral contribution arising from sonic line disturbances which are induced by the subsonic loading. Eqs A-23 to A-26 outline the inversion of the Laplace transforms, emphasizing the nature of the integration contours required within the Laplace transform complex variable (s) plane. Eqs A-27 to A-35 display some of the layered medium transfer functions which modify the Garrick and Rubinow solution. Appendix E contains a rigorous derivation of the results given by Eqs A-27 to A-35, for the case of two interior supersonic layers bounded by an external freestream. Also contained in Appendix E is the technique for

generalization to an arbitrary number of interior supersonic layers.

Appendix C contains a rigorous derivation of the particular integrals to the non-homogeneous, Laplace transformed, convected wave equations within the layers (Eq A-6, A-7, A-19, C-1). Besides the general theory, described by Eq (C-1 - C-5, C-13 to C-17), there is also presented an explicit set of results for the case where the sonic line disturbances vary linearly with, Z, the distance perpendicular to the layer. This solution would be a good approximation if several layers are used to describe the supercritical region, and could influence the choice of layering criteria for a computer program development. Also contained in Appendix C is a particular integral for the external freestream disturbances, wherein, the initial values are defined on an extension of the sonic line, which may be arbitrarily located for the convenience of the analysis without loss of generality, as long as the subsonic induction effects (see Eqs B-13 to B-15) are properly calculated.

Appendix E contains an explicit derivation of the particular integral contributions to the airfoil surface pressure which is displayed in Eqs (A-20) and Eqs (E-95, E-96, and E-97). The results of Eq E-97 show directly how the effects of the layer interface impedance changes influence the sonic line disturbance contribution to the airfoil surface pressure in the local supersonic region.

EXTENSIONS OF THE PRESENT WORK

Analytical Effort

The present theory is completed insofar as the problem is solved, in principle, provided the reader accepts the layered medium model as a useful mathematical and physical approximation to the total problem. Appendix B defines the necessary accounting procedure for calculating aerodynamic influence coefficients for aeroelastic analysis applications. It is clear that each matrix in Appendix B defines a computer subroutine requirement, and each of these subroutines may require subroutines.

It appears, therefore, that the primary analytical requirements will be as follows:

1. Write explicit equations for defining each element of each matrix defined in Appendix B.

2. Define and write explicit equations for subroutines to the matrices of Appendix B as required for the more complicated matrices of Appendix B.

3. Define and write explicit equations for subroutines to invert the Laplace transforms defined by Appendices A, C, D, and E, whose inverse Laplace transforms are necessary to define some of the matrices specified in Appendix B. This task might entail several subordinate subroutines such as:

   a. A subroutine to find poles of arbitrary transcendental functions of the complex variables. This may require a subordinated subroutine to find the zeroes of the complex function defining its denominator (see Eqs A-27 to A-33, for example).

   b. A subroutine to locate branch points in the complex plane for an arbitrary number of interior supersonic layers bounded by a subsonic external stream (see Eqs A-23 to A-26).

   c. A numerical quadrature subroutine to evaluate the line integrals along path segments on either side of branch cuts (line segments connecting branch points in the complex s plane. See Reference 42, Appendix A, Figure A-2 and Eqs A-23 to A-26).

   d. A subroutine to calculate complex values of residues at the poles defined by subroutine 3a, above. (See also Eq A-26.)

4. Write an interim analytical summary report covering the above items in sufficient detail such that, in conjunction with the present report, a computer programmer could work reasonably independently to begin coding the various subroutines with only occasional clarification by the theoretical aerodynamicist or aeroelastician.

5. Review and monitor computer programming effort; aid in check-out of subroutines.
6. Define sample problems to evaluate the sensitivity of the computer program to the number of layers, integration procedures, etc.
7. Write a final report according to NASA specifications.

Computer Programming Effort

Some of the computer programming effort could commence immediately, while some of the subroutine coding would await the detailed definition in the interim report defined as Item 4 under analytical effort. The programmer task activities are visualized as follows:

1. Construct a flow diagram based on Appendix B of the present report, and submit to analyst for review.
2. Code and check-out the various subroutines implied by Appendix B and the flow diagram, as soon as sufficient detail has been supplied by the analyst in the form of preliminary appendices to the interim analytical report.
3. Upon receipt of the interim analytical report, complete coding and checkout of the various subroutines defined above.
4. Assemble the entire computer program and write a separate report which would be a user-oriented document referring to the analyst's final report for the theory.
5. Review with NASA the format for the user document, and the software language compatibility requirements for various digital computer systems for which NASA may wish to compile the software.
6. Complete the user-oriented report documenting the computer program.
7. Submit computer program documentation report for review by analyst and NASA.

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Comments on Matrix Sizes and Computer Storage Requirements

Appendix B defines the matrix accounting system required for solution of the problem. Most of the arrays will be sized either by the number of chordwise locations where surface pressure and motion downwash are specified, or by the number of layers required to accurately model the supercritical region. It has already been suggested that layers might be defined at vertical positions having increments of steady flow Mach number of the order of .2 or less for greater accuracy. Examination of airfoil data shows that the upper range of local surface Mach number is about 1.4 to 1.6, for an embedded supercritical region within an exterior subsonic flow. This would suggest that 3 to 6 interface layers would certainly suffice, and possibly even the two-layer model, derived explicitly in Appendices A and E, might provide some useful guidance, at least for preliminary calculations.

The number of chordwise downwash control points is governed by two criteria:

1. A set of points required by solution of the classical subsonic lifting surface problem. This could range from as few as two points per chord (for rigid body motion) to 10 or 20 for structural vibration modes having appreciable camber bending. It would probably be judicious to define at least 10 chordwise control points on either side of the shock wave for the purposes of Eqs (B-3 to B-10). This would define 20 x 20 matrices for a strip theory treatment of a wing span station having local supercritical flow.

2. A second set of chordwise control points is needed to develop the surface pressure vs downwash relation on the upper wing surface beneath the layered supercritical region. This set of points must be sufficient to allow accurate evaluation of the convolution integral described by Eq A-22 and the inverse Laplace

transform of Eq (E-96). The inverse of Eq (E-96) can be evaluated at an arbitrary number of points by a contour integral similar to Eq (A-23). In evaluating the first term of Eq (A-22), enough chordwise points must be used for the accuracy which is required by the numerical integration scheme employed (e.g., Gaussian's quadrature, Simpson's rule, etc.). The required values of surface downwash at these points can be calculated by interpolation between the chordwise locations of the downwash control points used in the subsonic lifting surface solution. This set of integration control points for the supercritical region might range from 10 to 50 points depending on the sophistication of the quadrature method. In view of the complexity of the logic for the computer program as a whole, it may be preferable to begin with a simple quadrature scheme using a larger number of chordwise integration points.

This section will be concluded by an example, for a refined analytical model, of the size of some of the matrices in Appendix B. It is assumed that five layers will adequately describe the supercritical flow gradients, and that it is valid to assume a linear variation with Z of sonic line initial disturbances across each layer (see Appendix C, Eqs C-6 to C-12). Then as indicated in Appendix B, three quantities will be required to define the sonic line disturbance state vector for each layer interface (eqs B-13 to B-15 and B-35). For the assumption of ten downwash control points on each side of the shock (a refined description) then Table I defines the rank order of the largest sized matrices appearing in various equations in Appendix B. From the pattern in Table I one can easily deduce the ranks of the remaining matrices defined in Appendix B. It appears that the largest individual matrix partitions are of the order 10 x 15; therefore, it appears that even for a refined layering model, the present method is feasible within the storage capacity of many current digital computer systems. This also may include even the smaller storage allocations assigned to time-shared, remote terminal computer systems, provided the operations sequence is carefully programmed. It seems possible that a respectable accuracy could be achieved, with an active computer core storage capacity of the order of 16 000 words.

TABLE I.--- EXAMPLES OF MATRIX RANK SIZES
DEFINED BY EQUATIONS IN APPENDIX B

Assumes 5 layers and 10 downwash
points on each side of the shock wave

Equation Number	Number of Rows	Number of Columns	Type
B-10,11	10	10	square
B-12	10	5	rectangular
B-13-15	5	10	rectangular
B-16-19	5	5	square
B-20-21	10	5	square
B-22-25	10	10	square
B-26	15	15	square
B-27-29	5	10	square
B-30,31	5	1	column
B-32	25	15	rectangular
B-33	10	15	rectangular
B-34	15	15	square
B-35-37,39	15	1	column
B-38	15	10	square
B-40	15	15	square

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CONCLUDING REMARKS

A new two-dimensional strip theory for unsteady transonic flow has been presented for the calculation of unsteady aerodynamic influence coefficients (AICs). The method is valid for arbitrary, rigid body or aeroelastic deformations of a wing section whose steady flow field is characterized by a locally supersonic (supercritical) flow region which is adjacent to one side of the wing, and which is embedded in an exterior subsonic flow field. In the absence of supercritical flow effects the present theory reduces automatically to classical subsonic theory, which is desirable.

Based on empirical evidence for steady transonic supercritical flow past airfoils and wings, it is believed that the present theory, when fully developed for routine digital computation, will provide a valuable tool for the aeroelastician. It will be suitable for performing aeroelastic calculations for moderately swept, large aspect ratio, thick wings operating at high subsonic speeds, and will use simple sweep theory concepts to define the necessary equivalent two-dimensional freestream flow properties along the swept chord direction.

The present report also tentatively suggests empirical means for estimating finite span effects, using existing subsonic lifting surface theory, plus experimental data for wing pressures in steady flow. The suggested technique defines the chordwise and spanwise extent of supercritical flow from the measured steady flow wing pressures, and replaces the subsonic air loads by the present new theory for those spanwise stations sharing chordwise locally supercritical flow.

Various possible theoretical approaches in the literature were reviewed and rejected on various grounds, prior to selecting the present layered medium theory, which is believed to provide a middle ground in computational complexity, and has the advantage of providing good physical insight into the

crucial aspects of the problem. Merely by formulating the analysis by the present method yields important conclusions. For example, the following conclusions have been drawn from the present studies:

1. The usual anti-symmetry of lift above and below the wing is altered to an asymmetry of the upper and lower surface lift forward of the shock wave. These contributions must be separately calculated.

2. The air loads adjacent to locally subsonic regions can be calculated by the usual doublet procedure, if the air loads ahead of the shock, on the locally subsonic side of the wing are assigned a factor of one-half, and if additionally one accounts for subsonic downwash interference at the wing surface caused by disturbances transmitted through the shock wave from the supercritical region.

3. If one defines suitable subsonic flow field induction matrices, based on existing theory, one can couple the subcritical and supercritical regions via shock wave and sonic line disturbances which are then capable of algebraic elimination by their expression as linear combinations of wing surface downwash; hence, it is possible to explicitly derive the desired AICs required for aeroelastic application.

4. A preliminary assessment of the computational aspects indicates only a modest computer storage requirement for the present theory. The use of two to five interior supersonic layers may well suffice for many transonic flow problems of interest, where local wing surface Mach numbers seldom exceed 1.6.

5. The layering effects in the supercritical region are significant, as the impedance change between the wing surface and the freestream is proportional to the ratio of local to freestream absolute static pressure. This can vary by a factor of 3 for a typical local Mach number variation in the flow field between 0.75 and 1.6.

6. Unpublished numerical studies by the author, using acoustic methods have shown that for a wide range of conditions, a Mach number change

of 0.2 will produce a ten percent reflection coefficient for a single reflection from an interface in the local supersonic region. This result provides a simple physical basis for defining the layering criteria for a given steady supercritical flow field.

7. The nature of the local unsteady supersonic solution is easily understood in the present theory, wherein it is expressed in the form of the classical Garrick and Rubinow solution for unsteady flow disturbances to a uniform supersonic stream, times a layered medium multiple reflection factor, plus a particular integral term arising from sonic line disturbances. The sonic line disturbances are induced by the wing surface air loads adjacent to the subsonic region.

It is strongly urged that the present theory be implemented for digital computation, since it is inherently well oriented towards aeroelastic analysis requirements by the nature of the formulation.

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APPENDIX A

LAPLACE TRANSFORM SOLUTION TO THE FLOW FIELD IN THE MULTI-LAYERED, LOCALLY SUPERSONIC REGION A_U

The local supersonic region may be broken into layers of constant local Mach number with increments of about .2 in ΔM according to the results of studies already mentioned. The convected wave equation in each region is solved by Laplace transform on x, assuming initial values on the sonic line are given. A solution will be described for two interior supersonic layers. Layer 1 is defined by $(0 \leq Z \leq Z_1)$ and layer 2 is defined by $(Z_1 \leq Z \leq Z_2)$.

The velocity potential equation for harmonic motion is (omitting $e^{i\omega t}$):

$$\bar{\phi}_{1_{ZZ}} = B_1^2 \bar{\phi}_{1_{XX}} + \frac{2i\omega}{a_1} M_1 \bar{\phi}_{1X} - \frac{\omega^2}{a_1^2} \bar{\phi}_1 \qquad \text{(A-1)}$$

The surface boundary condition is

$$\bar{\phi}_{1_Z}(X,0) = \bar{W}_a(X); \quad \bar{W}_a(X) = W_a(X,t)e^{-i\omega t}. \qquad \text{(A-2)}$$

The pressure continuity at $Z = Z_1$ is:

$$\bar{p}_1(X,Z_1) = \bar{p}_2(X, Z_1)$$

$$-\rho_1 (U_1\bar{\phi}_{1X} + i\omega\bar{\phi}_1) = -\rho_2 (U_2\bar{\phi}_{2X} + i\omega\bar{\phi}_2) \qquad \text{(A-3)}$$

The streamline slope continuity at $Z = Z_1$ is:

$$\frac{1}{U_1} \bar{\phi}_{1_Z}(X, Z_1) = \frac{1}{U_2} \bar{\phi}_{2_Z}(X, Z_1) \qquad \text{(A-4)}$$

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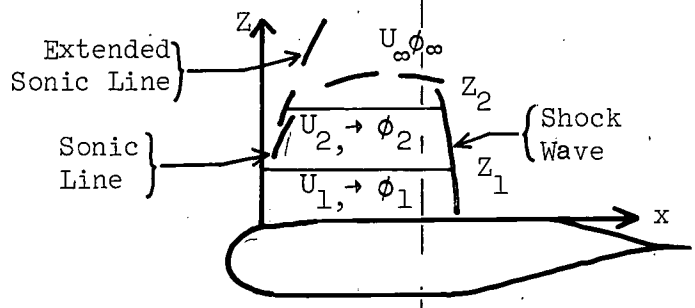
Where $\bar{\phi}_1$ and $\bar{\phi}_2$ are the velocity perturbation potentials in layers 1 and 2 separated by an interface at $Z = Z_1$ (see Fig. A-1)



$M_\infty = U_\infty / a_\infty < 1$

$M_2 = U_2 / a_2 ; B_2 = \sqrt{(M_2^2 - 1)}$

$M_1 = U_1 / a_1 ; B_1 = \sqrt{(M_1^2 - 1)}$

Fig. A-1 Two Supersonic Layer Model Bounded by an External Subsonic Free Stream

We next Laplace transform Equations (A1-A4) with respect to x and obtain

$$\tilde{\phi}(s,Z) = \int_0^\infty e^{-sx} \bar{\phi}(x,Z)\, dx \tag{A-5}$$

$$\tilde{\phi}_{1_{ZZ}} - K_1^2 \tilde{\phi}_1 = \tilde{G}_1(s,Z) \tag{A-6}$$

$$\tilde{\phi}_{2_{ZZ}} - K_2^2 \tilde{\phi}_2 = \tilde{G}_2(s,Z) \tag{A-7}$$

where

$$\tilde{G}_1(s,Z) = -\left\{ B_1^2 \left(s\phi_{SL}^{(1)} + \phi_{X,SL}^{(1)}\right) + 2i\frac{\omega}{a_1} M_1 \phi_{SL}^{(1)} \right\} \tag{A-8}$$

$$\tilde{G}_2(s,Z) = -\left\{ B_2^2 \left(s\phi_{SL}^{(2)} + \phi_{X,SL}^{(2)}\right) + 2i\frac{\omega}{a_2} M_2 \phi_{SL}^{(2)} \right\} \tag{A-9}$$

$$K_1^2 = \left\{ B_1^2 s^2 + \left[2M_1 i\left(\frac{\omega}{a_1}\right)s - \frac{\omega^2}{a_1^2} \right] \right\} \tag{A-10}$$

$$K_2^2 = \left\{ B_2^2 s^2 + \left[2M_2 i\left(\frac{\omega}{a_2}\right)s - \frac{\omega^2}{a_2^2} \right] \right\} \tag{A-11}$$

Here $\phi_{SL}^{(1)}$, $\phi_{X,SL}^{(1)}$, $\phi_{SL}^{(2)}$, $\phi_{X,SL}^{(2)}$ are the sonic line (initial) values of velocity potential and streamwise perturbation velocity at the sonic

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line for layers 1 and 2, respectively.

The boundary conditions are Laplace transformed as follows: At the airfoil surface

$$\tilde{\phi}_{1_Z}(s,0) = \tilde{W}_a(s) \tag{A-12}$$

At the first interface, pressure continuity yields

$$\rho_1 (U_1 s + i\omega)\, \tilde{\phi}(s,Z_1) = \rho_2 (U_2 s + i\omega)\, \tilde{\phi}_2(s,Z_1) \tag{A-13}$$

At the first interface, streamline slope equality yields

$$\frac{1}{U_1}\tilde{\phi}_{1_Z}(s,Z_1) = \frac{1}{U_2}\tilde{\phi}_{2_Z}(s,Z_1) \tag{A-14}$$

Solutions for $\tilde{\phi}_1(s,Z)$ and $\tilde{\phi}_2(s,Z)$ satisfying Equations(A6) and (A7) can be written

$$\tilde{\phi}_1(s,z) = C_1 \cosh(K_1 z) + D_1 \sinh(K_1 z) + \tilde{\phi}_{P,1}(s,z) \tag{A-15}$$

$$z = Z \text{ for } (0 \leq Z \leq Z_1)$$

$$\tilde{\phi}_2(s,z) = C_2 \cosh(K_2 z) + D_2 \sinh(K_2 z) + \tilde{\phi}_{P,2}(s,z) \tag{A-16}$$

$$z = Z - Z_1 \text{ for } (Z_1 \leq Z \leq Z_2)$$

Solutions of this type are written for each layer, except the disturbance to the free stream is of the form

$$\tilde{\phi}_\infty(s,z) = C_\infty e^{-K_\infty z} + \tilde{\phi}_{P_\infty}(s,z) \tag{A-17}$$

where

$$z = Z - Z_2 \tag{A-18}$$

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If M_∞ is subsonic, then $B_\infty = + i \sqrt{1-M_\infty^2}$, and K_∞ obeys rules of complex variable theory. We imagine the initial conditions given (by subsonic theory) on an extension of the sonic line (see Fig. A-1.). The coefficients $\{C_1\ C_2,\ D_1\ D_2, \ldots C_N,\ D_N,\ C_\infty\}$ are solved by applying the boundary conditions of the form(A13) and(A14) at the interfaces and by satisfying the surface downwash conditions.

The solutions, $\{\tilde{\phi}_{P,k}\}$, are particular integrals satisfying the non-homogeneous RHS of Equations(A6,A7, etc.).

$$\left(\tilde{\phi}_{P,k}\right)_{ZZ} - K_k^2\, \tilde{\phi}_{P,k}\,(s,Z) = \tilde{G}_k\,(s,Z) \qquad \text{(A-19)}$$

$\tilde{G}_k(s,Z)$ can probably be regarded as a constant or linear in Z over a narrow layer $(Z_{k1} \leq Z \leq Z_k)$ based on a curve fit to induced velocities on the sonic line from subsonic theory; therefore, solutions to(A19) can easily be found.

The Laplace transform for the upper surface lifting pressure can be written in the form

$$\Delta\tilde{p}\,(s,0) = \tilde{E}_{21}\,(s)\,\Delta\tilde{p}^{GR}\,(s,0) + \Delta\tilde{p}_{P,1}\,(s,0) \qquad \text{(A-20)}$$

where $\Delta\tilde{p}^{GR}(s,0)$ is the Garrick and Rubinow solution

$$\Delta\tilde{p}^{GR}\,(s,0) = -\rho_1\,(U_1 s + i\omega)\,\frac{\tilde{Wa}(s)}{K_1} \qquad \text{(A-21)}$$

and $\Delta\tilde{p}_{P,1}\,(s,0)$ is the additional surface pressure caused by sonic line disturbances.

The oscillatory surface pressure is obtained by the inverse Laplace transform of (A20).

$$\Delta\bar{p}\,(x,0) = \int_0^X \bar{E}_{21}\,(x-\xi)\,\Delta\bar{p}^{GR}\,(\xi,0)\,d\xi + \Delta\bar{p}_{P,1}\,(x,0) \qquad \text{(A-22)}$$

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Using the convolution principle (Ref. 42), the function $\bar{E}_{21}(x-\xi)$ is given by the Laplace transform inversion theorem

$$\bar{E}_{21}(x-\xi) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{E}_{21}(s) e^{s(x-\xi)} ds \qquad \text{(A-23)}$$

The inversion integral (A-23) is evaluated along the indented contour shown in Fig. A-2.

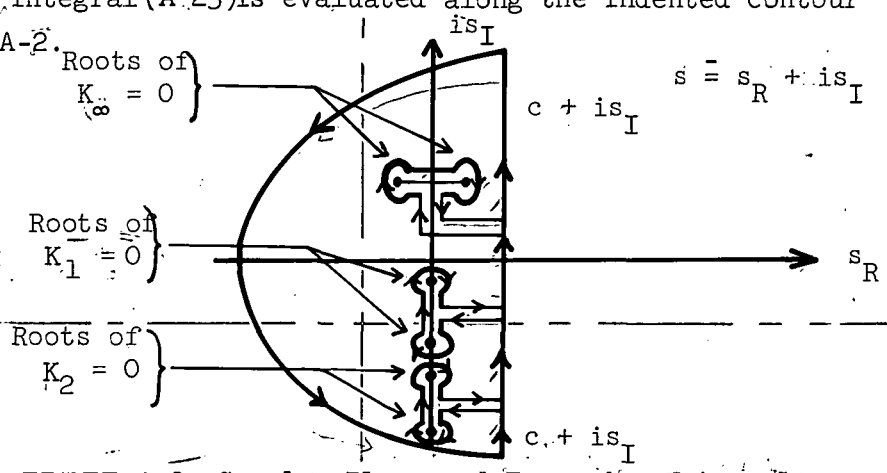

FIGURE A-2 Complex Plane and Inversion Integral Contour Showing Indentations Around

Branch Points: $(s = s_{k,1}, s_{k,2})$

Defined By: $K_k = 0 = \sqrt{(s-s_{k,1})(s-s_{k,2})}$

It can be seen from Equations (A10 & A11) that K_1^2 and K_2^2 are quadratics in s; therefore, one can write

$$K_1 = \pm\sqrt{(s-s_{1,1})(s-s_{1,2})} \qquad \text{(A-24)}$$

$$K_2 = \pm\sqrt{(s-s_{2,1})(s-s_{2,2})} \qquad \text{(A-25)}$$

and similar expressions for each layer. It turns out that the branch points ($s = s_{k,1}, s_{k2}$) lie on the imaginary axis for all layers which are locally supersonic, and for locally subsonic layers, the branch points are symmetric about the imaginary axis.

The integral(A23) may be evaluated by the calculus of residues (see Ref. 42) to yield

$$\bar{E}_{21}(x-\xi) = \frac{1}{2\pi i}\left\{2\pi i \sum_{j=1}^{N} e^{s_j(x-\xi)} \text{Residues } \tilde{E}_{21}(s_j) - \sum_k \int_{\text{Seg}k} e^{s(x-\xi)} \tilde{E}_{21}(s)ds\right\} \tag{A-26}$$

The last sum in(A26) represents the contribution of all the line integrals of $\tilde{E}_{21}(s)$ over various path segments shown in Fig.A-2. It can be shown that $\tilde{E}_{21}(s)$ is single valued everywhere in the complex s plane, except on opposite sides of the branch cuts (line segments connecting branch points s_{k1} and s_{k2} for each layer). It also turns out that the integrals around small circles surrounding the branch points vanish, and $\tilde{E}_{21}(s)$ is an even function of K_2 (or K_k in any layer separating adjacent supersonic regions k and k-1). Therefore, the only line integral segment contributions to Equation(A26) are from the first layer and the last layer adjacent to the subsonic free stream. This has been rigorously proven for the special case of two supersonic layers bounded by an external subsonic free stream.

The first sum in Equation(A26) represents residues at poles of $\tilde{E}_{21}(s)$ with contributions from each layer. $\tilde{E}_{21}(s)$ can be written in the form (numerator /denominator) and the denominator has the form

$$\tilde{D}_{E_{21}}(s) = \left\{1 - \tilde{H}_{21}(s) \tanh(K_1 Z_1)\right\} \tag{A-27}$$

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The required poles are zeroes of Equation(A27) which are countably infinite. Contributions from outer layers are contained in $\tilde{H}_{21}$ (s).

It is thus seen that the Laplace transform solution is equivalent to a multiple reflection solution but provides a systematic computation procedure.

Special Solution for Two Supersonic Layers Bounded by an External Subsonic Freestream

In this case

$$\tilde{E}_{21}(s) = \frac{N_{E_{21}}(s)}{\tilde{D}_{E_{21}}(s)} \tag{A-28}$$

where $\tilde{D}_{E_{21}}$ (s) is given by Equation(A27) and

$$\tilde{H}_{21}(s) = \tilde{Z}_{21}(s)\,\tilde{E}_2(s) \tag{A-29}$$

$$\tilde{N}_{21}(s) = \left\{ \tilde{H}_{21}(s) - \tanh(K_1 Z_1) \right\} \tag{A-30}$$

$$\tilde{Z}_{21}(s) = \frac{\rho_2 (U_2 s + i\omega)\, U_2 K_1}{\rho_1 (U_1 s + i\omega)\, U_1 K_2} \tag{A-31}$$

$$\tilde{E}_2(s) = \frac{\left\{ \tilde{Z}_{\infty,2} + \tanh(K_2 (Z_2 - Z_1)) \right\}}{\left\{ 1 + \tilde{Z}_{\infty,2} \tanh(K_2 (Z_2 - Z_1)) \right\}} \tag{A-32}$$

and

$$\tilde{Z}_{\infty,2} = \frac{\rho_\infty (U_\infty s + i\omega)\, U_\infty K_2}{\rho_2 (U_2 s + i\omega)\, U_2 K_\infty} \tag{A-33}$$

The quantities $\tilde{Z}_{21}(s)$ and $\tilde{Z}_{\infty,2}(s)$ are, in fact, the Laplace transforms of the impedance ratios across the first and second interfaces. In evaluation of K_∞ for the case, $M_\infty < 1$, one substitutes into Equations like(A10) and (A11)and obtains:

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$$K_\infty^2 = \left\{ B_\infty^2 s^2 + \left[2M_\infty \left(\frac{i\omega}{a_\infty}\right) s - \frac{\omega^2}{a_\infty^2} \right] \right\} \tag{A-34}$$

where

$$B_\infty^2 = -\beta_\infty^2 = -(1-M_\infty^2) \tag{A-35}$$

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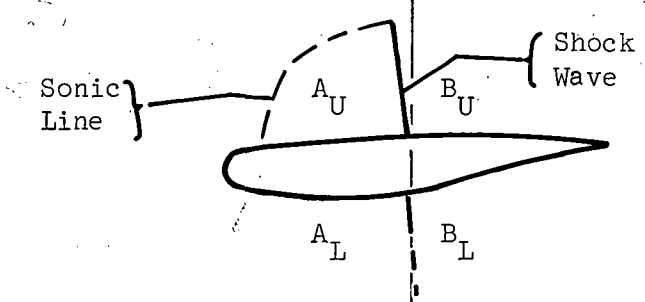
APPENDIX B

OVERALL SUBSONIC/TRANSONIC LIFTING SURFACE SOLUTIONS

Let the subscripts A_U and (B_U, B_L, A_L) denote, respectively, quantities associated with the local supersonic and subsonic regions, respectively (see Figure B-1).



Notes to B-1

A_U: $M_{A_U} \geq 1$

B_U: $M_{B_U} \leq 1$

B_L: $M_{B_L} < 1$

A_L: $M_{A_L} < 1$

Figure B-1 Designation of Regions of Locally Subsonic and Locally Supersonic Flow

The local supersonic velocity potential and lifting pressures are then expressed as:

$$\phi^{A_U} = \phi_H^{A_U} + \phi_P^{A_U} \tag{B-1}$$

$$p^{A_U} = p_H^{A_U} + p_P^{A_U} \tag{B-2}$$

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Where H denotes the homogeneous solution (due to the prescribed surface downwash) and P denotes the particular integral (due to sonic line disturbances, as discussed in Appendix A). ϕ^{A_U} and p^{A_U} include all of the multi-layer reflection effects discussed previously.

We next describe the surface downwash in the chordwise regions adjacent to A and B by column vector matrices $\{W_A\}$ and $\{W_B\}$. It is assumed that one has available a suitable subsonic kernel function/vortex lattice collocation type solution for the lifting surface pressure of the form

$$\left\{ \begin{array}{c} W_A \\ \hline W_B \end{array} \right\} = \left[\begin{array}{c|c} K_o^{AA} & K_o^{AB} \\ \hline K_o^{BA} & K_o^{BB} \end{array} \right] \left\{ \begin{array}{c} p_o^{A} \\ p_o^{B} \end{array} \right\} \tag{B-3}$$

In the purely subsonic case, the A and B distinction is omitted:

$$\{ W \} = [K_o] \{ p_o \} \tag{B-4}$$

The solution for $\{p_o\}$ in terms of $\{W\}$ is:

$$\{ p_o \} = [Q_{p_o}] \{ W \} \tag{B-5}$$

where

$$[Q_{p_o}] = [K_o]^{-1} \tag{B-6}$$

The object of the ensuing analysis is to replace the upper surface solution in region A_U by the results of the local supersonic solution and modify the local subsonic lift distribution to account for the interference in Region B and A_L caused by the presence of the local supersonic flow in Region A_U.

As a first step, define the lifting pressure as the difference between upper and lower surface pressure:

$$p = (p_L - p_U) \tag{B-7}$$

In purely subsonic flow,

$$p_{U_o} = p_{L_o} \tag{B-8}$$

and

$$p_o = 2p_{L_o} = -2p_{U_o} \tag{B-9}$$

When supercritical flow is present in Region A_U, this anti-symmetry property of the lifting pressure is destroyed, and one must calculate separately the upper and lower surface lift contributions ahead of the shock. Let us call A_L and B_L the lower wing regions forward and aft of the chordwise location of the upper wing surface shockwave location (see Fig. B-1).

It is still possible to calculate the lifting pressure in the subsonic regions B_U and B_L by assuming an anti-symmetric lifting doublet distribution; however, the doublet strength will be changed from its original subsonic theoretical values ~~(required by definition to match the motion downwash)~~. Three distinct sources of interference downwash arise:

(1) A change in doublet strength on the subsonic side in region A_L.

(2) The loss of one-half of the original subsonic doublet strength contributed by the upper side of the wing in the (now supersonic) region called A_U.

(3) The addition of an interference downwash along the airfoil surface in A_L, B_L, and B_U due to disturbances transmitted through the shock wave. These disturbances can be regarded as caused by a monopole source distribution distributed vertically along the shock wave front whose strength is everywhere equal to the component of velocity normal to the shock front on the downstream side (allowing for the possibility of a curved or oblique shock wave).

The strength of this velocity is given by Landahl's (Ref. 4) shock wave compatibility relations, suitably expressed in terms of the

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local supercritical solution (given by Eq.(B1)) and evaluated on the locus of the shock, $(X_{SH}(Z))$ on the upstream side.

The final form of the lifting pressure solution will be similar to Eq. (B5) in format:

$$\left\{p^A\right\} = \left[Q_p^{AA}\right]\left\{W_A\right\} + \left[Q_p^{AB}\right]\left\{W_B\right\} \qquad \text{(B-10)}$$

$$\left\{p^B\right\} = \left[Q_p^{BA}\right]\left\{W_A\right\} + \left[Q_p^{BB}\right]\left\{W_B\right\} \qquad \text{(B-11)}$$

it is necessary now to describe the matrices in more fundamental terms. From Eq.(B2) one can write the supersonic region surface pressure as:

$$\left\{p^A\right\} = \left[Q_{AA}^{HML}\right]\left\{W_A\right\} + \left[Q_X^{PML}\right]\left\{U_{SL}\right\} + \left[Q_Z^{PML}\right]\left\{W_{SL}\right\} + \left[Q_\phi^{PML}\right]\left\{\phi_{SL}\right\} \qquad \text{(B-12)}$$

where $\left\{U_{SL}\right\}$, $\left\{W_{SL}\right\}$ are the X and Z velocities on the sonic line (at each layer centerline) due to subsonic disturbances and $\left[Q_X^{PML}\right]$, $\left[Q_Z^{PML}\right]$ are the induction matrices for the effect on surface pressure due to the particular integrals arising from the local multi-layer supersonic solution. The term $\left[Q_\phi^{PML}\right]\left\{\phi_{SL}\right\}$ recognizes phase differences in velocity potential along the sonic line.

The sonic line disturbance potential and velocities are expressed in terms of subsonic induction effects

$$\left\{U_{SL}\right\} = \left[K_{o,A}^{XSL}\right]\left\{\tfrac{1}{2}\left(p_o^A + \Delta p^{A_L}\right)\right\} + \left[K_{o,B}^{XSL}\right]\left\{p_o^B + \Delta p^B\right\} \qquad \text{(B-13)}$$

$$\left\{W_{SL}\right\} = \left[K_{o,Z}^{ZSL}\right]\left\{\tfrac{1}{2}\left(p_o^A + \Delta p^{A_L}\right)\right\} + \left[K_{o,B}^{ZSL}\right]\left\{p_o^B + \Delta p^B\right\} \qquad \text{(B-14)}$$

$$\left\{\phi_{SL}\right\} = \left[K_{o,Z}^{\phi SL}\right]\left\{\tfrac{1}{2}\left(p_o^A + \Delta p^{A_L}\right)\right\} + \left[K_{o,B}^{\phi SL}\right]\left\{p_o^B + \Delta p^B\right\} \qquad \text{(B-15)}$$

In Eqs.(B13) and(B14) $\left\{\Delta p^{A_L}, \Delta p^B\right\}$ represent the interference loading in the subsonic local flow regions A_L, B_U, B_L due to the presence of locally supersonic flow in region A_U on the upper side of the wing. The factor

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1/2 multiplied by $\left\{ p_o^{A} + \Delta p^{A_L} \right\}$ accounts for the local supersonic region A_U not communicating with the subsonic region (except via the shock wave source distribution).

Next consider the disturbances transmitted from the supersonic region A_U through the shock wave

$$\left\{ U_{SH}^{B_U} \right\} = \left[T_{BA}^{XX,SH} \right] \left\{ U_{SH}^{A_U} \right\} + \left[T_{BA}^{XZ,SH} \right] \left\{ W_{SH}^{A_U} \right\} + \left[T_{BA}^{X\phi,SH} \right] \left\{ \phi_{SH}^{A_U} \right\} \tag{B-16}$$

$$\left\{ W_{SH}^{B_U} \right\} = \left[T_{BA}^{ZX,SH} \right] \left\{ U_{SH}^{A_U} \right\} + \left[T_{BA}^{ZZ,SH} \right] \left\{ W_{SH}^{A_U} \right\} + \left[T_{BA}^{Z\phi,SH} \right] \left\{ \phi_{SH}^{A_U} \right\} \tag{B-17}$$

$$\left\{ \phi_{SH}^{B_U} \right\} = \left[T_{BA}^{\phi X,SH} \right] \left\{ U_{SH}^{A_U} \right\} + \left[T_{BA}^{\phi Z,SH} \right] \left\{ W_{SH}^{A_U} \right\} + \left[T_{BA}^{\phi\phi,SH} \right] \left\{ \phi_{SH}^{A_U} \right\} \tag{B-18}$$

Equations (B16) to (B18) represent a generalized form of Landahl's shock compatibility relations for curved and oblique shocks. For a given shock geometry the <u>normal velocity</u> on the <u>downstream</u> side is

$$\left\{ V_{SH,n}^{B_U} \right\} = \left[n_x^{SH} \right] \left\{ U_{SH}^{B_U} \right\} + \left[n_z^{SH} \right] \left\{ W_{SH}^{B_U} \right\} \tag{B-19}$$

where $\vec{n}_{SH} = \vec{i}\, n_x^{SH} + \vec{k}\, n_z^{SH}$ is the unit normal vector to the shock front.

The surface downwash in the subsonic regions induced by the normal velocity (source distribution) on the downstream side of the shock can be expressed in terms of suitable Green's functions:

$$\left\{ \Delta W_{A_L}^{SH} \right\} = \left[G_{UBSH}^{WA} \right] \left\{ U_{SH}^{B_U} \right\} + \left[G_{WBSH}^{WA} \right] \left\{ W_{SH}^{B_U} \right\} + \left[G_{\phi SH}^{WA} \right] \left\{ \phi_{SH}^{B_U} \right\} \tag{B-20}$$

$$\left\{ \Delta W_{B_L}^{SH} \right\} = \left[G_{UBSH}^{WB} \right] \left\{ U_{SH}^{B_U} \right\} + \left[G_{WBSH}^{WB} \right] \left\{ W_{SH}^{B_U} \right\} + \left[G_{\phi SH}^{WB} \right] \left\{ \phi_{SH}^{B_U} \right\} \tag{B-21}$$

Because of Eqs.(B3),(B20), and(B21), we require that the modified subsonic doublet loading satisfy:

$$\left\{\Delta W_{A_L}^{SH}\right\} = \left[K_o^{AA}\right]\left\{\tfrac{1}{2}(\Delta p^{A_L} - p_o^{A})\right\} + \left[K_o^{AB}\right]\left\{\Delta p^{B}\right\} \tag{B-22}$$

$$\left\{\Delta W_{B_L}^{SH}\right\} = \left[K_o^{BA}\right]\left\{\tfrac{1}{2}(\Delta p^{A_L} - p_o^{A})\right\} + \left[K_o^{BB}\right]\left\{\Delta p^{B}\right\} \tag{B-23}$$

The terms involving $\left\{p_o^{A}\right\}$ Eqs.(B22) and(B23) are known and, therefore, are transposed to the left side to yield

$$\left(\left\{\Delta W_A^{SH}\right\} + \left[K_o^{AA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\}\right) = \left[K_o^{AA}\right]\left\{\tfrac{1}{2}\Delta p^{A_L}\right\} + \left[K_o^{AB}\right]\left\{\Delta p^{B}\right\} \tag{B-24}$$

$$\left(\left\{\Delta W_{B_L}^{SH}\right\} + \left[K_o^{BA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\}\right) = \left[K_o^{BA}\right]\left\{\tfrac{1}{2}\Delta p^{A_L}\right\} + \left[K_o^{BB}\right]\left\{\Delta p^{B}\right\} \tag{B-25}$$

Now it is necessary to express $\left\{\Delta W_{A_L}^{SH}\right\}$, $\left\{\Delta W_{B_L}^{SH}\right\}$ in terms of a linear combination of $\left\{\tfrac{1}{2}\Delta p^{A_L}\right\}$, $\left\{\Delta p^{B}\right\}$ plus other known quantities $\left\{p_o^{A}\right\}$, $\left\{p_o^{B}\right\}$ or $\left\{W_A\right\}$, $\left\{W_B\right\}$. First, Equations(B16), (B17), and(B18) are summarized as follows:

$$\begin{Bmatrix} \phi_{SH}^{B_U} \\ U_{SH}^{B_U} \\ W_{SH}^{B_U} \end{Bmatrix} = \left[T_{SH}^{BA}\right] \begin{Bmatrix} \phi_{SH}^{A_U} \\ U_{SH}^{A_U} \\ W_{SH}^{A_U} \end{Bmatrix} \tag{B-26}$$

From the local supersonic multi-layer solution, relations similar to Eqs.(B1) and(B2) may be written for any local supersonic quantity. These quantities may then be evaluated on the upstream side of the shock and expressed as follows:

$$\left\{\phi_{SH}^{A_U}\right\} = \left[G_{SH,A}^{\phi_W}\right]\left\{W_A + \Delta W_A^{P}\right\} + \left\{\phi_{P,SH}^{A_U}\right\} \tag{B-27}$$

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$$\left\{ U_{SH}^{A_U} \right\} = \left[G_{SH,A}^{UW} \right] \left\{ W_A + \Delta W_A^{P} \right\} + \left\{ U_{P,SH}^{A_U} \right\} \tag{B-28}$$

$$\left\{ W_{SH}^{A_U} \right\} = \left[G_{SH,A}^{WW} \right] \left\{ W_A + \Delta W_A^{P} \right\} + \left\{ W_{P,SH}^{A_U} \right\} \tag{B-29}$$

where from Eqs.(B1) and(B2) one can define

$$\left\{ U_{SH}^{A_U} \right\} = \left\{ \frac{\partial \phi}{\partial X}^{A_U} (X_{SH}(Z)) \right\} \tag{B-30}$$

$$\left\{ W_{SH}^{A_U} \right\} = \left\{ \frac{\partial \phi}{\partial Z}^{A_U} (X_{SH}(Z)) \right\} \tag{B-31}$$

Next, by analogy to Eq.(B12) one can express $\left\{ \Delta W_A^{P}, \phi_{P,SH}^{A_U}, W_{P,SH}^{A_U}, U_{P,SH}^{A_U} \right\}$ as a linear combination of $\left\{ \phi_{SL}, U_{SL}, W_{SL} \right\}$. Let us define a partitioned matrix relation.

$$\begin{Bmatrix} \Delta W_A^{P} \\ \phi_{P,SH}^{A_U} \\ U_{P,SH}^{A_U} \\ W_{P,SH}^{A_U} \end{Bmatrix} = \begin{bmatrix} T_{\phi SL}^{WAP} & T_{USL}^{WAP} & T_{WSL}^{WAP} \\ T_{\phi SL}^{\phi PSH} & T_{USL}^{\phi PSH} & T_{WSL}^{\phi PSH} \\ T_{\phi SL}^{UPSH} & T_{USL}^{UPSH} & T_{WSL}^{UPSH} \\ T_{\phi SL}^{WPSH} & T_{USL}^{WPSH} & T_{WSL}^{WPSH} \end{bmatrix} \begin{Bmatrix} \phi_{SL} \\ U_{SL} \\ W_{SL} \end{Bmatrix} \tag{B-32}$$

It is convenient to introduce shock front and sonic line "state vectors" such that:

$$\left\{ \Delta W_A^{P} \right\} = \left[T_{SL}^{WAP} \right] \left\{ \sigma_{SL} \right\} \tag{B-33}$$

$$\left\{ \sigma_{P,SH}^{A_U} \right\} = \left[T_{SL}^{APSH} \right] \left\{ \sigma_{SL} \right\} \tag{B-34}$$

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where the "state vectors" are defined as

$$\left\{\sigma_{SL}\right\} = \begin{Bmatrix} \phi_{SL} \\ U_{SL} \\ W_{SL} \end{Bmatrix} \tag{B-35}$$

$$\left\{\sigma_{P,SH}^{A_U}\right\} = \begin{Bmatrix} \phi_{P,SH}^{A_U} \\ U_{P,SH}^{A_U} \\ W_{P,SH}^{A_U} \end{Bmatrix} \tag{B-36}$$

Eqs.(B27) to(B29) can be regarded as defining a total shock entry "state vector"

$$\left\{\sigma_{SH}^{A_U}\right\} = \begin{Bmatrix} \phi_{SH}^{A_U} \\ U_{SH}^{A_U} \\ W_{SH}^{A_U} \end{Bmatrix} \tag{B-37}$$

From Eqs.(B27) through(B37)

$$\left\{\sigma_{SH}^{A_U}\right\} = \left[G_{WA}^{A_U SH}\right]\left(\left\{W_A\right\} + \left[T_{SL}^{WAP}\right]\left\{\sigma_{SL}\right\}\right) + \left[T_{SL}^{APSH}\right]\left\{\sigma_{SL}\right\} \tag{B-38}$$

One can rewrite Eq.(B26) in terms of a downstream shock "state vector"

$$\left\{\sigma_{SH}^{B_U}\right\} = \begin{Bmatrix} \phi_{SH}^{B_U} \\ U_{SH}^{B_U} \\ W_{SH}^{B_U} \end{Bmatrix} \tag{B-39}$$

Eq.(B26) then implies

$$\left\{\sigma_{SH}^{B_U}\right\} = \left[T_{SH}^{BA}\right]\left\{\sigma_{SH}^{A_U}\right\} \tag{B-40}$$

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Equations (B13), (B14), and (B15) can be rewritten in terms of a relationship between the sonic line "state vector" and the subsonic doublet loading

$$\left\{\sigma_{SL}\right\} = \left[K_{o,A}^{SL}\right]\left\{\tfrac{1}{2}\left(p_o^{A} + \Delta p^{A_L}\right)\right\} + \left[K_{o,B}^{SL}\right]\left\{p_o^{B} + \Delta p^{B}\right\} \quad \text{(B-41)}$$

One can by analogy to Eqs. (B2), (B27) through (B29) calculate the upper surface pressure in region A_U in terms of the sonic line "state vector"

$$\left\{p^{A_U}(Z=0)\right\} = \left\{p_s^{A_U}\right\} = \left[\begin{matrix} p_s^{A_U} \\ G_{WA} \end{matrix}\right]\left\{W_A + \Delta W_A^{P}\right\} + \left\{p_{P,s}^{A_U}\right\} \quad \text{(B-42)}$$

By analogy to Eq. (B32), the surface pressure due to the particular integral in the supersonic region may be written:

$$\left\{p_{P,s}^{A_U}\right\} = \left[\begin{matrix} p_{A_U,P} \\ T_{SL} \end{matrix}\right]\left\{\sigma_{SL}\right\} \quad \text{(B-43)}$$

Eqs. (B42) and (B43) combine to yield:

$$\left\{p^{A_U}(Z=0)\right\} = \left[\begin{matrix} p_s^{A_U} \\ G_{WA} \end{matrix}\right]\left(\left\{W_A\right\} + \left[\begin{matrix} WAP \\ T_{SL} \end{matrix}\right]\left\{\sigma_{SL}\right\}\right) + \left[\begin{matrix} p_{A_U,P} \\ T_{SL} \end{matrix}\right]\left\{\sigma_{SL}\right\} \quad \text{(B-44)}$$

Equations (B20) and (B21) can be rewritten as

$$\left\{\begin{matrix} \Delta W_{A_L}^{SH} \\ \hline \Delta W_{B_L}^{SH} \end{matrix}\right\} = \left[\begin{matrix} G_{BSH}^{WA_L} \\ \hline G_{BSH}^{WB_L} \end{matrix}\right]\left\{\sigma_{SH}^{B_U}\right\} \quad \text{(B-45)}$$

From Eqs. (B45), (B40), and (B38)

$$\left\{\begin{matrix} \Delta W_{A_L}^{SH} \\ \hline \Delta W_{B_L}^{SH} \end{matrix}\right\} = \left[\begin{matrix} G_{BSH}^{WA_L} \\ \hline G_{BSH}^{WB_L} \end{matrix}\right]\left[T_{SH}^{BA}\right]\left(\left(\left[G_{WA}^{A_U SH}\right]\left[T_{SL}^{WAP}\right] + \left[T_{SL}^{APSH}\right]\right)\left\{\sigma_{SL}\right\} + \left[G_{WA}^{A_U SH}\right]\left\{W_A\right\}\right) \quad \text{(46a)}$$

$$= \left[\begin{matrix} G_{SL}^{WA_L} \\ \hline G_{SL}^{WB_L} \end{matrix}\right]\left\{\sigma_{SL}\right\} + \left[\begin{matrix} G_{WA}^{WASH} \\ \hline G_{WA}^{WBSH} \end{matrix}\right]\left\{W_A\right\} \quad \text{(B-46b)}$$

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Substitution of Eq.(B41) $\{\sigma_{SL}\}$ in Eq.(B44) yields $\{p^{A_U}(Z=0)\}$ in terms of $\{\frac{1}{2}(p_o^{A} + \Delta p^{A_L})\}$, $\{\Delta p^{B}\}$. Substitution of Eq.(B41) for $\{\sigma_{SL}\}$ in (B-46b) yields $\{\Delta W_{A_L}^{SH}, \Delta W_{B_L}^{SH}\}$ as a linear function of $\{\frac{1}{2}(p_o^{A} + \Delta p^{A_L})\}$, $\{\Delta p^{B}\}$. Thus

$$\begin{Bmatrix} \Delta W_{A_L}^{SH} \\ \hline \Delta W_{B_L}^{SH} \end{Bmatrix} = \begin{bmatrix} G_{SL}^{WA_L} \\ \hline G_{SL}^{WB_L} \end{bmatrix} \left(\begin{bmatrix} K_{o;A}^{SL} \end{bmatrix} \begin{Bmatrix} \frac{1}{2}(p_o^{A} + \Delta p^{A_L}) \end{Bmatrix} + \begin{bmatrix} K_{o,B}^{SL} \end{bmatrix} \begin{Bmatrix} p_o^{B} + \Delta p^{B} \end{Bmatrix} \right) + \begin{bmatrix} G_{WA}^{WASH} \\ \hline G_{WA}^{WBSH} \end{bmatrix} \begin{Bmatrix} W_A \end{Bmatrix} \tag{B-47}$$

Substituting Eq.(B47) into Eqs.(B24) and(B25) leads to a well posed set of linear matrix algebraic equations for $\{\frac{1}{2}(p_o^{A} + \Delta p^{A_L})\}$ and $\{(\Delta p^{B})\}$ expressed as functions of $\{\frac{1}{2}p_o^{A}\}$ and $\{p_o^{B}\}$. These results may then be back substituted to get $\{p^{A_U}(Z=0)\}$ the upper surface supersonic pressure as a linear function of $\{W_A, \frac{1}{2}p_o^{A}, p_o^{B}\}$. Eq.(B47) may be expressed more compactly as

$$\begin{Bmatrix} \Delta W_{A_L}^{SH} \\ \hline \Delta W_{B_L}^{SH} \end{Bmatrix} = \left[\begin{array}{c|c} K_{AA}^{WASH} & K_{AB}^{WASH} \\ \hline K_{BA}^{WBSH} & K_{BB}^{WBSH} \end{array} \right] \begin{Bmatrix} \frac{1}{2}(p_o^{A} + \Delta p^{A_L}) \\ \hline (p_o^{B} + \Delta p^{B}) \end{Bmatrix} + \begin{bmatrix} G_{WA}^{WASH} \\ \hline G_{WA}^{WBSH} \end{bmatrix} \begin{Bmatrix} W_A \end{Bmatrix} \tag{B-49}$$

where

$$\begin{aligned} \left[K_{AA}^{WASH} \right] &= \left[G_{SL}^{WA_L} \right] \left[K_{o,A}^{SL} \right] \\ \left[K_{AB}^{WASH} \right] &= \left[G_{SL}^{WA_L} \right] \left[K_{o,B}^{SL} \right] \\ \left[K_{BA}^{WBSH} \right] &= \left[G_{SL}^{WB_L} \right] \left[K_{o,A}^{SL} \right] \\ \left[K_{BB}^{WBSH} \right] &= \left[G_{SL}^{WB_L} \right] \left[K_{o,B}^{SL} \right] \end{aligned} \tag{B-50}$$

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Putting Eq.(B49) into (B24) and (B25) yields

$$\left[K_{AA}^{WASH}\right]\left\{\tfrac{1}{2}(p_o^{A} + \Delta p^{A_L})\right\} + \left[K_{AB}^{WASH}\right]\left\{p_o^{B} + \Delta p^{B}\right\} + \left[K_o^{AA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\} + \left[G_{WA}^{WASH}\right]\left\{W_A\right\}$$

$$= \left[K_o^{AA}\right]\left\{\frac{\Delta p^{A_L}}{2}\right\} + \left[K_o^{AB}\right]\left\{\Delta p^{B}\right\} \qquad \text{(B-51)}$$

$$\left[K_{BA}^{WBSH}\right]\left\{\tfrac{1}{2}(p_o^{A} + \Delta p^{A_L})\right\} + \left[K_{BB}^{WBSH}\right]\left\{p_o^{B} + \Delta p^{B}\right\} + \left[K_o^{BA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\} + \left[G_{WA}^{WBSH}\right]\left\{W_A\right\}$$

$$= \left[K_o^{BA}\right]\left\{\frac{\Delta p^{A_L}}{2}\right\} + \left[K_o^{BB}\right]\left\{\Delta p^{B}\right\} \qquad \text{(B-52)}$$

The quantities $\left\{p_o^{A}\right\}$ and $\left\{p_o^{B}\right\}$ are known from the subsonic solution (Eqs.(B3) to(B6)); therefore, Eqs.(B51) and(B52) may be written in the following form:

$$\left[\bar{K}_{AA}\right]\left\{\frac{\Delta p^{A_L}}{2}\right\} + \left[\bar{K}_{AB}\right]\left\{\Delta p^{B}\right\} = \left[\Lambda^{AA}\right]\left\{\frac{p_o^{A}}{2}\right\} + \left[\Lambda^{AB}\right]\left\{p_o^{B}\right\} + \left[G_{WA}^{WASH}\right]\left\{W_A\right\} \qquad \text{(B-53)}$$

$$\left[\bar{K}_{BA}\right]\left\{\frac{\Delta p^{A_L}}{2}\right\} + \left[\bar{K}_{BB}\right]\left\{\Delta p^{B}\right\} = \left[\Lambda^{BA}\right]\left\{\frac{p_o^{A}}{2}\right\} + \left[\Lambda^{BB}\right]\left\{p_o^{B}\right\} + \left[G_{WA}^{WBSH}\right]\left\{W_A\right\} \qquad \text{(B-54)}$$

where

$$\left[\bar{K}_{AA}\right] = \left[K_o^{AA}\right] - \left[K_{AA}^{WASH}\right]$$

$$\left[\bar{K}_{AB}\right] = \left[K_o^{AB}\right] - \left[K_{AB}^{WASH}\right]$$

$$\left[\Lambda^{AA}\right] = \left[K_o^{AA}\right] + \left[K_{AA}^{WASH}\right] \qquad \text{(B-55)}$$

$$\left[\Lambda^{AB}\right] = \left[K_{AB}^{WASH}\right]$$

$$\left[\bar{K}_{BA}\right] = \left[K_o^{BA}\right] - \left[K_{BA}^{WASH}\right]$$

$$\left[\bar{K}_{BB}\right] = \left[K_o^{BB}\right] - \left[K_{BB}^{WASH}\right] \qquad \text{(B-56)}$$

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$$\left[\Lambda^{BA}\right] = \left[K_o^{BA}\right] + \left[K_{BA}^{WBSH}\right]$$

$$\left[\Lambda^{BB}\right] = \left[K_{BB}^{WBSH}\right]$$

Eqs.(B53) and(B54) may be more compactly written as:

$$\left[\bar{K}\right]\left\{\begin{matrix}\frac{1}{2}\Delta p^{A_L} \\ \Delta p^{B}\end{matrix}\right\} = \left[\Lambda\right]\left\{\begin{matrix}\frac{1}{2}p_o^{A} \\ p_o^{B}\end{matrix}\right\} + \left[\begin{matrix}G_{WA}^{WASH} \\ G_{WA}^{WBSH}\end{matrix}\right]\left\{W_A\right\} \qquad \text{(B-57)}$$

Solving Eq.(B57) yields finally the subsonic local flow interference pressure increment due to the presence of the supercritical flow region A_U on the upper side of the wing.

$$\left\{\begin{matrix}\frac{1}{2}\Delta p^{A_L} \\ \Delta p^{B}\end{matrix}\right\} = \left[\bar{K}\right]^{-1}\left(\left[\Lambda\right]\left\{\begin{matrix}\frac{1}{2}p_o^{A} \\ p_o^{B}\end{matrix}\right\} + \left[\begin{matrix}G_{WA}^{WASH} \\ G_{WA}^{WBSH}\end{matrix}\right]\left\{W_A\right\}\right) = \left[\begin{matrix}P_I^{A} \\ P_I^{B}\end{matrix}\right]\left\{\begin{matrix}\frac{1}{2}p_o^{A} \\ p_o^{B}\end{matrix}\right\} + \left[\begin{matrix}P_{I,WA}^{A} \\ P_{I,WA}^{B}\end{matrix}\right]\left\{W_A\right\} \qquad \text{(B-58)}$$

The interference pressure is seen to be proportional to the basic subsonic loading computed by classical subsonic kernel function or vortex lattice theory, plus an increment from the local supersonic solution.

Local Supersonic Upper Surface Pressure

From Eqs.(B44),(B41) and(B58) one can write the local supersonic pressure in the following form:

$$\left\{p^{A_U}(Z=0)\right\} = \left[G_{WA}^{p_sA_U}\right]\left\{W_A\right\} + \left(\left[G_{WA}^{p_sA_U}\right]\left[T_{SL}^{WAP}\right] + \left[T_{SL}^{p_sA_UP}\right]\right)\left(\left[K_{o,A}^{SL}\right]\left\{\frac{1}{2}(p_o^{A}+\Delta p^{A_L})\right\}\right.$$

$$\left. + \left[K_{o,B}^{SL}\right]\left\{p_o^{B} + \Delta p^{B}\right\}\right) \qquad \text{(B-59)}$$

Letting

$$\left[\bar{G}_{WA}^{p_sA_U}\right] = \left[G_{WA}^{p_sA_U}\right]\left[T_{SL}^{WAP}\right] + \left[T_{SL}^{p_sA_UP}\right] \qquad \text{(B-60a)}$$

and

$$\left[\Lambda_{G_{WA}}^{p_sA_U}\right] = \left[G_{WA}^{p_sA_U}\right] + \left[\bar{G}_{WA}^{p_sA_U}\right]\left[K_{o,A}^{SL}\right]\left[P_{I,WA}^{A}\right] \qquad \text{(B-60b)}$$

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and substituting(B58) in(B59) yields:

$$\left\{p^{A_U}(Z=0)\right\} = \left[G^{p_s A_U}_{WA}\right]\left\{W_A\right\} + \left[\bar{G}^{p_s A_U}_{WA}\right]\left(\left[K^{SL}_{o,A}\right]\left(\left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[P_I^{\,A}\right]\left\{\begin{matrix}\tfrac{1}{2}p_o^{\,A}\\ \hline p_o^{\,B}\end{matrix}\right\} + \left[P^{A}_{I,WA}\right]\left\{W_A\right\}\right)\right.$$
$$\left. + \left[K^{SL}_{o,B}\right]\left(\left\{p_o^{\,B}\right\} + \left[P_I^{\,B}\right]\left\{\begin{matrix}\tfrac{1}{2}p_o^{\,A}\\ \hline p_o^{\,B}\end{matrix}\right\} + \left[P^{B}_{I,WA}\right]\left\{W_A\right\}\right)\right) \qquad \text{(B-61)}$$

The separate contributions of the subsonic loading and supersonic downwash can be collected in Eq.(B61) to yield:

$$\left\{p^{A_U}(Z=0)\right\} = \left[\hat{G}^{p_s A_U}_{WA}\right]\left\{W_A\right\} + \left[G^{p_o A}_{p_s A_U}\right]\left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[G^{p_o B}_{p_s A_U}\right]\left\{p_o^{\,B}\right\} \qquad \text{(B-62)}$$

where

$$\left[G^{p_o A}_{p_s A_U}\right] = \left[\bar{G}^{p_s A_U}_{WA}\right]\left(\left[K^{SL}_{o,A}\right] + \left[K^{SL}_{o,A}\right]\left[P_I^{\,AA}\right] + \left[K^{SL}_{o,B}\right]\left[P_I^{\,BA}\right]\right) \qquad \text{(B-63)}$$

$$\left[G^{p_o B}_{p_s A_U}\right] = \left[\bar{G}^{p_s A_U}_{WA}\right]\left(\left[K^{SL}_{o,A}\right]\left[P_I^{\,AB}\right] + \left[K^{SL}_{o,B}\right] + \left[K^{SL}_{o,B}\right]\left[P_I^{\,BB}\right]\right) \qquad \text{(B-64}$$

and(B58)is further partitioned as follows:

$$\left[\begin{matrix}P_I^{\,A}\\ \hline P_I^{\,B}\end{matrix}\right] = \left[\begin{matrix}P_I^{\,AA} & P_I^{\,AB}\\ \hline P_I^{\,BA} & P_I^{\,BB}\end{matrix}\right] \qquad \text{(B-65)}$$

Lifting Pressure Forward of the Shock Wave

The lifting pressure in the region A forward of the shock can be obtained from Eqs.(B7),(B58) and(B62) as follows:

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$$\left\{p^A\right\} = \left\{p^{A_L} - p^{A_U}\right\}$$

$$= \left\{\tfrac{1}{2}(p_o^{\,A} + \Delta p^{A_L}) - p^{A_U}\right\} \tag{B-66}$$

$$= \left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[P_I^{\,AA}\right]\left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[P_I^{\,AB}\right]\left\{p_o^{\,B}\right\} + \left[P^A_{I,WA}\right]\left\{W_A\right\}$$

$$-\left(\left[\hat{G}^{p_s A_U}_{WA}\right]\left\{W_A\right\} + \left[G^{p_o A}_{p_s A_U}\right]\left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[G^{p_o B}_{p_s A_U}\right]\left\{p_o^{\,B}\right\}\right) \tag{B-67}$$

$$\left\{p^A\right\} = \left[\overline{P}_{AA}\right]\left\{\tfrac{1}{2}p_o^{\,A}\right\} + \left[\overline{P}_{AB}\right]\left\{p_o^{\,B}\right\} - \left[\hat{G}^{p_s A_U}_{WA}\right]\left\{W_A\right\} + \left[P^A_{I,WA}\right]\left\{W_A\right\} \tag{B-68}$$

where

$$\left[\overline{P}_{AA}\right] = \left(\left[1\right] + \left[P_I^{\,AA}\right] - \left[G^{p_o A}_{p_s A_U}\right]\right) \tag{B-69}$$

$$\left[\overline{P}_{AB}\right] = \left(\left[P_I^{\,AB}\right] - \left[G^{p_o B}_{p_s A_U}\right]\right) \tag{B-70}$$

Using Eqs.(B3) to(B6), Eq.(B68) can be put in the desired AIC format described by Eqs.(B10) and(B11) as follows:

$$\left\{p^A\right\} = \left[\overline{P}_{AA}\right]\left(\left[\tfrac{1}{2}Q^{AA}_{p_o}\right]\left\{W_A\right\} + \left[\tfrac{1}{2}Q^{AB}_{p_o}\right]\left\{W_B\right\}\right) + \left[P^A_{I,WA}\right]\left\{W_A\right\}$$

$$+ \left[\overline{P}_{AB}\right]\left(\left[Q^{BA}_{p_o}\right]\left\{W_A\right\} + \left[Q^{BB}_{p_o}\right]\left\{W_B\right\}\right) - \left[\hat{G}^{p_s A_U}_{WA}\right]\left\{W_A\right\} \tag{B-71}$$

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Therefore, the desired form of the aerodynamic loading influence coefficient, is finally obtained as:

$$\left\{p^{A}\right\} = \left[Q_p^{AA}\right]\left\{W_A\right\} + \left[Q_p^{AB}\right]\left\{W_B\right\} \tag{B-72}$$

where from(B71)

$$\left[Q_p^{AA}\right] = \left[\overline{P}_{AA}\right]\left[\tfrac{1}{2}Q_{p_o}^{AA}\right] + \left[\overline{P}_{AB}\right]\left[Q_{p_o}^{BA}\right]$$

$$= \left[\hat{G}_{WA}^{p_s A_U}\right] + \left[P_{I,WA}^{A}\right] \tag{B-73}$$

and

$$\left[Q_p^{AB}\right] = \left[\overline{P}_{AA}\right]\left[\tfrac{1}{2}Q_{p_o}^{AB}\right] + \left[\overline{P}_{AB}\right]\left[Q_{p_o}^{BB}\right] \tag{B-74}$$

Lifting Pressure Aft of the Shock Wave

From Eqs.(B3) through(B11) and(B58), the lifting pressure aft of the shock (in these purely subsonic regions) can be calculated as follows:

$$\left\{p^{B}\right\} = \left\{p^{B_U} - p^{B_L}\right\} = \left\{p_o^{B} + \Delta p^{B}\right\} \tag{B-75}$$

$$= \left\{p_o^{B}\right\} + \left[P_I^{BA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\} + \left[P_I^{BB}\right]\left\{p_o^{B}\right\} + \left[P_{I,WA}^{B}\right]\left\{W_A\right\}$$

$$= \left[\overline{P}_{BA}\right]\left\{\tfrac{1}{2}p_o^{A}\right\} + \left[\overline{P}_{BB}\right]\left\{p_o^{B}\right\} + \left[P_{I,WA}^{B}\right]\left\{W_A\right\} \tag{B-76}$$

where

$$\left[\overline{P}_{BA}\right] = \left[P_I^{BA}\right] \tag{B-77}$$

$$\left[\overline{P}_{BB}\right] = \left(\left[1\right] + \left[P_I^{BB}\right]\right) \tag{B-78}$$

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Expressing Eq.(B76) in terms of downwash via Eq.(B5), yields the desired AIC format given by Eqs.(B10) and(B11):

$$\left\{p^{B}\right\} = \left[Q_p^{BA}\right]\left\{W_A\right\} + \left[Q_p^{BB}\right]\left\{W_B\right\} \tag{B-79}$$

where from Eq.(B5) and(B76)

$$\left[Q_p^{BA}\right] = \left[\overline{P}_{BA}\right]\left[\tfrac{1}{2}Q_{p_o}^{AA}\right] + \left[\overline{P}_{BB}\right]\left[Q_{p_o}^{BA}\right] + \left[P_{I,WA}^{B}\right] \tag{B-80}$$

and

$$\left[Q_p^{BB}\right] = \left[\overline{P}_{BA}\right]\left[\tfrac{1}{2}Q_{p_o}^{AB}\right] + \left[\overline{P}_{BB}\right]\left[Q_{p_o}^{BB}\right] \tag{B-81}$$

Thus, Eqs.(B79) and(B72) display the desired form of the pressure/downwash AIC's with proper consideration for the interaction between the local subsonic and supersonic regions caused by transmission of disturbances through the shock wave and the sonic line. The resultant AIC's are perturbations of the classical subsonic AIC's (represented by $\left[Q_{p_o}\right]$) plus coupling terms (represented by $\left[\overline{P}\right]$) plus the direct supersonic surface downwash effect (- $\left[\hat{G}_{WA}^{p_s A_U}\right]$).

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APPENDIX C

PARTICULAR INTEGRAL REQUIRED BY THE LAPLACE TRANSFORM SOLUTION FOR THE MULTILAYERED SUPERSONIC REGION

In Appendix A, it is shown that in each supersonic layer, the velocity potential obeys the following non-homogeneous ordinary differential equation in z after performing a Laplace transformation with respect to X. In layers

$$\tilde{\phi}^{(k)}_{P_{zz}}(s,z) - K_k^2 \tilde{\phi}^{(k)}_P(s,z) = \tilde{G}_k(s,Z) \tag{C-1}$$

where

$$K_k^2 = \{B_k^2 s^2 + [2M_k \frac{i\omega}{a_k} s - (\omega/a_k)^2]\} \tag{C-2}$$

$$B_k = \sqrt{(M_k^2 - 1)} \text{ if } M_k > 1 \tag{C-3}$$

$$B_k \equiv i\sqrt{(1 - M_k^2)} \equiv i\beta_k \text{ , if } M_k < 1 \tag{C-4}$$

The functions $\tilde{G}_k(s,Z)$ arise from initial values of $\phi^{(k)}_{SL}$ and $\phi^{(k)}_{x,SL}$ (on the sonic line).

$$\tilde{G}_k(s,Z) = -\{B_k^2 (s\phi^{(k)}_{SL} + \phi^{(k)}_{x,SL}) + 2i(\omega/a_k) M_k \phi^{(k)}_{SL}\} \tag{C-5}$$

It will now be assumed that the initial values $\phi^{(k)}_{SL}$ and $\phi^{(k)}_{x,SL}$ are linear in Z for a narrow layer $\{Z_{k-1} \leq Z \leq Z_k\}$, and can be interpolated from numerical values at discrete points $\{Z_k : k=1,2,---,N\}$. These discrete numerical values are obtained from subsonic theory as indicated in Appendix B. Therefore, for $(Z_{k-1} \leq Z \leq Z_k)$, let

$$\tilde{G}_k(s,z) = \tilde{G}_{k_0}(s) + \tilde{G}_{k_1}(s)\, z \tag{C-6}$$

where,

$$z \equiv (Z - Z_{k-1}) \tag{C-7}$$

Then, Eqs (C1, C5 through (C7) imply

$$\tilde{G}_{k_o}(s) \equiv -\{B_k^2 (s\phi_{SL_o}^{(k)} + \phi_{x,SL_o}^{(k)}) + 2i(\frac{\omega}{a_k})M_k\phi_{SL_o}^{(k)}\} \tag{C-8}$$

$$\tilde{G}_{k_1}(s) \equiv -\{B_k^2 (s\phi_{SL_1}^{(k)} + \phi_{x,SL_1}^{(k)}) + 2i(\frac{\omega}{a_k})M_k\phi_{SL_1}^{(k)}\} \tag{C-9}$$

and where the linear approximations to $\phi_{SL}^{(k)}$ and $\phi_{x,SL}^{(k)}$ are

$$\phi_{SL}^{(k)}(Z) \equiv \phi_{SL_o}^{(k)} + \phi_{SL_1}^{(k)} z \tag{C-10}$$

$$\phi_{x,SL}^{(k)}(Z) \equiv \phi_{x,SL_o}^{(k)} + \phi_{x,SL_1}^{(k)} z \tag{C-11a}$$

$$\equiv \bar{u}_{SL_o}^{(k)} + \bar{u}_{SL_1}^{(k)} z \tag{C-11b}$$

Note that $\phi_{SL_o}^{(i)}$ can be set = 0 on the airfoil surface at the first layer; however, finite values exist for the other layers for transient flow. Next, consider solutions to Eq(C-1), with(C-6) substituted for the right hand side

$$\tilde{\phi}_{P_{zz}}^{(k)}(s,z) - K_k^2 \tilde{\phi}_P(s,z) = \tilde{G}_{k_o}(s) + z\,\tilde{G}_{k_1}(s) \tag{C-12}$$

The general solution of Eq(C-12)(or C-6) obtained by the method of variation of parameters, is (Ince, Ref 43 p.123)

$$\tilde{\phi}_P^{(k)}(s,z) = -\tilde{\phi}_{H_1}^{(k)} \int_0^z \frac{\tilde{\phi}_{H_2}^{(k)}(\zeta)\,\tilde{G}_k(s,\zeta)d\zeta}{\Delta(\tilde{\phi}_{H_1}\,\tilde{\phi}_{H_2})} + \tilde{\phi}_{H_2}^{(k)}(s,z) \int_0^z \frac{\tilde{\phi}_{H_1}^{(k)}(\zeta)\,\tilde{G}_k(s,\zeta)d\zeta}{\Delta(\tilde{\phi}_{H_1},\,\tilde{\phi}_{H_2})} \tag{C-13}$$

where the homogeneous functions are defined as solutions of Eq (C-12) when

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$\tilde{G}_k = 0$:

$$\tilde{\phi}^{(k)}_{H_1}(s,z) \equiv \cosh(K_k z) \tag{C-14}$$

$$\tilde{\phi}^{(k)}_{H_2}(s,z) \equiv \sinh(K_k z) \tag{C-15}$$

$\Delta(\tilde{\phi}_{H_1}, \tilde{\phi}_{H_2})$ is the Wronskian determinant:

$$\Delta(\tilde{\phi}_{H_1}, \tilde{\phi}_{H_2}) \equiv \begin{vmatrix} \tilde{\phi}^{(k)}_{H_1} & \tilde{\phi}^{(k)}_{H_2} \\ \dfrac{d\tilde{\phi}^{(k)}_{H_1}}{dz} & \dfrac{d\tilde{\phi}^{(k)}_{H_2}}{dz} \end{vmatrix} \tag{C-16a}$$

$$\equiv \begin{vmatrix} \cosh(K_k z) & \sinh(K_k z) \\ K_k \sinh(K_k z) & K_k \cosh(K_k z) \end{vmatrix} \tag{C-16b}$$

$$\Delta(\tilde{\phi}_{H_1}, \tilde{\phi}_{H_2}) = K_k\left(\cosh^2(K_k z) - \sinh^2(K_k z)\right) \tag{C-16c}$$

$$\equiv K_k \tag{C-16d}$$

Therefore, for any arbitrary $\tilde{G}_k(s,z)$

$$\tilde{\phi}^k_P(s,z) = \frac{-\cosh(K_k z)}{K_k}\int_0^z \sinh(K_k\zeta)\,\tilde{G}_k(s,\zeta)\,d\zeta + \frac{+\sinh(K_k z)}{K_k}\int_0^z \cosh(K_k\zeta)\,\tilde{G}_k(s,\zeta)\,d\zeta \tag{C-17}$$

For linear approximation to $\tilde{G}_k(s,z)$, given by Eq(C6), one obtains

$$\tilde{\phi}^k_P(s,z) = \frac{-\cosh(K_k z)}{K_k^2}\,\tilde{G}_{k_o}(s)\left[\cosh(K_k z) - 1\right] + \frac{+\sinh(K_k z)}{K_k^2}\,\tilde{G}_{k_o}(s)\,\sinh(K_k z) - \frac{1}{K_k}\left\{\cosh(K_k z)\int_0^z \sinh(K_k z)\,\zeta\,\tilde{G}_{k_1}(s)\,d\zeta\right\} + \frac{1}{K_k}\left\{\sinh(K_k z)\int_0^z \cosh(K_k\zeta)\,\zeta\,\tilde{G}_{k_1}(s)\,d\zeta\right\} \tag{C-18a}$$

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$$= \frac{\tilde{G}_{k_o}(s)}{K_k^2}\{-1 + \cosh(K_k z)\} - \frac{\tilde{G}_{k_1}(s)}{K_k}\{\tilde{I}_1(s,z)\cosh(K_k z)$$

$$- \tilde{I}_2(s,z)\sinh(K_k z)\} \qquad \text{(C-18b)}$$

The integrals, $\tilde{I}_1(s,z)$ and $\tilde{I}_2(s,z)$, are defined as

$$\tilde{I}_1(s,z) \equiv \int_0^z \zeta\sinh(K_k\zeta)d\zeta \qquad \text{(C-19a)}$$

$$\tilde{I}_2(s,z) \equiv \int_0^z \zeta\cosh(K_k\zeta)d\zeta \qquad \text{(C-19b)}$$

From elementary integral tables

$$\tilde{I}_1(s,z) \equiv \frac{1}{K_k^2}\left[(K_k z)\cosh(K_k z) - \sinh(K_k z)\right] \qquad \text{(C-20a)}$$

and

$$\tilde{I}_2(s,z) \equiv \frac{1}{K_k^2}\left[(K_k z)\sinh(K_k z) - \cosh(K_k z) + 1\right] \qquad \text{(C-20b)}$$

Substituting (C-20a) and (C-20b) into (C-18b) yields

$$\tilde{\phi}_P^{(k)}(s,z) = \frac{\tilde{G}_{k_o}(s)}{K_k^2}\{-1 + \cosh(K_k z)\}$$

$$- \frac{\tilde{G}_{k_1}(s)}{K_k^3}\{\cosh(K_k z)\left[(K_k z)\cosh(K_k z)\right.$$

$$\left.-\sinh(K_k z)\right] - \sinh(K_k z)\left[1 - \cosh(K_k z)\right.$$

$$\left.+(K_k z)\sinh(K_k z)\right]\} \qquad \text{(C-21a)}$$

Eq (C21a) may be further simplified to yield

$$\tilde{\phi}_P^{(k)}(s,z) = \frac{\tilde{G}_{k_o}(s)}{K_k^2}\{-1 + \cosh(K_k z)\} - \frac{\tilde{G}_{k_1}(s)}{K_k^3}\{(K_k z)$$

$$-\sinh(K_k z)\} \qquad \text{(C-21b)}$$

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Eq(C21b) represents the function required by the multilayer supersonic solution defined in Appendix A. Appendix A also requires $\phi_P^{(k)}$ and $\phi_{P,z}^{(k)}$ evaluated at $z = 0$ and at $z = (Z_k - Z_k) \equiv z_k$.

First, the derivative of(C21b) is

$$\frac{\partial \tilde{\phi}_P^{(k)}}{\partial z}(s,z) \equiv \tilde{\phi}_{P,z}^{(k)}$$

$$= \frac{\tilde{G}_{k_o}(s)}{K_k^2}\{K_k \sinh(K_k z)\} - \frac{\tilde{G}_{k_1}(s)}{K_k^3}\{K_k - \cosh(K_k z)K_k\} \qquad \text{(C-22a)}$$

$$\therefore \phi_{P,z}^{(k)}(s,z) = \frac{\tilde{G}_{k_o}(s)}{K_k}\sinh(K_k z) - \frac{\tilde{G}_{k_1}(s)}{K_k^2}\{1 - \cosh(K_k z)\} \qquad \text{(C-22b)}$$

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Eq(C22b) represents the interference downwash due to the particular integral in each layer $\{Z_k : k = 1, 2:N\}$. At $z = 0$ one has from Eq(C21b)

$$\tilde{\phi}_P^{(k)}(s,z = 0) \equiv 0 \qquad \text{(C-23)}$$

At $z = 0$ from Eq(C22b) implies

$$\tilde{\phi}_{P,z}^{(k)}(s,0) = 0 \qquad \text{(C-24)}$$

At $z = z_k \equiv (Z_k - Z_{k-1})$ Eqs(C21b) and(C22b) yield

$$\tilde{\phi}_P^{(k)}(s,z_k) = \frac{\tilde{G}_{k_o}(s)}{K_k^2}\{-1 + \cosh(K_k z_k)\}$$

$$- \frac{\tilde{G}_{k_1}(s)}{K_k^3}\{(K_k z_k) - \sinh(K_k z_k)\} \qquad \text{(C-25)}$$

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$$\tilde{\phi}_{P,z}^{(k)}(s,z_k) = \frac{G_{k_o}(s)}{K_k} \sinh(K_k z_k) - \frac{\tilde{G}_{k_1}(s)}{K_k^2}\{1 - \cosh(K_k z_k)\} \quad \text{(C-26)}$$

The quantities defined by Eqs(C23)through (C26)are needed for evaluation of layer interface boundary conditions defined in Appendix A.

Particular Integral for a Subsonic External Free Stream Disturbance

The Laplace transformed disturbance potential is governed by

$$\tilde{\phi}_{P_{zz}}^{(\infty)}(s,z) - K_\infty^2 \tilde{\phi}_P^{(\infty)}(s,z) = \tilde{G}_\infty(s,z) \quad \text{(C-27)}$$

where

$$K_\infty^2 = \{B_\infty^2 s^2 + 2M_\infty \left(\frac{i^{(1)}}{a_\infty}\right) s - \left(\frac{^{(1)}}{a_\infty}\right)^2\} \quad \text{(C-28)}$$

and, in general,

$$B_\infty^2 = M_\infty^2 - 1 \; ; \qquad M_\infty > 1 \quad \text{(C-29)}$$

$$= -\beta_\infty^2 \; ; \qquad M_\infty < 1$$

If $M_\infty < 1$ it is assumed that initial value disturbances are defined on an outward extension of the sonic line, and are related to subsonic flow disturbances in the manner described in Appendix B. If $M_\infty > 1$, then no initial disturbances need be considered; therefore, the particular integral is non-zero only when $M_\infty < 1$.

The right hand side of Eq(C27) like Eq(C5) is given by

$$\tilde{G}_\infty(s,z) \equiv -\{B_\infty^2(s\phi_{SL}^{(\infty)} + \phi_{x,SL}^{(\infty)}) + 2i\left(\frac{^{(1)}}{a_\infty}\right) M_\infty \phi_{SL}^{(\infty)}\} \quad \text{(C-30a)}$$

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$$= \{ \beta_\infty^2 (s\phi_{SL}^{(\infty)} + \phi_{x,SL}^{(\infty)}) - 2i(\frac{\omega}{a_\infty}) M_\infty \phi_{SL}^{(\infty)} \} \quad \text{(C-30b)}$$

It will be assumed that $\tilde{G}_\infty(s,Z)$ can be approximated in the form

$$\tilde{G}_\infty(s,Z) = \tilde{G}_{\infty,o} e^{-b_\infty z} \equiv \tilde{G}_\infty(s,Z_N) e^{-b_\infty z} \quad \text{(C-31)}$$

where for N interior layers

$$z = Z - Z_N \quad \text{(C-32)}$$

Linearly independent homogeneous solutions of Eq(C27) are of the form

$$\tilde{\phi}_{H_1}^{(\infty)}(s,z) = e^{-K_\infty z} \quad \text{(C-33a)}$$

$$\tilde{\phi}_{H_2}^{(\infty)}(s,z) = e^{+K_\infty z} \quad \text{(C-33b)}$$

By the method of variation of parameters, one can immediately write the particular integral to Eq(C27) in the form prescribed by Eq(C13).

$$\tilde{\phi}_P^{(\infty)}(s,z) = -e^{+K_\infty z} \int_z^\infty \frac{(e^{-K_\infty \zeta})(e^{-b_\infty \zeta})}{\Delta} \tilde{G}_{\infty,o} d\zeta + e^{-K_\infty z} \int_0^z \frac{e^{(K_\infty - b_\infty)\zeta} G_{\infty,o} d\zeta}{\Delta} \quad \text{(C-34)}$$

where Δ is the Wronskian determinant

$$\Delta = \begin{vmatrix} e^{-K_\infty z} & e^{K_\infty z} \\ -K_\infty e^{-K_\infty z} & K_\infty e^{K_\infty z} \end{vmatrix} = 2K_\infty \quad \text{(C-35)}$$

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$$\therefore \quad \phi_P^{(\infty)}(s,z) = \frac{-e^{K_\infty z}}{2K_\infty}(G_{\infty,o})\int_z^\infty e^{-(b_\infty + K_\infty)\zeta}\,d\zeta$$

$$\frac{+e^{-K_\infty z}}{2K_\infty}(G_{\infty,o})\int_o^z e^{(K_\infty - b_\infty)\zeta}\,d\zeta \qquad \text{(C-36)}$$

Carrying out the integration yields for $(b_\infty \neq K_\infty)$

$$\tilde{\phi}_P^{(\infty)}(s,z) = \frac{(\tilde{G}_{\infty,o})}{2K_\infty}\left\{ \frac{-e^{K_\infty z}\, e^{-(b_\infty + K_\infty)z}}{(K_\infty + b_\infty)} \quad \frac{+e^{-K_\infty z}\,(e^{(K_\infty - b_\infty)z} - 1)}{(K_\infty - b_\infty)} \right\} \qquad \text{(C-37)}$$

$$\therefore \quad \tilde{\phi}_P^{(\infty)}(s,z) = \frac{\tilde{G}_{\infty,o}}{2K_\infty}\left\{ \frac{-e^{-b_\infty z}}{(K_\infty + b_\infty)} \quad \frac{+e^{-b_\infty z}}{(K_\infty - b_\infty)} \right\}$$

$$- \frac{\tilde{G}_{\infty,o}\, e^{-K_\infty z}}{2K_\infty}\left\{ \frac{1}{(K_\infty - b_\infty)} \right\} \qquad \text{(C-38)}$$

In the special case $b_\infty = K_\infty$ then Eq(C36) yields

$$\tilde{\phi}_P^{(\infty)}(s,z) = \frac{\tilde{G}_{\infty,o}}{2K_\infty}\left\{ \frac{-e^{-K_\infty z}}{2K_\infty} + z e^{-K_\infty z} \right\}$$

$$= \frac{(\tilde{G}_{\infty,o})}{2K_\infty^2}\, e^{-K_\infty z}\left\{ (K_\infty z) - \frac{1}{2} \right\} \qquad \text{(C-39)}$$

For large $K_\infty z$ the first term dominates and the disturbance decay rate is slower than for the case $K_\infty \neq b_\infty$. Since Eq(C27) also governs the subsonic lifting surface theory (when $\tilde{G}_\infty(s,z) = 0$) one could expect the sonic line disturbances due to subsonic doublets to behave like $e^{-K_\infty z}$ at large $K_\infty z$; therefore, Eq(C39) is probably the relevant solution.

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APPENDIX D

SHOCK WAVE TRANSFER MATRICES

The following analysis is restricted to normal shock waves (shock front perpendicular to the airfoil surface). This is a good approximation to many cases of interest in transonic flow, at least as far as the final transition from supersonic to subsonic flow, even when there is a preliminary supersonic oblique shock compression near the end of the supercritical zone (above a so-called "supersonic bubble"; see Piercey Reference 24 Section 11 and Blackwell, Reference 24 Section 21).

The following analysis uses results of Landahl (Reference 5, p. 113) and expresses these results in the format required by Appendix B. Landahl writes the relation between velocity potential and streamwise perturbation velocity (backwash) across a normal shock as follows:

$$\bar{\phi}_x^{A_U} - i\frac{\alpha}{b}\bar{\phi}^{A_U} = -\bar{\phi}_x^{B_U} - i\frac{\alpha}{b}\bar{\phi}^{B_U} \qquad \text{(D-1)}$$

where,

$$\alpha = 2k_A M_{A_U}^2 / (1-M_{B_U}^2) \qquad \text{(D-2)}$$

The reduced frequency k_A is defined in terms of a reference length b and the upstream flow velocity U_{A_U}

$$k_A = \omega b/U_{A_U} \qquad \text{(D-3)}$$

Here we use A_U and B_U to denote conditions upstream and downstream of the shock, following Appendix B. The tangential velocity component, W, (upwash) must be continuous across the shock wave; therefore,

$$W_{SH}^{A_U} = W_{SH}^{B_U} \qquad \text{(D-4)}$$

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or,

$$\left(\frac{\partial\bar{\phi}^{A_U}}{\partial z}\right)_{SH} = \left(\frac{\partial\bar{\phi}^{B_U}}{\partial z}\right)_{SH} \tag{D-4b}$$

For a single layer, the relations(D1) to(D4) can be expressed in state vector form (as in Appendix B). Let

$$\left\{\bar{\sigma}^{A_U}\right\} = \begin{Bmatrix} \bar{\phi}^{A_U}_{SH} \\ \bar{U}^{A_U}_{SH} \\ \bar{W}^{A_U}_{SH} \end{Bmatrix} \tag{D-5}$$

where,

$$\bar{U}^{A_U}_{SH} = \left(\frac{\partial\bar{\phi}^{A_U}}{\partial x}\right)_{x_{SH}} \tag{D-6}$$

Then, Eqs(D1) to(D4) can be written in matrix form

$$\begin{bmatrix} -i\alpha/b & 1 & 0 \\ \partial/\partial x & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{\phi}^{A_U}_{SH} \\ \bar{U}^{A_U}_{SH} \\ \bar{W}^{A_U}_{SH} \end{Bmatrix} = \begin{bmatrix} \frac{-i\alpha}{b} & -1 & 0 \\ \partial/\partial x & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{\phi}^{B_U}_{SH} \\ \bar{U}^{B_U}_{SH} \\ \bar{W}^{BU}_{SH} \end{Bmatrix} \tag{D-7}$$

The operator $\partial/\partial x$ becomes s after Laplace transformation, and all harmonic, x dependent (barred) quantities become ($\sim$) quantities, dependent on s, the Laplace transform complex variable.

Thus, Eq(D7) transforms to

$$\begin{bmatrix} -i\alpha/b & 1 & 0 \\ s & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \tilde{\phi}^{A_U}_{SH} \\ \tilde{U}^{A_U}_{SH} \\ \tilde{W}^{A_U}_{SH} \end{Bmatrix} = \begin{bmatrix} -i\alpha/b & -1 & 0 \\ s & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \tilde{\phi}^{B_U}_{SH} \\ \tilde{U}^{B_U}_{SH} \\ \tilde{W}^{B_U}_{SH} \end{Bmatrix} \tag{D-8}$$

Thus, one can write the transfer matrix partitions for change of the state vector across a normal shock at the centerline of layer k as follows:

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$$\left\{\tilde{\sigma}^{B_U}_{SH}\right\}_k = \left\{\begin{matrix}\tilde{\sigma}^{B_U}_{SH}\\ \tilde{U}^{B_U}_{SH}\\ \tilde{W}^{B_U}_{SH}\end{matrix}\right\}_k \tag{D-9}$$

$$\left\{\tilde{\sigma}^{B_U}_{SH}\right\} = \left[T^{(k)SH}_{BA}\right]\left\{\tilde{\sigma}^{A_U}_{SH}\right\} \tag{D-10}$$

$$\left[T^{(k)SH}_{BA}\right] = \begin{bmatrix} -i\alpha_k/b & -1 & 0\\ s & -1 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix} -i\alpha_k/b & 1 & 0\\ s & -1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{D-11}$$

The inversion can be accomplished by partitioning. First it is noted that the upper left hand two by two partition inverts as follows (dropping the subscript k):

$$\begin{bmatrix} -i\alpha/b & -1\\ s & -1\end{bmatrix}^{-1} = \frac{1}{(i\alpha/b+s)}\begin{bmatrix} -1 & 1\\ -s & -i\alpha/b\end{bmatrix} \tag{D-12}$$

Evaluating the upper left hand partition of (D-11) yields

$$\begin{bmatrix} -i\alpha/b & -1\\ s & -1\end{bmatrix}^{-1}\begin{bmatrix} -i\alpha/b & 1\\ s & -1\end{bmatrix} =$$

$$\frac{1}{\left(\frac{i\alpha}{b}+s\right)}\begin{bmatrix} -1 & 1\\ -s & -i\alpha/b\end{bmatrix}\begin{bmatrix} -i\alpha/b & 1\\ s & -1\end{bmatrix} \tag{D-13}$$

$$= \frac{1}{\left(\frac{i\alpha}{b}+s\right)}\begin{bmatrix} (i\alpha/b+s) & -2\\ 0 & (-s+i\alpha/b)\end{bmatrix} \tag{D-14}$$

Therefore, Eq (D-11) reduces to

$$\left[T^{(k)SH}_{BA}\right] = \begin{bmatrix} 1 & -2/(s+i\alpha_k/b) & 0\\ 0 & (-s+i\alpha_k/b)/(s+i\alpha_k/b) & 0\\ 0 & 0 & 1\end{bmatrix} \tag{D-15}$$

Since the shock wave transfer function is a point function there is no coupling between layers in the larger matrix implied by Eq (B-26).

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APPENDIX E

EXPLICIT SOLUTION FOR TWO INTERIOR SUPERSONIC LAYERS BOUNDED BY AN EXTERNAL FREESTREAM AND TECHNIQUE FOR GENERALIZATION TO N INTERIOR LAYERS

From Appendix A the Laplace transformed boundary conditions at interface 1 and 2 are summarized as follows: At the interface $Z = Z_1$

$$\tilde{\phi}_z^{(1)}(s,Z_1)/U_1 = \tilde{\phi}_z^{(2)}(s,Z_1)/U_2$$

$$\tilde{p}^{(1)}(s,Z_1) = \tilde{p}^{(2)}(s,Z_1)$$

At $Z = Z_2$:

$$\tilde{\phi}_z^{(2)}(s,Z_2)/U_2 = \tilde{\phi}^{(\infty)}(s,Z_2)/U_\infty \qquad \text{(E-1)}$$

$$\tilde{p}^{(2)}(s,Z_2) = \tilde{p}^{(\infty)}(s,Z_2)$$

Using Eqs (15) through (17) of appendix A, Eq.(E-1) becomes(at $Z=Z_1; z=z_1=Z_1$):

$$\frac{K_1}{U_1}\left[C_1 \sinh(K_1 z_1) + D_1 \cosh(K_1 z_1)\right] + \frac{1}{U_1}\tilde{\phi}_{Pz}^{(1)}(s,z_1) = \frac{K_2 D_2}{U_2} \qquad \text{(E-2a)}$$

$$-\rho_1 (U_1 s + i\omega)\left\{\left[C_1 \cosh(K_1 z_1) + D_1 \sinh(K_1 z_1)\right] + \tilde{\phi}_P^{(1)}(s,z_1)\right\} = \qquad \text{(E-2b)}$$

$$= -\rho_2 (U_2 s + i\omega)\, C_2$$

At the external freestream interface $(Z = Z_2,\ z = z_2 = (Z_2 - Z_1))$:

$$\frac{K_2}{U_2}\left[C_2 \sinh(K_2 z_2) \pm D_2 \cosh(K_2 z_2)\right] + \frac{1}{U_2}\tilde{\phi}_{Pz}^{(2)}(s,z_2) = \qquad \text{(E-2c)}$$

$$= \frac{1}{U_\infty}\left\{-K_\infty C_\infty + \tilde{\phi}_{P_z}(s,z=0)\right\}$$

$$-\rho_2 (U_2 s + i\omega)\left\{\left[C_2 \cosh(K_2 z_2) + D_2 \sinh(K_2 z_2)\right] + \tilde{\phi}_P^{(2)}(s,z_2)\right\} =$$

$$= -\rho_\infty (U_\infty s \mp i\omega)\left\{C_\infty + \tilde{\phi}_P^{(\infty)}(s,0)\right\} \qquad \text{(E-2d)}$$

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Next, we write the Laplace transformed airfoil surface normal velocity (upwash) boundary condition

$$\tilde{W}_a(s) = \left\{ K_1 D_1 + \tilde{\phi}_{Pz}^{(1)}(s,0) \right\} \tag{E-3}$$

Eqs.(E2a-E2d)and(E3) provide five equations for the five unknowns, C_1, D_1, C_2, D_2, C_∞.

Each of the interface pressure and streamline slope continuity boundary conditions can be put in the form of a transfer matrix.

$$\left\{ F_{P_1}^{(1)} \right\} + \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix} \begin{Bmatrix} C_1 \\ D_1 \end{Bmatrix} = \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \\ B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix} \begin{Bmatrix} C_2 \\ D_2 \end{Bmatrix} + \left\{ F_{P_1}^{(2)} \right\} \tag{E-4}$$

where for $Z = Z_j$, $k = j$ or $j + 1$ we define vectors $\left\{ F_{P_j}^{(k)} \right\}$ such that

$$\left\{ F_{P,j}^{(k)} \right\} = \left\{ \begin{matrix} 1/U_k \left(\partial \tilde{\phi}_P^{(k)}(s,Z_j)/\partial z \right) \\ \tilde{p}_{P_j}^{(k)}(s,Z_j) \end{matrix} \right\} \tag{E-5}$$

$$\tilde{p}_{P_j}^{(k)} = -\rho_k (U_k s + i\omega)\, \tilde{\phi}_P^{(k)}(s,Z_j) \tag{E-6}$$

The vectors $\left\{ F_{P,j}^{(k)} \right\}$ define forcing functions arising from the particular integrals (defined in Appendix C) which, in turn, are caused by the sonic line initial value disturbances.

The external freestream interface conditions (E-2c) and E-2d) can be written in similar format

$$\left\{ F_{P_2}^{(2)} \right\} + \left[A^{(2)} \right] \begin{Bmatrix} C_2 \\ D_2 \end{Bmatrix} = \left\{ B^{(\infty)} \right\} C_\infty + \left\{ F_{P_2}^{(\infty)} \right\} \tag{E-7}$$

where, (∞) implies the use of external freestream steady flow field conditions for fluid property $(U_\infty, \rho_\infty, a_\infty)$ evaluations. The two by two transfer matrices

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$\left[A^{(k)}\right]$ and $\left[B^{(k)}\right]$ are defined as follows from Eqs (E-2)

$$A_{11}^{(k)} = \frac{K_k}{U_k} \sinh (K_k z_k)$$

$$A_{12}^{(k)} = \frac{K_k}{U_k} \cosh (K_k z_k)$$

$$A_{21}^{(k)} = -\rho_k (U_k s + i\omega) \cosh(K_k z_k) \qquad \text{(E-8)}$$

$$A_{22}^{(k)} = -\rho_k (U_k + i\omega) \sinh(K_k z_k)$$

and,

$$B_{11}^{(k+1)} = 0$$

$$B_{12}^{(k+1)} = A_{12}^{(k+1)} (s, z_{k+1} = 0) = K_{k+1}/U_{k+1}$$

$$B_{21}^{(k+1)} = A_{21}^{(k+1)} (s, z_{k+1} = 0) = -\rho_{k+1} (U_{k+1} s + i\omega) \qquad \text{(E-9)}$$

$$B_{22}^{(k+1)} = A_{22}^{(k+1)} (s, z_{k+1} = 0) = 0$$

Thus, the elements of $\left[A^{(k)}\right]$ and $\left[B^{(k+1)}\right]$ satisfy a recursion relationship which is helpful for computation.

The column vector $\left\{\beta^{(\infty)}\right\}$ defined in Eq (E-7) is associated with the interface between the outermost interior layer and the external freestream (valid whether M_∞ is either supersonic or subsonic). $\left\{\beta^{(\infty)}\right\}$ is thus defined by Eqs (E-7) and (E-2d) by an obvious generalization from 2 to N supersonic layers.

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$$\left\{\beta^{(\infty)}\right\} = \begin{bmatrix} B_{11}^{(\infty)} & B_{12}^{(\infty)} \\ B_{21}^{(\infty)} & B_{22}^{(\infty)} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} B_{11}^{(\infty)} + B_{12}^{(\infty)} \\ B_{21}^{(\infty)} + B_{22}^{(\infty)} \end{Bmatrix} \tag{E-10}$$

The superscript (∞) implies substitution of freestream fluid properties for the $(N + 1)$ layer in Eq(E9). Thus,

$$B_{11}^{(\infty)} = 0 \; ; \quad B_{12}^{(\infty)} = -K_\infty/U_\infty$$

$$B_{21}^{(\infty)} = -\rho_\infty(U_\infty s + i\omega) \; ; \quad B_{22}^{(\infty)} = 0 \tag{E-11}$$

Therefore, (E-10) and (E-11) yield

$$\left\{\beta^{(\infty)}\right\} = \begin{Bmatrix} -K_\infty/U_\infty \\ -\rho_\infty(U_\infty s + i\omega) \end{Bmatrix} \tag{E-12}$$

The two by two transfer matrices $\left[A^{(k)}\right]$ and $\left[B^{(k+1)}\right]$ allow one to easily solve for the coefficients $\begin{Bmatrix} C_k \\ D_k \end{Bmatrix}$ including especially C_1 and D_1 which define the surface pressure. First, it is necessary to eliminate C_∞ by use of Eq (E-7) which can be re-expressed as a constraint between C_2 and D_2 (or C_N and D_N in the general case of N interior layers).

$$C_N = \alpha_{NN} D_N + C_{NP} \tag{E-13}$$

where, from Eqs (E-2c) and E-2d)

$$A_{11}^{(N)} C_N + A_{12}^{(N)} D_N + (1/U_N)\tilde{\phi}_{Pz}^{(N)}(s, z_N) = \frac{1}{U_\infty}\left(-K_\infty C_\infty + \tilde{\phi}_{Pz}^{(\infty)}(s, 0)\right) \tag{E-14a}$$

$$A_{21}^{(N)} C_N + A_{22}^{(N)} D_N - \rho_N(U_N s + i\omega)\tilde{\phi}_P^{(N)}(s, z_N) = -\rho_\infty(U_\infty s + i\omega)\left(C_\infty + \tilde{\phi}_P^{(\infty)}(s, 0)\right) \tag{E-14b}$$

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Solve Eqs (E-14) for C_∞

$$A_{21}^{(N)} C_N + A_{22}^{(N)} D_N + \tilde{p}_{P,N}^{(N)}(s, z_N)$$

$$= \tilde{p}_{P,N}^{(\infty)}(s,0) - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty}{K_\infty} \left[(\delta_{P,N}^{(\infty)} - \delta_{P,N}^{(N)}) \right. \tag{E-15}$$

$$\left. - A_{11}^{(N)} C_N - A_{12}^{(N)} D_N \right]$$

Therefore, (E-15) can be written

$$\gamma_{11}^{(N)} C_N + \gamma_{12}^{(N)} D_N = g_N^{(\infty)} \tag{E-16}$$

where,

$$\gamma_{11}^{(N)} = A_{21}^{(N)} - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty}{K_\infty} A_{11}^{(N)} \tag{E-17a}$$

$$\gamma_{12}^{(N)} = A_{22}^{(N)} - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty}{K_\infty} A_{12}^{(N)} \tag{E-17b}$$

and,

$$g_N^{(\infty)} = (p_{P,N}^{(\infty)}(s,o) - p_{P,N}^{(N)}(s,z_N)) - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty}{K_\infty} (\delta_{P,N}^{(\infty)} - \delta_{P,N}^{(N)}) \tag{E-18}$$

and the particular integral contributions to interface streamline slopes are defined as

$$\delta_{P,N}^{(N)} = \frac{1}{U_N} \tilde{\phi}_{Pz}^{(N)}(s, z_N)$$

$$\delta_{P,N}^{(\infty)} = \frac{1}{U_\infty} \tilde{\phi}_{Pz}^{(\infty)}(s, 0) \tag{E-19}$$

From Eqs (E-13) and(E-16) through(E-19)

$$\alpha_{NN} = -\gamma_{12}^{(N)} / \gamma_{11}^{(N)} \tag{E-20}$$

and

$$C_{N,P} = g_N^{(\infty)} / \gamma_{11}^{(N)} \tag{E-21}$$

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Having defined the constraint Eq (E-13) for the external stream effect we solve for (C_k, D_k; k = 1,2) From Eqs (E-3), (E-4) and (E-13) one obtains

$$\begin{Bmatrix} C_1 \\ D_1 \end{Bmatrix} = \left[A^{(1)}\right]^{-1}\left(\left[B^{(2)}\right]\begin{Bmatrix} C_2 \\ D_2 \end{Bmatrix} + \begin{Bmatrix} F^{(2)}_{P,1} \end{Bmatrix} - \begin{Bmatrix} F^{(1)}_{P,1} \end{Bmatrix}\right) \tag{E-22}$$

$$= \left[A^{(1)}\right]^{-1}\left(\left[B^{(2)}\right]\begin{Bmatrix} \alpha_{22}D_2 + C_{2,P} \\ D_2 \end{Bmatrix} + \begin{Bmatrix} F^{(2)}_{P,1} \end{Bmatrix} - \begin{Bmatrix} F^{(1)}_{P,1} \end{Bmatrix}\right) \tag{E-23}$$

$$= \begin{Bmatrix} E_1^{(1,2)} \\ E_2^{(1,2)} \end{Bmatrix} D_2 + \begin{Bmatrix} \Delta E_1^{(1,2)} \\ \Delta E_2^{(1,2)} \end{Bmatrix} C_{2,P} +$$

$$+ \left[A^{(1)}\right]^{-1}\left(\begin{Bmatrix} F^{(2)}_{P,1} \end{Bmatrix} - \begin{Bmatrix} F^{(1)}_{P,1} \end{Bmatrix}\right) \tag{E-24}$$

Define the inverse of $\left[A^{(k)}\right]$ as $\left[I_A^{(k)}\right]$

$$\left[I_A^{(k)}\right] = \left[A^{(k)}\right]^{-1} \tag{E-25}$$

Then

$$\begin{Bmatrix} E^{(1,2)} \\ 1 \\ E_2^{(1,2)} \end{Bmatrix} = \left[A^{(1)}\right]^{-1}\begin{bmatrix} 0 & B_{12}^{(2)} \\ B_{21}^{(2)} & 0 \end{bmatrix}\begin{bmatrix} 0 & \alpha_{22} \\ 0 & 1 \end{bmatrix} =$$

$$\left[I_A^{(1)}\right]\begin{bmatrix} 0 & B_{12}^{(2)} \\ 0 & B_{21}^{(2)}\alpha_{22} \end{bmatrix} = \left[I_A^{(1)}\right]\begin{Bmatrix} B_{12}^{(2)} \\ B_{21}^{(2)}\alpha_{22} \end{Bmatrix} \tag{E-26}$$

Since D_1 is known from Eq (E-3), then the second row of Eq (E-24) can be solved to express D_2 in terms of D_1

$$D_1 = E_2^{(1,2)} D_2 + I^{(1)}_{A_{21}}\left(\delta^{(2)}_{P,1} - \delta^{(1)}_{P,1}\right) + I^{(1)}_{A_{22}}\left(\tilde{p}^{(2)}_{P,1} - \tilde{p}^{(1)}_{P,1}\right)$$

$$+ \Delta E_2^{(1,2)} C_{2,P} \tag{E-27}$$

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Then,

$$D_2 = \frac{1}{E_2^{(1,2)}} (D_1 + \Delta D_{1,P}) \tag{E-28}$$

where $\Delta D_{1,P}$ is defined by

$$\Delta D_{1,P} = I_{A_{21}}^{(1)} (\delta_{P,1}^{(1)} - \delta_{P,1}^{(2)}) + I_{A_{22}}^{(1)} (\tilde{p}_{P,1}^{(1)} - \tilde{p}_{P,1}^{(2)}) + \Delta E_2^{(1,2)} C_{2,P} \tag{E-29}$$

From Eq (E-26)

$$E_1^{(1,2)} = I_{A_{11}}^{(1)} B_{12}^{(2)} + I_{A_{12}}^{(1)} \alpha_{22} B_{21}^{(2)} \tag{E-30a}$$

$$E_2^{(1,2)} = I_{A_{21}}^{(1)} B_{12}^{(2)} + I_{A_{22}}^{(1)} \alpha_{22} B_{21}^{(2)} \tag{E-30b}$$

$$\Delta E_1^{(1,2)} = I_{A_{12}}^{(1)} B_{21}^{(2)} \tag{E-30c}$$

$$\Delta E_2^{(1,2)} = I_{A_{22}}^{(1)} B_{21}^{(2)} \tag{E-30d}$$

Next, solve the first row of Eq (E-24) for C_1, since D_1 is already known from Eq (E-3)

$$C_1 = E_1^{(1,2)} D_2 + \Delta E_1^{(1,2)} C_{2,P} + I_{A_{11}}^{(1)} (\delta_{P,1}^{(2)} - \delta_{P,1}^{(1)}) + I_{A_{12}}^{(1)} (\tilde{p}_{P,1}^{(2)} - \tilde{p}_{P,1}^{(1)}) \tag{E-31}$$

$$= \frac{E_1^{(1,2)}}{E_2^{(1,2)}} (D_1 + \Delta D_{1,P}) + \Delta E_1^{(1,2)} C_{2,P} + I_{A_{11}}^{(1)} (\delta_{P,1}^{(2)} - \delta_{P,1}^{(1)} + I_{A_{12}}^{(1)} (p_{P,1}^{(2)} - p_{P,1}^{(1)})) \tag{E-32}$$

Some general simplifications occur as a result of results in Appendix C concerning the particular integrals $\tilde{\phi}_P^{(k)}(s,z)$. These are expressed as

$$\tilde{\delta}_{P,1}^{(2)}(s,Z_1) = \delta_{P,1}^{(2)}(s,0) = \frac{1}{U_2} \tilde{\phi}_{Pz}^{(2)}(s,0) = 0 \tag{E-33}$$

$$\tilde{\phi}_P^{(2)}(s,0) = \tilde{\phi}_P^{(2)}(s,Z_1) = 0 \tag{E-34}$$

$$\tilde{p}_P^{(2)}(s,Z_1) = \tilde{p}_{P,1}^{(2)}(s,0) = -\rho_2 (U_2 s + i\omega) \tilde{\phi}_{P,1}^{(2)}(s,0) = 0 \tag{E-35}$$

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Likewise, Eq(E-3) for D_1 simplifies

$$\delta_p^{(1)}(s,Z = z = 0) = \frac{1}{U_1}\tilde{\phi}_{Pz}^{(1)}(s,0) = 0 \qquad \text{(E-36)}$$

and

$$\tilde{p}_P^{(1)}(s,0) = \tilde{p}_{P,1}^{(1)}(s,0) = -\rho_1(U_1 s + i\omega)\tilde{\phi}_P^{(1)}(s,0) = 0 \qquad \text{(E-37)}$$

Then from Eqs (E-3) and (E-36)

$$D_1 = \tilde{W}_a(s)/K_1 = D_1^{GR} \qquad \text{(E-38)}$$

which agrees with Reference 2, p. 366, for the Laplace transform solution to the Garrick and Rubinow case of an unbounded uniform supersonic stream.

Then using Eqs (E-34) to (E-38) one obtains from (E-29) that $\Delta D_{1,P}$ simplifies to

$$\Delta D_{1,P} = I_{A_{21}}^{(1)}\delta_{P,1}^{(1)} + I_{A_{22}}^{(1)}\tilde{p}_{P,1}^{(1)} + \Delta E_2^{(1,2)}C_{2,P} \qquad \text{(E-39)}$$

$$C_1 = \frac{E_1^{(1,2)}}{E_2^{(1,2)}}(D_1 + \Delta D_{1,P}) + \Delta E_1^{(1,2)}C_{2,P} - (I_{A_{11}}^{(1)}\delta_{P,1}^{(1)} + I_{A_{12}}^{(1)}\tilde{p}_{P,1}^{(1)}) \qquad \text{(E-40)}$$

In the 2 interior layer case, N = 2 and Eq (E-21) yields

$$C_{2P} = g_2^{(\infty)}/\gamma_{11}^{(2)} \qquad \text{(E-41)}$$

where from (E-18)

$$g_2^{(\infty)} = (\tilde{p}_{P,2}^{(\infty)}(s,0) - \tilde{p}_{P,2}^{(2)}(s,z_2)) - \rho_\infty\frac{(U_\infty s + i\omega)}{K_\infty}U_\infty(\delta_{P,2}^{(\infty)} - \delta_{P,2}^{(2)}) \qquad \text{(E-42)}$$

$$\delta_{P,2}^{(2)} = (1/U_2)\tilde{\phi}_{Pz}^{(2)}(s,z_2) \qquad \text{(E-43)}$$

$$\delta_{P,2}^{(\infty)} = (1/U_\infty)\tilde{\phi}_{Pz}^{(\infty)}(s,z = 0) \qquad \text{(E-44)}$$

From Eq (E-17a)

$$\gamma_{11}^{(2)} = A_{21}^{(2)} - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty A_{11}^{(2)}}{K_\infty} \tag{E-45}$$

From (E-17b) and E-20)

$$\gamma_{12}^{(2)} = A_{22}^{(2)} - \rho_\infty (U_\infty s + i\omega) \frac{U_\infty}{K_\infty} A_{12}^{(2)} \tag{E-46}$$

$$\alpha_{22}^{(2)} = -\gamma_{12}^{(2)}/\gamma_{11}^{(2)} \tag{E-47}$$

The matrix $\left[I_A^{(k)}\right]$ defined by Eq (E25) is

$$\left[I_A^{(k)}\right] = \left[\left[A^{(k)}\right]^{-1}\right] = \left[\begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}^{-1}\right]$$

$$= \frac{1}{\Delta^{(k)}} \left[\begin{bmatrix} A_{22}^{(k)} & -A_{12}^{(k)} \\ -A_{21}^{(k)} & A_{11}^{(k)} \end{bmatrix}\right] \tag{E-48}$$

where

$$\Delta^{(k)} = A_{11}^{(k)} A_{22}^{(k)} - A_{12}^{(k)} A_{21}^{(k)} \tag{E-49}$$

For the case k = 1, using Eqs (E-8) it is found that

$$\Delta^{(1)} = K_1 \rho_1 (U_1 s + i\omega)/U_1 \tag{E-50}$$

$$I_{A_{11}}^{(1)} = -(U_1/K_1)\sinh(K_1 z_1) \tag{E-51a}$$

$$I_{A_{12}}^{(1)} = \frac{-\cosh(K_1 z_1)}{\rho_1 (U_1 s + i\omega)} \tag{E-51b}$$

$$I_{A_{21}}^{(1)} = (U_1/K_1)\cosh(K_1 z_1) \tag{E-51c}$$

$$I_{A_{22}}^{(1)} = \frac{\sinh(K_1 z_1)}{\rho_1 (U_1 s + i\omega)} \tag{E-51d}$$

Forming the matrix one finds

$$\left[I_A^{(1)} \right] = \begin{bmatrix} \dfrac{-U_1 \sinh(K_1 z_1)}{K_1} & \dfrac{-\cosh(K_1 z_1)}{\rho_1 (U_1 s + i\omega)} \\ \left(\dfrac{U_1}{K_1}\right) \cosh(K_1 z_1) & \dfrac{\sinh(K_1 z_1)}{\rho_1 (U_1 s + i\omega)} \end{bmatrix} \tag{E-52a}$$

$$= \frac{1}{\rho_1 (U_1 s + i\omega)} \begin{bmatrix} \tilde{Z}_1(s) \sinh(K_1 z_1) & -\cosh(K_1 z_1) \\ -\tilde{Z}_1(s) \cosh(K_1 z_1) & \sinh(K_1 z_1) \end{bmatrix} \tag{E-52b}$$

where we define the Laplace transformed interface impedance in layer 1 as

$$\tilde{Z}_1(s) = -\rho_1 (U_1 s + i\omega) U_1 / K_1 \tag{E-53}$$

Airfoil Surface Pressure in the Supercritical Region A_U

As defined in Appendix B the local surface pressure in A_U (where $M_{L_{A_U}} > 1$) is

$$\tilde{p}_s^{A_U} = \tilde{p}^{A_U}(z = 0)$$

$$= -\rho_1 (U_1 s + i\omega) \left(C_1 \cosh(K_1 z) + D_1 \sinh(K_1 z) + \tilde{\phi}_P^{(1)}(s,z) \right)_{z = 0}$$

$$= -\rho_1 (U_1 s + i\omega) \left(C_1 + \tilde{\phi}_P^{(1)}(s,0) \right) \tag{E-54}$$

But from Eq (E-37) $\tilde{\phi}_P^{(1)}(s,0) = 0$; therefore,

$$\tilde{p}_s^{A_U} = \tilde{p}^{A_U}(z = 0) = -\rho_1 (U_1 s + i\omega)\, C_1 \tag{E-55}$$

Using Eqs (E40 - E53) for C_1 yields

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$$C_1 = \frac{E_1^{(1,2)}}{E_2^{(1,2)}} \left(\frac{\tilde{W}_a(s)}{K_1}\right) + \frac{E_1^{(1,2)}}{E_2^{(1,2)}} \Delta D_{1,P} + \Delta E_1^{(1,2)} C_{2,P}$$

$$+ \frac{1}{\rho_1(U_1 s + i\omega)} \left(-\tilde{Z}_1(s)\sinh(K_1 z_1)\delta_{P,1}^{(1)} + \cosh(K_1 z_1)\tilde{p}_{P,1}^{(1)}\right) \quad \text{(E56)}$$

Now the ratio $E_1^{(1,2)}/E_2^{(1,2)}$ is the quantity called $\tilde{E}_{21}(s)$ in Appendix A. From Eqs (E-30a, E-30b, E-52a & E-52b we first define $E_1^{(1,2)}$ and $E_2^{(1,2)}$.

$$E_1^{(1,2)} = \frac{1}{\rho_1(U_1 s + i\omega)} \left(\tilde{Z}_1(s) B_{12}^{(2)} \sinh(K_1 z) - \alpha_{22} B_{21}^{(2)} \cosh(K_1 z_1)\right) \quad \text{(E-57)}$$

$$E_2^{(1,2)} = \frac{1}{\rho_1(U_1 s + i\omega)} \left(-\tilde{Z}_1(s) B_{12}^{(2)} \cosh(K_1 z_1) + \alpha_{22} B_{21}^{(2)} \sinh(K_1 z_1)\right) \quad \text{(E-58)}$$

From Eqs (E-8, E-9, E-45-E-51)

$$\gamma_{11}^{(2)} = -\rho_2(U_2 s + i\omega)\cosh(K_2 z_2) - \rho_\infty(U_\infty s + i\omega)\frac{U_\infty K_2}{K_\infty U_2}\sinh(K_2 z_2) \quad \text{(E-59)}$$

$$\gamma_{12}^{(2)} = -\rho_2(U_2 s + i\omega)\sinh(K_2 z_2) - \rho_\infty(U_\infty s + i\omega)\frac{U_\infty K_2}{K_\infty U_2}\cosh(K_2 z_2) \quad \text{(E-60)}$$

$$\alpha_{22} = -\gamma_{12}^{(2)}/\gamma_{11}^{(2)} \quad \text{(E-61)}$$

$$= \frac{-\left(\sinh(K_2 z_2) + \frac{\rho_\infty(U_\infty s + i\omega)U_\infty K_2}{\rho_2(U_2 s + i\omega)K_\infty U_2}\cosh(K_2 z_2)\right)}{\left(\cosh(K_2 z_2) + \frac{\rho_\infty(U_\infty s + i\omega)U_\infty K_2}{\rho_2(U_2 s + i\omega)U_2 K_\infty}\sinh(K_2 z_2)\right)}$$

$$\alpha_{22} = \frac{-\left(\sinh(K_2 z_2) + \frac{\tilde{Z}_\infty(s)}{\tilde{Z}_2(s)}\cosh(K_2 z_2)\right)}{\left(\cosh(K_2 z_2) + \frac{\tilde{Z}_\infty(s)}{\tilde{Z}_2(s)}\sinh(K_2 z_2)\right)} \quad \text{(E-62)}$$

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$$\alpha_{22} = -\frac{(\sinh(K_2 z_2) + \tilde{Z}_{\infty,2}(s)\cosh(K_2 z_2))}{(\cosh(K_2 z_2) + \tilde{Z}_{\infty,2}(s)\sinh(K_2 z_2))} \tag{E-63}$$

where the impedances, $\tilde{Z}_2(s)$ and $\tilde{Z}_\infty(s)$ are defined by analogy to Eq (E-57)

$$\tilde{Z}_2^{(s)} = -\rho_2(U_2 s + i\omega)U_2/K_2 \tag{E-64}$$

$$\tilde{Z}_\infty^{(s)} = -\rho_\infty(U_\infty s + i\omega)U_\infty/K_\infty \tag{E-65}$$

and the ratios

$$\tilde{Z}_{\infty,2}(s) = \tilde{Z}_\infty(s)/\tilde{Z}_2(s)$$

$$\tilde{Z}_{2,1}(2) = \tilde{Z}_2(s)/\tilde{Z}_1(s) \tag{E-66}$$

From Eqs (E-57 - E-66) and Eq (E-9)

$$E_1^{(1,2)} = \frac{1}{\rho_1(U_1 s + i\omega)}\left(\tilde{Z}_1(s)\frac{K_2}{U_2}\sinh(K_1 z_1) + \alpha_{22}\rho_2(U_2 s + i\omega)\cosh(K_1 z_1)\right) \tag{E-67}$$

$$E_2^{(1,2)} = \frac{1}{\rho_1(U_1 s + i\omega)}\left(-\tilde{Z}_1(s)\frac{K_2}{U_2}\cosh(K_1 z_1) - \alpha_{22}\rho_2(U_2 s + i\omega)\sinh(K_1 z_1)\right) \tag{E-68}$$

We now form the ratio

$$\tilde{E}_{21}^{(s)} = E_1^{(1,2)}(s)/E_2^{(1,2)}(s) \tag{E-69}$$

$$\tilde{E}_{21}(s) = \frac{\left(\frac{\tilde{Z}_1}{\tilde{Z}_2}\sinh(K_1 z_1) - \alpha_{22}\cosh(K_1 z_1)\right)}{\left(-\tilde{Z}_1/\tilde{Z}_2)\cosh(K_1 z_1) + \alpha_{22}\sinh(K_1 z_1)\right)}$$

$$= \frac{-(\tanh(K_1 z_1) - \tilde{Z}_{2,1}(s)\alpha_{22})}{(1 - \alpha_{22}\tilde{Z}_{2,1}(s)\tanh(K_1 z_1))} \tag{E-70}$$

Now, from Eq (E-63), α_{22} can be written in the form

$$\alpha_{22} = \frac{-(Z_{\infty,2}(s) + \tanh(K_2 z_2))}{(1 + Z_{\infty,2}\tanh(K_2 z_2))} = \tilde{E}_2(s) \qquad \text{(E-71)}$$

where $\tilde{E}_2(s)$ is the quantity defined in Eq(A-32)Appendix A.

If one defines as in Eq (A-29) of Appendix A

$$\tilde{H}_{21}(s) = \tilde{Z}_{2,1}(s)\tilde{E}_2(s) \qquad \text{(E-72)}$$

Then Eqs (E-70 - E-72) yield

$$\tilde{E}_{21}(s) = \frac{-(\tanh(K_1 z_1) - \tilde{H}_{21}(s))}{(1 - \tilde{H}_{21}(s)\tanh(K_1 z_1))} \qquad \text{(E-73)}$$

This is precisely the same as Eqs (A-27) through(A-33) of Appendix A. To further establish the correspondence with Appendix A we note that Eq (A-33) of Appendix A equals Eq (A-71) and from Eqs (A-27) and (A-30) of Appendix A and Eq (A-78) it is seen that

$$\tilde{N}_{21}(s) = (\tilde{H}_{21}(s) - \tanh(K_1 z_1)) \qquad \text{(E-74)}$$

$$\tilde{D}_{21}(s) = (1 - \tilde{H}_{21}(s)\tanh(K_1 z_1)) \qquad \text{(E-75)}$$

where from Eqs (E-66 - E-71, E-72 and E-73)

$$\tilde{H}_{21}(s) = -\frac{\rho_2(U_2 s + i\omega)\, U_2 K_1 (\tilde{Z}_{\infty,2}(s) + \tanh(K_2 z_2))}{\rho_1(U_1 s + i\omega)\, U_1 K_2 (1 + Z_{\infty,2}(s)\tanh(K_2 z_2))} \qquad \text{(E-76)}$$

Particular Integral Contribution to
Local Supersonic Surface Pressure

It can be seen from Eqs (E-54 - E-56 E-69 - E-76) that the term $\tilde{E}_{21}(s)\,\tilde{W}_a(s)/K_1$ represents the direct contribution of the local surface upwash, $\tilde{W}_a(s)$ in the local supersonic region A_U. The remaining terms in Eq (E-56) represent surface pressure effects due to sonic line disturbances. Following Eq (B-42) of Appendix B.

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$$\left\{ p^{A_U}(z = 0) \right\} = \left\{ p_s^{A_U} \right\} = \left[\begin{matrix} p_s^{A_U} \\ G_{WA} \end{matrix} \right] \left\{ W_A + \Delta W_A^{P} \right\} + \left\{ p_{P,s}^{A_U} \right\} \tag{E-77}$$

Eq (E-77) represents, in matrix form the numerical evaluation of the convolution integral, which is the required inverse Laplace transform of Eq (E-55) with Eq (E-56) substituted for C_1.

Eq (E-56) can be split into two terms

$$C_1 = C_{1,H} + C_{1,P} \tag{E-78}$$

where the first term is the homogeneous solution for the multilayer supersonic region.

$$C_{1,H} = \tilde{E}_{21}(s)\, \tilde{W}_a(s)/K_1 \tag{E-79}$$

and the second term of (E-78) is the particular integral contribution.

$$C_{1,P} = E_{21}(s)\, \Delta D_{1,P} + \Delta E_1^{(1,2)} C_{2,P}$$

$$+ \frac{1}{\rho_1(U_1 s + i\omega)} \left(-\tilde{Z}_1(s) \sinh(K_1 z_1)\, \delta_{P,1}^{(1)} + \tilde{p}_{P,1}^{(1)} \cosh(K_1 z_1) \right) \tag{E-80}$$

By comparing Eq (A-54) with Eq (A-22) of Appendix A and Eq (E-77) above, it can be seen that the Laplace transform of the particular integral contribution to the surface pressure in the local supersonic region is

$$\Delta\tilde{p}_{P,1}(s,0) = L\left(p_{P,s}^{A_U}\right) \tag{E-81}$$

$$= -\rho_1(U_1 s + i\omega) C_{1,P}$$

Since, by Appendix C, there is no direct contribution to the airfoil surface downwash from the particular integral in the first layer, we have

$$\tilde{\phi}_{Pz}^{(1)}(s,0) = \Delta\tilde{W}_{AP}(s) = 0 \tag{E-82}$$

Then from Eqs (E-55, E-78 and E-79)

$$\tilde{p}^{A}U(s,z=0) = -\rho_1(U_1 s + i\omega)(\tilde{E}_{21}(s)\frac{\tilde{W}_a(s)}{K_1} + C_{1,P}) \qquad \text{(E-83)}$$

Therefore, by the convolution principle

$$\bar{p}^{A}U(x,z=0) = \int_0^x E_{21}(x-\xi)\Delta p^{GR}(\xi)d\xi + L^{-1}(-\rho_1(U_1 s + i\omega)C_{1,P}) \qquad \text{(E-84)}$$

$$\Delta\bar{p}^{GR}(\xi) = L^{-1}(-\rho_1(U_1 s + i\omega)\tilde{W}_a(s)/K_1) \qquad \text{(E-85)}$$

An alternative form of Eq (E-84) shows the direct comparison with Eq (E-83)

$$p^{A}U(x,z=0) = \int_0^x G_{WA}^{p_s A}U(x-\xi)\tilde{W}_a(\xi)d\xi + L^{-1}(-\rho_1(U_1 s + i\omega)C_{1,P}) \qquad \text{(E-87)}$$

where by the Laplace transform inversion theorem

$$G_{WA}^{p_s A}U(x-\xi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(-\rho_1(U_1 s + i\omega)\frac{E_{21}(s)}{K_1(s)})e^{s(x-\xi)}ds \qquad \text{(E-88)}$$

Thus, it is seen that Eq (E-77) is a matrix statement of a numerical integration scheme which is equivalent to the analytical solution expressed by Eq (E-87). Therefore, Eq (E-87) represents the basis for calculating the matrix elements required by Eq (B-41) of appendix B.

We next consider the inversion of $C_{1,P}$. First, Eq (E-80) must be simplified. Using Eqs (E-52a,E-52b,E-51 and E-29) one finds

$$\Delta D_{1,P} = I_{A21}^{(1)}\delta_{P,1}^{(1)} + I_{A22}^{(1)}\tilde{p}_{P,1}^{(1)} + \Delta E_2^{(1,2)}C_{2,P}$$

$$= \frac{1}{\rho_1(U_1 s + i\omega)}(-\tilde{Z}_1(s)\cosh(K_1 z_1)\,\delta_{P,1}^{(1)} + \tilde{p}_{P,1}\sinh(K_1 z_1))$$

$$+ \Delta E_2^{(1,2)}C_{2,P} \qquad \text{(E-89)}$$

From Eqs (E-41 - E-59)

$$C_{2,P} = g_2^{(\infty)}/\gamma_{11}^{(2)} \tag{E-90}$$

$$= \frac{(\tilde{p}_{P,2}^{(\infty)}(s,0) - \tilde{p}_{P,2}^{(2)}(s,z_2)) - \tilde{Z}_\infty(s)(\delta_{P,2}^{(\infty)} - \delta_{P,2}^{(2)})}{-\rho_2(U_2 s + i\omega)(\cosh(K_2 z_2) + Z_{\infty,2}^{(s)}\sinh(K_2 z_2))}$$

From Eqs (E-34b, E-56b and E-9)

$$\Delta E_1^{(1,2)} = I_{A12}^{(1)} B_{21}^{(2)}$$

$$= \frac{-\rho_2(U_2 s + i\omega)}{\rho_1(U_1 s + i\omega)}(-\cosh(K_1 z_1)) \tag{E-91}$$

Substituting Eqs (E-89, E-90, and E-91 into E-80) provides the particular integral contribution to the local surface pressure in the supersonic region which is represented by Eq (A-20) of Appendix A

$$\Delta\tilde{p}_{P,1}(s,o) = -\rho_1(U_1 s + i\omega)C_{1,P} \tag{E-92}$$

$$= -\tilde{E}_{21}(s)\,(-\tilde{Z}_1\cosh(K_1 z_1)\delta_{P,1}^{(1)} + \sinh(K_1 z_1)\tilde{p}_{P,1}^{(1)}(s,z_1))$$

$$- (-\tilde{Z}_1\cosh(K_1 z_1)\delta_{P,1}^{(1)} + \tilde{p}_{P,1}^{(1)}(s,z_1)\sinh(K_1 z_1)) + (-\tilde{E}_{21}(s)\sinh(K_1 z_1) +$$

$$+ \cosh(K_1 z_1))\frac{((\tilde{p}_{P,2}^{(\infty)} - \tilde{p}_{P,2}^{(2)}) - \tilde{Z}_\infty(\delta_{P,2}^{(\infty)} - \delta_{P,2}^{(2)}))}{(\cosh(K_2 z_2) + \tilde{Z}_{\infty,2}\sinh(K_2 z_2))} \tag{E-93}$$

We next simplify Eq (E-93). It is convenient to represent Eq (E-93) as follows:

$$\Delta\tilde{p}_{P,1}(s,0) = \tilde{A}_1\delta_{P,1}^{(1)} + \tilde{B}_1 p_{P,1}^{(1)}$$

$$+ \tilde{A}_2\,((\tilde{p}_{P,2}^{(\infty)} - \tilde{p}_{P,2}^{(2)}) - \tilde{Z}_\infty(\delta_{P,2}^{(\infty)} - \delta_{P,2}^{(2)})) \tag{E-94}$$

where by comparison of Eqs (E-93 and E-94)

$$\tilde{A}_1 = \tilde{Z}_1(s)\ (1 + \tilde{E}_{21}(s))\ \cosh\ (K_1 z_1) \tag{E-95a}$$

$$\tilde{B}_1 = -\ \sinh(K_1 z_1)\ (1 + \tilde{E}_{21}(s)) \tag{E-95b}$$

$$\tilde{A}_2 = \frac{(\cosh(K_1 z_1) - \tilde{E}_{21}(s)\sinh(K_1 z_1))}{(\cosh(K_2 z_2) + Z_{\infty,2}\sinh(K_2 z_2))} \tag{E-95c}$$

Eqs (E-92) to (E-95) display the effects of relfections, back to the airfoil surface pressure, of the contributions of the particular integrals in regions (1, 2 and ∞) to the streamline slopes and pressures at the layer interfaces at $z = z_1 = Z_1$ and at $Z = Z_2$ (where $z_2 = Z_2 - Z_1$). The last term in Eq (E-94) displays the impedance mismatch feature encountered in crossing the outer layer adjacent to the freestream flow. The inversion of the Laplace transforms implied by Eqs (E-93) to (E-95) will be discussed in a later study.

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