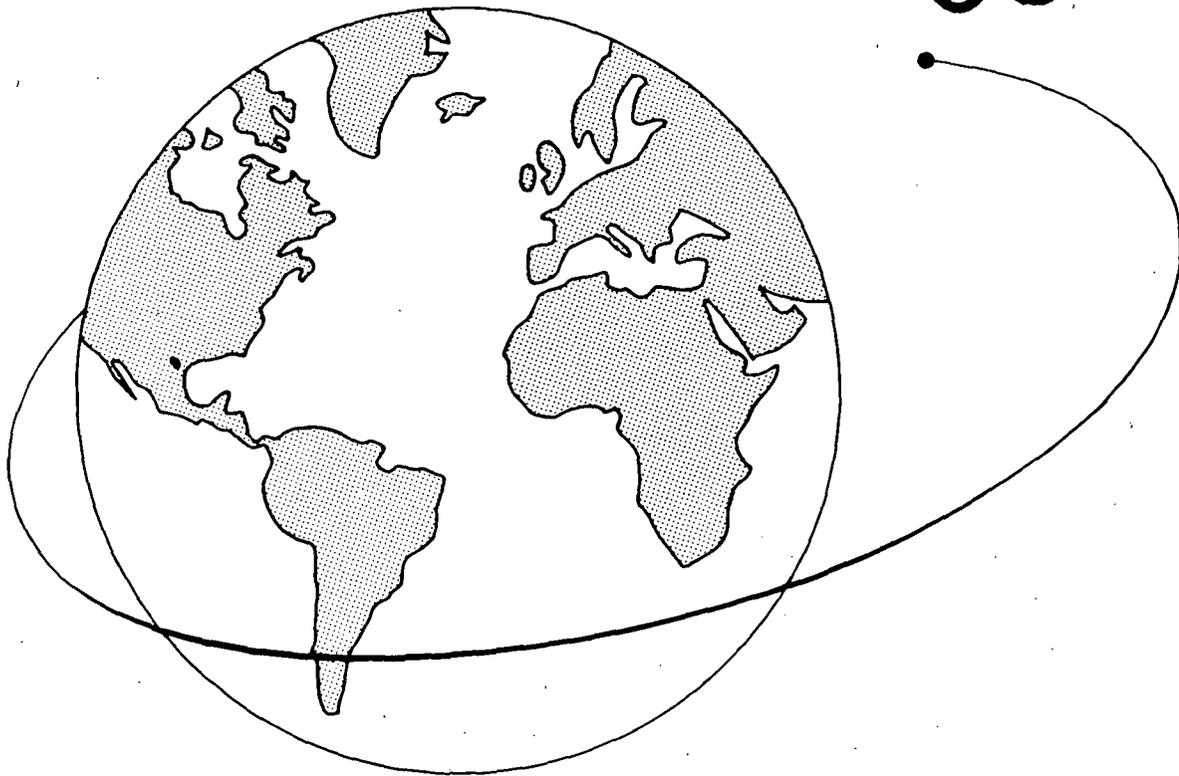


SAMPLING FUNCTIONS FOR GEOPHYSICS

G. E. O. GIACAGLIA and C. A. LUNDQUIST

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ABSTRACT

A set of spherical sampling functions is defined such that they are related to spherical-harmonic functions in the same way that the sampling functions of information theory are related to sine and cosine functions. An orderly distribution of $(N + 1)^2$ sampling points on a sphere is given, for which the $(N + 1)^2$ spherical sampling functions span the same linear manifold as do the spherical-harmonic functions through degree N . In this case, the transformations between the spherical sampling functions and the spherical-harmonic functions are given by recurrence relations. The spherical sampling functions of two arguments are extended to three arguments and to nonspherical reference surfaces. Typical applications of this formalism to geophysical topics are sketched.

RESUME

On définit un ensemble de fonctions sphériques telles qu'elles sont reliées aux fonctions sphériques-harmoniques de la même manière que les fonctions d'échantillonnage de la théorie de l'information le sont aux fonctions sinus et cosinus. On donne une distribution ordonnée de $(N + 1)^2$ points d'échantillonnage sur une sphère, pour lesquels les $(N + 1)^2$ fonctions sphériques d'échantillonnage recouvrent le même espace linéaire que le font les fonctions sphériques-harmoniques jusqu'au degré N . Dans ce cas, les transformations entre les fonctions sphériques d'échantillonnage et les fonctions sphériques-harmoniques sont données par des relations de récurrence. Les fonctions sphériques d'échantillonnage pour deux arguments sont étendues à trois arguments et à des surfaces de référence non sphériques. On ébauche des applications typiques de ce formalisme aux problèmes géophysiques.

КОНСПЕКТ

Определяется система сферических пробных функций таковых что они связаны со сферическими-гармоническими функциями таким же образом как и пробные функции теории информации связаны с функциями синуса и косинуса. Приводится организованное распределение $(N+1)^2$ пробных точек по сфере, для которых $(N+1)^2$ сферические пробные функции охватывают то же самое линейное многообразие как и сферические гармонические функции степени N . В этом случае преобразование между сферическими пробными функциями и сферическими гармоническими функциями приводится с помощью рекуррентных отношений. Сферические пробные функции двух аргументов распространяются для трех аргументов и на несферические контрольные площади. Набрасываются типичные применения этого формализма к геофизическим темам.

SAMPLING FUNCTIONS FOR GEOPHYSICS

G. E. O. Giacaglia and C. A. Lundquist

1. MOTIVATION AND HISTORY

The mathematical formalism in the following sections arose initially as a candidate technique for the analysis of satellite-to-ocean altitudes (Lundquist, Giacaglia, Hebb, and Mair, 1969). Many authors have noted the potential geophysical value of accurate measurements of the height of a spacecraft above the ocean (e.g., Frey, Harrington, and Von Arx, 1965; Godbey, 1965; Moore, 1965; Lundquist, 1967a), and various review panels have long recommended implementation of such measurements in the space program of the United States (e.g., Pierson and panel, 1965; Stewart and panel, 1965; Woollard and panel, 1966; Doyle and panel, 1969; Kaula, 1970). Indeed, the program of the National Aeronautics and Space Administration now does include flights of spacecraft altimeters during the 1970s (Pierson and Mehr, 1972; Stanley, Roy, and Martin, 1972). Because preparations for analysis of the altitude data may require as much lead time as does fabrication of the spacecraft hardware, we were motivated to investigate appropriate analytical tools.

The formalism summarized here evolved over several years. In 1967, we first examined the requirements for the analysis of satellite-to-ocean altitudes (Lundquist, 1967a, b). This led to a judgment that an alternative mathematical representation of the geopotential would facilitate treatment of altitude data (Giacaglia and Lundquist, 1968; Lundquist *et al.*, 1969). The latter reference defined in general terms a set of functions, differing from spherical harmonics, in which the earth's potential can

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be expanded. The most important feature of this formalism is the property that the coefficients in the expansion are exactly a tabulation of the values of the potential at a grid of points on the earth.

In the 1969 documents, several important mathematical topics were not yet treated. One such topic is the realization that the functions defined in those references are a natural extension of the sampling functions of information theory (Lundquist and Giacaglia, 1972a). Another related topic is the utilization of the sampling-function technique for geophysical quantities other than the gravitational potential (Giacaglia and Lundquist, 1971).

The main objective of the present work is a comprehensive mathematical development of the sampling-function formalism for representing geophysical quantities around a spherical or a nearly spherical body such as the earth. The discussion here unifies and largely supersedes the mathematical developments in the earlier references. The final section concerns some geophysical applications of the sampling-function formalism that augment the applications discussed in the 1971 references.

Motivated by the satellite-altimetry expectations or by other circumstances in which a spherical-harmonic representation is awkward, various authors have offered still further alternative representations for the geopotential. These include distributions of point masses (Sjogren, Muller, and Gottlieb, 1971; Balmino, 1972; Kaplan and Kunciw, 1972; Needham, 1970) and surface layers of varying mass density (Vinti, 1971; Croopnick, 1970; Koch and Morrison, 1970; Koch and Witte, 1971; Koch, 1972). For a critical review of alternative representations, see Kaula (1970).

The principal motivation for an expansion in sampling functions instead of spherical harmonics arises when a representation containing much short-wavelength detail is required. If, for example, physical features of 1° extent are to be represented in the geoid, a spherical-harmonic representation through degree and order 180 must be employed. The value of the radius to the geoid is then calculated at any point as the sum of $(180+1)^2 = 32,761$ terms. If features of 0.1° extent are considered, the spherical-harmonic representation would have $(1800+1)^2 = 3,243,601$ terms! With the equivalent sampling-function expansion, only a few terms contribute significantly to the value at any point.

It is a fundamental property of the sampling functions that they exist in one-to-one correspondence with a grid of sampling points on a sphere or similar reference figure. At each of these sampling points, only the single corresponding sampling function has a nonzero value, and this function multiplied by its coefficient makes the dominant contribution to the sum in the neighborhood of the sampling point. Thus, the coefficients of the terms in a sampling-function expansion are a tabulation of the value of the represented quantity at the grid of sampling points.

On the other hand, the spherical-harmonic expansion through some degree and order N and the sampling-function expansion that spans the same linear manifold are completely equivalent. In both cases, there are $(N + 1)^2$ independent functions and terms in the truncated series. The spherical-harmonic functions and the sampling functions are related by a linear transformation in the linear manifold. Further, the transformation and its inverse can both be expressed analytically for any N . Hence, it is simple to transform to a spherical-harmonic representation when that is most convenient, or to a sampling-function representation when the tabular characteristic is useful.

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2. GENERAL PROPERTIES OF SAMPLING FUNCTIONS

2.1 Introduction

In this section, we address the concept of sampling functions appropriate to the representation of periodic functions over a well-defined domain. The main interest is in linear, spherical, and nearly spherical domains.

To achieve a reasonably self-contained discussion and to establish notation, it is convenient to restate the definitions of well-known quantities and operations. (The reader is referred to the many standard references – for example, Sansone, 1959.)

A function $f(x)$ that is square integrable (in the Lebesgue sense) over a certain domain $D\{x\}$ will be indicated by

$$f(x) \in L_2(D) .$$

All the L_2 functions in D constitute a Hilbert space $H(D)$. As usual, in this space we denote the scalar (inner) product of $f, g \in L_2(D)$ by

$$(f, g) = \int_D f(x) g(x) dx , \quad (2.1)$$

which is a real number since we shall consider only real valued functions. Also, the positive number

$$\|f\| = (f, f)^{1/2} \quad (2.2)$$

is the norm of f . The distance between two functions $f, g \in L_2(D)$ is the norm of $f - g$. In general, if the distance is small, the difference $f - g$ need not be small everywhere in D .

A sequence of functions $\{f_n\} \in L_2(D)$, finite or countably infinite in number, is orthonormal in D if

$$(f_i, f_k) = \delta_{ik} \quad (2.3)$$

for all i, k . The series

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) + \dots, \quad (2.4)$$

where

$$c_k = (f, f_k),$$

is the generalized Fourier series of $f(x)$. The c_k are the generalized Fourier coefficients of $f(x)$ with respect to the set f_1, f_2, \dots . In general, nothing can be said about pointwise convergence (with respect to the defined norm) of series (2.4) to the function $f(x)$, but it can be shown that (2.4) has the least-squares deviation (distance) from $f(x)$ and that the Bessel's inequality

$$\sum_n c_n^2 \leq \|f\|^2 \quad (2.5)$$

holds.

The sequence $\{f_n\}$ of orthonormal functions is complete in $L_2(D)$ if, for every $f(x) \in L_2(D)$, the corresponding generalized Fourier series converges to f in the mean, i. e.,

$$\lim_{n \rightarrow \infty} \left\| f - \sum_{k=1}^n c_k f_k \right\| = 0. \quad (2.6)$$

The Parseval relation

$$\sum_{k=1}^{\infty} c_k^2 = \|f\|^2 \quad (2.7)$$

is a necessary and sufficient condition for completeness. Other conditions are also available, but this will suffice for our needs. It should be noted that if $f, g \in L_2(D)$ possess the same Fourier series with respect to a complete system, then they are equal (almost everywhere) in D .

Uniform convergence with respect to an orthonormal set implies uniform convergence (almost everywhere) to a function f .

It will be necessary to consider the concept of completeness for nonorthogonal systems. If $\{f_n\} \in L_2(D)$ is a sequence of functions, linearly independent in D (i. e., every finite number of terms of the sequence is a linearly independent system), the sequence is complete in D if any $f \in L_2(D)$ can be approximated in the mean by linear combinations of f_n to any accuracy.

Another concept of importance is the following: A subset Ω of a space S is called a linear manifold in S if it has the property that when the elements $f_1, f_2, \dots, f_n \in \Omega$, any linear combination $c_1 f_1 + c_2 f_2 + \dots + c_n f_n \in \Omega$. If Ω is closed in S , Ω is called a subspace of S . If Ω is a subspace of $L_2(D)$, then each element can be uniquely represented in the form

$$f = g + h \quad ,$$

where $g \in \Omega$ and where h is orthogonal to all elements of Ω . The element g is called the projection of f into Ω .

If there is an m such that $\{f_1, f_2, \dots, f_m\}$ is the maximum number of orthogonal functions, linearly independent in Ω , then Ω is a finite dimensional linear manifold, or the span of $\{f_1, f_2, \dots, f_m\}$.

With respect to the problem of fitting a function with a series or a polynomial (which is actually a projection operation), it is important to remember the fundamental theorem: Let $f_1, f_2, \dots, f_n, \dots \in L_2(D)$ be a given orthonormal sequence and let $f \in L_2(D)$. The generalized Fourier series of $f(x)$ is (2.4), where

$$c_k = (f, f_k) = \int_D f f_k \, dx . \quad (2.8)$$

Let a positive integer N and an orthonormal system

$$f_1(x), f_2(x), \dots, f_N(x)$$

be given. Then, from all functions of the form

$$k_1 f_1 + k_2 f_2 + \dots + k_N f_N ,$$

the one that has the least distance from f is series (2.4), with the c_k 's given by equation (2.8). It is of paramount importance to note that this theorem holds, in general, only for orthonormal systems.

For a finite dimensional linear space, the concept of completeness loses its meaning, in the sense that any maximal set of linearly independent functions is a basis of the linear space. In this case, the necessary and sufficient condition for the functions $\{f_1, f_2, \dots, f_n\}$ to be linearly independent in D is the nonvanishing of the Gram's determinant

$$G = |(f_i, f_j)| , \quad (2.9)$$

but, of course, this condition also applies to linear independence of any subset of an infinite sequence $\{f_n\}$.

Finally, it is important to observe that, although the inner product has been defined without any weighting function, introducing a weighting function does not affect any of the previous statements. Weighting functions should nevertheless $\in L_2(D)$ and be nonnegative in D .

2.2 Sampling-Function Concepts

Before we deal with the concept of sampling functions, it is helpful to describe their origin, which can largely be identified with the theory of interpolation and Lagrange polynomials. From this point of view, the literature is quite abundant on one-dimensional or n-dimensional spaces R^n , but very sparse on other types of spaces.

Let $S(D)$ be an N-dimensional subspace of a Hilbert space $H(D)$, and f_1, f_2, \dots, f_N be an orthonormal basis in S . Any function $f \in S$ is uniquely written as a linear combination of f_1, f_2, \dots, f_N . On the other hand, any $g \in H$ is best fit in S by the projection of g on $S(D)$ over the orthonormal basis $\{f_1, f_2, \dots, f_N\}$, so that for any set of real numbers k_1, k_2, \dots, k_N ,

$$\left\| g - \sum_{j=1}^N (g, f_j) f_j \right\| \leq \left\| g - \sum_{j=1}^N k_j f_j \right\| \quad (2.10)$$

on D (e.g., Sansone, 1959; Rektorys, 1969).

This is the classical problem of interpolation. Generalizations to a best fitting of f and its derivatives through a certain order are widely discussed in the literature, including some early papers by Hermite (1878).

Now, consider another set of functions $\varphi_1, \varphi_2, \dots, \varphi_N \in S$, such that

$$\varphi_j(x) = \sum_{i=1}^N C_{ji} f_i(x) \quad , \quad j = 1, 2, \dots, N \quad , \quad (2.11)$$

and let $C = \{C_{ij}\}$ be a nonsingular matrix. By definition,

$$C_{ij} = (\varphi_i, f_j)$$

and

$$(\varphi_i, \varphi_j) = \sum_{\ell=1}^N C_{i\ell} (f_\ell, \varphi_j) = \sum_{\ell=1}^N C_{i\ell} C_{\ell j} ,$$

so that the matrix

$$\{(\varphi_i, \varphi_j)\} = C^T C \quad (2.12)$$

is nonsingular. It follows that $\varphi_1, \varphi_2, \dots, \varphi_N$ is a basis for S .

Now let $Q(x_1, x_2, \dots, x_m)$ be a point of D , the domain of definition of all $g \in H(D)$. If the functions $\varphi_k(Q)$, $k = 1, 2, \dots, N$, can be defined in such a way that for a set of N points $Q_j \in D$ we can verify

$$\varphi_k(Q_j) = \delta_{kj} , \quad (2.13)$$

then such functions are classically called collocation functions and are particular cases of sampling functions for the subspace S . In the case $S \equiv H$, we shall have a denumerable infinite number of sampling functions. The completeness of these with respect to all $g \in H(D)$ cannot be assumed a priori.

The functions $f_k \in S$ are uniquely represented everywhere in D by

$$f_k = \sum_{j=1}^N a_{kj} \varphi_j , \quad k = 1, 2, \dots, N ,$$

where $\{a_{kj}\} = \{C_{nm}\}^{-1}$. Since the a_{kj} are real constants,

$$f_k(Q_\ell) = \sum_{j=1}^N a_{kj} \varphi_j(Q_\ell) = a_{k\ell} ,$$

so that

$$f_k(Q) = \sum_{j=1}^N f_k(Q_j) \varphi_j(Q)$$

uniquely. Consider any other function $f \in S$. It can be written uniquely as

$$\begin{aligned} f(Q) &= \sum_{k=1}^N b_k f_k(Q) = \sum_{k=1}^N b_k \sum_{j=1}^N f_k(Q_j) \varphi_j(Q) \\ &= \sum_{j=1}^N \sum_{k=1}^N \{b_k f_k(Q_j)\} \varphi_j(Q) = \sum_{j=1}^N f(Q_j) \varphi_j(Q) , \end{aligned}$$

so that, for all $f \in S$, $Q \in D$,

$$f(Q) = \sum_{j=1}^N f(Q_j) \varphi_j(Q) , \quad (2.14)$$

which is the basic justification for the name of sampling functions given to the $\varphi_j(Q)$. Analogously, any $g \in H(D)$ has an expansion in S given by

$$g_S(Q) = \sum_{j=1}^N g(Q_j) \varphi_j(Q) , \quad Q \in D ,$$

but since the set of φ_j is not, in general, orthonormal in the integral sense, such an expansion cannot be assumed to be the best fitting of f in the sense already defined. It is obvious that, in an N -dimensional subspace, the N conditions (eq. (2.13)) completely define φ_k . Therefore, orthogonality of these, in the sense

$$(\varphi_k, \varphi_j) = \int_D \varphi_k(x) \varphi_j(x) \omega(x) dx = 0 , \quad (2.15)$$

for $k \neq j$, can be achieved, if possible at all, only by a particular choice of the sampling points Q_j ($j = 1, 2, \dots, N$) $\in D$ or a proper choice of a weighting function $\omega(x) \geq 0$ in D . In some instances, orthogonality can be obtained or will follow automatically as a basic property of the subspace.

In general, sampling functions satisfy the following properties:

A. Orthogonality over the set of sampling points,

$$[\varphi_i, \varphi_j] = \sum_{k=1}^N \varphi_i(Q_k) \varphi_j(Q_k) = \delta_{ij} . \quad (2.16)$$

B. Normality over the set of sampling points,

$$\sum_{k=1}^N \varphi_k(Q_j) = \sum_{j=1}^N \varphi_k(Q_j) = 1 . \quad (2.17)$$

C. Normality everywhere in D , i. e.,

$$\sum_{k=1}^N \varphi_k(Q) = 1 , \quad Q \in D . \quad (2.18)$$

This last property follows immediately from the fundamental property (2.14). In fact, suppose

$$\sum_{k=1}^N \varphi_k(Q) = f(Q) \in S(D) ;$$

then

$$f(Q) = \sum_{j=1}^N f(Q_j) \varphi_j(Q) ,$$

and, therefore, $f(Q_j) = 1$ for $j = 1, 2, \dots, N$, so that, necessarily, $f(Q) = \text{constant} = 1$.

2.3 Sampling Functions and Fourier Analysis

It is not necessary to discuss in this report the nature and properties of sampling functions and how they are defined in their most general form. However, in view of the problem we originally proposed to solve, it is instructive to mention the role of these functions in information and communication theory. Historically, sampling functions are related to a generalization of the process of interpolation introduced by E. T. Whittaker (1915) under the name of cardinal functions. The basic statement, sometimes referred to as the Shannon principle but actually proved by J. M. Whittaker and several others, is the following: If a function $f(t)$ contains no frequencies higher than ω radians per second, $f(t)$ can be recovered from periodic samples $f(t_n)$ taken $2\pi/2\omega$ seconds apart. More general results are available, such as the recovery of $f(t)$ from nonperiodic samples, deletion of a finite set of samples, conditions for obtaining $f(t)$ when only the past is sampled, and a minimum mean-square error estimate of $f(t)$ based on any (eventually nonperiodic) set of samples (Beutler, 1961). The function $f(t)$ can be a random process, a periodic process affected by noise, or a function generally supposed to be in a Hilbert space. Functions containing frequencies not higher than a certain value are called band-limited functions.

The nature of our original motivation is contained in the above description — namely, the recovery of the mean sea level from samples corresponding to satellite-borne altimeter measurements. Nevertheless, we are facing a much more general problem, for no sampling functions, in the above sense, have been introduced for functions defined on a sphere, spheroid, or ellipsoid. Take, for example, the mean sea level; it can be represented as a series of spherical harmonics:

$$r(\zeta, \lambda) = R_0 \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\bar{a}_{nm} \bar{X}_{nm}(\zeta, \lambda) + \bar{b}_{nm} \bar{Y}_{nm}(\zeta, \lambda) \right], \quad (2.19)$$

where the \bar{X}_{nm} and \bar{Y}_{nm} are defined in equation (2.52). If $r(\zeta, \lambda) \in L_2$ (sphere), the series (2.19) will converge in the mean. Moreover, in truncating the series to some degree $n = N$, i. e., writing

$$r_N(\zeta, \lambda) = R_0 \sum_{n=0}^N \sum_{m=0}^n \left(\bar{a}_{nm} \bar{X}_{nm} + \bar{b}_{nm} \bar{Y}_{nm} \right), \quad (2.20)$$

we know that this projection on the linear manifold defined by the finite set of spherical harmonics of maximum degree N has the property of least-squares deviation from $r(\zeta, \lambda)$ if the coefficients \bar{a}_{nm} and \bar{b}_{nm} are defined by

$$\bar{a}_{nm} = \frac{1}{4\pi R_0} \int_{-1}^1 d\zeta \int_0^{2\pi} r(\zeta, \lambda) \bar{X}_{nm}(\zeta, \lambda) d\lambda \quad (2.21)$$

and

$$\bar{b}_{nm} = \frac{1}{4\pi R_0} \int_{-1}^1 d\zeta \int_0^{2\pi} r(\zeta, \lambda) \bar{Y}_{nm}(\zeta, \lambda) d\lambda. \quad (2.22)$$

In a generalized sense, $r_N(\zeta, \lambda)$ is a band-limited function in two variables, $\zeta = \cos \theta$ and λ , which are, respectively, the geocentric colatitude and longitude. In more detail, let a grid of longitude and latitude circles be drawn on a sphere. To this grid, there correspond a minimum and a maximum value of wavelength both in longitude and in latitude.

In a geoid representation, the grid spacing should be chosen to correspond to the resolution of geoidal undulations attainable with the observation instrument. This is pertinent in the case of satellite-borne altimeters or in the case of the resolution of harmonics above the noise level in the perturbations introduced by the geopotential in

the orbits of satellites. These measurements correspond, respectively, to 0.1 wavelength for the geoidal undulations ultimately measured by an altimeter and about 10° wavelength for the geopotential variations sensed by satellite orbits.

In the usual terminology of information theory, the quantities we want to describe – mainly the geoid and the geopotential – are treated as band-limited functions both in longitude and in latitude and therefore can possibly be recovered from samples taken at maximum intervals corresponding in some sense to half the wavelength. Specific relations corresponding to these quantities will be given in Section 6, where applications are discussed.

In one dimension, the mathematical correspondent to the Shannon principle can be stated simply. If $f(t)$ is band-limited, i. e., if

$$f(t) = \int_{-\Omega}^{+\Omega} e^{-i\omega t} g(\omega) d\omega , \quad (2.23)$$

where $g(\omega)$ is integrable, then $f(t)$ has the series expansion

$$f(t) = \sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi} . \quad (2.24)$$

On the other hand, if $f(t)$ is periodic, it is evident that equation (2.24) reduces to a finite number of terms. If $f(t)$ is not periodic, the sampling theorem can be used from the point of view of interpolation; i. e., we require that, given a suitably restricted sequence of numbers, we construct a function $f(t)$ defined by the (finite) series (2.24) that interpolates values between these numbers. Evidently, in doing so, we reverse the situation and force $f(t)$ to be a band-limited function, for its spectrum will vanish outside the frequency interval $[-\Omega, +\Omega]$. Interpolation in these terms and corresponding bounds for truncations in the one-dimensional case are well known (e. g., Beutler, 1961; Zakai, 1965). We only point out here that bounds can be defined for an

interpolatory process based on sampling functions, and in some instances the sequence of truncated series

$$f_N(t) = \sum_{n=-N}^{+N} f\left(\frac{n\pi}{\Omega}\right) \frac{\sin(\Omega t - n\pi)}{\Omega t - n\pi}$$

can be proved to converge (at least in the mean) to $f(t)$ as $N \rightarrow \infty$.

The connection between the sampling theorem and Fourier analysis is easily established. Consider the infinite series

$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) , \quad (2.25)$$

which converges under the Dirichlet conditions on $f(t)$. Let $f_N(t)$ denote the partial sum

$$f_N(t) = \frac{1}{2} a_0 + \sum_{n=1}^N (a_n \cos n\omega t + b_n \sin n\omega t) ,$$

which can be written

$$f_N(t) = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(\tau) K_N[\omega(\tau - t)] d\tau , \quad (2.26)$$

where

$$K_N(x) = \frac{\sin [(2N + 1)(x/2)]}{2 \sin (x/2)} . \quad (2.27)$$

If we extend the interval 2π to the entire real axis, it is readily verified that equation (2.27) reduces to sampling functions, and equation (2.26) to the corresponding theorem. For periodic functions, they represent the same result.

Finally, it is important to observe that the most general form of the sampling theorem corresponds to random processes that are band-limited, in terms of expected values, while the form of the theorem remains the same with the condition that $f(t)$ is a distribution. Assuming the response to a pulse to be affected by a random noise, the application of the sampling theorem should also provide a method of recovering the pulse by assuming band limits and therefore eliminating noise problems (Campbell, 1968; Zakai, 1965; Balakrishnan, 1957; Bennett, 1956).

2.4 Classical Examples of Sampling Functions

A typical example, and probably one of the oldest, is given by Lagrange interpolation polynomials. Consider all polynomials $R(x)$ of maximum degree N defined on the segment $a \leq x \leq b$. Let x_i ($i = 0, 1, \dots, N$) be $N + 1$ points on the segment, with $x_0 = a$ and $x_N = b$. The pertinent sampling functions are Lagrange polynomials:

$$L_k(x) = \left[\prod_{\substack{i=1 \\ i \neq k}}^N (x - x_i) \right] / \left[\prod_{\substack{i=1 \\ i \neq k}}^N (x_k - x_i) \right], \quad k = 0, 1, \dots, N \quad (2.28)$$

On the other hand, $L_k(x)$ are not an orthogonal set, and in fact, as is well known, they do not satisfy the condition for best fitting of $L_2(a, b)$ functions.

This idea can be generalized by considering all trigonometric polynomials $T(\lambda)$ of maximum degree N , defined on the line segment $0 \leq \lambda < 2\pi$. If λ_i ($i = 0, 1, \dots, 2N$) are $2N + 1$ points on the segment, the pertinent sampling functions are the $2N + 1$ trigonometric polynomials

$$p_k(\lambda) = \frac{\left[\prod_{\substack{i=0 \\ i \neq k}}^{2N} \sin \frac{1}{2} (\lambda - \lambda_i) \right]}{\left[\prod_{\substack{i=0 \\ i \neq k}}^{2N} \sin \frac{1}{2} (\lambda_k - \lambda_i) \right]}, \quad (2.29)$$

$k = 0, 1, \dots, 2N .$

In the particular case where the λ_i are uniformly distributed,

$$\lambda_i = \frac{2\pi i}{2N+1}, \quad i = 0, 1, \dots, 2N, \quad (2.30)$$

we obtain the well-known sampling functions

$$q_k(\lambda) = \frac{1}{2N+1} \left[1 + 2 \sum_{j=1}^N \cos j (\lambda - \lambda_k) \right]$$

$$= \frac{1}{2N+1} \frac{\sin \{ [(2N+1)/2] (\lambda - \lambda_k) \}}{\sin (1/2) (\lambda - \lambda_k)}, \quad k = 0, 1, \dots, 2N. \quad (2.31)$$

All these functions and their properties are described in detail in the literature (see, e. g., Brillouin, 1962, pp. 93-111). They can easily be generalized to two or higher dimensions, for squares, cubes, or hypercubes, both for polynomials of several variables and for multiperiodic functions. It is interesting to note that the $q_k(\lambda)$ are also orthogonal in the integral sense:

$$(q_k, q_j) = \int_0^{2\pi} q_k(\lambda) q_j(\lambda) d\lambda = \frac{2\pi}{2N+1} \delta_{kj} .$$

The functions

$$\bar{q}_k(\lambda) = \sqrt{\frac{2N+1}{2\pi}} q_k(\lambda) \quad (2.32)$$

are a set of orthonormal sampling functions in $[0, 2\pi]$ and, therefore, are best fitting for any function $f(\lambda) \in L_2(0, 2\pi)$; that is, they satisfy an equality of type (2.10).

The examples we have presented so far are well established, and there is no need for specific derivation. Nevertheless, in general, given a linear manifold defined by the space of a finite number of functions and given an equal number of sampling points on the support of the manifold, there seems to be no broadly recognized definition of an equivalent basis constituted by sampling functions. This will be done in the following section.

2.5 Recursive Algorithm Defining a Finite Set of Sampling Functions

Consider a set of functions $f_1(Q), f_2(Q), \dots, f_N(Q)$ spanning an N -dimensional subspace L of a Hilbert space $H(D)$, $Q \in D$. Let a set of N sampling points $Q_1, Q_2, \dots, Q_N \in D$ be given. Consider the problem of defining N functions $S_1(Q), S_2(Q), \dots, S_N(Q)$ spanning the same linear manifold and having the properties of being orthonormal over the set of points Q_1, Q_2, \dots, Q_N , i. e.,

$$[S_i, S_j] = \sum_{k=1}^N S_i(Q_k) S_j(Q_k) = \delta_{ij} \quad (2.33)$$

Following the philosophy of the Gram-Schmidt process, let

$$\begin{aligned} S_1 &= \alpha_1^1 f_1 \\ S_2 &= \alpha_2^2 f_2 + \alpha_2^1 S_1 \\ S_3 &= \alpha_3^3 f_3 + \alpha_3^2 S_2 + \alpha_3^1 S_1 \\ &\vdots \\ S_k &= \alpha_k^k f_k + \alpha_k^{k-1} S_{k-1} + \dots + \alpha_k^1 S_1 \\ &\vdots \\ S_N &= \alpha_N^N f_N + \alpha_N^{N-1} S_{N-1} + \dots + \alpha_N^1 S_1 \end{aligned}$$

By imposing, in succession, conditions (2.33), we find

$$\alpha_k^j = -\alpha_k^k \frac{[f_k, S_j]}{[S_j, S_j]} , \quad k = 1, 2, \dots, N; \quad j = 1, 2, \dots, k-1 , \quad (2.34)$$

and therefore

$$S_k = \alpha_k^k \left\{ f_k - \sum_{j=1}^{k-1} \frac{[f_k, S_j]}{[S_j, S_j]} S_j \right\} , \quad k = 1, 2, \dots, N . \quad (2.35)$$

The procedure is applicable, of course, if none of the inner products $[S_j, S_j]$ is zero, which sets a restriction on the distribution of the sampling points Q_1, Q_2, \dots, Q_N in D . If the distribution is nonsingular, α_k^k can be defined so that $[S_k, S_k] = 1$, and therefore

$$S_k = \alpha_k^k \left\{ f_k - \sum_{j=1}^{k-1} [f_k, S_j] S_j \right\} , \quad k = 1, 2, \dots, N . \quad (2.36)$$

The α_k^k are also obtained by recurrence by using $[S_k, S_k] = 1$, i.e.,

$$\left(\alpha_k^k \right)^2 \left\{ [f_k, f_k] - \sum_{j=1}^{k-1} [f_k, S_j]^2 \right\} = 1 , \quad k = 1, 2, \dots, N . \quad (2.37)$$

For example,

$$\begin{aligned} \left(\alpha_1^1 \right)^2 [f_1, f_1] &= 1 \\ \left(\alpha_2^2 \right)^2 \left\{ [f_2, f_2] - [f_2, S_1]^2 \right\} &= 1 \\ \dots & \end{aligned}$$

The above method is quite important in the problem of constructing for a linear manifold a base that is orthogonal over a set of points, i. e., orthogonal in a discrete sense. If orthogonality both in the integral and in the discrete sense can be achieved by a proper choice of a grid of points, we have an optimum solution to the problem.

Let us take, as a first example, the sequence $1, x, x^2, x^3, \dots$, which is complete in $L_2(a, b)$ for every closed interval $[a, b]$. Orthogonalization leads to Legendre polynomials $P_0(x), P_1(x), P_2(x), \dots$, and the normalization can be set at will. Now consider the following discrete formulation of the problem, where the initial sequence is truncated at the N th degree. In other words, we look at the linear manifold defined by

$$1, x, x^2, \dots, x^N, \quad (2.38)$$

so that any polynomial of the N th degree at most can evidently be written uniquely as a linear combination of the set (2.38), or of the set $P_0(x), P_1(x), \dots, P_N(x)$. Take $N + 1$ equally spaced points x_j ($j = 0, 1, \dots, N$), with $x_0 = a$ and $x_N = b$, so that

$$\begin{aligned} x_j &= a + jh \\ h &= \frac{b - a}{N} \\ j &= 0, 1, \dots, N \end{aligned} \quad (2.39)$$

One can construct the $N + 1$ polynomials $\phi_0(j, N), \phi_1(j, N), \dots, \phi_N(j, N)$ orthogonal over the set of points $j = 0, 1, \dots, N$, with the property that

$$[\phi_s(j, N), F_{s-1}(j)] \equiv \sum_{j=0}^N \phi_s(j, N) F_{s-1}(j) = 0, \quad (2.40)$$

where $[\phi, F]$ now plays the role of an inner product, and $F_{s-1}(j)$ is any polynomial of degree $s - 1$ or less. This leads to

$$\begin{aligned}\bar{\phi}_0 &= 1 \\ \bar{\phi}_1 &= \frac{2j}{N} - 1 \\ \bar{\phi}_2 &= \frac{12j^2 + 2N(N-1) - 12Nj}{2N(N-1)} \\ &\dots\end{aligned}$$

under the normalization

$$\sum_{j=0}^N \bar{\phi}_n(j, N) \bar{\phi}_{n'}(j, N) = \delta_{nn'} \quad (2.41)$$

The substitution

$$j = \frac{N}{2} (1 + y) \quad ,$$

which makes $j = 0$, ($x = a$), correspond to $y = -1$ and $j = N$, ($x = b$), to $y = +1$, leads to

$$\begin{aligned}\bar{\phi}_0 &= P_0^*(y, N) = 1 \\ \bar{\phi}_1 &= P_1^*(y, N) = y \\ \bar{\phi}_2 &= P_2^*(y, N) = \frac{3Ny^2 - (N+2)}{2(N-1)} \\ &\dots\end{aligned} \quad (2.42)$$

It follows that any polynomial of maximum degree N of y defined in $[-1, +1]$ can be uniquely written as a linear combination of $P_0^*, P_1^*, P_2^*, \dots, P_N^*$. The important property is that for $N \rightarrow \infty$, it can be verified that

$$P_n^*(y, N) \rightarrow P_n(y) \quad ;$$

i. e., the usual Legendre polynomials are obtained. This is an example of the construction of a base for a linear manifold from a discrete point of view, leading, in the limit, to a complete sequence in a Hilbert space.

As a second example, consider the sequence $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$, which is orthogonal and complete in $L_2(-\pi, \pi)$. Any $f(x) \in L_2(-\pi, \pi)$ as a convergent (in the mean) Fourier series, as is well known. Moreover, if $f(x)$ and $f'(x)$ are piecewise continuous in $[-\pi, \pi]$, the series coincides with $f(x)$ everywhere except at the points of discontinuity, where it gives the value $(1/2)[f(x+0) + f(x-0)]$. This produces the well-known Gibbs phenomenon.

Again, consider a finite set

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos Nx, \sin Nx \quad (2.43)$$

of $2N + 1$ functions. In this case, the formulation for a discrete number of points is immediate since the set (2.43) is also orthogonal over the set of $2N + 1$ points,

$$x_j = j \frac{2\pi}{2N + 1}, \quad j = 0, 1, \dots, 2N,$$

and any trigonometric polynomial of maximum degree N , defined in $[-\pi, \pi]$, is exactly and uniquely given as a linear combination of the set (2.43).

A third example, very important for our future problems, employs the sequence

$$X_{00}, X_{10}, X_{11}, Y_{11}, X_{20}, X_{21}, Y_{21}, X_{22}, Y_{22}, \dots,$$

where

$$\begin{aligned} X_{nm} &= P_{nm}(\zeta) \cos m\lambda, & m = 0, 1, \dots, n; \quad n = 0, 1, 2, \dots \\ Y_{nm} &= P_{nm}(\zeta) \sin m\lambda, & m = 1, 2, \dots, n; \quad n = 0, 1, 2, \dots \end{aligned} \quad (2.44)$$

are the usual surface spherical harmonics. They constitute an orthogonal and complete set in $[-1 \leq \zeta \leq 1; 0 \leq \lambda < 2\pi]$; i. e., the unit sphere (σ) for all functions $f(\zeta, \lambda) \in L_2(\sigma)$. Any f of this space has a convergent (in the mean) Fourier series. In general, the discrete formulation of the present case is to our knowledge an unsolved problem. In fact, considering the linear manifold defined by the set

$X_1, X_2, \dots, X_{(N+1)^2}$ of spherical surface harmonics of maximum degree $n = N$, we are faced with the question of how to distribute $(N + 1)^2$ points on the sphere so that the Gram-Schmidt discrete orthogonalization process does not lead to zero divisors.

More explicitly, assume that a point distribution $Q_j(\zeta_j, \lambda_j)$, $j = 1, 2, \dots, M$ (with $M = (N + 1)^2$), is given. As in the previous examples, discrete analogies of the above finite set spanning the same linear manifold can be defined by the relations (2.33) through (2.37). In this specific case,

$$\begin{aligned}
 S_1 &= \alpha_1^1 X_1 \\
 S_2 &= \alpha_2^2 X_2 + \alpha_2^1 S_1 \\
 &\vdots \\
 S_k &= \alpha_k^k X_k + \alpha_k^{k-1} S_{k-1} + \dots + \alpha_k^1 S_1 \\
 &\vdots \\
 S_M &= \alpha_M^M X_M + \alpha_M^{M-1} S_{M-1} + \dots + \alpha_M^1 S_1,
 \end{aligned} \tag{2.45}$$

where, imposing the discrete orthogonality conditions over the set of Q_j ,

$$\begin{aligned}
 [S_2, S_1] &= 0 \\
 [S_3, S_2] &= [S_3, S_1] = 0 \\
 &\dots,
 \end{aligned}$$

we find

$$\alpha_k^j = -\alpha_k^k \frac{[X_k, S_j]}{[S_j, S_j]}, \quad k = 1, 2, \dots, M; \quad j = 1, 2, \dots, k-1. \tag{2.46}$$

Evidently, the set Q_k is acceptable if the inner products $[S_j, S_j]$ are nonzero for $j = 1, 2, \dots, (N+1)^2$. The discussion of such problems is far from trivial, and we shall say more about them in Sections 3 and 4.

After the functions $S_k(Q)$ are obtained, the definition of the sampling functions is immediate. In fact, we look for a set of N sampling functions W_1, W_2, \dots, W_N spanning the same linear manifold and satisfying the conditions

$$W_j(Q_k) = \delta_{jk} \quad . \quad (2.47)$$

If the W 's span the same space as do the S 's (and the f 's), then, certainly, we can write uniquely

$$W_k(Q) = \sum_{j=1}^N \beta_k^j S_j(Q) \quad . \quad (2.48)$$

Multiplying equation (2.48) by $S_i(Q)$ and summing over all sampling points, we get

$$[W_k, S_i] = \sum_{j=1}^N \beta_k^j [S_j, S_i] \quad .$$

Since

$$[W_k, S_i] = S_i(Q_k)$$

and

$$[S_j, S_i] = \delta_{ij} \quad ,$$

it follows that

$$\beta_k^j = S_j(Q_k) \quad , \quad (2.49)$$

so that

$$W_k(Q) = \sum_{j=1}^N S_j(Q_k) S_j(Q) \quad , \quad (2.50)$$

which solves the problem. Reciprocally,

$$S_k(Q) = \sum_{j=1}^N S_k(Q_j) W_j(Q) . \quad (2.51)$$

2.6 Properties of Sampling Functions on a Sphere

Since our objective is to define sampling functions spanning the same linear manifold defined by the set of $(N + 1)^2$ spherical harmonics up to a given degree N , we shall deal at present only with properties pertinent to this manifold. The problem of point distribution will be treated in the next section, so here we will describe properties derivable from the sole assumption that a set of sampling points exists, in the sense that the recursive algorithm described in Section 2.5 is not singular.

We shall eventually use spherical (surface) harmonics normalized according to the following relations:

$$\begin{aligned} \bar{X}_{nm}(\zeta, \lambda) &= \sqrt{2\epsilon_m} \bar{P}_{nm}(\zeta) \cos m\lambda , & m = 0, 1, \dots, n \\ \bar{Y}_{nm}(\zeta, \lambda) &= 2\bar{P}_{nm}(\zeta) \sin m\lambda , & m = 1, 2, \dots, n \end{aligned} \quad (2.52)$$

(see eq. (6.6) for the relation to other normalizations) for $n = 0, 1, 2, \dots, N$, where $\epsilon_0 = 1$ and $\epsilon_m = 2$, $m > 0$. Also, by the usual definitions,

$$\bar{P}_{nm}(\zeta) = \left[\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_{nm}(\zeta) \quad (2.53)$$

and

$$P_{nm}(\zeta) = \frac{(1-\zeta^2)^{m/2}}{2^n n!} \frac{d^{n+m}}{d\zeta^{n+m}} (\zeta^2 - 1)^n . \quad (2.54)$$

To simplify notation, let any of the spherical harmonics be represented by $\bar{X}_k(\zeta, \lambda)$, $k = 1, 2, \dots, (N+1)^2$. The necessary formulas to implement such single-index

notation will be given in Section 4.1. The orthonormality conditions are written

$$(\overline{X}_k, \overline{X}_j) = \frac{1}{4\pi} \int_{-1}^1 d\zeta \int_0^{2\pi} \overline{X}_k(\zeta, \lambda) \overline{X}_j(\zeta, \lambda) d\lambda = \delta_{kj} .$$

For unnormalized spherical harmonics X_{nm}, Y_{nm} (or X_k), we have $(X_{nm}, Y_{n'm'}) = 0$ and

$$\left. \begin{aligned} (X_{nm}, X_{n'm'}) \\ (Y_{nm}, Y_{n'm'}) \end{aligned} \right\} = \frac{\epsilon_m (n+m)!}{(2n+1)(n-m)!} \delta_{nn'} \delta_{mm'} . \quad (2.55)$$

Any function contained in the linear manifold defined by the $\overline{X}_k(\zeta, \lambda)$ can, by definition, be written uniquely as

$$f(\zeta, \lambda) = \sum_{k=1}^{(N+1)^2} \overline{a}^k \overline{X}_k(\zeta, \lambda) , \quad (2.56)$$

where

$$\begin{aligned} \overline{a}^k &= \frac{1}{4\pi} \int_0^{2\pi} d\lambda \int_{-1}^1 f(\zeta, \lambda) \overline{X}_k(\zeta, \lambda) d\zeta \\ &= (f, \overline{X}_k) . \end{aligned} \quad (2.57)$$

The definition of a pertinent set of sampling functions corresponding to the finite number of wavelengths represented by the $(N+1)^2$ spherical harmonics rests on the hypothesis that there exists at least one set of $(N+1)^2$ points (ζ_k, λ_k) such that $(N+1)^2$ functions $W_k(\zeta, \lambda)$ can be defined that satisfy

$$W_k(\zeta_j, \lambda_j) = \delta_{kj}$$

and span the same linear manifold; that is, the Gram determinant $G = |(W_k, W_j)|$ should not vanish.

We must initially verify the relations

$$\bar{X}_k(\zeta, \lambda) = \sum_{j=1}^M \bar{X}_k(\zeta_j, \lambda_j) W_j(\zeta, \lambda) , \quad (2.58)$$

where $M = (N + 1)^2$ and where the matrix

$$\{\bar{X}_{kj}\} \equiv \{\bar{X}_k(\zeta_j, \lambda_j)\}$$

should not be singular, so that equation (2.58) can be inverted. It follows that

$$W_j(\zeta, \lambda) = \sum_{k=1}^M \bar{A}_{jk} \bar{X}_k(\zeta, \lambda) , \quad (2.59)$$

where

$$\bar{A} = \{\bar{A}_{jk}\} = \{\bar{X}_{kj}\}^{-1} .$$

From equation (2.58), assuming $\bar{X}_0 = \bar{P}_{00} = 1$, it follows that the normalization condition already mentioned holds everywhere on the sphere; i. e.,

$$\sum_{j=1}^M W_j(\zeta, \lambda) = 1 . \quad (2.60)$$

From equation (2.59), it also follows that

$$(W_i, W_j) = \sum_{k=1}^M \bar{A}_{ik} \bar{A}_{jk} ,$$

and therefore

$$\{(W_i, W_j)\} = \bar{A} \bar{A}^T, \quad (2.61)$$

so that the Gram's determinant is not zero. Reciprocally, it follows from equation (2.61) that if $G \neq 0$, $|\bar{A}| \neq 0$ and therefore \bar{A}^{-1} exists.

When we use the unnormalized spherical harmonics, we shall define

$$(X_k, X_j) = \sigma_k \delta_{kj}, \quad (2.62)$$

the inner product always being considered with the weight $1/4\pi$, the area of the unit sphere.

We shall now obtain a general form of the sampling functions $W_k(\zeta, \lambda)$ appropriate to the discussion of properties unrelated to the point distribution. We shall also freely alternate from the notation X_k or (X_{nm}, Y_{nm}) for unnormalized and \bar{X}_k or $(\bar{X}_{nm}, \bar{Y}_{nm})$ for normalized spherical harmonics, whichever happens to be more appropriate. At all times, M will mean $(N + 1)^2$.

From

$$W_j(\zeta, \lambda) = \sum_{k=1}^M A_{jk} X_k(\zeta, \lambda),$$

it follows that

$$(W_j, X_k) = \sum_{\ell=1}^M A_{j\ell} (X_\ell, X_k) = A_{jk} \sigma_k \quad (2.63)$$

and therefore

$$W_j(\zeta, \lambda) = \sum_{k=1}^M \frac{1}{\sigma_k} (W_j, X_k) X_k(\zeta, \lambda). \quad (2.64)$$

Also, from

$$X_k(\zeta, \lambda) = \sum_{j=1}^M X_k(\zeta_j, \lambda_j) W_j(\zeta, \lambda)$$

it can be seen that

$$(X_k, W_j) = \sum_{\ell=1}^M X_k(\zeta_\ell, \lambda_\ell) (W_\ell, W_j) ;$$

then, substitution into equation (2.64) yields

$$W_j(\zeta, \lambda) = \sum_{k=1}^M \left[\sum_{\ell=1}^M \frac{1}{\sigma_k} X_k(\zeta_\ell, \lambda_\ell) v_{j\ell} \right] X_k(\zeta, \lambda) , \quad (2.65)$$

where

$$v_{j\ell} = (W_j, W_\ell) .$$

Introducing the explicit expressions for $X_k(\zeta, \lambda)$ and σ_k into equation (2.65), we obtain

$$\begin{aligned} W_j(\zeta, \lambda) &= \sum_{n=0}^N \sum_{m=0}^n \frac{\epsilon_m (2n+1)(n-m)!}{(n+m)!} \sum_{\ell=1}^M v_{j\ell} \\ &\times P_{nm}(\zeta_\ell) P_{nm}(\zeta) \cos m(\lambda - \lambda_\ell) . \end{aligned} \quad (2.66)$$

Let the cosine of the angle between the sampling point $Q_\ell(\zeta_\ell, \lambda_\ell)$ and a generic point $Q(\zeta, \lambda)$ on the unit sphere be represented by γ_ℓ , so that

$$\gamma_\ell = \zeta_\ell \zeta + \sqrt{1 - \zeta_\ell^2} \sqrt{1 - \zeta^2} \cos(\lambda_\ell - \lambda) .$$

By the addition theorem of Legendre, it follows from equation (2.66) that

$$W_j(\zeta, \lambda) = \sum_{\ell=1}^M v_{j\ell} \sum_{n=0}^N (2n+1) P_n(\gamma_\ell) \quad (2.67)$$

or, in view of a well-known identity,

$$W_j(\zeta, \lambda) = (N+1) \sum_{\ell=1}^M \frac{v_{j\ell}}{1-\gamma_\ell} [P_N(\gamma_\ell) - P_{N+1}(\gamma_\ell)] , \quad (2.68)$$

which is nonsingular at $\gamma_\ell = 1$, since this is a simple root of the quantity in brackets. For computational purposes, equation (2.68) is of no use since the quantities $v_{j\ell}$ are not known unless all W_j are defined. Nevertheless, the equation is important for the study of the condition under which the point distribution leads to orthogonality with unit weight, that is, under what conditions

$$(W_j, W_k) = v_k \delta_{kj} . \quad (2.69)$$

If this relation were satisfied, equation (2.68) would give

$$W_j(\zeta, \lambda) = v_j \frac{N+1}{1-\gamma_j} [P_N(\gamma_j) - P_{N+1}(\gamma_j)] , \quad (2.70)$$

and after

$$W_j(\zeta_\ell, \lambda_\ell) = \delta_{j\ell}$$

is imposed, it is found that

$$P_N(\gamma_{j\ell}) = P_{N+1}(\gamma_{j\ell}) , \quad j \neq \ell ; j, \ell = 1, 2, 3, \dots, M . \quad (2.71)$$

The set of equations (2.71) gives the necessary angular spacing, $\arccos \gamma_{j\ell}$, between the sampling points if the W's are orthogonal (in the integral sense) with unit weight. However, it does not give the location of the points on the sphere. The location is obviously not arbitrary, for we require the matrix $\{\bar{X}_k(\zeta_j, \lambda_j)\}$ to be nonsingular. Because of the trivial solution $\gamma_{j\ell} = 1$, equation (2.71) is actually an equation of the Nth degree in $\gamma_{j\ell}$. Since it can be shown that all roots are real, we conclude that a set of $(N + 1)^2$ sampling points leading to orthogonal sampling functions with unit weight cannot have more than N different angles between the points.

The simplest case, $N = 1$, corresponding to $(N + 1)^2 = 4$ points, leads to a very interesting result. For $N = 1$, equation (2.71) gives

$$\gamma_{j\ell} = \frac{3}{2} \gamma_{j\ell}^2 - \frac{1}{2}, \quad (2.72)$$

whose solutions are $\gamma_{j\ell} = 1$ (trivial) and $\gamma_{j\ell} = -1/3$. The angle between any two of the four points must be the same and equal to about $109^\circ 5'$. This is precisely a regular tetrahedron. Indeed, a tetrahedron with a vertex at the north pole leads to a set of four orthogonal sampling functions spanning the same linear manifold as $\{1, \zeta, \sqrt{1 - \zeta^2} \cos \lambda, \sqrt{1 - \zeta^2} \sin \lambda\}$.

For $N = 2$, that is, nine points, the possible angles are two, whose cosines are given by $\gamma_{j\ell} = (-1 \pm \sqrt{6})/5$. We find no configuration that satisfies such conditions. For $N \geq 3$, the only practical way of solving the problem is by numerical means, but experiments in this sense have given no clue to a possible solution. Nothing can be asserted for general values of N . Moreover, it may be possible that by the introduction of a weight, orthogonality can be achieved among the W's for a given point distribution. Another condition of orthogonality is easily obtained from the foregoing equations. Now assuming normalized spherical harmonics ($\sigma_k = 1$), we find

$$\sum_{\ell=1}^M \nu_\ell \bar{X}_k(\zeta_\ell, \lambda_\ell) \bar{X}_j(\zeta_\ell, \lambda_\ell) = \delta_{kj}, \quad (2.73)$$

but this condition is of no use, because the v_ℓ 's are not known. It gives, nevertheless, a geometric interpretation to the constants v_ℓ : The v_ℓ are weights necessary to render the set of spherical harmonics orthogonal over the set of sampling points.

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3. SAMPLING-POINT DISTRIBUTIONS

3.1 General Considerations of the Point Distribution

The problem of distributing in the most regular way a given number of points on a sphere has been studied for centuries and, in general, is still unsolved. A good account is given by Coxeter (1962), who discusses finding $N(\varphi)$, the greatest possible number of small circles of angular radius φ that can be packed on the surface of a sphere without overlapping. Such a number has an upper bound, attained by only a few specific values of φ and given by

$$N(\varphi) \leq N_U(\varphi) = \frac{2\alpha}{\alpha - (\pi/6)} \quad , \quad (3.1)$$

where $\sin \alpha = (1/2) \sec \varphi$. A lower bound has also been established:

$$N(\varphi) \geq N_L(\varphi) = \frac{2\pi}{\sqrt{3}} \left[\operatorname{cosec}^2 \varphi - \left(\frac{3}{2} \operatorname{cosec}^2 \varphi \right)^{3/2} \right] \quad , \quad (3.2)$$

where N_L is positive only for $\varphi < 41^\circ 48'$ (i. e., $\operatorname{cosec} > 3/2$). The most uniform point distribution is seemingly provided by the centers of the aforementioned circles. For $N = 3, 4, 6$, and 12 , it is found that $N = N_U$, with φ given by 60° , $54^\circ 44'$ (regular tetrahedron), 45° , and $31^\circ 43'$, respectively.

In the problem of sampling functions for a sphere, the number of points has to be a perfect square, $(N + 1)^2$, where N is the maximum degree of the spherical harmonics considered. This fact sets one important constraint. A second, and far more important, constraint is that the matrix $\left\{ \bar{X}_{kj} \right\} = \left\{ \bar{X}_k(\zeta_j, \lambda_j) \right\}$ should not be singular.

It is all too easy to construct point distributions for which $\{\bar{x}_{kj}\}$ is singular. Numerical experiments indicate that it is difficult to find an orderly, equatorially symmetric distribution that yields a nonsingular $\{x_{kj}\}$. No useful general characterization of singular or nonsingular cases emerged from our brief examination of this problem. However, the distribution discussed below seems to be one useful example of a convenient distribution of sampling points on a sphere.

3.2 The Recommended 1-3-5 Point Distribution

Lacking a general theory of how best to distribute $(N + 1)^2$ points uniformly on a sphere, we wish to produce a distribution that is, in some sense, a generalization of the tetrahedron configuration. In one orientation of this configuration, one point is at the north pole and three points are on a circle south of the equator. Since spherical harmonics of maximum degree N are expected, by wavelength arguments, to correspond to a net of at most $N + 1$ latitude circles (parallels) and $2N + 1$ longitude circles (meridians), it is natural to consider the pole and N different latitudes corresponding, respectively, to $\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_N$. A total number of $(N + 1)^2$ points is obtained if we set one point at ζ_0 (north pole), three at ζ_1 , five at ζ_2, \dots , and $2N + 1$ at ζ_N . This distribution will be called the 1-3-5 distribution.

Now it must be decided what the values of $\zeta_1, \zeta_2, \dots, \zeta_N$ are and how the $2k + 1$ points are to be distributed on each circle of latitude specified by ζ_k . Again, because of the natural wavelength in longitude λ of the spherical harmonics, apparently the points on each circle should be equally spaced so that at each ζ_k we associate $2k + 1$ points of longitudes,

$$\lambda_{kj} = j \frac{2\pi}{2k + 1} + \left(\left\langle \frac{N + 1 - k}{2} \right\rangle - 2 \left\langle \frac{N + 1 - k}{4} \right\rangle \right) \pi, \quad j = 0, 1, 2, \dots, 2k,$$

where $\langle \quad \rangle$ indicates the integer part of the enclosed argument. This choice is made so that at subsequent latitudes we shift the longitude origin to 180° , and thus there is no accumulation of points at $\lambda = 0$. In view of the results of Section 2.6, regardless

of how the circles are situated and the points distributed on the circles, this 1-3-5 distribution cannot with unit weight yield orthogonality in the integral sense among the W's. This can be illustrated by the case $N = 2$ discussed in Section 2, for which the cosine of the angle between sampling points must be $(-1 \pm \sqrt{6})/5$ for orthogonality. If θ_1 and θ_2 are the colatitudes corresponding to

$$\cos \theta_{1,2} = -\frac{1}{5} \mp \frac{\sqrt{6}}{5} = \begin{cases} -0.69 \\ 0.29 \end{cases}, \quad \theta_1 \approx 133^\circ.7, \quad \theta_2 \approx 73^\circ.1,$$

the cosine of the angular distance of three points on a circle at θ_1 is given by

$$\cos \alpha = \cos^2 \theta_1 + \cos \frac{2\pi}{3} \sin^2 \theta_1 = 0.214,$$

and that of five points on a circle at θ_2 by

$$\cos \beta = \cos^2 \theta_2 + \cos \frac{2\pi}{5} \sin^2 \theta_2 = 0.367.$$

Thus, approximately, $\alpha = 78^\circ.8$, $\beta = 68^\circ.5$, which is significantly different from θ_2 .

If there is a general proof for the existence or nonexistence of a set of sampling points leading to orthogonal sampling functions, it would probably have to be based on an approach similar to that indicated in Section 2.

The point distribution we propose here is given by

$$\begin{aligned} \zeta_k &= (-1)^k \cos \frac{k\pi}{2N+1}, \quad k = 0, 1, \dots, N \\ \lambda_{kj} &= \frac{2j\pi}{2k+1} + \left(\left\langle \frac{N+1-k}{2} \right\rangle - 2 \left\langle \frac{N+1-k}{4} \right\rangle \right) \pi, \quad j = 0, 1, \dots, 2k, \end{aligned} \quad (3.3)$$

where $\langle \ \rangle$ is the integer part of the argument, so that any point has coordinates (ζ_k, λ_{kj}) . For this reason, we initially attach two indices to specify the sampling functions; i. e.,

$$W_{kj}(\zeta, \lambda) = \sum_{n=0}^N \sum_{m=0}^n \left[\bar{A}_{kj}^{nm} \bar{X}_{nm}(\zeta, \lambda) + \bar{B}_{kj}^{nm} \bar{Y}_{nm}(\zeta, \lambda) \right]. \quad (3.4)$$

The distribution (3.3) is reasonably uniform, as shown in Figure 1. The cosine of the angular distance between points (ζ_k, λ_{kj}) and (ζ_p, λ_{pq}) is given by

$$\cos \psi_{kj, pq} = \left(1 - \zeta_k^2\right)^{1/2} \left(1 - \zeta_p^2\right)^{1/2} + \zeta_k \zeta_p \cos(\lambda_{kj} - \lambda_{pq}). \quad (3.5)$$

Figure 2 shows how the maximum and the minimum separations of two sampling points change with $(N+1)^2$ and compares the upper and lower bound curves defined by equations (3.1) and (3.2).

Now there remains the problem of whether the 1-3-5 point distribution is everywhere dense in the limit $N \rightarrow \infty$. Given a positive and arbitrary ϵ and a point $(\bar{\theta}, \bar{\lambda})$ on the sphere, consider a spherical rectangle with its center $(\bar{\theta}, \bar{\lambda})$ and sides ϵ . Then, for N sufficiently large, we should be able to prove that however small ϵ may be, there is at least one point of the distribution lying inside the above defined rectangle.

Consider $\epsilon_1 \leq \epsilon/2$ and $\epsilon_2 \leq \epsilon/2$. For some N, k, j , with $0 \leq k \leq N$ and $0 \leq j \leq 2k$, we must seek to verify the following conditions:

$$\begin{aligned} \bar{\theta} - \epsilon_1 &< \theta_k < \bar{\theta} + \epsilon_1 \\ \bar{\lambda} - \epsilon_2 &< \lambda_{kj} < \bar{\lambda} + \epsilon_2, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} \theta_k &= k \frac{2\pi}{2N+1}, \quad k = 0, 1, \dots, N \\ \lambda_{kj} &= j \frac{2\pi}{2k+1} + \theta_{Nk}, \quad j = 0, 1, \dots, 2k. \end{aligned} \quad (3.7)$$

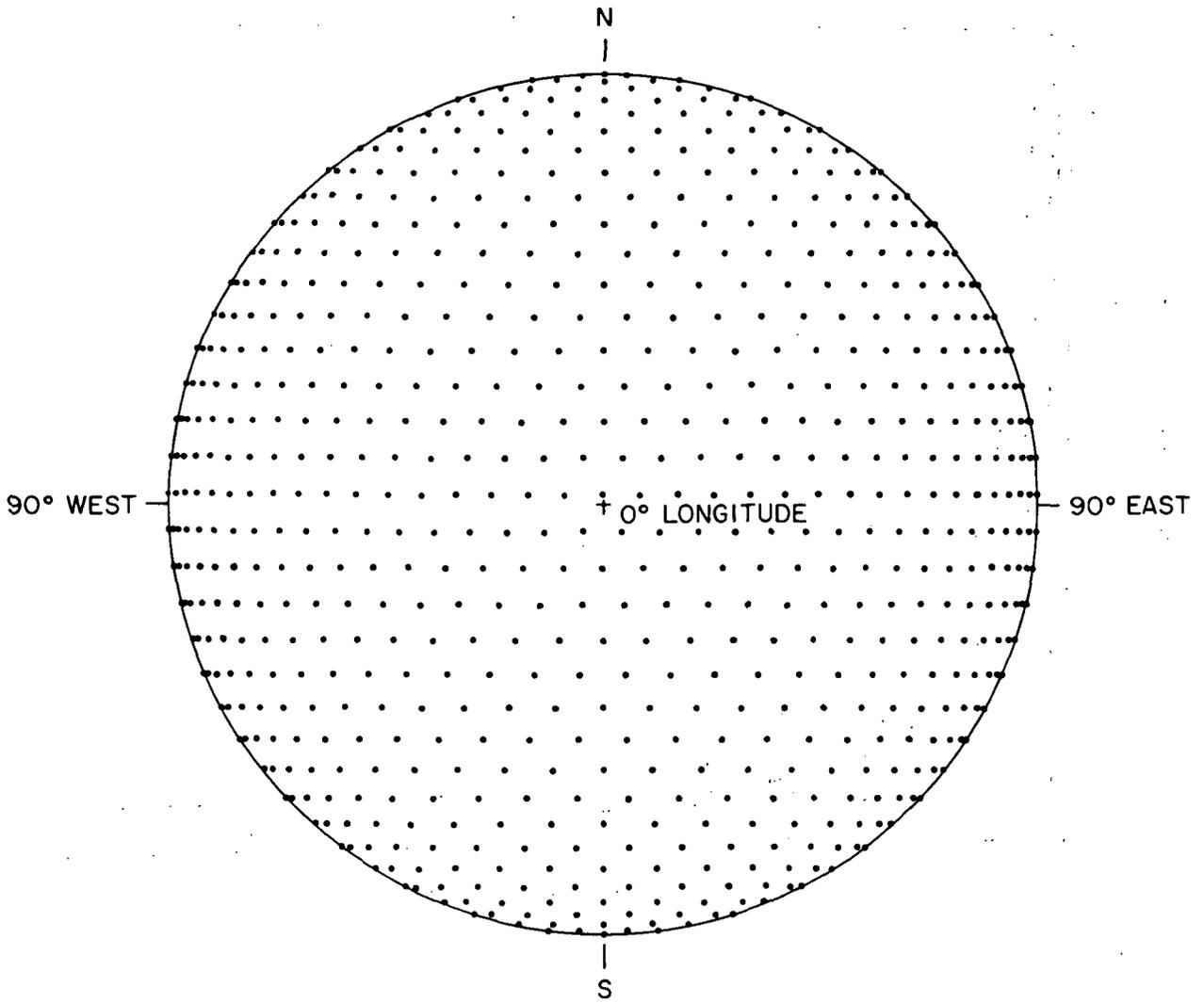


Figure 1. Distribution of sampling points for $N = 36$.

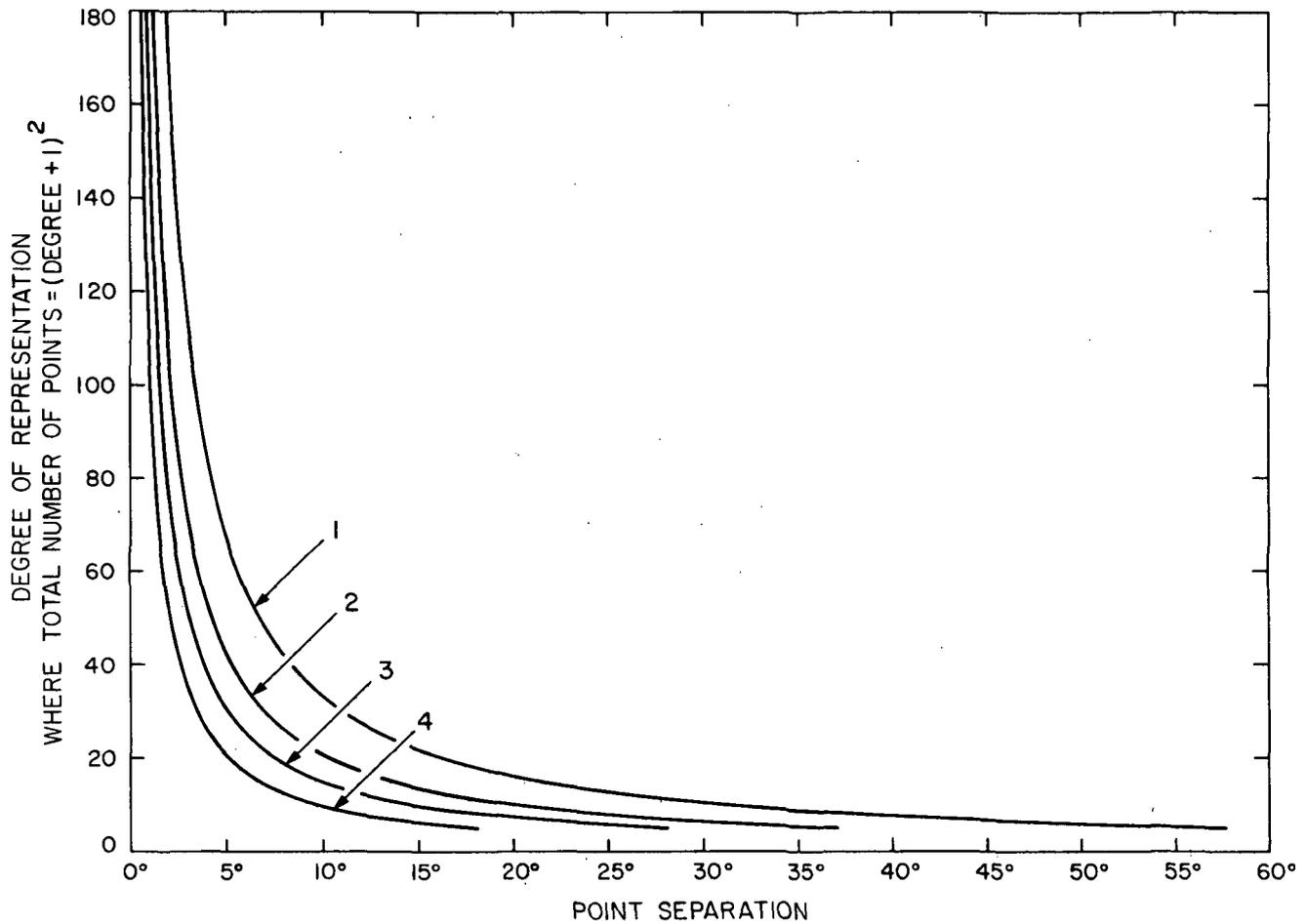


Figure 2. Curve 1 is the maximum and curve 3 is the minimum spacing between points for the 1-3-5 distribution on a sphere. Curve 2 is the upper bound and curve 4 the lower bound on the number of points as a function of separation, as given by equations (3.1) and (3.2), respectively.

(For convenience in this proof, we use a slightly different subscript convention from that in equation (3.3)). The coefficient β_{Nk} is 0 or 1. It follows that k, j have to satisfy the inequalities

$$(2N + 1) \frac{\bar{\theta} - \epsilon_1}{2\pi} < k < (2N + 1) \frac{\bar{\theta} + \epsilon_1}{2\pi} \quad (3.8)$$

and

$$\frac{2k + 1}{2\pi} (\bar{\lambda} - \beta_{Nk}\pi - \epsilon_2) < j < \frac{2k + 1}{2\pi} (\bar{\lambda} + \beta_{Nk}\pi + \epsilon_2) \quad (3.9)$$

under the restrictions (3.7) on k and j . Given ϵ and $\bar{\theta}, \bar{\lambda}$, the above conditions can be satisfied if N is sufficiently large and k, j are conveniently chosen. Actually, $\bar{\theta} + \epsilon_1$ cannot be larger than π , and $\bar{\theta} - \epsilon_1$ cannot be smaller than zero, so that whatever $\bar{\theta}, \epsilon$ are, we find

$$0 < k < N + \frac{1}{2} ,$$

which is always verified. On the other hand, through the same reasoning,

$$0 < j < 2N + 1 ,$$

which is also within the allowed limits.

Obviously, since $k \leq N$ and $j \leq 2k$, we obtain from the relations (3.8) and (3.9) the limits

$$(2N + 1) \frac{\bar{\theta} + \epsilon_1}{2\pi} \leq N \quad (3.10)$$

$$(2k + 1) \frac{\bar{\lambda} + \beta_{Nk}\pi + \epsilon_2}{2\pi} \leq 2k .$$

The first of these gives, excluding the value $\bar{\theta} + \epsilon_1 = \pi$,

$$2N \geq \frac{\bar{\theta} + \epsilon_1}{\pi - (\bar{\theta} + \epsilon_1)}, \quad (3.11)$$

and the second gives

$$2N \geq 2k \geq \frac{\bar{\lambda} + \pi + \epsilon_2}{\pi - (\bar{\lambda} + \epsilon_2)}, \quad (3.12)$$

so that

$$2N \geq \max \left[\frac{\bar{\lambda} + \pi + \epsilon_2}{\pi - (\bar{\lambda} + \epsilon_2)}, \frac{\bar{\theta} + \epsilon_1}{\pi - (\bar{\theta} + \epsilon_1)} \right]. \quad (3.13)$$

Once N is found from equation (3.13), k can be chosen to satisfy relation (3.8), and thus condition (3.9) defines j . It is obvious that if $\bar{\theta} = \pi$ and $\epsilon_1 \rightarrow 0$, then $N \rightarrow \infty$, as should be expected since the south pole is reached only in that limit.

The distribution will be uniform in the limiting sense if the following is verified.

Let n be the number of points falling inside an area ΔS of the sphere (unit radius), however chosen, and M be the total number of points. Then we must have

$$\lim_{N \rightarrow \infty} \frac{n}{M} = \frac{\Delta S}{4\pi}.$$

For the 1-3-5 distribution, the above uniformity condition is an open question.

4. SAMPLING FUNCTIONS FOR A SPHERE

4.1 Explicit Form of the Sampling Functions for a Sphere

In Section 2 we gave the properties of sampling functions that are independent of the choice of a point distribution. Numerical tests indicate that the 1-3-5 distribution of Section 3 is nonsingular for any N tested, of which $N = 36$ was the highest. A general proof that it is nonsingular for any N is at present not available. Nevertheless, the nature of the 1-3-5 distribution makes it unlikely that it would lead to a singular matrix $\{\bar{X}_k(\zeta_\ell, \lambda_\ell)\}$ mainly because of its rather complete lack of symmetry. Of course, this is a weak mathematical justification. With this reservation in mind, we will proceed to derive expressions for the sampling functions of two arguments corresponding to the 1-3-5 distribution of sampling points on a sphere.

In order to obtain the explicit form of the functions $W_{kj}(\zeta, \lambda)$, written as

$$W_{kj}(\zeta, \lambda) = \sum_{n=0}^N \sum_{m=0}^n \left[\bar{A}_{kj}^{nm} \bar{X}_{nm}(\zeta, \lambda) + \bar{B}_{kj}^{nm} \bar{Y}_{nm}(\zeta, \lambda) \right], \quad (4.1)$$

it is convenient to proceed in the following steps:

- A. Reduce all double-index coefficients to a single-index notation.
- B. Solve the recurrence relations defining the intermediary set of "spherical harmonics" $S_p(\zeta, \lambda)$ that satisfy the condition $[S_p, S_q] = \delta_{pq}$ (as in Section 2.5).
- C. Obtain single-index coefficients corresponding to equation (4.1) from these recurrence relations and from the definition of the W 's.
- D. Revert to double-index coefficients and recover the A 's and B 's.

Step A. Single-coefficient specification of spherical harmonics and points of distribution.

We first transform the set $\{\bar{X}_{nm}, \bar{Y}_{nm}\}$ to the set $\{\bar{X}_p\}$ by the following relations:

$$\begin{aligned} \bar{X}_{nm} &\rightarrow \bar{X}_p, & p = n^2 + 2m + 1; & m = 0, 1, 2, \dots, n \\ \bar{Y}_{nm} &\rightarrow \bar{X}_p, & p = n^2 + 2m; & m = 1, 2, \dots, n, \end{aligned} \quad (4.2)$$

where

$$n = 0, 1, 2, \dots, N; \quad p = 1, 2, \dots, M; \quad M = (N + 1)^2.$$

Conversely,

$$\bar{X}_p \rightarrow \begin{cases} \bar{X}_{nm} \\ \bar{Y}_{nm} \end{cases}, \quad (4.3)$$

where

$$\langle \sqrt{p} \rangle^2 - p \begin{cases} \neq 0, & \text{Case 1 below} \\ = 0, & \text{Case 2 below.} \end{cases}$$

Again, $\langle \quad \rangle$ indicates the integer part of the argument.

For Case 1,

$$n = \langle \sqrt{p} \rangle; \quad p - n^2 = \begin{cases} \text{even: } m = \frac{1}{2}(p - n^2) \rightarrow \bar{Y}_{nm} \\ \text{odd: } m = \frac{1}{2}(p - n^2 - 1) \rightarrow \bar{X}_{nm} \end{cases}.$$

For Case 2,

$$n = \sqrt{p} - 1 \rightarrow \bar{X}_{nm}.$$

Next we transform the set (ζ_k, λ_{kj}) into (ζ_q, λ_q) by the relations

$$q = k^2 + j + 1$$

$$\langle \sqrt{q} \rangle^2 - q \begin{cases} \neq 0, & \text{Case 1' below} \\ = 0, & \text{Case 2' below} \end{cases} ,$$

where for Case 1'

$$k = \langle \sqrt{q} \rangle , \quad j = q - k^2 - 1 ,$$

and for Case 2'

$$k = \sqrt{q} - 1 , \quad j = q - k^2 - 1 .$$

Then the single-coefficient sampling functions can be written

$$W_q(\zeta, \lambda) = \sum_{p=1}^M \bar{A}_{qp} \bar{X}_p(\zeta, \lambda) . \quad (4.4)$$

We now have

$$W_q(\zeta_s, \lambda_s) = \delta_{qs} , \quad s, q = 1, 2, \dots, M . \quad (4.5)$$

For notational convenience, it is useful to define the following matrices:

$$\begin{aligned} W &= \{W_{qs}\} \equiv \{W_q(\zeta_s, \lambda_s)\} = I \quad (\text{identity}) \\ \bar{X} &= \{\bar{X}_{ps} = \bar{X}_p(\zeta_s, \lambda_s)\} \\ \bar{A} &= \{\bar{A}_{qp}\} \\ \bar{A}\bar{X} &= I . \end{aligned} \quad (4.6)$$

Step B. Intermediary spherical harmonics.

As we saw in Section 2.5, it is convenient to introduce the functions

$$S_p(\zeta, \lambda) = \sum_{q=1}^p a_{pq} \bar{X}_q(\zeta, \lambda) , \quad (4.7)$$

with

$$[S_p, S_q] = \sum_{s=1}^M S_p(\zeta_s, \lambda_s) S_q(\zeta_s, \lambda_s) = \delta_{pq} .$$

The recurrence relations defining the S's were obtained as

$$S_p(\zeta, \lambda) = \alpha_p^p \left\{ \bar{X}_p(\zeta, \lambda) + \sum_{j=1}^{p-1} \alpha_p^j S_j(\zeta, \lambda) \right\} , \quad (4.8)$$

where

$$\alpha_p^j = - [\bar{X}_p, S_j] , \quad j = 1, 2, \dots, p-1$$

$$\alpha_p^p = \left[\bar{X}_p + \sum_{j=1}^{p-1} \alpha_p^j S_j , \bar{X}_p + \sum_{j=1}^{p-1} \alpha_p^j S_j \right]^{-1/2} .$$

As an example, we list the first few of these spherical harmonics:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= \sqrt{3} \zeta \\ S_3 &= \sqrt{3} \sqrt{1 - \zeta^2} \sin \lambda \\ S_4 &= \sqrt{3} \sqrt{1 - \zeta^2} \cos \lambda \\ S_5 &= \sqrt{5} (3\zeta^2 - 1)/2 \\ S_6 &= \sqrt{15} \zeta \sqrt{1 - \zeta^2} \sin \lambda \\ S_7 &= \sqrt{15} \zeta \sqrt{1 - \zeta^2} \cos \lambda \\ S_8 &= \sqrt{15/4} (1 - \zeta^2) \sin 2\lambda \\ S_9 &= \sqrt{15/4} (1 - \zeta^2) \cos 2\lambda . \end{aligned}$$

Step C. Single-index coefficients.

All constants α_p^j ($p = 1, 2, \dots, M$; $j = 1, 2, \dots, p$) and $S_j(\zeta_k, \lambda_k) = S_{jk}$ ($j = 1, 2, \dots, M$; $k = 1, 2, \dots, M$) are needed to implement the computation of the sampling functions $W_p(\zeta, \lambda)$. These constants can be computed sequentially, as follows:

$$\begin{aligned}\alpha_1^1 &= (N+1)^{-1} \\ S_{1q} &= (N+1)^{-1}, \quad q = 1, 2, \dots, M \\ \alpha_2^1 &= -(N+1)^{-1} \sum_{s=1}^M \bar{X}_{2s} \\ \alpha_2^2 &= \left\{ \sum_{s=1}^M \left(\bar{X}_{2s} + \alpha_2^1 S_{1s} \right)^2 \right\}^{-1/2} \\ S_{2q} &= \alpha_2^2 \left(\bar{X}_{2q} + \alpha_2^1 S_{1q} \right)\end{aligned}$$

For $i = 3, 4, \dots, M$; $j = 1, 2, \dots, i-1$,

$$\begin{aligned}\alpha_i^j &= - \sum_{s=1}^M \bar{X}_{is} S_{js} \\ \alpha_i^i &= \left\{ \sum_{s=1}^M \left(\bar{X}_{is} + \sum_{j=1}^{i-1} \alpha_i^j S_{js} \right)^2 \right\}^{-1/2}\end{aligned}$$

and

$$S_{iq} = \alpha_i^i \left(\bar{X}_{iq} + \sum_{j=1}^{i-1} \alpha_i^j S_{jq} \right).$$

In summary, the α 's and S 's can be generated sequentially in the following order:

$$\begin{aligned}
& \alpha_1^1 \\
& S_{11}, S_{12}, \dots, S_{1M} \\
& \alpha_2^1, \alpha_2^2 \\
& S_{21}, S_{22}, \dots, S_{2M} \\
& \alpha_3^1, \alpha_3^2, \alpha_3^3 \\
& S_{31}, S_{32}, \dots, S_{3M} \\
& \vdots \\
& \alpha_k^1, \alpha_k^2, \dots, \alpha_k^k \\
& S_{k1}, S_{k2}, \dots, S_{kM}
\end{aligned}$$

for $k = 1, 2, \dots, M$.

By successive substitution of the recurrence relation

$$S_n = \alpha_n^n \left(\bar{X}_n + \sum_{j=1}^{n-1} \alpha_n^j S_j \right), \quad 1 \leq n \leq M,$$

into itself, we arrive at

$$\begin{aligned}
S_n = \alpha_n^n \left\{ \bar{X}_n + \sum_{k=1}^{n-1} \sum_{j_k=1}^{n-k} \left(\sum_{j_{k-1}=j_k+1}^{n-k+1} \sum_{j_{k-2}=j_k+2}^{n-k+2} \right. \right. \\
\left. \left. \dots \sum_{j_1=j_k+k-1}^{n-1} \alpha_n^{j_1} \alpha_{j_1}^{j_1} \alpha_{j_1}^{j_2} \alpha_{j_1}^{j_2} \dots \alpha_{j_{k-1}}^{j_k} \alpha_{j_{k-1}}^{j_k} \right) \bar{X}_{j_k} \right\}. \quad (4.9)
\end{aligned}$$

If we accumulate terms $j_k=1, j_k=2, \dots, j_k=n-1$, it follows that the coefficients of $S_n(\zeta, \lambda)$ in terms of $\bar{X}_p(\zeta, \lambda)$, as given in equation (4.7), are

$$\begin{aligned}
a_{nn} &= \alpha_n^n \\
a_{np} &= \alpha_n^n \left\{ \alpha_n^p \alpha_p^p + \sum_{k=2}^{n-p} \left(\sum_{j_{k-1}=p+1}^{n-k+1} \sum_{j_{k-2}=p+2}^{n-k+2} \right. \right. \\
&\quad \left. \left. \dots \sum_{j_1=p+k-1}^{n-1} \alpha_n^{j_1} \alpha_{j_1}^{j_1} \alpha_{j_1}^{j_2} \dots \alpha_{j_{k-1}}^p \alpha_p^p \right) \right\} .
\end{aligned} \tag{4.10}$$

These have the recurrence relation

$$a_{np} = a_{pp} \sum_{j=p+1}^n a_{nj} \alpha_j^p . \tag{4.11}$$

Step D. The sampling functions.

As we have already shown,

$$W_p(\zeta, \lambda) = \sum_{q=1}^M S_{qp} S_q(\zeta, \lambda) ,$$

so that if we use equation (4.7),

$$W_p(\zeta, \lambda) = \sum_{s=1}^M \left\{ \sum_{q=s}^M S_{qp} a_{qs} \right\} \bar{X}_s(\zeta, \lambda) ,$$

and therefore the coefficients \bar{A}_{qp} of equation (4.4) are given by

$$\bar{A}_{qp} = \sum_{j=p}^M S_{jq} a_{jp} , \quad q = 1, 2, \dots, M ; p = 1, 2, \dots, M . \tag{4.12}$$

The index q corresponds to the pair (k, j) , and p to the pair (n, m) , so that \bar{A}_{kj}^{nm} , \bar{B}_{kj}^{nm} are easily recovered from \bar{A}_{qp} .

4.2 Structure of Sampling Functions for a Sphere

The previous section gave explicit formulas for $W_{kj}(\zeta, \lambda)$ in a form appropriate for computations. Another derivation of these relations follows, but in a form that exposes more of the general properties of the $W_{kj}(\zeta, \lambda)$.

From equation (4.1), it follows that

$$\begin{aligned} W_{kj}(\zeta_k, \lambda) &= \sum_{m=0}^N \left\{ \left[\sum_{n=m}^N \bar{A}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) \right] \cos m\lambda \right. \\ &\quad \left. + \left[\sum_{n=m}^N \bar{B}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) \right] \sin m\lambda \right\} \\ &= \sum_{m=0}^N \left(a_{kj}^m \cos m\lambda + b_{kj}^m \sin m\lambda \right). \end{aligned} \quad (4.13)$$

We require that

$$W_{kj}(\zeta_k, \lambda_{k\ell}) = \delta_{j\ell},$$

which defines uniquely a set of $2k + 1$ sampling functions $q_{kj}(\lambda)$ on the segment $[0, 2\pi]$; that is, according to equation (2.31),

$$W_{kj}(\zeta_k, \lambda) = q_{kj}(\lambda) = \frac{1}{2k+1} \left[1 + 2 \sum_{m=1}^k \cos m(\lambda - \lambda_{kj}) \right], \quad \begin{array}{l} k = 1, 2, \dots, N; \\ j = 0, 1, \dots, 2k. \end{array} \quad (4.14)$$

It follows that

$$\begin{aligned}
\sum_{n=0}^N \bar{A}_{kj}^{n0} \bar{P}_{n0}(\zeta_k) &= \frac{1}{2k+1} \\
\sum_{n=m}^N \bar{A}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) &= \frac{2}{2k+1} \cos m\lambda_{kj} \\
\sum_{n=m}^N \bar{B}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) &= \frac{2}{2k+1} \sin m\lambda_{kj}
\end{aligned} \tag{4.15}$$

for $k = 1, 2, \dots, N$; $j = 0, 1, \dots, 2k$; $m = 1, 2, \dots, k$; and that

$$\begin{aligned}
\sum_{n=m}^N \bar{A}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) &= 0 \\
\sum_{n=m}^N \bar{B}_{kj}^{nm} \bar{P}_{nm}(\zeta_k) &= 0
\end{aligned} \tag{4.16}$$

for $k = 1, 2, \dots, N$; $j = 0, 1, \dots, 2k$; $m = k+1, k+2, \dots, N$.

Since ζ_k , $k \neq 0$, is not in general a zero of $\bar{P}_{nm}(\zeta_k)$, it follows that

$$\bar{A}_{kj}^{nm} = \bar{B}_{kj}^{nm} = 0 \tag{4.17}$$

for $m = k+1, k+2, \dots, N$; $n = m, m+1, \dots, N$.

It follows further that we can write

$$W_{kj}(\zeta, \lambda) = \sum_{m=0}^k \left[T_{kj}^m(\zeta) \cos m\lambda + S_{kj}^m(\zeta) \sin m\lambda \right], \tag{4.18}$$

where

$$T_{kj}^m(\zeta) = \sum_{n=m}^N \bar{A}_{kj}^{nm} \bar{P}_{nm}(\zeta) \quad (4.19)$$

$$S_{kj}^m(\zeta) = \sum_{n=m}^N \bar{B}_{kj}^{nm} \bar{P}_{nm}(\zeta) .$$

(Note: $S_{kj}^m(\zeta)$ is not the same function as $S_p(\zeta, \lambda)$ of the previous section.)

We next consider two situations leading to the definition and specification of additional types of sampling functions appropriate for certain linear manifolds.

We consider first the set

$$1, \zeta, \zeta^2, \zeta^3, \dots, \zeta^N ,$$

or the equivalent set

$$\bar{P}_{00}, \bar{P}_{10}(\zeta), \dots, \bar{P}_{N0}(\zeta) .$$

We wish to define the pertinent set of sampling functions corresponding to the points

$$\zeta_k = (-1)^k \cos \frac{k\pi}{2N+1} , \quad k = 0, 1, \dots, N , \quad (4.20)$$

that is, the values of ζ defining the 1-3-5 point distribution.

The sampling functions are easily found to be

$$C_k(\zeta) = \frac{\epsilon_k}{2N+1} \left[1 + 2 \sum_{j=1}^N \tau_j(\zeta_k) \tau_j(\zeta) \right] , \quad (4.21)$$

where $\tau_j(\zeta) = \cos j\theta$, $\zeta = \cos \theta$, and, as usual, $\epsilon_0 = 1$, $\epsilon_k = 2$, $k > 0$.

Consider next the set of associated Legendre functions of the first kind and of given order m :

$$\bar{P}_{mm}(\zeta), \bar{P}_{m+1,m}(\zeta), \dots, \bar{P}_{N,m}(\zeta) .$$

As is well known, this is an orthonormal set and defines a finite dimensional (dimension = $N - m + 1$) space. Again, for the point distribution (4.20), we wish to define the pertinent set of sampling functions. The solution of this problem, which is not trivial in detail but is simple in essence, is given by application of the method presented in Section 2.5.

We begin by constructing a system orthogonal over the set of points ζ_k , $k = 0, 1, \dots, N$. Evidently, there are m points in excess, which must be excluded; for reasons given below, we exclude the set $\zeta_0, \zeta_1, \dots, \zeta_{m-1}$. Specifically, the problem is to find, for any given N and m , a set of $N - m + 1$ functions,

$$G_{lm}(\zeta) = \sum_{n=m}^N g_{\ell}^{nm} \bar{P}_{nm}(\zeta) , \quad (4.22)$$

that satisfy the $N - m + 1$ conditions

$$G_{lm}(\zeta_i) = \delta_{li} , \quad \ell, i = m, m+1, \dots, N . \quad (4.23)$$

The intermediary functions with discrete orthogonality properties over the set $\zeta_m, \zeta_{m+1}, \dots, \zeta_N$ are given by

$$H_{mm}(\zeta) = \alpha_{mm} \bar{P}_{mm}(\zeta)$$

$$H_{lm}(\zeta) = \alpha_{lm} \left\{ \bar{P}_{lm}(\zeta) - \sum_{j=m}^{\ell-1} \left[\bar{P}_{lm}, H_{jm} \right] H_{jm}(\zeta) \right\}, \quad \ell = m+1, m+2, \dots, N , \quad (4.24)$$

where α_{lm} is chosen so that

$$[H_{\ell m}, H_{\ell m}] = 1 .$$

We can now write

$$G_{\ell m}(\zeta) = \sum_{n=m}^N h_{\ell}^{nm} H_{nm}(\zeta) ,$$

multiply by $H_{jm}(\zeta)$,

$$H_{jm}(\zeta) G_{\ell m}(\zeta) = \sum_{n=m}^N h_{\ell}^{nm} H_{jm}(\zeta) H_{nm}(\zeta) ,$$

set $\zeta = \zeta_s$, $m \leq s \leq N$, and sum over s . This gives

$$[H_{jm}, G_{\ell m}] = \sum_{n=m}^N h_{\ell}^{nm} [H_{jm}, H_{nm}]$$

or

$$H_{jm}(\zeta_{\ell}) = h_{\ell}^{jm} ,$$

and finally,

$$G_{\ell m}(\zeta) = \sum_{n=m}^N H_{nm}(\zeta_{\ell}) H_{nm}(\zeta) , \tag{4.25}$$

which, combined with equation (4.24), gives the desired result. In this way, we find the functions $H_{nm}(\zeta)$ and therefore the $G_{\ell m}(\zeta)$ in a recurrent way.

The form (4.22) can also be obtained directly. In fact, if the properties of the Lagrange interpolation polynomials are used first, we can easily see that

$$G_{\ell m}(\zeta) = \frac{\left(1 - \zeta^2\right)^{m/2} \prod_{s=m}^{N'} (\zeta - \zeta_s)}{\left(1 - \zeta_\ell^2\right)^{m/2} \prod_{s=m}^{N'} (\zeta_\ell - \zeta_s)}, \quad (4.26)$$

where the prime indicates $s \neq \ell$. To transform the numerator in terms of the associated Legendre functions, we use the orthogonality properties of these for a fixed order. After some algebra, we obtain for the coefficients of equation (4.22)

$$g_\ell^{nm} \left(1 - \zeta_\ell^2\right)^{m/2} \prod_{s=m}^{N'} (\zeta_\ell - \zeta_s) = \sum_{j=n-m}^{(N+n-m)/2} a_{2j+n-m}^{\ell m} b_m^{2j+n-m, n-m}, \quad (4.27)$$

where

$$b_m^{2j+n-m, n-m} = \left[\frac{(2n+1)(n-m)! 2}{(n+m)!} \right] \frac{1}{n!} \\ \times \sum_{t=0}^{(n-m)/2} \frac{(2n-2t)! (-1)^t \binom{n}{t} 2^{j-t} (j+n-m-t)!}{(n-m-2t)! (2j+2n-2t+1)!!}$$

and

$$a_{N-k}^{\ell m} = 1$$

$$a_{N-k-1}^{\ell m} = - \sum_{s=m}^{N'} \zeta_s$$

$$a_{N-k-2}^{\ell m} = \sum_{s_1=m}^{N'} \sum_{s_2=m}^{N'} \zeta_{s_1} \zeta_{s_2}$$

\vdots

$$a_0^{\ell m} = (-1)^{N-m} \prod_{s=m}^{N'} \zeta_s.$$

We now return to the study of the functions $W_{kj}(\zeta, \lambda)$. First, we observe that for $k = 0$, since $\bar{P}_{nm}(1) = 0$, $m \neq 0$, we can write

$$W_{00} = \sum_{n=0}^N \bar{A}_{00}^{n0} \bar{P}_{n0}(\zeta) ,$$

with the $N + 1$ requirements

$$W_{00}(\zeta_r) = \delta_{0r} , \quad r = 0, 1, \dots, N .$$

This uniquely defines the sampling function $W_{00}(\zeta)$ and identifies it with $C_0(\zeta)$, defined by equation (4.21); i. e.,

$$W_{00}(\zeta) = \frac{1}{2N+1} \left[1 + 2 \sum_{j=1}^N \tau_j(\zeta) \right] = C_0(\zeta) . \quad (4.28)$$

Another important relation is found by considering that

$$\bar{P}_{n0}(\zeta) = \sum_{k=0}^N \sum_{j=0}^{2k} \bar{P}_{n0}(\zeta_k) W_{kj}(\zeta, \lambda)$$

and also

$$\bar{P}_{n0}(\zeta) = \sum_{k=0}^N \bar{P}_{n0}(\zeta_k) C_k(\zeta) ,$$

so that, for $k > 0$,

$$\sum_{j=0}^{2k} W_{kj}(\zeta, \lambda) = C_k(\zeta) = \frac{2}{2N+1} \left[1 + 2 \sum_{j=1}^N \tau_j(\zeta) \tau_j(\zeta_k) \right] . \quad (4.29)$$

Recalling the form of $W_{kj}(\zeta, \lambda)$ given by equation (4.18) and taking equation (4.19) into account, we can write

$$T_{kj}^m(\zeta) = \sum_{p=m}^N T_{kj}^m(\zeta_p) G_{pm}(\zeta)$$

$$S_{kj}^m(\zeta) = \sum_{p=m}^N S_{kj}^m(\zeta_p) G_{pm}(\zeta) ,$$

so that

$$W_{kj}(\zeta, \lambda) = \sum_{m=0}^k \sum_{p=m}^N G_{pm}(\zeta) \left[T_{kj}^m(\zeta_p) \cos m\lambda + S_{kj}^m(\zeta_p) \sin m\lambda \right] . \quad (4.30)$$

The problem is now reduced to obtaining the numbers

$$T_{kj}^m(\zeta_p), S_{kj}^m(\zeta_p) , \quad \begin{array}{l} k = 1, 2, \dots, N ; m = 0, 1, \dots, k ; \\ j = 0, 1, \dots, 2k ; p = m, m+1, \dots, N . \end{array} \quad (4.31)$$

To do so, it is convenient to consider three cases: 1) $\zeta = \zeta_k$,
2) $\zeta = \zeta_r$, $r > k$. and 3) $\zeta = \zeta_r$, $r < k$.

1) Let $\zeta = \zeta_k$. We must have

$$W_{kj}(\zeta_k, \lambda_{kp}) = \sum_{m=0}^k \sum_{n=m}^N \bar{P}_{nm}(\zeta_k) \left[A_{kj}^{nm} \cos m\lambda_{kp} + B_{kj}^{nm} \sin m\lambda_{kp} \right] = \delta_{pj} .$$

Multiplying successively by $\cos \alpha \lambda_{kp}$, $\sin \alpha \lambda_{kp}$ and summing over p ($p = 0, 1, \dots, 2k$), we find

$$\sum_{n=m}^N \bar{P}_{nm}(\zeta_k) A_{kj}^{nm} = \frac{\varepsilon_m}{2k+1} \cos m\lambda_{kj}$$

$$\sum_{n=m}^N \bar{P}_{nm}(\zeta_k) B_{kj}^{nm} = \frac{2}{2k+1} \sin m\lambda_{kj} ,$$
(4.32)

so that

$$\begin{aligned}
T_{kj}^m(\zeta_k) &= \frac{\varepsilon_m}{2k+1} \cos m\lambda_{kj} \\
S_{kj}^m(\zeta_k) &= \frac{2}{2k+1} \sin m\lambda_{kj} ,
\end{aligned} \tag{4.33}$$

which, in fact, verifies equation (4.14).

2) Consider now $\zeta = \zeta_r$, $r > k$. We must have

$$\sum_{m=0}^k \sum_{n=m}^N \bar{P}_{nm}(\zeta_r) \left[A_{kj}^{nm} \cos m\lambda_{rs} + B_{kj}^{nm} \sin m\lambda_{rs} \right] = 0 \quad , \quad \begin{array}{l} r = k+1, k+2, \dots, N ; \\ s = 0, 1, \dots, 2r . \end{array} \tag{4.34}$$

Let $m' = 0, 1, 2, \dots, k$. Then, multiplying equation (4.34) successively by $\cos m'\lambda_{rs}$ and $\sin m'\lambda_{rs}$ and finally summing over s give, respectively,

$$\begin{aligned}
\sum_{n=m}^N \bar{P}_{nm}(\zeta_r) A_{kj}^{nm} &= 0 \\
\sum_{n=m}^N \bar{P}_{nm}(\zeta_r) B_{kj}^{nm} &= 0 \quad ,
\end{aligned} \quad m = 0, 1, \dots, k ; r = k+1, k+2, \dots, N \quad . \tag{4.35}$$

Then,

$$\begin{aligned}
T_{kj}^m(\zeta_r) &= 0 \\
S_{kj}^m(\zeta_r) &= 0 \quad , \quad r > k .
\end{aligned} \tag{4.36}$$

It is important to note that the results (eq. (4.35)) are true only because in this case

$$m + m' \leq 2k < 2r < 2r + 1 \quad . \tag{4.37}$$

In another situation ($r < k$), the result would not be the same. In that case, the summation over s is not trivial, since there are divisors of the type

$$\sin \frac{m + m'}{2r + 1} \pi \quad (4.38)$$

that may be zero if $m + m' \geq 2r + 1$. Equations (4.36) also show that

$$W_{kj}(\zeta_r, \lambda) = 0 \quad , \quad (4.39)$$

for any λ , for $r > k$.

3) Finally, let $\zeta = \zeta_r$, $r < k$. In this case, the above procedure does not apply. Equation (4.34) is still valid, however. Let us observe that the point distribution on every circle is such that

$$\cos m\lambda_{kj} = \cos m\lambda_{k, 2k-j+1} \quad (4.40)$$

$$\sin m\lambda_{kj} = -\sin m\lambda_{k, 2k-j+1} \quad , \quad j = 1, 2, \dots, k \quad ,$$

for any k . Therefore, substituting $2r - s + 1$ for s into equation (4.34), we get

$$\sum_{m=0}^k \sum_{n=m}^N \bar{P}_{nm}(\zeta_r) \left[A_{kj}^{nm} \cos m\lambda_{rs} - B_{kj}^{nm} \sin m\lambda_{rs} \right] = 0 \quad , \quad (4.41)$$

and from equations (4.34) and (4.41), we find

$$\sum_{m=0}^k \sum_{n=m}^N \bar{P}_{nm}(\zeta_r) A_{kj}^{nm} \cos m\lambda_{rs} = 0 \quad (4.42)$$

$$\sum_{m=0}^k \sum_{n=m}^N \bar{P}_{nm}(\zeta_r) B_{kj}^{nm} \sin m\lambda_{rs} = 0 \quad ,$$

or

$$\sum_{m=0}^k T_{kj}^m(\zeta_r) \cos \lambda_{rs} = 0 \quad (4.43)$$

$$\sum_{m=1}^k S_{kj}^m(\zeta_r) \sin \lambda_{rs} = 0, \quad s = 0, 1, \dots, r; \quad r = 0, 1, \dots, k=1$$

Also, using equation (4.40) in equation (4.32), we find, in general,

$$A_{kj}^{nm} = A_{k, 2k-j+1}^{nm} \quad (4.44)$$

$$B_{kj}^{nm} = -B_{k, 2k-j+1}^{nm};$$

that is,

$$T_{kj}^m(\zeta) = T_{k, 2k-j+1}^m(\zeta) \quad (4.45)$$

$$S_{kj}^m(\zeta) = -S_{k, 2k-j+1}^m(\zeta),$$

so that the number of unknown T's and S's is reduced by half (plus one, corresponding to $j = 0$).

Other general important relations are obtained from

$$\sum_{j=0}^{2k} W_{kj}(\zeta, \lambda) = C_k(\zeta);$$

that is,

$$\sum_{j=0}^{2k} T_{kj}^0(\zeta) = C_k(\zeta)$$

$$\sum_{j=0}^{2k} T_{kj}^m(\zeta) = 0, \quad m \neq 0 \quad (4.46)$$

$$\sum_{j=0}^{2k} S_{kj}^m(\zeta) = 0, \quad m \neq 0.$$

In particular, for $r \neq k$, $C_k(\zeta_r) = 0$, and using equation (4.45) with equation (4.46), we get

$$T_{k0}^m(\zeta_r) + 2 \sum_{j=1}^k T_{kj}^m(\zeta_r) = 0 \quad (4.47)$$

$$S_{k0}^m(\zeta_r) = 0, \quad r = 0, 1, \dots, k-1; \quad m = 0, 1, \dots, k.$$

The system to be solved is equation (4.43) plus equation (4.47). But for a given pair (k, j) , we have the system (4.43), which alone can be solved by recurrence. In fact, we find

$$\sum_{m=r}^{\leq k} T_{k\ell}^m(\zeta_r) + \sum_{m=r+1}^{\leq k} T_{k\ell}^m(\zeta_r) = 0$$

$$\sum_{m=r-1}^{\leq k} T_{k\ell}^m(\zeta_r) + \sum_{m=r+2}^{\leq k} T_{k\ell}^m(\zeta_r) = 0 \quad (4.48)$$

$$\vdots$$

$$\sum_{m=0}^{\leq k} T_{k\ell}^m(\zeta_r) = 0$$

for $r = 0, 1, \dots, k-1$, and the summations are over m modulus $2r + 1$.

The analogous system for the S 's is given by

$$\sum_{m=r}^{\leq k} S_{k\ell}^m(\zeta_r) - \sum_{m=r+1}^{\leq k} S_{k\ell}^m(\zeta_r) = 0 \quad (4.49)$$

$$\vdots$$

$$\sum_{m=1}^{\leq k} S_{k\ell}^m(\zeta_r) - \sum_{m=2r}^{\leq k} S_{k\ell}^m(\zeta_r) = 0$$

for $r = 0, 1, \dots, k-1$, and again the summations are over m modulus $2r + 1$.

These two systems are obtained by recognizing that only $r + 1$ different values of $\frac{\cos}{\sin} m\lambda_{rs}$ can occur – that is, for example,

$$\cos p\lambda_{rs} = \cos (p + 2r + 1) \lambda_{rs} = \dots = \cos [p + j(2r + 1)] \lambda_{rs} .$$

Similar relations exist for the sines, with only an alternate change of sign.

Finally, the values of $T_{kj}^m(\zeta_p)$ and $S_{kj}^m(\zeta_p)$ needed to complete equation (4.30) are computed in sequence ($m = k, m = k-1, \dots$), using equations (4.33) and (4.36) with (4.49).

4.3 Approximation Error in Sampling-Function Representation

In previous sections, we have shown that quantities generally represented by series of spherical harmonics can instead be expressed in a series of sampling functions. One fact, though, should always be kept in mind: Any function defined on a sphere enjoys the important property of least-squares deviation (and therefore convergence in the mean) when expanded in a series of spherical harmonics or in any other orthonormal complete set in L_2 (sphere). The coefficients of this series are uniquely defined.

For truncated series, the same can be said for any sequence spanning the same linear manifold of the spherical harmonics. This is a basic motivation for using the sampling functions W_{kj} we have defined.

Consider first the now familiar form of sampling functions for a sphere:

$$W_\ell(\zeta, \lambda) = \sum_{i=1}^M \bar{A}_{\ell i} \bar{X}_i(\zeta, \lambda) ,$$

and, reciprocally,

$$\bar{X}_k(\zeta, \lambda) = \sum_{\ell=1}^M \bar{X}_k(\zeta_\ell, \lambda_\ell) W_\ell(\zeta, \lambda) .$$

For spherical harmonics normalized according to equation (2.52), we have

$$\begin{aligned} (\bar{X}_k, \bar{X}_j) &= \delta_{kj} = \sum_{\ell=1}^M \bar{X}_k(\zeta_\ell, \lambda_\ell) (W_\ell, \bar{X}_j) = \sum_{\ell=1}^M \bar{X}_k(\zeta_\ell, \lambda_\ell) \sum_{i=1}^M \bar{A}_{\ell i} (\bar{X}_i, \bar{X}_j) \\ &= \sum_{\ell=1}^M \bar{X}_k(\zeta_\ell, \lambda_\ell) \bar{A}_{\ell j} \quad , \end{aligned}$$

and therefore

$$\sum_{\ell=1}^M \bar{A}_{\ell j} \bar{X}_k(\zeta_\ell, \lambda_\ell) = \delta_{kj} \quad . \quad (4.50)$$

These relations show that the set of constants $\bar{A}_{\ell j}$ is orthogonal to the set of spherical harmonics over the ensemble of sampling points. This property is very important, for if the sampling-point distribution in the limit $N \rightarrow \infty$ is everywhere dense on the sphere, we can conclude that

$$\lim_{N \rightarrow \infty} \sum_{\ell=1}^M \bar{A}_{\ell j} \bar{X}_k(\zeta_\ell, \lambda_\ell) = (\bar{X}_j, \bar{X}_k) \quad , \quad (4.51)$$

where, of course, $(\zeta_\ell, \lambda_\ell)$ goes into a generic point on the sphere.

Next, consider the harmonic Fourier series for continuous functions F defined on the sphere:

$$F(\zeta, \lambda) = \sum_{n=1}^{\infty} E_n \bar{X}_n(\zeta, \lambda) \quad , \quad (4.52)$$

where $E_n = (F, \bar{X}_n)$. The Parseval equality holds in this case; i. e.,

$$\sum_{n=1}^{\infty} E_n^2 = \|F\|^2 \quad . \quad (4.53)$$

On the other hand, consider the truncated representation

$$\begin{aligned}\hat{F}(\zeta, \lambda) &= \sum_{k=1}^M F(\zeta_k, \lambda_k) W_k(\zeta, \lambda) = \sum_{n=1}^M \left[\sum_{k=1}^M F(\zeta_k, \lambda_k) \bar{A}_{kn} \right] \bar{X}_n(\zeta, \lambda) \\ &= \sum_{n=1}^M D_n(N) \bar{X}_n(\zeta, \lambda) .\end{aligned}\quad (4.54)$$

From equations (4.52) and (4.54), we obtain

$$F - \hat{F} = \sum_{n=1}^M [E_n - D_n(N)] \bar{X}_n(\zeta, \lambda) + \sum_{n=M+1}^{\infty} E_n \bar{X}_n(\zeta, \lambda)$$

and therefore

$$\begin{aligned}\|F - \hat{F}\|^2 &= \frac{1}{4\pi} \int_{-1}^1 d\zeta \int_0^{2\pi} (F - \hat{F})^2 d\lambda = \sum_{n=1}^M [E_n - D_n(N)]^2 + \sum_{n=M+1}^{\infty} E_n^2 \\ &= \sum_{n=1}^{\infty} E_n^2 - \sum_{n=1}^M [2E_n D_n(N) - D_n^2(N)] ,\end{aligned}\quad (4.55)$$

or, using the proper definitions,

$$\begin{aligned}\|F - \hat{F}\|^2 &= \|F\|^2 - 2 \sum_{n=1}^M (F, \bar{X}_n) \sum_{k=1}^M \bar{A}_{kn} F(\zeta_k, \lambda_k) \\ &\quad + \sum_{n=1}^M \left[\sum_{k=1}^M \sum_{\ell=1}^M \bar{A}_{kn} F(\zeta_k, \lambda_k) \bar{A}_{\ell n} F(\zeta_\ell, \lambda_\ell) \right] ,\end{aligned}$$

which, in view of equation (4.51), leads to

$$\lim_{N \rightarrow \infty} \|F - \hat{F}\|^2 = \|F\|^2 - 2 \sum_{n=1}^{\infty} (F, \bar{X}_n)(F, \bar{X}_n) + \sum_{n=1}^{\infty} (F, \bar{X}_n)(F, \bar{X}_n) = 0 .$$

It follows that if in the limit $N \rightarrow \infty$ the sampling-point distribution is everywhere dense on the sphere, the limit of the distance $\|F - \hat{F}\|$ as $N \rightarrow \infty$ is zero, which guarantees convergence in the mean of $\hat{F} \rightarrow F$. The truncation error can be obtained after precise bounds are found for the sampling functions or, equivalently, for the coefficients of their linear representation in terms of spherical harmonics. These have well-known estimates (see, e.g., Sansone, 1959, p. 202).

5. SAMPLING FUNCTIONS OF THREE ARGUMENTS

5.1 Spherical Reference Surface

For the 1-3-5 distribution of points on a sphere or of directions in space, Section 4 developed a set of sampling functions $W_p(\zeta, \lambda)$ that are linear combinations of the surface spherical harmonics. For some applications, these are the appropriate functions; but for others, a corresponding set of functions is needed, a set that incorporates dependence on radial distance r as well as on ζ and λ . A typical example is the representation of the gravitational potential about a body such as the earth.

Generalization to three dimensions is simple if a sphere is chosen as the reference figure on which the sampling points are distributed according to equations (3.3). The coefficients \bar{A}_{qp} in equation (4.12) are still applicable, and the three-dimensional sampling functions are defined by a formula analogous to equation (4.1), namely,

$$Z_k(r, \zeta, \lambda) = \sum_{n=0}^N \sum_{m=0}^n \left(\frac{a}{r}\right)^{n+1} \left[\bar{A}_k^{nm} \bar{X}_{nm} + \bar{B}_k^{nm} \bar{Y}_{nm} \right], \quad (5.1)$$

where a is the radius of the reference sphere. Conversely,

$$\left(\frac{a}{r}\right)^{n+1} \bar{X}_{nm}(\zeta, \lambda) = \sum_{k=1}^M \bar{X}_{nm}(\zeta_k, \lambda_k) Z_k(r, \zeta, \lambda) \quad (5.2)$$

As an example, the representation of the geopotential illustrates that definition (5.1) achieves the desired end. The potential exterior to a body is conventionally written as

$$V = -\frac{\mu}{R} \sum_{n=0}^{\infty} \sum_{m=0}^n \left[\bar{C}^{nm} \left(\frac{R}{r}\right)^{n+1} \bar{X}_{nm}(\zeta, \lambda) + \bar{S}^{nm} \left(\frac{R}{r}\right)^{n+1} \bar{Y}_{nm}(\zeta, \lambda) \right] . \quad (5.3)$$

Truncating this expression at degree N , and using equation (2.58), we can rewrite equation (5.3) in the form

$$V = -\frac{\mu}{R} \sum_{k=1}^M \left[\sum_{n=0}^N \left(\frac{R}{r}\right)^{n+1} L_n^k \right] W_k(\zeta, \lambda) , \quad (5.4)$$

where

$$L_n^k = \sum_{m=0}^n \left[\bar{C}^{nm} \bar{X}_{nm}(\zeta_k, \lambda_k) + \bar{S}^{nm} \bar{Y}_{nm}(\zeta_k, \lambda_k) \right] . \quad (5.5)$$

On the other hand, if there are $(N+1)^2$ functions $Z_k(r, \zeta, \lambda)$, $k = 1, 2, \dots, M=(N+1)^2$, defined by equation (5.1) such that, in general,

$$Z_k(r_\ell, \zeta_\ell, \lambda_\ell) = \delta_{k\ell} \quad (5.6)$$

for points $(r_\ell, \zeta_\ell, \lambda_\ell)$ on the sphere $r_\ell = a$, then we can write (5.4) in the form

$$V = -\frac{\mu}{R} \sum_{k=1}^M L^k Z_k(r, \zeta, \lambda) , \quad (5.7)$$

where

$$-\frac{\mu}{R} L^k = V(r_k, \zeta_k, \lambda_k) . \quad (5.8)$$

The form (5.4) can be used with the right side of equation (5.7), yielding

$$L^k = \sum_{n=0}^N \left(\frac{R}{a}\right)^{n+1} L_n^k = \sum_{n=0}^N \left(\frac{R}{a}\right)^{n+1} \sum_{m=0}^n \left[\bar{C}^{nm} \bar{X}_{nm}(\zeta_k, \lambda_k) + \bar{S}^{nm} \bar{Y}_{nm}(\zeta_k, \lambda_k) \right] \quad (5.9)$$

If the spherical-harmonic coefficients \bar{C}^{nm} , \bar{S}^{nm} are given, the corresponding sampling-function coefficients result from equation (5.9). If the sampling-function coefficients L^k are given, then substitution of equation (5.1) into equation (5.7) yields the expressions for the spherical-harmonic coefficients:

$$\bar{C}^{nm} = \sum_{k=1}^M L^k \bar{A}_k^{nm} \quad (5.10)$$

$$\bar{S}^{nm} = \sum_{k=1}^M L^k \bar{B}_k^{nm}$$

5.2 Nonspherical Reference Surface

The surface of the earth differs from a sphere by about 1 part in 298, and for some purposes, a sampling-point distribution on a reference ellipsoid might be preferable to a reference sphere. In other circumstances, a more detailed reference surface that better approximates an equipotential surface of the earth might be useful.

The objective, of course, is to formulate a set of sampling functions $Z_k^*(r, \zeta, \lambda)$ defined for sampling points on the reference surface and spanning the linear manifold determined by the solid spherical harmonics truncated at some degree N . In principle, the procedures described in Section 2.5 are applicable. In this approach, the solid spherical harmonics are the starting point, rather than the surface harmonics as in Section 4.1. This approach may be more or less practical, depending on how complex the reference surface may be and on how the points are distributed on it.

The inverse of equation (5. 13) is

$$Z_j^{(1)}(r, \zeta, \lambda) = (1 - \delta_{j1}) Z_j(r, \zeta, \lambda) + \left\{ \frac{\delta_{j1} [1 + Z_j(r_1^*, \zeta_1^*, \lambda_1^*)] - Z_j(r_1^*, \zeta_1^*, \lambda_1^*)}{Z_1(r_1^*, \zeta_1^*, \lambda_1^*)} \right\} Z_1(r, \zeta, \lambda) \quad (5. 14)$$

The coefficients of the $Z_1(r, \zeta, \lambda)$ in this equation can be evaluated from the known expressions for $Z_j(r, \zeta, \lambda)$. Thus, the $Z_j^{(1)}(r, \zeta, \lambda)$ can now be treated as known functions and the same steps repeated, with a second point from the sphere being moved to the nonspherical reference surface.

If the $Z_j^{(h-1)}(r, \zeta, \lambda)$ are the sampling functions after $h - 1$ such steps, then for the hth step the formulas are

$$Z_k^{(h-1)}(r, \zeta, \lambda) = (1 - \delta_{hk}) Z_k^{(h)}(r, \zeta, \lambda) + Z_k^{(h-1)}(r_h^*, \zeta_h^*, \lambda_h^*) Z_h^{(h)}(r, \zeta, \lambda) \quad (5. 15)$$

and

$$Z_j^h(r, \zeta, \lambda) = (1 - \delta_{jh}) Z_j^{(h-1)}(r, \zeta, \lambda) + \left\{ \frac{\delta_{jh} [1 + Z_j^{(h-1)}(r_h^*, \zeta_h^*, \lambda_h^*)] - Z_j^{(h-1)}(r_h^*, \zeta_h^*, \lambda_h^*)}{Z_h^{(h-1)}(r_h^*, \zeta_h^*, \lambda_h^*)} \right\} Z_h^{(h-1)}(r, \zeta, \lambda) \quad (5. 16)$$

In matrix form, the transformation (5. 16) is simply a unit matrix with the h column replaced by the coefficients of the second term on the right side. The successive application of M such transformations carries the functions $Z_k(r, \zeta, \lambda)$ into the functions $Z_k^*(r, \zeta, \lambda)$.

If the nonspherical reference surface were simple enough, the M transformations might be compounded analytically into a single transformation. A substantial simplification can also result from choosing the sampling points on the nonspherical surface such that $\zeta_k = \zeta_k^*$ and $\lambda_k = \lambda_k^*$.

Of course, in choosing the new sampling points, attention must be given to avoiding singular cases of the transformations (5.15) and (5.16). Clearly, one necessary condition is that

$$Z_h^{(h-1)}(r_h^*, \zeta_h^*, \lambda_h^*) \neq 0 \quad .$$

For points P_h^* near P_h , this condition should be easily satisfied, since

$$Z_h^{(h-1)}(r_h, \zeta_h, \lambda_h) = 1 \quad .$$

Similarly, for P_h^* near P_h , the numerator of the second term of equation (5.16) can be expected to be small if $j \neq h$, since

$$Z_j^{(h-1)}(r_h, \zeta_h, \lambda_h) = 0 \quad .$$

Thus, for the case P_h^* near P_h , transformation (5.16) differs only slightly from an identity matrix.

6. APPLICATIONS

6.1 General Considerations for Applications

As noted in Section 1, this formulation of sampling functions started in anticipation of the need to handle satellite-to-ocean altitude measurements. This application has been discussed elsewhere (Lundquist *et al.*, 1969; Lundquist and Giacaglia, 1972b) and is addressed again later in this section. It is a specific example of a larger class of possible geophysical applications.

These applications occur when it is appropriate to aggregate measurements of some geophysical quantity into a set of representative values for localized geographical regions and yet maintain the ability to relate these representative values conveniently to a global representation of the physical quantity. For satellite-to-ocean altitudes, the quantity to be so treated is the geocentric radius to mean sea level.

The sampling-function representation is, of course, formulated such that the tabulated values at the sampling points have a clear relation to a spherical-harmonic global representation. In a function-fitting sense, the dependence of many geophysical quantities on position can be approximated well by a spherical-harmonic expansion, and hence equivalently by a sampling-function expansion. Further, the spherical-harmonic expansion in three dimensions is manifestly appropriate for fields such as the gravity potential of the earth, because the spherical harmonics are solutions of the Laplace equation. Since the sampling functions were required to be linear combinations of the spherical harmonics, they, too, are solutions of the Laplace equation. If this requirement had not been imposed, difficulties could arise when the gravity potential is expanded in functions that are not solutions of this equation.

In the following, each specific application falls naturally into one of two groups, depending on whether a two- or a three-dimensional model is required. Sections 6.2 and 6.3 are concerned with examples of the former group, and Section 6.4, with the latter.

For whatever geophysical quantity is to be represented in a sampling-function expansion, the question naturally arises as to how values of this quantity will be calculated at points other than the sampling points. An important objective of the sampling-function expansion is, of course, to avoid calculating the sum of very many terms of a series.

For the two-variable case of a quantity defined on a sphere or similar reference surface, the simplest procedure for intermediate points is the use of some elementary interpolation formula between the grid of values at the sampling points. For some quantities, there may be a particular interpolation formula that is conventionally used for finding values between tabulated values at a grid of points. If this is the case, the conventional procedure would probably be preferable.

The general behavior of the sampling functions suggests an interpolation formula that can be applied if there is no reason to prefer some other particular formula. Let α_i be the angular distance from the intermediate point to the *i*th sampling point. Then the value of a function $f(\zeta, \lambda)$ can be approximated by

$$f(\zeta, \lambda) = \frac{\sum_{i \in \{\Omega\}} f(\zeta_i, \lambda_i) q_0(\alpha_i)}{\sum_{i \in \{\Omega\}} q_0(\alpha_i)}, \quad (6.1)$$

where

$$\alpha_i = \zeta_i \zeta + \sqrt{1 - \zeta_i} \sqrt{1 - \zeta} \cos(\lambda_i - \lambda)$$

$$q_0(\alpha) = \frac{1}{2N+1} \frac{\sin \left\{ \left[\frac{(2N+1)}{2} \right] \alpha \right\}}{\sin \left[\frac{(1/2)\alpha \right]},$$

from equation (2.31), and $\{\Omega\}$ is the set of sampling points near point (ζ, λ) .

The situation in three dimensions is more involved and should be considered in detail for specific applications.

6.2 The Geoid and Mean Sea Level

The geoid is that equipotential surface for the earth (including in the geopotential a term yielding centrifugal force) that best approximates mean sea level. Functionally, the geoid is specified when the radial distance to it from the center of mass is given as a function of direction – that is, when an expression for $r(\zeta, \lambda)$ is provided. Thus, the representation of the geoid is an example in which the two-variable formulation of sampling functions (Section 4) can be applied.

A convenient representation of the geoid is central to the treatment of satellite-to-ocean altitude measurements, because after appropriate processing, these measurements lead to determinations of the geocentric radius to mean sea level. Thus, a satellite altimeter over the open ocean traces essentially the profile of an equipotential surface to the extent that sea level approximates such a surface. From oceanographic studies, it seems that mean sea level and an equipotential surface differ by at most a few meters. If, however, an altimeter has an accuracy in the decimeter range, it can be expected to provide a representation of mean sea level to be compared with a geoid from some other source.

Over land masses, information about the geoid is obtained from level measurements by surveying techniques and gravity measurements. This information is often compiled into geoid contour maps for the continental areas measured. These maps provide a source of data, complementary to ocean-altitude measurements, for inclusion in a comprehensive geoid representation.

By means of fundamental mathematical theorems, complete knowledge of an equipotential surface enclosing the earth is equivalent to complete specification of the earth's external gravitational field, and conversely. Thus, the geopotential representation obtained from analysis of satellite perturbations also provides information about the geoid – particularly about its longer wavelength features.

In its simplest outline, a procedure for refinement of the geoid representation could progress as follows: First, the best available spherical-harmonic description of the geopotential, presumably from satellite orbit analyses, would be used to

produce a global geoid that could be expressed as an expansion of $r(\zeta, \lambda)$ in surface spherical harmonics. Second, for this geoid, the value of $r(\zeta_i, \lambda_i)$ at the sampling points for some suitably high value of N would be evaluated. It would also be possible to calculate these values directly from the three-dimensional geopotential representation if this were computationally more efficient than the two-step procedure. Third, either geoid-map data for continental areas or altimeter data for ocean areas would be averaged or aggregated appropriately to generate refined geoid radii at the sampling points. Where no new data are available, the radii derived from the geopotential representation would be retained. Fourth, the refined geoid representation in terms of sampling functions could be transformed into the equivalent representation in spherical harmonics. Further, the geoid representation could be transformed into the equivalent geopotential representation. The analytical tools for the use of sampling functions for these four steps were developed in previous sections.

The first and fourth steps involve the auxiliary problem of the relation between the coefficients for an expansion of the geopotential in solid spherical harmonics and those for an expansion of the geoid in surface spherical harmonics. This question has been reviewed by Rapp (1970) and described in some detail by Burša (1968, 1969). Our discussion below follows Burša, who excludes terms of size 10^{-9} and smaller.

Assume that at the surface of the earth

$$U_{\text{grav}} + U_{\text{rot}} = \frac{GM}{R_0} \quad , \quad (6.2)$$

where

$$U_{\text{grav}} = \frac{GM}{r} \sum_{n=0}^N \sum_{m=0}^n \left(\frac{a_e}{r}\right)^n [C_{nm}^* X_{nm}^* + S_{nm}^* Y_{nm}^*]$$

$$U_{\text{rot}} = \frac{1}{2} q \frac{GM}{r} \left(\frac{r}{a_e}\right)^3 (1 - \zeta^2)$$

$$q = \frac{\omega^2 a_e^2}{GM} \quad .$$

The quantities a_e , GM , and ω are assumed constant, and the normalization of the spherical harmonics X^* , Y^* is specified below. We need to adjust R_0 , C_{nm}^* , and S_{nm}^* .

We consider the following expressions for the radius of the geoid:

$$r = R_0 \sum_{p=1}^M \bar{r}_p W_p(\zeta, \lambda) \quad (6.3)$$

or

$$r = R_0 \sum_{p=1}^M \bar{a}_p \bar{X}_p(\zeta, \lambda) = R_0 \sum_{n=0}^N \sum_{m=0}^n (\bar{a}_{nm} \bar{X}_{nm} + \bar{b}_{nm} \bar{Y}_{nm}) \quad , \quad (6.4)$$

where

$$\bar{r}_p = \frac{1}{R_0} r(\zeta_p, \lambda_p)$$

Given \bar{r}_p , we can compute \bar{a}_p ; and given \bar{a}_p , we can compute \bar{r}_p by using the formulas of Section 4.

We note that in the two-index notation,

$$\bar{r}_p \rightarrow \bar{r}_{kj}$$

$$\bar{a}_p \rightarrow \bar{a}_{nm} \text{ or } \bar{b}_{nm}$$

Here, the normalization factor we adopt for $\bar{X}_{nm}, \bar{Y}_{nm}$ (see eq. (2.52)) is

$$\bar{X}_{nm} = \sigma_{nm} P_{nm}(\zeta) \cos m\lambda$$

$$\bar{Y}_{nm} = \sigma_{nm} P_{nm}(\zeta) \sin m\lambda \quad ,$$

where

$$\sigma_{nm}^2 = \frac{2(2n+1)(n-m)!}{\epsilon_m (n+m)!} \quad (6.5)$$

$$\epsilon_0 = 1, \quad \epsilon_m = 2 \quad \text{for } m > 0,$$

and we get

$$\frac{1}{4\pi} \int_0^{2\pi} d\lambda \int_{-1}^{+1} d\zeta \overline{X}_{nm}^2(\zeta, \lambda) = 1.$$

We note that a normalization often used (see, e.g., Lundquist and Veis, 1966, vol. 1) is

$$X_{nm}^* = \omega_{nm} P_{nm}(\zeta) \cos m\lambda$$

$$Y_{nm}^* = \omega_{nm} P_{nm}(\zeta) \sin m\lambda,$$

where

$$\omega_{nm}^2 = \frac{2(2n+1)(n-m)!}{(n+m)!}, \quad (6.6)$$

so that

$$\frac{1}{4\pi} \int_0^{2\pi} d\lambda \int_{-1}^{+1} d\zeta (X_{nm}^*)^2(\zeta, \lambda) = \begin{cases} \frac{1}{2}, & m = 0 \\ 1, & m > 0 \end{cases},$$

an asymmetry we should like to avoid.

We consider first the case in which a set of coefficients $\bar{a}_{nm}, \bar{b}_{nm}$ is given for a geoid representation. As a first step, we express this set in an unnormalized form,

$$a_{nm} = \sigma_{nm} \bar{a}_{nm}$$

$$b_{nm} = \sigma_{nm} \bar{b}_{nm} \quad . \quad (6.7)$$

To calculate C_{nm} , S_{nm} in terms of the a_{nm} , b_{nm} , it is convenient to define the following quantities:

$$A = a_{00} - 1$$

$$B = 1 - a_{00} - a_{20}$$

$$C = 3A - \frac{54}{5}A^2 + \frac{6}{5}B^2 - \frac{3}{5}AB$$

$$D = B - \frac{54}{35}A^2 - \frac{34}{35}B^2 - \frac{18}{35}AB$$

$$\beta = 3A - \frac{6}{5}C^2 + \frac{6}{5}D^2 - \frac{1}{5}CD - \frac{1944}{35}A^3 - \frac{216}{35}A^2B$$

$$\gamma = B - \frac{54}{35}C^2 - \frac{34}{35}D^2 - \frac{18}{105}CD - \frac{432}{35}A^3 - \frac{108}{35}A^2B \quad .$$

(6.8)

Then the parameters in a geopotential representation are

$$R_0 = \left(\frac{GM\beta}{\omega^2} \right)^{1/3}$$

$$J_2 = \frac{\gamma}{a_e^2} R_0^2$$

$$\alpha = \frac{a_e}{R_0}$$

(6.9)

$$J_4 = \frac{1}{\alpha^4} \left(-a_{40} + \frac{6}{35}\beta^2 + \frac{6}{35}\beta\gamma - \frac{36}{35}\gamma^2 + \frac{216}{385}\beta^3 + \frac{192}{385}\beta^2\gamma \right)$$

(eq. (6.9) continued on next page)

$$J_6 = \frac{1}{\alpha} \left(-a_{60} - \frac{8}{77} \beta^3 - \frac{12}{77} \beta^2 \gamma \right)$$

$$J_n = -\frac{1}{\alpha} a_{n0} \quad , \quad n = 3, 5; 7, 8, 9, \dots, N$$

$$C_{nm} = \frac{1}{\alpha} a_{nm} \quad ,$$

$$n = 3, 4, 5, \dots, N; m = 1, 2, 3, \dots, n \quad .$$

$$S_{nm} = \frac{1}{\alpha} b_{nm} \quad ,$$

(6.9)

If desired, these can be expressed in the conventional normalization:

$$C_{nm}^* = \frac{1}{\omega_{nm}} C_{nm}$$

$$S_{nm}^* = \frac{1}{\omega_{nm}} S_{nm} \quad .$$

For the case in which the geopotential coefficients are given and the geoid coefficients must be found, we first express C_{nm}^* , S_{nm}^* in unnormalized form:

$$C_{nm} = \omega_{nm} C_{nm}^* \quad ,$$

$$m \neq 0 \quad .$$

$$S_{nm} = \omega_{nm} S_{nm}^* \quad ,$$

We then define

$$q = \frac{\omega_e^2 a_e^3}{GM}$$

$$\alpha = \frac{a_e}{R_0}$$

$$\beta = \frac{q}{\alpha^3}$$

$$\gamma = \alpha^2 J_2 \quad .$$

(6.10)

The coefficients in the geoid representation are thus given by

$$\begin{aligned}
 a_{00} &= 1 + \frac{1}{3} \beta + \frac{2}{5} \beta^2 + \frac{24}{35} \beta^3 + \frac{1}{15} \beta \gamma + \frac{8}{35} \beta^2 \gamma - \frac{2}{5} \gamma^2 \\
 a_{20} &= -\frac{1}{3} \beta - \frac{4}{7} \beta^2 - \frac{8}{7} \beta^3 - \frac{5}{21} \beta \gamma - \frac{4}{7} \beta^2 \gamma - \gamma - \frac{4}{7} \gamma^2 \\
 a_{40} &= -\alpha^4 J_4 + \frac{6}{35} \beta^2 + \frac{6}{35} \beta \gamma - \frac{36}{35} \gamma^2 + \frac{216}{385} \beta^3 + \frac{192}{385} \beta \gamma \\
 a_{60} &= -\alpha^6 J_6 - \frac{8}{77} \beta^3 - \frac{12}{77} \beta^2 \gamma \\
 a_{10} &= a_{11} = b_{11} = a_{21} = b_{21} = 0 \\
 a_{n0} &= -\alpha^n J_n, \quad n = 3, 5, 7, 8, \dots, N \\
 a_{nm} &= \alpha^n C_{nm}, \\
 b_{nm} &= \alpha^n S_{nm}, \quad n = 2, 3, \dots, N; m = 1, 2, \dots, n
 \end{aligned} \tag{6.11}$$

Finally, these can be normalized in the convention adopted previously:

$$\bar{a}_{nm} = \frac{a_{nm}}{\sigma_{nm}}$$

$$\bar{b}_{nm} = \frac{b_{nm}}{\sigma_{nm}}.$$

Relations (6.9) and (6.11) were obtained by Burša (1969) and do not include terms 10^{-9} or smaller. More precise relations have been derived and will be published in a separate article.

6.3 Topography of the Earth

The topography of the earth is another geophysical quantity for which a mathematical representation is sometimes needed (Lee and Kaula, 1967). This is a further case in which a representation in sampling functions of two arguments can be used.

Several interrelated entities might be considered. The first is the geocentric radius to the mean surface of the solid earth. A second is the elevation of the mean solid surface above or below mean sea level. These two entities are related, of course, through the mean-sea-level representation discussed in Section 6.2.

If topographic maps of land areas and bathymetric maps of the sea floor are the initial sources of data, the mean elevation of the area around each sampling point could be averaged to give the coefficients in a sampling-function representation of the profile relative to mean sea level. These coefficients could then be added to the mean-sea-level coefficients to get the coefficients for the geocentric radius to the mean surface of the solid earth. Either of these representations of topography could easily be transformed into an equivalent spherical-harmonic representation by the formalisms in the previous chapters.

The profile of the solid and ocean surface of the earth might also be desired, for example, as the lower boundary of the atmosphere (see, e.g., Verniani, 1966). In this case, the coefficients for land areas can be taken from the representation of the solid earth; those for the ocean areas, from the representation of mean sea level; and those for coastal areas, from appropriate averaging of the solid-earth and sea-level coefficients. Here, the advantage of the sampling-function representation is evident, because the required coefficients can be selected individually on the basis of the geographical situation in the neighborhood of the sampling points.

6.4 The Geopotential

The gravitational potential around the earth is an important example of a quantity that requires description in functions of three arguments. The formalism of Section 5 was presented with this in mind. Section 6.2 discussed the important relations between

the geoid and the geopotential. If a spherical reference surface of radius R is used, following the formalism of Section 5.1, the potential can be written

$$V = -\frac{\mu}{R} \sum_{n=0}^N \sum_{m=0}^n \left(\frac{R}{r}\right)^{n+1} \left[\bar{C}^{nm} \bar{X}_{nm}(\zeta, \lambda) + \bar{S}^{nm} \bar{Y}_{nm}(\zeta, \lambda) \right] , \quad (6.12)$$

or equivalently as

$$V = -\frac{\mu}{R} \sum_{k=1}^M L^k Z_k(r, \zeta, \lambda) . \quad (6.13)$$

Then the coefficients are related by

$$L^k = \sum_{n=0}^N \sum_{m=0}^n \left[\bar{C}^{nm} \bar{X}_{nm}(\zeta_k, \lambda_k) + \bar{S}^{nm} \bar{Y}_{nm}(\zeta_k, \lambda_k) \right] , \quad (6.14)$$

and inversely,

$$\begin{aligned} \bar{C}^{nm} &= \sum_{k=1}^M L^k A_k^{nm} \\ \bar{S}^{nm} &= \sum_{k=1}^M L^k B_k^{nm} , \end{aligned} \quad (6.15)$$

with the A_k^{nm} and B_k^{nm} as derived in Section 4. Thus, a sampling-function representation can be used effectively whenever it may be a convenient alternative to an expansion in spherical harmonics.

Other alternatives to a spherical-harmonic expansion have been advanced in the geophysical literature, and their relationship to a sampling-function representation

can be examined. Several methods employ adjustable mass distributions on a spherical shell to achieve a detailed representation of the geopotential (e. g., Vinti, 1971; Croopnick, 1970; Koch and Morrison, 1970; Koch and Witte, 1971; Koch, 1972). Normally, the leading inverse-square term, the oblateness term, and perhaps a few other low-degree terms would be represented in the usual spherical-harmonic form, and the surface mass distribution would represent only the remaining short-wavelength detail in the potential.

The basic theoretical idea behind this method is the consideration that a spherical shell of surface mass density

$$\chi(\zeta, \lambda) = \frac{\mu}{4\pi R^2} \sum_{n=0}^{\infty} \sum_{m=0}^n (2n+1) \left[\bar{C}^{nm} \bar{X}_{nm}(\zeta, \lambda) + \bar{S}^{nm} \bar{Y}_{nm}(\zeta, \lambda) \right] \quad (6.16)$$

produces an external potential of the form (6.12), where

$$\begin{aligned} \bar{C}^{nm} &= \frac{4\pi R^2}{(2n+1)\mu} (\chi, \bar{X}_{nm}) \\ \bar{S}^{nm} &= \frac{4\pi R^2}{(2n+1)\mu} (\chi, \bar{Y}_{nm}) \quad , \end{aligned} \quad (6.17)$$

and as usual,

$$4\pi(\chi, \bar{X}_{nm}) = \int_{-1}^1 d\zeta \int_0^{2\pi} \chi(\zeta, \lambda) \bar{X}_{nm}(\zeta, \lambda) d\lambda \quad , \quad (6.18)$$

and similarly for \bar{Y}_{nm} .

If the spherical surface is divided into a grid of areas S_k , the surface integral (6.18) can be written as a sum of integrals over these areas. For simplicity, the density in each area can be assumed to be a constant, χ_k . Then,

$$4\pi(\chi, \bar{X}_{nm}) = \sum_k \chi_k \bar{X}_{nm}(S_k) \quad , \quad (6.19)$$

where

$$\bar{X}_{nm}(S_k) = \int_{S_k} \bar{X}_{nm} dS \quad . \quad (6.20)$$

The integrals $\bar{X}_{nm}(S_k)$ depend only on the spherical harmonics and on the way the sphere is divided into the areas S_k . These integrals are constants that can be easily calculated and tabulated.

Substituting equation (6.19) into equation (6.17) gives the coefficients for the spherical-harmonic expansion of the potential due to the specified mass distribution χ_k . This will be an infinite set of coefficients, which, of course, can be suitably truncated. However, a desired objective is to avoid calculating the spherical harmonics; this can be accomplished by an approximation that replaces each area of surface mass by a centered point mass (Sjogren et al., 1971; Balmino, 1972; Kaplan and Kunciw, 1972; Needham, 1970),

$$m_k = \chi_k S_k \quad . \quad (6.21)$$

The potential of such a distribution of point masses is

$$V = -\mu \sum_k \frac{m_k}{\Delta_k} \quad , \quad (6.22)$$

where

$$\Delta_k^2 = (x - x_k)^2 + (y - y_k)^2 + (z - z_k)^2 \quad ,$$

(x_k, y_k, z_k) being the coordinates of the point masses.

If one should wish to adopt this sort of formalism, use of the sampling functions can still effect some simplifications. The first step is to recognize that the wavelength of the meaningful detail in the potential is limited by the degree at which equation (6. 16) is truncated, by the spacing of the areas S_k , or by the spacing of the point masses. Thus, it is appropriate to pick the degree N corresponding to the desired short-wavelength limit and to work within the linear manifold defined by the spherical harmonics through degree N .

The second step is to expand $\chi(\zeta, \lambda)$ in sampling functions:

$$\chi(\zeta, \lambda) = \sum_{i=1}^M \chi(\zeta_i, \lambda_i) W_i(\zeta, \lambda) \quad . \quad (6. 23)$$

Given a set of coefficients \bar{C}^{nm} , \bar{S}^{nm} , the values of $\chi(\zeta_i, \lambda_i)$ follow simply from the evaluation of equation (6. 16) at the sampling points. Presumably, the values $\chi(\zeta_i, \lambda_i)$ would be subject to an adjustment that makes use of some sort of observational data.

The next step is to find the coefficients \bar{C}^{nm} , \bar{S}^{nm} that correspond to the improved $\chi(\zeta_i, \lambda_i)$. Substituting equation (6. 23) into equation (6. 16) gives

$$\bar{C}^{nm} = \frac{4\pi R^2}{(2n+1)\mu} \sum_{i=1}^M \chi(\zeta_i, \lambda_i) \left(W_i(\zeta, \lambda), \bar{X}_{nm}(\zeta, \lambda) \right) \quad . \quad (6. 24)$$

But

$$\left(W_i(\zeta, \lambda), \bar{X}_{nm}(\zeta, \lambda) \right) = A_i^{nm} \quad , \quad (6. 25)$$

so

$$\bar{C}^{nm} = \frac{4\pi R^2}{(2n+1)\mu} \sum_{i=1}^M \chi(\zeta_i, \lambda_i) A_i^{nm}$$

$$\bar{S}^{nm} = \frac{4\pi R^2}{(2n+1)\mu} \sum_{i=1}^M \chi(\zeta_i, \lambda_i) B_i^{nm} .$$
(6.26)

These are the desired equations for calculating the potential coefficients from the model values of the surface density.

In summary, if it is desired to use a surface-density model for the geopotential, it is necessary to use only the $(N+1)^2$ density values at the defined sampling points as the model parameters. The geopotential coefficients for a spherical-harmonic expansion (or a sampling-function expansion) follow directly from those values.

It is also possible to take the further step to a mass-point representation. The area S_k associated with each sampling point (ζ_k, λ_k) can easily be specified. A mass m_k can hence be associated with the point by

$$m_k = \int_{S_k} \chi(\zeta, \lambda) dS ,$$
(6.27)

or, with lesser accuracy, by

$$m_k = \chi(\zeta_k, \lambda_k) S_k .$$
(6.28)

These can be used in a potential expression of the form of equation (6.22).

6.5 Other Applications

Many other applications of sampling-function formalism could be discussed. Perhaps the representation of vector fields is the most important. Nevertheless, these other applications, in general, parallel the examples of the previous sections and thus will not be treated here. We leave such discussions until they are needed in conjunction with a specific application.

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BIOGRAPHICAL NOTES

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From 1966 to 1970, he was professor of General Mechanics at the University of São Paulo, and from 1970 to 1972, Visiting Professor of Aerospace Engineering at the University of Texas. In 1972, he was appointed Dean of the Institute of Astronomy and Geophysics and Director of the Astronomical Observatory at the University of São Paulo. He is jointly on the scientific staff of the Smithsonian Astrophysical Observatory, specializing in research in celestial mechanics.

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From 1956 to 1960, he was Chief of the Physics and Astrophysics Section, Research Projects Laboratory, Army Ballistic Missile Agency, and from 1960 to 1962, he held concurrent positions as Director of the Supporting Research Office and Chief of the Physics and Astrophysics Branch of the Research Projects Division at the Marshall Space Flight Center.

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NOTICE

This series of Special Reports was instituted under the supervision of Dr. F. L. Whipple, Director of the Astrophysical Observatory of the Smithsonian Institution, shortly after the launching of the first artificial earth satellite on October 4, 1957. Contributions come from the Staff of the Observatory.

First issued to ensure the immediate dissemination of data for satellite tracking, the reports have continued to provide a rapid distribution of catalogs of satellite observations, orbital information, and preliminary results of data analyses prior to formal publication in the appropriate journals. The Reports are also used extensively for the rapid publication of preliminary or special results in other fields of astrophysics.

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