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A SOLUTION OF THE GEODETIC BOUNDARY VALUE PROBLEM TO ORDER e^3

R. S. MATHER

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Geodynamics Branch

Geodynamics Program Division

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*On leave of absence from the University of New South Wales, Sydney, Australia.

GODDARD SPACE FLIGHT CENTER

Greenbelt, Maryland

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by

R. S. Mather*

Geodynamics Branch

SUMMARY

A solution is obtained for the geodetic boundary value problem which defines height anomalies to ±5 cm, if the Earth were rigid. The solution takes into account the existence of the Earth's topography, together with its ellipsoidal shape and atmosphere.

A relation is also established between the commonly used solution of Stokes and a development correct to order e^3. The data requirements call for a complete definition of gravity anomalies at the surface of the Earth and a knowledge of elevation characteristics at all points exterior to the geoid. In addition, spherical harmonic representations must be based on geocentric rather than geodetic latitudes.

No unique solution is possible in theory at the present time due to the nature of the Earth's atmosphere and the limited knowledge of its structure. Practical solutions which are only marginally in error with respect to the estimates of

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accuracy given above, are possible if an adequate model were adopted for the atmosphere.

A quick-look analysis based on statistical considerations of the Earth's gravity field, indicates that a definition which would meet the requirements given above for studies of sea surface topography, is afforded by a global grid with a 10 km spacing in non-mountainous and undisturbed regions, provided such information were

(a) controlled by a global gravity standardization network of ±50 μgal accuracy; and

(b) elevations were based on a correlation of all the major continental datums with errors kept below ±15 cm.

Any predictions that are necessary must be based only on the height correlation characteristics over limited distances.

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A SOLUTION OF THE GEODETIC BOUNDARY VALUE PROBLEM TO ORDER e^3

1. INTRODUCTION

1.1 Preamble

Until recently, it would have appeared rather inconsequential to spend time formulating a solution of the geodetic boundary value problem to order e^3 for a variety of reasons. In the first instance, others have published developments with this end in view (e.g., Zagrebin 1952; Molodenskii et al 1962, p. 53 et seq.; Bjerhammar 1962). Secondly, it seemed highly unlikely that such determinations could ever be put to any practical use. Further, the development of laser tracking systems which promise ranges to objects in near Earth orbit with an internal precision of a few cm, tends to obviate any reason for carrying out the burdensome task implicit in the very accurate solution of the boundary value problem, on the basis of geodetic considerations on continents alone. The role of such solutions is the definition of ellipsoidal elevations, through the height anomaly, and hence geocentric position to a few cm. Accuracy of this type is called for only when studying secular variations in geodetic position which, at the time of writing, should be more conveniently obtained either from the ranges to satellites from a truly global network of tracking stations when adequate systems are operational, or from Very Long Baseline Interferometry (VLBI).

Interest in this problem has been revived by two recent developments. Firstly, the definition of sea surface topography to optimum levels for oceanographic analysis requires a solution of the geodetic boundary value problem to two orders of magnitude better than that afforded by Stokes' integral (e.g., Heiskanen & Moritz 1967, p. 94) alone or one order of magnitude better than those solutions which took into account the effect of the topography (e.g., Molodenskii et al 1962, p. 118; Moritz 1966; Mather 1971b). Secondly recent developments in metrology promise that greater precision may well be achieved in the definition of the Earth's gravity field, enabling the establishment of a global gravity standardization network with an absolute accuracy which is an order better than is possible at present.

These proposed investigations of sea surface topography also have great geodetic significance in view of the commonplace departures of "Mean Sea Level" from an equipotential surface, as defined from the results of geodetic levelling, among other factors. The magnitudes of the stationary departures of sea surface topography, as measured at coastlines, from an equipotential surface, appear to be as large as 2 m along the north-east coastline of Australia (Roelse et al 1971), while discrepancies of a lesser though nevertheless significant magnitude, have been reported in the United States (Sturges 1972). As will be shown in section 4, a preliminary definition of the sea surface topography must precede the evaluation of geopotential differences with respect to the geoid, at points on the surface of the Earth, if an accurate solution of the boundary value problem is to be obtained, free from serious systematic error.

The goal of programs for the mapping of the sea surface topography from space seek the resolution of those characteristics with wave lengths of 200 km to ±10 cm (Weiffenbach 1972). This, in turn, calls for the definition of the equipotential surface corresponding to "Mean Sea Level" to this same order of accuracy over the oceans. The following development, along with all other means for assessing the problem, indicates that the solution of the geodetic boundary value problem is the most promising method available for tackling this problem with the accuracy quoted in the title of this paper on the basis of the technology available at the present time.

While several second order solutions are available, most of these efforts have concentrated on amending the reference surface from a sphere to an ellipsoid of revolution and defining the relevant correction terms. None of the solutions consider the effect of the Earth's atmosphere. Also neglected are certain marginal conditions in the inter-relationship between the gravity anomaly and the disturbing potential which are of significance in defining the height anomaly to ±5 cm (i.e., $o\{e^3 h_d\}$). The equivalent precision required in the definition of the gravity anomaly can be seen from equations 13 to be ±50 μgal. This figure is about four times smaller than the absolute accuracy of any of the stations included in the International Gravity Standardization Net 1971 (Morelli et al. 1971). This however does not imply that the individual values defining the gravity field have to be established with this precision when solving the boundary value problem by quadratures. This is discussed in greater detail in section 4.6.

A further important consideration is the preservation of geocentric characteristics of the gravitational solution. If the solution is not referred directly to the geocenter (Earth's center of mass), it must nevertheless be possible to relate the origin of the coordinate system used, to the geocenter without ambiguity and to the desired accuracy.

The development presented in the following sections, endeavors to define a solution of the geodetic boundary value problem with a resolution of ±5 cm in the height anomaly, taking into account, the effect of the atmosphere and, at the same time, using spherical harmonic expansions only when the function concerned satisfies Laplace's equation to the requisite precision. To emphasize the point, spherical harmonic functions are <u>not</u> used as a convenient three dimensional representation, but only when physically justified. Any exceptions to this rule are carefully qualified. In addition, the development is biased towards relating solutions obtained by the use of Stokes' integral alone, to that which is correct to $o\{e^3\}$. This would imply that only correction terms need be evaluated to completely define the solution if sufficient precision were maintained in the calculation of Stokes' integral. These terms are formulated on the assumption that the solution is iterative, requiring the Stokesian term computation as a pre-requisite for evaluation. This procedure seems difficult to avoid in any solution with pretensions to accuracy, except at the expense of loss of definition in the context of Earth space. Section C of the Appendix shows the equation which needs to be solved if an iterative process is to be avoided.

The development also investigates techniques for optimizing the solution of the integral and the representations required for the global gravity field.

1.2 Notation

The symbols adopted have been designed to minimize confusion. To achieve this end, subscripts have been used to differentiate between quantities which have similar characteristics. Thus the symbol V is used to represent a potential whose magnitude is usually small. V_d is the disturbing potential, while V_a is the potential of the atmosphere. Similarly the symbol h is used to represent ellipsoidal elevation, while h_d is the height anomaly. The subscript d also traces a common thread, namely, quantities which are a consequence of the distortion of the Earth from an ellipsoidal reference model.

1.2.1 Symbols

A = constant associated with azimuth

A_n = surface harmonic of degree n in the spherical harmonic representation of disturbing potential

a = equatorial radius of reference ellipsoid

C_{nm} = surface harmonic $\equiv P_{nm}(\sin\phi_c)\ [C_{1_{nm}} \cos m\lambda + C_{2_{nm}} \sin m\lambda]$

$$c_R = f\left(\frac{1}{3} - \sin^2\phi_c\right) + \frac{h}{R_m} + o\{f^2\} \qquad \cdots\cdots \text{(A-6)}$$

$$c_r = \left(\frac{\Delta R}{r_0}\right)^2 + c_{RP} + c_R + o\{f^2\} \qquad \cdots\cdots \text{(A-14)}$$

$$c_x = \frac{\cos(1/2\,\psi - \theta)}{\cos(1/2\,\psi + \theta + \delta)} - 1 \qquad \cdots\cdots \text{(A-27)}$$

$$c_\Delta = \frac{1 + 2\dfrac{dR}{R}}{1 + c_{\bar{r}}} - 1 \qquad \cdots\cdots \text{(65)}$$

$c_\phi = f + m - 3 f \sin^2 \phi_c$ ...... (A-37)

$dR = h_{max} - h + f \sin^2 \phi_c + o\{f^2 R\}$ ...... (56)

dS = element of surface area at the physical surface of the Earth

$dS' = dS \cos \beta = R^2 \, d\sigma$

dV = element of volume

$d\sigma$ = element of solid angle

$E\{\Delta g\}_n$ = error of representation of gravity anomalies for a $n° \times n°$ square

e = eccentricity of the meridian ellipse $= 2f - f^2$

$F(\psi) = f(\psi) \sin \psi$

f = flattening of the meridian ellipse

$f(\psi)$ = Stokes' function $= \operatorname{cosec} 1/2\,\psi + 1 - 5 \cos \psi - 6 \sin 1/2\,\psi -$

$3 \cos \psi \log [\sin 1/2\,\psi\,(1 + \sin 1/2\,\psi)]$ ..... (82)

G_n = n-th degree surface harmonic in the representation of $\Delta g'$ at the surface of the Earth

g = observed gravity at the surface of the Earth

h = ellipsoidal elevation

h_d = height anomaly

h_n = normal height

h' = orthometric height

K = constant for evaluation of Stokes' integral by quadratures

$= 1.58 \times 10^{-2}$ cm mgal^{-1} (degrees)$^{-2}$

k = gravitational constant

M = mass of the Earth

$M\{X\}$ = global mean value of X

$m = \frac{a\omega^2}{\gamma_e}$

N = elevation of geoid above ellipsoid

R = distance from geocenter to a point at the Earth's surface

$\bar{R}$ = radius of minimum geocentered sphere which encloses the solid earth

R_m = mean radius of the Earth

r = distance from the element of surface area dS to the point of computation P at the Earth's surface

$\bar{r} = 2\bar{R} \sin 1/2\, \psi$ ······(61)

$r_0 = 2R_m \sin 1/2\, \psi$ ······(A-12)

U = spheropotential due to the reference system

U_0 = U on the surface of the reference ellipsoid, which is defined as an equipotential surface

V_a = potential due to the atmosphere

V_d = disturbing potential

$V'_d = V_d - V_a$

W = geopotential

W_0 = potential of the geoid

X_i = geocentric rectangular Cartesian coordinate system $X_1X_2X_3$

x_i = local rectangular Cartesian coordinate system $x_1x_2x_3$ with x_3 axis along local normal, the x_1x_2 plane defining the local horizon and completing the local Laplacian triad

α = azimuth

β = ground slope

β_1 = term of order f in the formula

$$\gamma_0 = \gamma_e \; [1 + \beta_1 \sin^2 \phi_c + \beta_2 \sin^4 \phi_c]$$

for normal gravity

γ = normal gravity

Δg = gravity anomaly at the surface of the Earth

$$\Delta g' = \Delta g_c - 2 \frac{W_0 - U_0}{R} - dR \frac{\partial \Delta g}{\partial R} \qquad \cdots\cdots (83)$$

$$\Delta g_c = \Delta g + 2 \frac{V_d}{R_m} [f + m - 3f \sin^2 \phi_c] + 2 \frac{V_a}{R_m} + \frac{\partial V_a}{\partial h} - \frac{1}{2} g\zeta^2 + dR \frac{\partial \Delta g}{\partial h} + o\{f^2 \Delta g\}$$

$$= \Delta g + o\{10^{-2} \Delta g\}$$

$$\Delta R = R_m [c_{Rp} - c_R] = f(\sin^2 \phi_c - \sin^2 \phi_{cp}) + \frac{h_p - h}{R_m} + o\{f^2\} \qquad \cdots\cdots \text{(A-11)}$$

ΔW = difference in geopotential between the geoid and a point at the Earth's surface

$\delta = \text{f} \sin 2\phi_c \cos \alpha_\sigma + o\{f^2\} \qquad \cdots\cdots$ (A-10)

η = prime vertical component of the deflection of the vertical ($= \xi_2$)

$\theta = \frac{\Delta R}{2R_m} \cot \frac{1}{2} \psi - \delta + o\{f^2\}$ for $\psi > 10°$ $\qquad \cdots\cdots$ (A-26)

λ = longitude, positive east

ξ = meridian component of the deflection of the vertical ($= \xi_1$)

ζ = deflection of the vertical, positive if the vertical lies north, east of the outward normal

ρ = density

$$\Phi = \frac{2R_m (1 + c_R)}{r_0^2 (1 + c_r)} [(1 + c_R) \cos \delta - (1 + c_{Rp}) \cos (\psi + \delta)] - 1 \qquad \cdots\cdots \text{(A-17)}$$

ϕ_c = geocentric latitude, positive north

ψ = angle between geocentric radii to the element of surface area dS and the point of computation P

ω = angular velocity of rotation of the Earth

1.2.2 Conventions

$a = b + o\{b^2\}$ = terms whose order of magnitude are equivalent to or less than b^2 are neglected ($b < 1$)

$x_\alpha y_\alpha = x_1 y_1 + x_2 y_2$

$\frac{x^i}{i!} d^i y = x\,dy + \frac{x^2}{2!} d^2 y + \frac{x^3}{3!} d^3 y + \ldots\ldots\ldots\ldots\ldots$, i taking all possible values

$x_i = a_{ij} b_j \equiv x_i = a_{i1}\ b_1 + a_{i2}\ b_2 + \ldots\ldots$, there being as many equations as possible values of i

$a \underline{\Omega}\, c$ a has the same order of magnitude as c

$a \doteqdot c$ a is approximately equal to c

1.2.3 Subscripts

a = assumed values, usually either astronomical or with reference to a regional geodetic datum

c = geocentric; correction to free air term

d = disturbance value between physical and reference systems

e = equatorial value

g = geodetic values referred to the geocentric ellipsoid

$_m$ = global mean value

$_p$ = evaluated at the point of computation P

$_\sigma$ = evaluated at the element of surface area dS

$\bar{X}$ = value of "x" on the surface of the minimum geocentered sphere which encloses the Solid Earth.

2. BASIC DEFINITIONS

2.1 Gravitational Potential

Gravitational potential is defined as the scalar W such that the acceleration vector g due to the gravitational field is defined by the relation

$$\vec{g} = -\vec{\nabla} W \tag{1}$$

where

$$\vec{\nabla} = \frac{\partial}{\partial X_i} \vec{i} \tag{2}$$

the X_i axis system being a geocentric Cartesian frame whose Earth space location is defined by the unit vectors $\underline{i}$ along the X_i axes.

Along the equipotential surface W = Constant,

$$\frac{dW}{ds} = 0 \tag{3}$$

if s is a linear displacement on the surface. If the latter is defined by the vector $\vec{R}$ given by

$$\vec{R} = X_i \, \vec{i},$$

equation 3 can be expressed as

$$\frac{dW}{ds} = \frac{\partial W}{\partial X_i}\frac{dX_i}{ds} = \vec{\nabla} W \cdot \frac{d\vec{R}}{ds} = 0 \tag{4}$$

As the vector $d\vec{R}$, as shown in figure 1, lies entirely in the equipotential surface, it follows that the vector $\vec{\nabla} W$ which equals -g from equation 1, is normal to the equipotential surface.

The significant conclusion is that the vector g is always normal to the related equipotential surface. The incremental normal displacement is called an increment in orthometric elevation.

2.2 The disturbing potential

The disturbing potential V_{dp} at a point P in Earth space is defined by the relation

$$V_{dp} = W_p - U_p \tag{5}$$

where W is the geopotential due to the rotating Earth and its atmosphere, and U is the spheropotential due to the system of reference which arises from a gravitating ellipsoid of revolution rotating with the same angular velocity as the Earth. This definition implies a rigid Earth and deviations from this model are discussed in section 2.4. The subscript p refers to evaluation at the point P.

The definition of V_d at the Earth's surface is not achieved in circumstances identical with those at satellite altitudes. In the latter case, V_d is determined directly from observations. In such a case, the position of the point P at which

V_d has been defined is therefore known. This is not so at the Earth's surface where the situation is more accurately described in figure 2. The observed quantities specifying the Earth space location of a point P at the Earth's surface are

(a) the assumed latitude ϕ_a and longitude λ_a; and

(b) the difference in geopotential ΔW between the geoid and P as determined by geodetic levelling.

ϕ_a and λ_a can be either observed astronomically or else defined with respect to some regional geodetic datum. The telluroid has been defined as the locus of points $Q(\phi_a, \lambda_a, U_0 + \Delta W)$ on the reference system where the first two coordinates are astronomical values, U_0 being the potential on the surface of the reference ellipsoid (Mather 1968, p. 518). In the context of the geodesy of the 70's, it is more likely that ϕ_a and λ_a are coordinates on the regional geodetic system, which differ from the equivalent values (ϕ_g, λ_g) on a geocentric system by up to 5 arcsec. More about this is section 2.3. The elevation h of P above the ellipsoid is not known, but that of Q is. U_p which is therefore unknown, can be related to the value U_Q at both Q and P', situated at the intersection of the normal through P and the equipotential surface $U = U_Q$, by the Taylor series

$$U_p = U_Q + \frac{h_d^i}{i!} \frac{\partial^i U}{\partial h^i} \tag{6}$$

where h_d is the height anomaly at P (= PP'), measured along the spherop normal at P. The latter deviates from the ellipsoid normal by an angle whose magnitude is of order f^2 (Mather 1971b, p. 80). It follows that the effect of curvature of the normals introduces linear errors into elevation whose maximum order of magnitude is $f^4 \times 10$ km. The linear equivalent is 10^{-2} mm and of no consequence in the present development.

As h_d is therefore normal to the spherop $U = U_Q$, it follows from equation 1 that

$$\frac{\partial U}{\partial h} = -\gamma \tag{7}$$

where γ is normal gravity at $P'(\phi_g, \lambda_g, U_0 + \Delta W)$. It should be noted that

$$\begin{aligned} \gamma &= \gamma_Q - (\phi_a - \phi_g)\, \gamma_e\, \beta_1 \sin 2\phi_c \\ &= \gamma_Q + \Delta\xi\, \gamma_e\, \beta_1 \sin 2\phi_c + 0\{1\ \mu\text{gal}\} \end{aligned} \tag{8}$$

γ_e and β_1 being the relevant terms in the formula for normal gravity (e.g., Heiskanen & Moritz 1967, p. 78), while $\Delta\xi$ is the correction to the meridian component of the deflection of the vertical due to the departure of the regional geodetic datum from a geocentric location (e.g., Mather 1971a, p. 63). Equation 8 is of relevance only if a world geodetic system is not available. As the definition of γ is only required to ± 50 μgal, it would suffice if $\Delta\xi$ were resolved to approximately ± 1 arcsec (as a prerequisite to a complete solution) in such a case.

The second derivative is well known to be

$$\frac{\partial^2 U}{\partial h^2} = -\frac{\partial \gamma}{\partial h} \doteq 0.3 \text{ mgal m}^{-1}.$$

Thus the term obtained in equation 6 when i = 2 has a magnitude of $o\{10^{-3}$ kgal m$\}$ and can be neglected in the present study. Equation 6 can therefore be expressed as

$$U_p = U_0 + \Delta W - \gamma h_d + o\{10^{-3} \text{ kgal m}\} \tag{9}$$

The geopotential W_p at P is unknown because the potential W_0 of the geoid has not been established. Also

$$W_p = W_0 + \Delta W \tag{10}$$

The use of equations 5, 9 and 10 give

$$V_{dp} = W_0 - U_0 + \gamma h_d + 0\{10^{-3} \text{ kgal m}\} \tag{11}$$

In summary, the height anomaly h_d is the linear displacement along the spherop normal, of the geop $W = W_p$ passing through P at the Earth's surface, from the associated spherop $U = U_Q$ which has the same difference of potential with respect to the reference ellipsoid $U = U_0$ as $W = W_p$ has in relation to the geoid $W = W_0$. Thus

$$U_Q - U_0 = W_p - W_0 = \Delta W \cdot$$

If $W_0 = U_0$, $W_p = U_Q$. This cannot however be assumed to be the case at this stage.

Notes

(i) γ is the value of normal gravity at a point on the associated spherop $U = U_Q$. Its relationship to the value of normal gravity at an equivalent point on the reference ellipsoid is defined by equation 96, the required precision being $o\{e^3 \gamma\}$.

(ii) The term $W_0 - U_0$ is indeterminate from gravitational considerations alone. It can be evaluated if a geometrical relation is established independently between the geoid and the ellipsoid. Its magnitude has been estimated at 2.7 kgal m (Mather 1971b, p. 98). Else it can be assumed to be zero on the basis that kM has been determined to $o\{4 \times 10^{12}\ \text{cm}^3\ \text{sec}^{-2}\}$. Present day determinations (e.g., Esposito 1972) claim an accuracy of $500 \times 10^{12}\ \text{cm}^3\ \text{sec}^{-2}$ and hence fall short of the precision required for satisfying this condition at the time of writing.

2.3 The gravity anomaly

The gravity anomaly at the surface of the Earth Δg is defined as the difference between observed gravity g_p at P on the Earth's surface, and situated on the spherop $W = W_p$, and normal gravity γ at the point P' on the associated spherop $U = U_Q$, as shown in figure 2. Thus equation 8 gives

$$\Delta g = g_p - \gamma = g_p - \gamma_Q - \Delta\xi\, \gamma_e\, \beta_1 \sin 2\phi_c$$

or

$$\Delta g = \Delta g_a - \Delta\xi\, \gamma_e\, \beta_1 \sin 2\phi_c \tag{12}$$

where Δg_a is the gravity anomaly calculated using geodetic coordinates referred to the local geodetic datum. $\Delta\xi$ is given by (Mather 1971a, p. 63)

$$\Delta\xi = \frac{1}{\rho + h} [\Delta\xi_0 (\rho_0 + h_0) [\cos\phi_0 \cos\phi + \sin\phi_0 \sin\phi \cos\delta\lambda]$$

$$+ \Delta\eta_0 (\nu_0 + h_0) \sin\phi \sin\delta\lambda -$$

$$- \Delta N_0 [\sin\phi_0 \cos\phi - \cos\phi_0 \sin\phi \cos\delta\lambda]$$

where ρ, ν are the radii of curvature of the reference ellipsoid in the meridian and prime vertical directions, while $\Delta\xi_0$, $\Delta\eta_0$ and ΔN_0 are the corrections to the deflections of the vertical and the ellipsoid elevation at the origin of the regional geodetic datum, on conversion to geocentric values, the subscript 0 referring to values at the origin, and $\delta\lambda = \lambda - \lambda_0$. As V_d is defined as

$$V_d = W - U,$$

differentiation along the spherop normal gives

$$\frac{\partial V_d}{\partial h} = \frac{\partial W}{\partial h} - \frac{\partial U}{\partial h},$$

all values referring to P at the Earth's surface. From equations 1 and 7,

$$-\frac{\partial U}{\partial h} = \gamma_p ,$$

while observed gravity

$$g = -\frac{\partial W}{\partial h'} ,$$

where differentiation in this case is along the local vertical. Small changes in h' are quantities which can be observed, but h' itself, which is the orthometric elevation, is unknown in the absence of knowledge of the stratification of matter exterior to the geoid. Thus

$$\frac{\partial V_d}{\partial h} = -g \cos \zeta + \gamma_p ,$$

where ζ is the deflection of the vertical. γ_p is not a known quantity while γ, as defined in equation 7, is. γ_p can be related to γ by a Taylor's series

$$\gamma_p = \gamma + \frac{h_d^i}{i!} \frac{\partial^i \gamma}{\partial h^i}$$

$\partial^2\gamma/\partial h^2$ is given by (e.g., Heiskanen & Moritz 1967, p. 79)

$$\frac{\partial^2 \gamma}{\partial h^2} \doteqdot \frac{6\gamma}{a^2} = o\{2 \times 10^{-14} \text{ cm}^{-1} \text{ sec}^{-2}\} .$$

The term obtained when i = 2 will have a maximum magnitude for the largest possible value of h_d (= o {10^2 m}), given by

$$\frac{1}{2} h_d^2 \frac{\partial^2 \gamma}{\partial h^2} = o\{1\ \mu gal\}.$$

Thus

$$\frac{\partial V_d}{\partial h} = -g \left[1 - \frac{1}{2}\zeta^2\right] + \gamma + h_d \frac{\partial \gamma}{\partial h} + o\{1\ \mu gal\}$$

$$= -\Delta g + h_d \frac{\partial \gamma}{\partial h} + \frac{1}{2} g \zeta^2 + o\{1\ \mu gal\} \qquad (13)$$

as $\zeta = 0\{3 \times 10^{-4}$ rad$\}$ in mountainous country.

Notes

(i) In estimating magnitudes of quantities, Δg should be assigned $o\{10^2$ mgal$\}$. Thus $e^3 \Delta g = o\{5 \times 10 \mu gal\}$. The contribution of the term $1/2\ g\zeta^2$ holds the same sign at all locations with a maximum magnitude of $o\{5 \times 10\ \mu gal\}$ and must therefore be treated as a systematic effect. It will be retained in all formulae for the present.

2.4 The Boundary Value Condition

The formulation of the boundary value condition is freely available in the literature (e.g., Moritz 1965; Mather 1968). Derivations stem from Green's third identity (e.g., Heiskanen & Moritz 1967, p. 11). If r is the distance of the relevant element of surface area dS or volume dV_i interior to the bounding surface S, from a point P on S, the scalar ϕ satisfies the equation

$$\iiint_{V_i} \frac{1}{r} \vec{\nabla}^2 \phi \, dV_i = -2\pi\phi_p + \iint_S \left\{\frac{1}{r} \vec{\nabla}\cdot\vec{N}\phi - \phi\vec{\nabla}\cdot\vec{N}\frac{1}{r}\right\} dS \qquad (14)$$

where $\vec{N}$ is the unit vector defining the outward normal to S, $\vec{\nabla}$ being defined by equation 2. No approximations are involved in the formulation of equation 14 apart from assumptions implicit in qualifying the existence of the relevant integrals. $\vec{\nabla} \cdot \vec{N}$ is the derivative of the scalar as evaluated along the outward normal and must exist exterior to and on the surface. The geopotential W is given by

$$W = V_e + V_a + V_r \tag{15}$$

where V_e is the attractive potential due to the solid Earth and oceans, hereafter referred to as that of the solid Earth, V_a is the attractive potential due to the atmosphere, and V_r is the rotational potential. As V_e satisfies Laplace's equation at all points exterior to the physical surface of the Earth S, while V_a does likewise at all points in V_i within S. To simplify the application of equation 14 to V_e with V_i exterior to S gives

$$2\pi V_{ep} = - \iint_S \left[\frac{1}{r} \vec{\nabla} \cdot \vec{N} V_e - V_e \vec{\nabla} \cdot \vec{N} \frac{1}{r}\right] dS \tag{16}$$

Similar application to V_r with V_i interior to S gives

$$2\omega^2 \iiint_{V_i} \frac{1}{r} dV_i + 2\pi V_{rp} = \iint_S \left[\frac{1}{r} \vec{\nabla} \cdot \vec{N} V_r - V_r \vec{\nabla} \cdot \vec{N} \frac{1}{r}\right] dS \tag{17}$$

where ω is the angular velocity of rotation of the Earth, which is assumed constant, implying a rigid Earth. It also follows that

$$2\pi(V_{ep} - V_{rp}) = - \iint_S \left[\frac{1}{r}\vec{\nabla}\cdot\vec{N}\,(V_e + V_r) - (V_e + V_r)\,\vec{\nabla}\cdot\vec{N}\,\frac{1}{r}\right] dS$$

$$+ 2\omega^2 \iiint_{V_i} \frac{1}{r}\, dV_i \qquad (18)$$

Similar application of equation 14 to the spheropotential U due to the gravitating reference system which has the same rotation characteristics as the Earth, and defined by

$$U = U_e + V_r \qquad (19)$$

gives

$$2\pi(U_{ep} - V_{rp}) = - \iiint_S \left[\frac{1}{r}\vec{\nabla}\cdot\vec{N}(U_e + V_r) - (U_e + V_r)\,\vec{\nabla}\cdot\vec{N}\,\frac{1}{r}\right] dS$$

$$+ 2\omega^2 \iint_{V_i} \frac{1}{r}\, dV_i \qquad (20)$$

As the integrations in both equations 18 and 20 are taken over the same surface, it follows that appropriate differencing gives

$$2\pi(W_p - U_p - V_a) = - \iint_S \left[\frac{1}{r}\vec{\nabla}\cdot\vec{N}\,(V_e - U_e) - (V_e - U_e)\,\vec{\nabla}\cdot\vec{N}\,\frac{1}{r}\right] dS \qquad (21)$$

on using equations 15 and 19. Further combination with equation 5 gives

$$2\pi(V_{dp} - V_{ap}) = - \iint_S \left[\frac{1}{r}\vec{\nabla}\cdot\vec{N}\,V_d' - V_d'\,\vec{\nabla}\cdot\vec{N}\,\frac{1}{r}\right] dS \qquad (22)$$

where

$$V'_d = V_d - V_a \tag{23}$$

Notes

(i) V'_d is the disturbing potential due to the solid Earth and oceans. It is of importance as it satisfies Laplace's equation at all points exterior to the Earth's surface provided that the ellipsoid lies within the former at all points (i.e., the ellipsoid is smaller than the geoid by at least the maximum negative geoid undulation). This would require in theory, an ellipsoid which is approximately 100 m smaller than that of best fit. Under these conditions

$$\vec{\nabla}^2 V'_d = 0,$$

and hence V'_d can be represented by a solution in spherical harmonics of the form

$$V'_d = \frac{kM}{R} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^n \sum_{m=0}^{n} C_{nm} \tag{24}$$

where

$$C_{nm} = P_{nm}(\sin \phi_c)\ [C1_{nm} \cos m\lambda + C2_{nm} \sin m\lambda] \tag{25}$$

(ii) The harmonic of degree one can be excluded from equation 24. This would imply that the reference ellipsoid were centered at the center of mass of the solid Earth which will not coincide with the geocenter unless the mass distribution of the atmosphere has no first degree harmonic on a geocentric coordinate system. Further, no unique solution of the boundary value problem is possible unless the density distribution of the atmosphere were known. As this is constantly varying, the definition of a model for the atmosphere is called for.

Let G_e and G_a be the centers of mass of the solid Earth and atmosphere respectively, as shown in figure 3. If the X_i axis system is centered on the geocenter G, it is possible to define the coordinates $\overline{X}_{ei}$ of G_e in terms of those ($\overline{X}_{ai}$) of G_a, on the assumption that the model for the atmosphere is capable of formulation from direct measurement more readily than that of the solid Earth.

If dV_e is an element of the volume V_e exterior to the solid Earth which contains the atmosphere with density ρ_a, then

$$M_a \overline{X}_{ai} = \iiint_{V_e} \rho_a X_i \, dV_e \tag{26}$$

where M_a is the total mass of the atmosphere. A similar consideration of the solid Earth, of total mass M_e, contained within the volume V_i, gives

$$M_e \overline{X}_{ei} = \iiint_{V_i} \rho_e X_i \, dV_i \tag{27}$$

ρ_e being the density of matter contained in the element of volume dV_i.

As the X_i axis system has its origin at the geocenter,

$$M \times 0 = \int\int\int_{V_e} \rho_a X_i \, d V_e + \int\int\int_{V_i} \rho_e X_i \, d V_i = M_a \overline{X}_{ai} + M_e \overline{X}_{ei},$$

where M is the mass of the Earth. Thus

$$\overline{X}_{ei} = -\frac{M_a}{M_e} \overline{X}_{ai} \tag{28}$$

In summary, the boundary condition set out in equation 22, is built around the disturbing potential for the solid Earth (V_d') which has the advantage of satisfying Laplace's equation and hence being expressible in spherical harmonics. This representation would not have any terms of degree 1 if the atmospheric potential, referred to a geocentric coordinate system, also had no first degree terms. If this is not the case, as seems likely, the spherical harmonic representation is referred to a coordinate system based on the center of mass of the solid Earth, the relationship to the geocenter being given by equations 26 and 28. This problem will not be considered further.

(iii) The inclusion of equation 17 in equations 18 and 20 implies that ω is a constant independent of position within the surface of the Earth. This is not so in practice due to departures of the Earth from a rigid body, variations in the rate of rotation and polar motion. The first effect is allowed for as the well known correction for Earth tides to observed gravity, with a magnitude o $\{10^2 \mu\,\text{gal}\}$. The gravitational effect of the rotation g_r is given by

$$g_r = p\,\omega^2$$

where

$$p = (X_1^2 + X_2^2)^{1/2},$$

the X_i being defined as in section 1.2.1. The change δg_r in g_r due to polar motion can be interpreted as a consequence of changes dX_i in X_i and $d\omega$ in ω, the relevant relation being

$$\delta g_r = \frac{X_1\, d X_1 + X_2\, d X_2}{p}\,\omega^2 + 2\,p\,\omega\, d\,\omega$$

$$= \left[\frac{X_\alpha\, d X_\alpha}{p^2} + 2\,\frac{d\,\omega}{\omega}\right] g_r .$$

As $d\omega/\omega = o\,\{3 \times 10^{-7}\}$ and $dX/X = o\,\{10^{-6}\}$ at mid latitudes, the effect on $g = o\,\{1\mu\ \text{gal}\}$. The limited magnitude makes it possible for this effect to be neglected for the present development, even though it is dominated by a set of even zonal harmonics.

(iv) The validity for adopting spherical harmonic representations for functions on S is discussed in section B3 of the Appendix.

3. SOLUTION OF THE BOUNDARY EQUATION

Equation 22 can be written as

$$V_d = V_a + \frac{1}{2\pi} \iint_S \left[V_d' \,\vec{\nabla}\cdot\vec{N}\,\frac{1}{r} - \frac{1}{r}\,\vec{\nabla}\cdot\vec{N}\,V_d' \right] dS \tag{29}$$

the surface integral being taken over the physical surface of the Earth. The latter can be represented by the telluroid without introducing errors in excess of $o\{f^2\}$ and hence smaller than the accuracy sought in the present study. On adopting a local x_i axis system at the element dS, with the x_3 axis oriented along the local spherop normal and the x_1 and x_2 axes oriented north and east respectively, it can be shown (Mather 1971b, p. 80) that

$$\underline{N} = \cos\beta\,[-\tan\beta_\alpha\,\vec{\alpha} + \vec{3}] \tag{30}$$

where β is the slope of the topography at dS while β_1 and β_2 are the components of the ground slope in the north and east directions respectively. Thus

$$\vec{\nabla}\cdot\vec{N}\,\frac{1}{r} = \frac{\cos\beta}{r^3}\,[x_\alpha \tan\beta_\alpha - x_3] \tag{31}$$

and

$$\vec{\nabla}\cdot\vec{N}\,V_d' = \cos\beta \left[-\frac{\partial V_d'}{\partial x_\alpha}\tan\beta_\alpha + \frac{\partial V_d'}{\partial x_3} \right] \tag{32}$$

On considering equations 11 and 23,

$$V_d' = V_d - V_a = (W_0 - U_0) - V_a + \gamma h_d + o\,\{5 \times 10^{-2}\ \mathrm{kgal\,m}\} \tag{33}$$

The use of equations 13 and 23 gives

$$\frac{\partial V_d'}{\partial x_3} = \frac{\partial V_d'}{\partial h} = \frac{\partial V_d}{\partial h} - \frac{\partial V_a}{\partial h}$$

$$= - \Delta g + h_d \frac{d\gamma}{dh} + \frac{1}{2} g \zeta^2 - \frac{\partial V_a}{\partial h} + o\{1 \mu\, gal\} \tag{34}$$

The last term in equation 34 is the attraction of the atmosphere. It follows that observed gravity must be numerically increased for the attraction of the atmosphere before computing the gravity anomaly. Assuming a density of 10^{-3} gm cm^{-3} for a 20 km thick layer, treated as a Bouguer plate, the magnitude of this term is

$$\frac{\partial V_a}{\partial h} = o\{10^{-3}\, gal\} .$$

If standard concepts of the nature of the lower atmosphere are accepted, (e.g., Smithsonian Meteorological Tables 1958, p. 267), this correction will be correlated with elevation and is more than likely to have a first degree harmonic as discussed earlier in note (ii) to section 2.4. For a treatment of the atmosphere consisting of a series of nearly ellipsoidal shells, see IAG 1970, p. 62 et seq.

The use of equations 31, 32 and 34 in equation 29 gives

$$V_{dp} = V_{ap} + \frac{1}{2\pi} \iint_S \left[V_d' \left(x_\alpha \tan \beta_\alpha - x_3 \right) \frac{1}{r^3} + \frac{1}{r} \left(\frac{\partial V_d'}{\partial x_\alpha} \tan \beta_\alpha - \frac{\partial V_d'}{\partial h} \right) \right] \cos \beta \, dS \tag{35}$$

Cos β dS is the projection of the element of surface area dS from the telluroid, onto the associated spherop (Mather 1971b, p. 81). The projected surface area is related to the element of solid angle $d\sigma$ by the relations

$$d\,S \cos\beta = R^2\, d\,\sigma = R^2 \cos\phi_c\, d\,\phi_c\, d\,\lambda \tag{36}$$

where (R, ϕ_c, λ) are coordinates on a geocentric spherical system.

The basic equation at 35 can be written without approximation as

$$V_{dp} = V_{ap} + I_A + I_B \tag{37}$$

where

$$I_A = \frac{1}{2\pi} \iint_S \left[\frac{R^2}{r}\left(-\frac{\partial V_d'}{\partial h} - V_d' \frac{x_3}{r^2}\right)\right] d\,\sigma \tag{38}$$

and

$$I_B = \frac{1}{2\pi} \iint_S \left[\frac{R^2}{r}\left(\frac{x_\alpha \tan\beta_\alpha}{r^2} V_d' + \frac{\partial V_d'}{\partial x_\alpha} \tan\beta_\alpha\right)\right] d\,\sigma \tag{39}$$

The integral I_A contains the standard Stokesian term, but masked by

(a) the ellipticity of the meridians;

(b) the undulations of the topography; and

(c) the gravitational effect of the Earth's atmosphere.

1/r can be expressed as the standard zonal harmonic series (e.g., Jeffreys and Jeffreys 1962, p. 634)

$$\frac{1}{r} = \frac{1}{R} \sum_{n=0}^{\infty} \left(\frac{R_p}{R}\right)^n P_{no}\,(\cos\psi) \tag{40}$$

As V_d' satisfies Laplace's equation at all points exterior to and, in the limit, on the Earth's surface, as discussed in section B3 of the Appendix, V_d' can be expressed by the series

$$V_d' = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}}, \quad n \neq 1 \tag{41}$$

The exclusion of the first degree harmonic places the reference ellipsoid at the center of mass of the solid Earth, which does not coincide with the geocenter. The resulting consequences have been explained in section 2.4. It follows that

$$\begin{aligned} \frac{\partial V_d'}{\partial h} &= \frac{\partial V_d'}{\partial R} \sec(\delta\phi - \delta\phi_c) \\ &= -\sum_{n=0}^{\infty} (n+1) \frac{A_n}{R^{n+2}} + o\left\{f^2 \frac{\partial V_d'}{\partial h}\right\} \end{aligned} \tag{42}$$

where $\delta\phi$ and $\delta\phi_c$ are defined in figure A1. The expression for the gravity anomaly Δg in terms of spherical harmonics is obtained from equation 34, on defining the vertical gradient of normal gravity at the surface of the Earth. This can be related to the equivalent value $((\partial\gamma/\partial h)_0)$ at the ellipsoid by the Taylor series

$$\frac{\partial \gamma}{\partial h} = \left(\frac{\partial \gamma}{\partial h}\right)_0 + h \frac{\partial^2 \gamma}{\partial h^2} + \cdots\cdots$$

The first term on the right is well known to be (e.g., Heiskanen and Moritz 1967, p. 293)

$$\left(\frac{\partial \gamma}{\partial h}\right)_0 = -\frac{2\gamma_0}{a}\ (1 + f + m - 2\ f\ \sin^2 \phi_c + o\ \{f^2\}),$$

where all quantities are as defined in section 1.2.1. As γ_0 on the reference ellipsoid is related to γ at the surface of the Earth by the relation

$$\gamma_0 = \gamma\left(1 + 2\frac{h}{a} + o\ \{f^2\}\right),$$

and a is related to R through equations A3 and A4 as

$$a = R_0\ (1 + f\ \sin^2 \phi_c + o\ \{f^2\}) = R\ \left(1 - \frac{h}{a} + f\ \sin^2 \phi_c + o\ \{f^2\}\right),$$

it follows that

$$\frac{\partial \gamma}{\partial h} = -\frac{2\gamma_0}{a}\left(1 + f + m - 3\frac{h}{a} - 2\ f\ \sin^2 \phi_c\right) = -\frac{2\gamma}{R}\ (1 + f + m - 3\ f\ \sin^2 \phi_c + o\ \{f^2\}) \tag{43}$$

as $\partial^2 \gamma/\partial h^2$ is given in section 2.3 as $6\gamma/a^2$.

The combination of equation 43 with equations 33 and 34 gives

$$\Delta g = -\frac{\partial V_d'}{\partial h} - \frac{2}{R}(V_d' + V_a - (W_0 - U_0))(1 + f + m - 3\ f\ \sin^2 \phi_c) + \frac{1}{2}\ g\ \zeta^2 - \frac{\partial V_a}{\partial h}$$

$$= \left(-\frac{\partial V_d'}{\partial h} - \frac{2\ V_d'}{R}\right) - \frac{2\ V_d'}{R}\ (f + m - 3\ f\ \sin^2 \phi_c) + \frac{2}{R}(W_0 - U_0)(1 + f + m - 3\ f\ \sin^2 \phi_c)$$

$$-\left(\frac{2\ V_a}{R} + \frac{\partial V_a}{\partial h}\right) + \frac{1}{2}\ g\ \zeta^2 + o\ \{f^2\ \Delta g\} \tag{44}$$

The contribution of the second set of terms is o $\{100\mu\,\text{gal}\}$ while that of the third is o $\{1\ \text{mgal}\}$ if $(W_0 - U_0)$ is o $\{10\ \text{kgal m}\}$. The terms of $o\{f\}$ in this set can therefore be disregarded without introducing errors in excess of $o\{10^1\mu\,\text{gal}\}$. The use of equations 41 and 42 gives

$$\Delta g = \sum_{n=0}^{\infty} (n-1)\,\frac{A_n}{R^{n+2}} - \frac{2\,V_d'}{R}\,c_\phi + 2\,\frac{W_0 - U_0}{R} - \left(\frac{2\,V_a}{R} + \frac{\partial V_a}{\partial h}\right)$$

$$+ \frac{1}{2}\,g\,\zeta^2 + o\,\{f^2\,\Delta g\}, \qquad n \neq 1 \tag{45}$$

where c_ϕ is given by equation A37.

On using equations A15, 41 and 42, the integral I_A at 38 can be written as

$$I_A = I_{A1} + I_{A2} \tag{46}$$

where

$$I_{A1} = \frac{1}{2\pi}\iint \frac{R^2}{r}\left(\sum_{n=0}^{\infty}(n+1)\,\frac{A_n}{R^{n+2}} - \frac{1}{2R}\sum_{n=0}^{\infty}\frac{A_n}{R^{n+1}}\right) d\sigma, \qquad n \neq 1$$

$$= \frac{1}{2\pi}\iint \frac{R^2}{r}\sum_{n=0}^{\infty}\frac{2n+1}{2}\,\frac{A_n}{R^{n+2}}\,d\sigma, \qquad n \neq 1 \tag{47}$$

and

$$I_{A2} = -\frac{1}{4\pi}\iint \frac{R}{r}\,\Phi\,V_d'\,d\sigma \tag{48}$$

The second integral deals with quantities which are due either the topography or the ellipticity of the meridian. The contribution from this integral is f times smaller than that from I_{A1} for the same region, except when ψ is small and in those instances when $(h_p - h)$ is large in comparison to r. Even in regions of rugged topography situated very close to the point of computation P, the contribution of I_{A2} is at least an order smaller than that of I_{A1} as can be seen from the discussion linking equations A17, A18 and A19. Equation 47 contains the major contribution to I, including that due to the well known integral of Stokes. The spherical harmonic model adopted for V_d' in equation 41 is a necessary intermediate in the combination of the effects of the gravity anomaly and the disturbing potential. This provides an effective technique for obtaining a first approximation for V_d' through Stokes' integral with adequate accuracy, hence permitting the use of an iterative method of solution of the geodetic boundary value problem. For further discussion on the possibility of using non-iterative procedures, see section C of the Appendix.

The conventional procedure due to Stokes cannot be followed when solving equation 47 without introducing approximations due to the following reasons.

1. R varies with $d\sigma$.

2. The spherical harmonic expansion

$$\sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{A_n}{R^{n+2}}, \quad n \neq 1$$

only holds at and exterior to the surface of the Earth and is defined at all points on the latter. In this case, the values at the Earth's surface are defined for limited ranges of R given by $R = R_m + o\,\{f\}$ such that points on it are completely defined by the set $\{\phi_c, \lambda\}$.

The standard Stokesian practice calls for the replacement of the spherical harmonic term by a set of surface harmonics G_n, implying the relation

$$\Delta g' = \sum_{n=0}^{\infty} G_n = \sum_{n=0}^{\infty} \frac{(n-1)\, A_n}{R^{n+2}}, \qquad n \neq 1 \tag{49}$$

This technique which is valid on the surface of a sphere, next equates individual harmonics of degree n by the relation

$$A_n = \frac{R^{n+2}}{n-1} G_n, \qquad n \neq 1 \tag{50}$$

The method breaks down at the physical surface of the Earth as the variations of R from R_m, though of o {f}, are nevertheless functions of the set $\{\phi_c, \lambda\}$. In this case, $\Delta g'$, defined at equation 49, is related to the gravity anomaly through equation 45 as

$$\Delta g' = \sum_{n=0}^{\infty}{}' (n-1) \frac{A_n}{R^{n+2}}, \qquad n \neq 1 = \Delta g + \frac{2\, V_d'}{R} c_\phi - c_{\Delta g} \tag{51}$$

c_ϕ being given by equation A37 and $c_{\Delta g}$ by equation A44. Note that $\Delta g'$ is not defined for all $\{\phi_c, \lambda\}$, given R, unless $R > \bar{R}$, where $\bar{R}$ is the radius of the minimum sphere which is exterior to the solid Earth, with its center at the center of mass of the latter.

It can be concluded that the replacement defined in equation 50 is valid if the surface harmonic expansion of $\Delta g'$ (a) refers to the sphere of radius R, and not the physical surface of the Earth; and (b) does not exist in those regions where R is less than the geocentric radius to the local topography.

Any other interpretation would result in a loss of definition, causing approximation errors of order fh_d, which is unacceptable in terms of the accuracy estimates specified for this paper. As V_d' can be expressed by equation 41 at all points exterior to the physical surface of the Earth, it follows that $\Delta g'$ also exists everywhere in this same region, being defined by the first equality at 51. It also has the characteristic of taking values at the physical surface of the Earth defined by the second equality at 51. These values can be deduced from observations at this surface, where $\Delta g'$ differs from the gravity anomaly Δg by magnitudes which are of order 1 mgal.

Equation 47 can therefore be written as

$$I_{A1} = \frac{1}{2\pi} \iint \frac{R^2}{r} \sum_{n=0}^{\infty} \frac{2n+1}{2(n-1)} G_n \, d\sigma, \qquad n \neq 1 \tag{52}$$

without approximation. It should also be noted that

$$I_{A1} = \frac{1}{2\pi} \iint \frac{R^2}{r} \left(\Delta g' + \frac{3V_d'}{2R} \right) d\sigma \tag{53}$$

again without approximation. The Stokesian procedure calls for the expansion of $1/r$ in a set of zonal harmonics, using equation 40 and the use of the orthogonal properties of surface harmonics when I_{A1} can be transformed to

$$I_{A1} = \frac{1}{4\pi} \int\int R \sum_{n=0}^{\infty} \frac{2n+1}{n-1} \left(\frac{R_p}{R}\right)^n P_{no} (\cos\psi) \, \Delta g' \, d\sigma \tag{54}$$

This manipulation is not possible in the case of equation 52, without introducing errors of order fI_{A1} for the following reasons.

(1) R does not remain constant as $\{\phi_c, \lambda\}$ varies over the Earth's surface.

(2) While surface harmonics retain orthogonal properties on integration over any closed surface as illustrated in section B2 of the Appendix, this does not apply to G_n, which may not exist for certain $\{\phi_c, \lambda\}$ as explained in the discussion following equation 51.

The conditions for the recovery of Stokes' integral I_S, correct to $o\{f^2 I\}$, can be obtained as follows. As $\Delta g'$ and V_d' exist in the domain exterior to the physical surface of the Earth, satisfying equations 51 and 41 respectively, let $\Delta g'$ and V_d' take values $\overline{\Delta g'}$ and $\overline{V_d'}$ on $\bar{R}$, where

$$\bar{R} = a + h_{max} \tag{55}$$

h_{max} being any number marginally larger than the maximum ellipsoidal elevation possible. The displacement dR along the geocentric radius between a point at the Earth's surface and the sphere of radius $\bar{R}$ is given by

$$dR = \bar{R} - R = h_{max} - h + a f \sin^2 \phi_c + o\{f^2 R\} = o\{f R\} \tag{56}$$

$\overline{V_d'}$ and $\overline{\Delta g}$ are related to V_d' and $\Delta g'$ at the surface of the Earth by the Taylor series

$$\overline{\Delta g'} = \Delta g' + \frac{(d R)^i}{i\,!} \frac{d^i (\Delta g')}{d R^i} = \Delta g' + c_g \tag{57}$$

and

$$\overline{V'_d} = V'_d + \frac{(d R)^i}{i\,!} \frac{d^i (V'_d)}{d R^i} = V'_d + c_V \tag{58}$$

The use of equations 57 and 58 in 53 gives

$$I_{A1} = \frac{1}{2\pi}\iint \frac{\overline{R}^2}{\overline{r}} (1 + c_\Delta) \left[\overline{\Delta g'} + \frac{3\overline{V}'_d}{2\overline{R}} - \left(c_g + \frac{3\, c_V}{2\overline{R}} \right) + \frac{3 V'_d}{2\overline{R}} \left(\frac{d R}{\overline{R}} + o\{f^2\} \right) \right] d\sigma \tag{59}$$

where

$$c_\Delta = \frac{R^2}{r} \frac{\overline{r}}{\overline{R}^2} - 1 \tag{60}$$

and

$$\overline{r} = 2\overline{R} \sin \frac{1}{2} \psi \tag{61}$$

Equation 59 can be further partitioned according to the relations

$$I_{A1} = I_S + I_C \tag{62}$$

where

$$I_S = \frac{1}{2\pi}\iint \frac{\overline{R}^2}{\overline{r}} \left(\overline{\Delta g'} + \frac{3\,\overline{V}_d}{2\overline{R}} \right) d\sigma \tag{63}$$

and

$$I_C = \frac{1}{2\pi}\iint \frac{\overline{R}^2}{\overline{r}} \left[c_\Delta \left(\Delta g' + \frac{3 V'_d}{2 R} \right) - \left(c_g + \frac{3\, c_V}{2 R} - \frac{3 V'_d}{2\overline{R}} \frac{d R}{\overline{R}} + o\{f^2\} \right) \right] d\sigma + o\{f\, I_c\} \tag{64}$$

On appreciating the analogy that exists between equations A12 and 61, c_Δ can be shown to be

$$c_\Delta = \frac{1 + 2\frac{d\,R}{R} + o\,\{f^2\}}{(1 + c_{\bar{r}})^{1/2}} - 1 \tag{65}$$

where

$$c_{\bar{r}} = \left(\frac{\Delta\,R}{\bar{r}}\right)^2 - \frac{d\,R + d\,R_p}{R} + o\,\{f^2\} \tag{66}$$

on lines similar to those used in the derivations of equation A14.

The first set of terms on the right of equation 64 contribute $o\,\{fI_s\}$ except when ψ is small and $\Delta\,R$ is dominated by the magnitude of $h_p - h$. In such a case, considerations similar to those expressed in formulating equations A17 to A19 apply.

The second set of terms is also of order $\{f\,\Delta\,g\}$. While this is obvious in the case of the last of the terms in this set, it is also apparent in the case of the other two. The largest contribution to c_g is due to the term obtained in equation 57 when $i = 1$.

$$c_g = d\,R\,\frac{\partial\Delta g'}{\partial R} + o\left\{(dR)^2\,\frac{\partial^2\,(\Delta g)}{\partial R^2}\right\}. \tag{67}$$

where $\partial\Delta g'/\partial R$ is well known to be given by (e.g., Heiskanen and Moritz 1967, p. 117)

$$\frac{\partial\Delta g'}{\partial\,R} = -\,\gamma\left[\sum_{\alpha=1}^{2}\frac{\partial\,\xi_\alpha}{\partial\,x_\alpha} - \frac{\xi\,\tan\phi_c}{R} - \frac{2\,h_d}{R^2} + o\left\{f\,\frac{\partial\,\Delta\,g}{\partial\,R}\right\}\right] \tag{68}$$

The magnitude of this term is $o\{10^{-9}\ \text{gal cm}^{-1}\}$, giving $c_g = o\{1\ \text{mgal}\}$ if $\partial \xi_\alpha / \partial x_\alpha = o\{1\ \text{arcsec}\ (10^2\ \text{km})^{-1}\}$. In disturbed areas, this could be one order of magnitude larger. In all circumstances, terms involving c_g need only be evaluated to the order of the flattening, to meet the accuracy requirements called for in the present development.

Similarly, c_V is given by

$$c_V = d R \frac{\partial V_d'}{\partial h} + o\left\{ (d R)^2 \frac{\partial^2 V_d'}{\partial h^2} \right\} \tag{69}$$

$(\partial V_d' / \partial h)$ is given by equation 34 and the effect of the term containing c_V is f times smaller than the contribution of $(\partial V_d' / \partial h)$ in the Stokesian term. Equation 64 can therefore be written as

$$I_C = \frac{1}{2\pi} \iint \frac{\overline{R}^2}{\overline{r}} \left(\Delta g' \left(c_\Delta + \frac{3}{2} \frac{d R}{R} \right) + \frac{3 V_d'}{2 R} \left(c_\Delta + 3 \frac{d R}{R} \right) - d R \frac{\partial \Delta g}{\partial R} + o\{f^2 \Delta g'\} \right) d\sigma \tag{70}$$

The solution of equation 63 is well known but will be traced out here for completeness. From equations 41 and 51, I_S can be written as

$$I_S = \frac{1}{2\pi} \iint \frac{\overline{R}^2}{\overline{r}} \sum_{n=0} \frac{2n+1}{2} \frac{A_n}{\overline{R}^{n+2}} \qquad n \neq 1 \tag{71}$$

As

$$\overline{\Delta g'} = (\Delta g')_{R=\overline{R}} = \sum_{n=0}^{\infty} (n-1) \frac{A_n}{\overline{R}^{n+2}} = \sum_{n=0}^{\infty} \overline{G}_n, \qquad n \neq 1 \tag{72}$$

the replacement

$$A_n = \frac{\bar{R}^{n+2}}{n-1} \bar{G}_n \tag{73}$$

is valid and I_S can be written as

$$I_S = -\frac{\bar{R}}{4\pi} \bar{G}_0 \int_0^{2\pi} \int_0^{\pi} \frac{2 \sin \frac{1}{2} \psi \cos \frac{1}{2} \psi \, d\psi \, d\alpha}{2 \sin \frac{1}{2} \psi} + \frac{\bar{R}^2}{4\pi} \iint \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \bar{G}_n$$

$$\times \frac{1}{\bar{R}} \sum_{i=0}^{\infty} \left(\frac{R_p}{\bar{R}} \right)^i P_{i0} (\cos \psi) \, d\sigma \tag{74}$$

on using equation 40, R_p being equal to $\bar{R}$ from the definition of $\bar{r}$ at equation 61. As $\bar{R}$, R_p and $\bar{G}_n$ exist over the range of integration at all points on the surface of the sphere of radius $\bar{R}$, the use of the orthogonal properties of surface harmonics gives

$$I_S = -\bar{R} \, (M \{\Delta g'\})_{R=\bar{R}} + \frac{\bar{R}}{4\pi} \iint f(\psi) \, \overline{\Delta g'} \, d\sigma \tag{75}$$

where

$$f(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R_p}{R} \right)^n P_{n0} (\cos \psi) \tag{76}$$

$\overline{\Delta g'}$ is related to the gravity anomaly Δg through equations 51, 57 and 67 as

$$\overline{\Delta g'} = \Delta g + \frac{2 V_d'}{R} c_\phi - c_{\Delta g} + d R \frac{\partial \Delta g}{\partial R} + o \{f^2 \Delta g\} \tag{77}$$

$$= \Delta g + o \{1 \text{ mgal}\} \text{ in undisturbed regions}$$

The Stokesian manipulation is well known to be obtained as follows

$$f(\psi) = t_1 + t_2 \tag{78}$$

where

$$t_1 = 2\sum_{n=2}^{\infty}\left(\frac{R_p}{R}\right)^n P_{n0}(\cos\psi) = 2\,\bar{R}\left(\frac{1}{\bar{r}} - \frac{1}{\bar{R}} - \frac{R_p}{\bar{R}^2}\cos\psi\right) \tag{79}$$

and

$$t_2 = 3\sum_{n=2}^{\infty}\frac{1}{n-1}\left(\frac{R_p}{R}\right)^n P_{n0}(\cos\psi) = \frac{3}{\bar{R}}\int_R^{\infty}\sum_{n=2}^{\infty}\left(\frac{R_p}{\bar{R}}\right)^n P_{n0}(\cos\psi)\,d\bar{R}$$

$$= \frac{3}{\bar{R}}\int_R^{\infty}\bar{R}\left(\frac{1}{\bar{r}} - \frac{1}{\bar{R}} - \frac{R_p}{\bar{R}^2}\cos\psi\right)d\bar{R} \tag{80}$$

on using equation 40. As

$$\int_R^{\infty}\frac{\bar{R}\,d\bar{R}}{\bar{r}} = \int_R^{\infty}\frac{\bar{R} - R_p\cos\psi}{\bar{r}}\,d\bar{R} + R_p\cos\psi\int_R^{\infty}\frac{\bar{r} + \bar{R} - R_p\cos\psi}{\bar{r}\,(\bar{r} + \bar{R} - R_p\cos\psi)}\,d\bar{R}$$

$$= [\bar{r}]_R^{\infty} + R_p\cos\psi\int_R^{\infty}\frac{d\bar{R} + d\bar{r}}{\bar{r} + \bar{R} - R_p\cos\psi}$$

$$= [\bar{r} + R_p\cos\psi\log(\bar{r} + \bar{R} - R_p\cos\psi)]_{\bar{R}}^{\infty},$$

$$t_2 = \frac{3}{\bar{R}}\left[(\bar{r} - \bar{R}) + R_p\cos\psi\log\frac{\bar{r} + \bar{R} - R_p\cos\psi}{\bar{R}}\right]_{\bar{R}}^{\infty}$$

Further,

$$\lim_{\bar{R}\to\infty} (\bar{r} - \bar{R}) = \lim_{\bar{R}\to\infty} \left[\bar{R} \left(1 + \left(\frac{R_p}{\bar{R}}\right)^2 - 2\frac{R_p}{\bar{R}} \cos\psi \right)^{1/2} - \bar{R} \right] = - R_p \cos\psi,$$

and

$$\lim_{\bar{R}\to\infty} \left[\log \frac{\bar{r} + \bar{R} - R_p \cos\psi}{\bar{R}} \right] = \log 2.$$

Thus

$$t_2 = \frac{3}{\bar{R}} \left[\bar{R} - R_p \cos\psi - \bar{r} - R_p \cos\psi \log \left(\frac{\bar{r} + \bar{R} - R_p \cos\psi}{2\bar{R}} \right) \right] \tag{81}$$

and $f(\psi)$ is obtained from equations 78, 79 and 81 as

$$\begin{aligned} f(\psi) &= 2\bar{R} \left(\frac{1}{\bar{r}} - \frac{1}{\bar{R}} - \frac{R_p}{\bar{R}^2} \cos\psi \right) + \frac{3}{\bar{R}} \left(\bar{R} - R_p \cos\psi - \bar{r} - R_p \cos\psi \log \left[\frac{\bar{r} + \bar{R} - R_p \cos\psi}{2\bar{R}} \right] \right) \\ &= \operatorname{cosec} \frac{1}{2}\psi + 1 - 5\cos\psi - 6\sin\frac{1}{2}\psi - 3\cos\psi \log \left(\sin\frac{1}{2}\psi \left[1 + \sin\frac{1}{2}\psi \right] \right) \end{aligned} \tag{82}$$

as $R_p = \bar{R}$ and $\bar{r}$ is given by equation 61. As $W_0 - U_0$ is unknown, it is preferable to separate it from $\Delta g'$ before computations. This is convenient as Stokes' function is insensitive to zero degree effects. On defining Δg_c as

$$\Delta g_c = \overline{\Delta g'} - 2\frac{W_0 - U_0}{\bar{R}} \tag{83}$$

where $\overline{\Delta g'}$ is given by equation 77, I_S can be written from equation 75 as

$$I_S = 2\,(W_0 - U_0) - \bar{R}\,M\,\{\Delta\,g_c\} + \frac{\bar{R}}{4\,\pi}\iint f\,(\psi)\,\Delta\,g_c\,d\,\sigma \tag{84}$$

without introducing any approximations in the surface integral term, where

$$\Delta\,g_c = \Delta\,g + \frac{2\,V_d'}{R}\,c_\phi - c_{\Delta g} + d\,R\,\frac{\partial\,\Delta\,g}{\partial\,R} + o\,\{f^2\,\Delta\,g\} \tag{85}$$

from equation 77, c_ϕ being given by equation A37 while $c_{\Delta g}$ is obtained from equation A44 as

$$c_{\Delta g} = \frac{1}{2}\,g\,\zeta^2 - 2\,\frac{V_a}{R} - \frac{\partial\,V_a}{\partial\,h} + o\,\{f^2\,\Delta\,g\} \tag{86}$$

Notes:

(1) $\Delta\,g_c$ is approximately equal to the gravity anomaly Δg. The second and third terms in equation 85 are both of order $f\Delta g$ and hence do not have magnitudes in excess of 300 μ gal. The magnitude of the final term depends on the variability of the Earth's gravitational field in the locality. On the average, its magnitude is of order 1 mgal, though it could be one order larger in regions of rapid change. As this term cannot be evaluated until the terms $(\partial\,\xi_a/\partial x_a)$ are known, the contribution of the Stokesian term is therefore determined by iteration. The solution need be iterated only once and a convenient set of formulae for this purpose is given in section 4.1.

(2) No difficulty should be encountered in computing the atmospheric corrections whose magnitudes are controlled by the model adopted for the Earth's atmosphere. This correction, which approaches 1 mgal (IAG 1970, p. 62) should be applied as part of the routine when computing the gravity anomaly.

(3) It would not be adequate to use the free air reduction (e.g., 0.3086 mgal m^{-1}) in computing the gravity anomaly. Instead, the relation

$$\gamma = \gamma_0 - \frac{2\gamma_0 h_n}{a}\left[1 + f + m - \frac{3}{2}\frac{h}{R}h_n - 2 f \sin^2\phi_c + o\{f^2\}\right] \tag{87}$$

should be utilized when computing normal gravity at the associated spherop. The quantity h_n, called the normal height, is obtained from the observed differences in geopotential ΔW using the equation (e.g., Heiskanen and Moritz 1967, p. 171)

$$h_n = \frac{\Delta W}{\gamma_0}\left[1 + (1 + f + m - 2 f \sin^2\phi_c)\frac{\Delta W}{a\gamma_0} + \left(\frac{\Delta W}{a\gamma_0}\right)^2 + o\{f^3\}\right] \tag{88}$$

ΔW having the same significance as in section 2.4.

(4) The first iteration for h_d will be obtained from Stokes' integral as before. This contribution, equivalent in significance to that provided by the free air geoid in present day solutions to the order of the flattening, need be calculated only once if (a) the gravity anomaly Δg, computed to $o\{e^3 \Delta g\}$ using equations 87 and 88, were corrected for the atmospheric

effects prior to evaluation; and (b) the radius of the sphere were taken as $\bar{R}$ and not R_m.

(5) The initial iteration should also include the evaluation of the components of the deflection of the vertical, using the Vening Meinesz integrals (e.g., ibid, p. 111). The second iteration need only be the correction terms which are more conveniently included elsewhere, as shown in section 4.1, as the order of magnitude does not exceed 0.3 kgal m.

(6) It is tempting to introduce a function of the type

$$V_d'(1 - \delta) = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}}, \qquad n \neq 1$$

in an attempt to combine the effects of I_{A1} and I_{A2} to give an integral of the form

$$I_A = \frac{R_m}{4\pi} \iint f(\psi)\, \Delta g_a \, d\sigma,$$

where

$$\Delta g_a = \Delta g\,(1 - \delta').$$

This technique is not unfamiliar in the literature but is not used in the present development for two reasons.

(1) There is no physical validity for defining V_d' $(1 - \delta)$ as a spherical harmonic series unless V_d' $(1 - \delta)$ satisfies Laplace's equation. This cannot however be proved to be the case at the physical surface

of the Earth where the height anomalies are to be defined. Further δ is a function of both the topography and the Earth's ellipticity. It has no definition except at the surface of the Earth. Thus while δ can be completely defined by a set of surface harmonics, it is invalid to equate it to a set of spherical harmonics. The differentiation between the two cases is important as the definition of the gravity anomaly from the disturbing potential is implicitly based on the existence of radial derivatives of the latter. This follows only in circumstances where the spherical harmonic representation has a physical basis.

(2) Difficulties are posed in interpreting the location of the center of the reference ellipsoid as the harmonic $n = 1$ is inadmissible in the solution. Also see section 4.2.

4. SOLUTION SUMMARY AND DISCUSSION

4.1 Equations for a Solution to Order e^3

From the development in section 3, the height anomaly h_d is obtained from equations 11, 37, 38, 48, 70 and 84 as

$$h_{dp} = N_{fp} + N_{cp} \tag{89}$$

where

$$N_{fp} = \frac{W_0 - U_0}{\gamma_p} - \overline{R}\,\frac{M\{\Delta g_c\}}{\gamma_p} + \frac{\overline{R}}{4\pi\,\gamma_p}\iint f(\psi)\,\Delta g_c\,d\sigma \tag{90}$$

and

$$N_{cp} = \frac{V_a}{\gamma_p} + \frac{1}{2\pi\gamma_p}\iint \frac{R^2}{r}\left[\frac{\partial V'_d}{\partial x_\alpha}\tan\beta_\alpha + V'_d\left(\frac{x_\alpha\tan\beta_\alpha}{r^2} + \frac{1}{2R}\left[3\left(c_\Delta + 3\frac{dR}{R}\right) - \Phi\right]\right)\right.$$

$$\left. - dR\frac{\partial\Delta_g}{\partial h} + \Delta g'\left(c_\Delta + \frac{3}{2}\frac{dR}{R}\right) + 0\{e^3\,\Delta g\}\right] d\sigma \tag{91}$$

Certain terms have been adjusted in equation 91 on the understanding that $N_{cp} = o\,\{10^{-1}\ N_{fp}\}$, thereby ignoring effects of order fR in its formulation. The constituent terms in equations 89 to 91 are defined by the following relations, the equation numbers referring to their identification in the text.

$$V'_d = V_d - V_a \tag{33}$$

while

$$\Delta g_c = \Delta g_1 + \Delta g_2 \tag{92}$$

where the use of equations 85 and 86 gives

$$\Delta g_1 = \Delta g + \frac{\partial V_a}{\partial h} + 2\frac{V_a}{R_m} \tag{93}$$

and

$$\Delta g_2 = \frac{2V'_d}{R_m}c_\phi - \frac{1}{2}g\zeta^2 + dR\frac{\partial\Delta g}{\partial h} + o\{e^3\,\Delta g\} \tag{94}$$

Also

$$\frac{\partial\Delta g}{\partial h} = -\gamma\left[\sum_{\alpha=1}^{2}\frac{\partial\xi_\alpha}{\partial x_\alpha} - \frac{\xi_1\tan\phi_c}{R_m} - 2\frac{N_f}{R_m^2} + o\left\{f\frac{\partial\Delta g}{\partial h}\right\}\right] \tag{67}$$

and

$$c_\phi = f + m - 3f \sin^2 \phi_c + o\{f^2\} \tag{A-37}$$

m being defined in section 1.2.1. The angle ψ is obtained from

$$\psi = \cos [\sin \phi_c \sin \phi_{cp} + \cos \phi_c \cos \phi_{cp} \cos d\lambda] \tag{A-8}$$

where

$$d\lambda = \lambda - \lambda_p \tag{A-7}$$

The geocentric distance R is related to the mean radius of the Earth R_m and that of the minimum sphere enclosing all topography (radius $\bar{R}$) by the equations

$$R = R_m (1 + c_R) \tag{A-5}$$

where

$$c_R = \frac{h}{R_m} + f \left(\frac{1}{3} - \sin^2 \phi_c\right) + o\{f^2\} \tag{A-6}$$

and

$$\bar{R} = R + dR$$

where

$$dR = h_{max} - h + f \sin^2 \phi_c + o\{f^2 R\} \tag{56}$$

Also

$$r = r_0 (1 + c_r)^{1/2} \tag{A-13}$$

where

$$r_0 = 2R_m \sin \frac{1}{2} \psi \tag{A-12}$$

and

$$c_r = \left(\frac{\Delta R}{r_0}\right)^2 + c_{Rp} + c_R + o\{f^2\} \tag{A-14}$$

In addition,

$$\bar{r} = 2\bar{R}\sin\frac{1}{2}\psi \tag{61}$$

while

$$c_\Delta = \frac{1 + 2\dfrac{dR}{R}}{(1 + c_{\bar{r}})^{1/2}} - 1 \tag{65}$$

$$c_{\bar{r}} = \left(\frac{\Delta R}{\bar{r}}\right)^2 - \frac{dR + dR_p}{R_m} + o\{f^2\} \tag{66}$$

and

$$\Delta R = c_{Rp} - c_R \tag{A-11}$$

The other quantities requiring definition are

$$\Phi = \frac{2R}{r^2}\left[R - R_p\cos(\psi + \delta)\right] - 1 \tag{A-17}$$

$$\frac{\partial V_d'}{\partial x_\alpha}\tan\beta_\alpha = -\gamma\xi_\alpha\tan\beta_\alpha + N_f\frac{\partial\gamma}{\partial x_1}\tan\beta_1 + o\{e^2\Delta g\} \tag{A-32}$$

$$\frac{x_\alpha}{r^2}\tan\beta_\alpha = \frac{R}{r^2}(1 + c_x)\sin\psi\frac{dh}{dr} \tag{A-29}$$

where

$$c_x = \frac{\cos\left(\dfrac{1}{2}\psi - \theta\right)}{\cos\left(\dfrac{1}{2}\psi + \theta + \delta\right)} - 1 \tag{A-27}$$

$$\theta = \frac{\Delta R}{2R_m} \cot \frac{1}{2} \psi - \delta + o\{f^2\} \quad \text{if} \quad \psi > 5° \tag{A-26}$$

$$\delta = f \sin 2\phi_c \cos \alpha_\sigma \tag{A-10}$$

and

$$\frac{dh}{dr} = \cos \alpha_\sigma \tan \beta_1 + \sin \alpha_\sigma \tan \beta_2 \tag{95}$$

The quantity Δg in equation 93 is given by the relations

$$\Delta g = g - \gamma,$$

where g is the value of gravity observed at the surface of the Earth, while γ is defined by

$$\gamma = \gamma_0 + \Delta\gamma,$$

where γ_0 is normal gravity on the reference ellipsoid and

$$\Delta\gamma = -\frac{2\gamma_0 h_n}{a}\left[1 + f + m - \frac{3h_n}{2a} - 2f \sin^2 \phi_c + o\{f^2\}\right] \tag{87}$$

the normal height h_n being obtained from the difference ΔW in geopotential between the geoid and the point at the Earth's surface by the relation

$$h_n = \frac{\Delta W}{\gamma_0}\left[1 + \frac{\Delta W}{a\gamma_0}\left(1 + f + m + \frac{\Delta W}{a\gamma_0} - 2f \sin^2 \phi_c\right)\right] + o\{f^3\} \tag{88}$$

Alternately,

$$\gamma = \gamma_0 - \frac{2\Delta W}{a}\left[1 + f + m - \frac{1}{2}\frac{\Delta W}{a\gamma_0} - 2f\sin^2\phi_c + o\{f^2\}\right] \tag{96}$$

Notes:

(i) A rigorous solution is obtained only if the reference ellipsoid always lies within the physical surface of the Earth. Such a figure is smaller than the figure of best fit by approximately 100 m. If the values of normal gravity were then based on this figure plus an independently determined value of kM, all gravity anomalies will be too small by $o\{2 \times 10^1 \text{ mgal}\}$. The linear effect in N_{fp} is contained entirely in the first two terms of equation 90 as Stokes' integral is insensitive to effects of zero degree.

(ii) $\Delta g'$ is defined by equation 51. In the context of the note to equation 91, it would suffice if $\Delta g'$ were taken to be equal to the gravity anomaly Δg for purposes of evaluation to order $e^3 h_d$.

4.2 Procedure for obtaining a numerical solution

The equations summarized in section 4.1 completely define the solution of the geodetic boundary value problem to the order of the cube of the eccentricity. The form of these equations and the discussion in section C of the Appendix indicate that a non-iterative approach to the solution is not possible as the evaluation of N_c at equation 91 requires a knowledge of V_d' which is obtained from h_d using equation 33, and the components ξ_a of the deflection of the vertical ζ.

It is well known that N_f contributes over 90% of the magnitude of h_d (e.g. Mather 1971b, p. 89). This is equivalent to the free air geoid in solutions to the order of the flattening. In determinations to order $e^3 h_d$, the same contribution is obtained by the use of Δg_1, defined by equation 93, in Stokes' integral, as expressed at 90. Let the numerical value so obtained be N_{f1} while the value deduced from equation 33 for V'_d be V_{d1}.

The only other contributions with magnitude greater than fh_d arise from the terms at A17 and A32, the former being of significance only when large topographical undulations occur near the point of computation (ibid, p. 86). As

$$V'_d = V_{d1} + o\{10^{-1}\, V'_d\},$$

the use of V_{d1} in lieu of V'_d when computing these topographical corrections will result in a value N_{c1} for N_c which is correct to $o\,\{10^{-1}\, N_c\}$ (i.e., to ±1m). Let

$$V_{d2} = V_{d1} + \frac{1}{\gamma} N_{c1} = V'_d + o\,\{10^{-2}\, V'_d\}$$

The computation of Δg_2, defined by equation 94, using V_{d2} and the equivalent values of ξ_α (ibid, p. 88) and its use in equation 90 will give the balance contribution to h_d from the expression for N_f, the magnitude being estimated at fh_d, though it could be as large as $10^{-2}\, h_d$ in mountainous regions. If this magnitude is N_{f2}, define

$$V_{d3} = V_{d2} + \frac{1}{\gamma} N_{f2} = V'_d + o\,\{10^{-2}\, V'_d\}.$$

The use of either V_{d2} or V_{d3} or in lieu of V_d' when evaluating equation 91 will result in value N_{e2} which is correct to $o\{10^{-2} N_c\}$ (i.e., to ±10 cm). Defining

$$V_{d4} = V_{d3} + \frac{1}{\gamma} N_{c2} = V_d' + o\{e^3 V_d'\},$$

equation 91 should be iterated a third time using V_{d4}' for V_d' to give the final value of N_{c3} for N_c. Hence

$$h_d = N_{f1} + N_{f2} + N_{c3} + o\{e^3 h_d\} \tag{97}$$

These evaluations must be completed on a global basis. No solutions of the geodetic boundary value problem to order $e^3 h_d$ can therefore be obtained from data restricted to a local region.

Notes:

(1) A complete solution requires the evaluation of N_c to be iterated three times. As pointed out in section C of the Appendix, it is not possible to avoid the iterative procedure. Considerable economy could be effected if the number of iterations could be reduced by obtaining a more accurate estimate of V_d' after the first iteration. Unfortunately this cannot be achieved by the analysis of the orbital perturbations of near Earth satellites as results obtained to date indicate a lack of sensitivity to the topographical effects.

(2) As pointed out in the introduction, the principal need for an accurate geoid solution is in the study of sea surface topography. A resolution to ±10 cm can be obtained by just 2 iterations of equation 91.

(3) It has been assumed that $\partial^2 \Delta g / \partial h^2$ has a negligible magnitude. This would be a reasonable assumption over oceanic areas, but may be a limitation in mountainous and gravitationally disturbed regions. Such an effect is of consequence only if it holds the same sign over considerable extents as discussed in section 4.3. It would not be unreasonable to assume that the net effect is negligible for studies of the sea surface topography.

(4) The magnitude of $\partial \xi_\alpha / \partial x_\alpha$ has been assumed to be of order ±1 arcsec $(10^2\ \text{km})^{-1}$, when the contribution to Δg_2 is of order 5×10^{-1} mgal. This magnitude can be considered to be an average value (e.g., Mather, Barlow & Fryer 1971, figs. 3.2 and 3.3) though it could be a factor of 10 greater. In such disturbed regions, which are characterized by short wavelengths in ξ_α, both positive and negative values are equally likely. The overall effect is therefore small unless the disturbed regions lie close to the point of computation. It should also be noted that such regions invariably occur in areas of rugged topography. On the other hand, the Australian data referred to above indicates a significant number of these disturbed regions are not correlated with any topographical feature.

(5) The evaluations of N_{f1} and N_{f2} should be based on algorithms seeking a precision of 5 parts in 10^4 in the final values.

4.3 The Representation of the Gravity Field

It must be established that the global gravity field is capable of definition with adequate precision to afford the determination of h_d to $o\{e^3 h_d\}$. There are, in general, two techniques available for this purpose. The first is the determination of gravity anomalies at the surface of the Earth by direct determinations of g. The second is the determination of the disturbing potential V_d from the analysis of the orbital perturbations of near Earth satellites. Accuracies attained at the present time in the determination of g are controlled by the global gravity standardization network. It is expected that all gravity holdings will be converted in the near future to values referred to the International Gravity Standardization Network (IGSN 1971) whose absolute accuracy is estimated at ±0.2 mgal (Morelli et al 1971, p. 5). This figure is a factor of 4 inferior to the ±50 μ gal figure implicit in the formulae listed in section 4.1. Individual gravity ties to stations in IGSN 1971 can be made to ±0.1 mgal. This figure will be shown to be acceptable if the density of stations in the gravity standardization net is sufficiently high.

To investigate this further, it is necessary to analyze the computational procedures adopted in evaluating the major contribution called N_{f1} in section 4.2. For simplicity, this will be called the Stokesian contribution even though this is not strictly so in the case of a second order solution. On excluding the terms of zero degree and adopting the system of quadratures for the evaluation of the Stokesian term, equation 90 can be written as

$$N_f^{(cm)} = \frac{\bar{R}^{(cm)} \times \pi^2}{4\pi\gamma \times 3.24 \times 10^4} \sum_i n_i^2 \sum_j \mu_{ij}\; f(\psi_{ij})\; \Delta g_{1ij}^{(mgal)}$$

$$= K \sum_i n_i^2 \sum_j \mu_{ij}\; f(\psi_{ij})\; \Delta g_{1ij}^{(mgal)} \tag{98}$$

where Δg_{1ij} is the value of the gravity anomaly representing an $n_i^\circ \times n_i^\circ$ square,

$$K \doteq 1.58 \times 10^{-2} \tag{99}$$

and $\mu = \cos\,\phi_c$ or $\sin\,\psi$, depending on the system of coordinates adopted.

It is required that the errors e_N in N_f due to the adoption of the quadratures technique be kept to within the ±5 cm limit. The errors in each of the individual products being summed, could be of two types. The first is of an accidental nature, characterized by the subscript a and the second is systematic in character, denoted by the subscript s. It is well know that the magnitude of the latter per individual term in the summation, should be significantly smaller than the former as it holds the same sign over a considerable number of terms.

In the case of the total accidental error e_{Na} in N_f obtained from N_t summations, the contribution e_{Nta} from a product of the form

$$t = k\,\mu\,f(\psi)\,\Delta g\,n_i^2 \tag{100}$$

should not exceed $e_{Na} / \sqrt{N_t}$. The total systematic error e_{Ns} in N_f due to contributions e_{Nts} from each of the terms at 100 bears the relation

$$e_{Nts} \leq o\,\{e_{Ns}/N_t\}$$

if the systematic error persists with the same magnitude and sign for all N_t terms. If $N_t = o\,\{10^6\}$, then $e_{Nts} = o\,\{10^{-3}\, e_{Nta}\}$. In practice, it is more likely that e_{Nts} exhibits systematic error characteristics over some subset N_t' of N_t, behaving as an accidental error over the $N\ell$ larger subsets, where

$$o\,\{N\ell\} = o\left\{\frac{N_t}{N_t'}\right\}.$$

The evaluation of a surface integral by quadratures calls for the subdivision of the surface into infinitesimally small elements, the evaluation of the kernel of the integral at each of these elements, and the summation of each of these magnitudes. In evaluating equation 100, it is necessary to adopt values for μ, $f(\psi)$ and Δg to represent the $n_i \times n_i$ degree square mentioned in equation 98. If current practice were followed, the value of n_i and hence N_t, would depend on the following factors.

(1) The error of representation $E\,\{\Delta g\}_n$ of a $n^\circ \times n^\circ$ square, as defined in section D to the Appendix. This is a measure of the variability of Δg within a square of a given size. Individual values of $E\,\{\Delta g\}_n$ are well known to depend on topographical variations in the case of the gravity

anomaly but the magnitude of $E\{\Delta g\}_n$ can in general be assumed to be representative of a given value of n for purposes of statistical estimation.

(2) Stokes' function $f(\psi)$ should vary linearly over the region. If a (ψ, α) system of coordinates were used, $\mu = \sin\psi$ and it is convenient to define

$$F(\psi) = f(\psi)\sin\psi \tag{101}$$

which is more stable than $f(\psi)$ for small ψ.

(3) No correlation should exist between the variations in $F(\psi)$ and Δg from the value adopted for the representation of the square.

Consider in the first instance the representation of Δg for the $n^\circ \times n^\circ$ square as afforded by the mean value $\overline{\Delta g}$ for the square, situated at the square center at which point the value of $F(\psi)$ is $\overline{F(\psi)}$. If each $n^\circ \times n^\circ$ area were subdivided into N $m^\circ \times m^\circ$ equal area sub-divisions, let the individual values of the gravity anomaly and $F(\psi)$ be related to $\overline{\Delta g}$ and $\overline{F(\psi)}$ by the relations

$$\Delta g_i = \overline{\Delta g} + c_{gi} \quad \text{and} \quad F(\psi_i) = \overline{F(\psi)} + c_{\psi i}.$$

The total contribution to the final integral by the $n^\circ \times n^\circ$ area is given by

$$t = Km^2 \sum_{i=1}^{N} (\overline{\Delta g} + c_{gi})(\overline{F(\psi)} + c_{\psi i})$$

$$= Kn^2\, \overline{\Delta g}\, \overline{F(\psi)} + m^2 K\, \overline{F(\psi)} \sum_{i=1}^{N} c_{gi} + m^2 K\, \overline{\Delta g} \sum_{i=1}^{N} c_{\psi i} + m^2 K \sum_{i=1}^{N} c_{gi}\, c_{\psi i} \tag{102}$$

where

$$N = (n/m)^2 \tag{103}$$

The first term in equation 102 is the contribution due to the adoption of the area mean at the center of the $n^\circ \times n^\circ$ square while the second and third are zero by definition. The final term will tend to zero if there is no correlation between Δg and $F(\psi)$ as mentioned above. This possibility can be lessened further by restricting n so that variations in $f(\psi)$ are linear over the area to the desired precision. In the present case it is desired to keep e_N down to $o\{e^3 h_d\} = o\{+5 \text{ cm}\}$. Hence the departures of $F(\psi)$ from linearity over the $n^\circ \times n^\circ$ should not exceed $e^3 F(\psi)$. The magnitude of variations in $f(\psi)$ and $F(\psi)$ are functions of ψ. Table 1 gives the relationship between the square size n and ψ which satisfy the linearity relation.

The use of Stokes' function $f(\psi)$ to evaluate equation 98 for all ψ would involve approximately two million summations if the above limits were adhered to and the effect of representation errors from Stokes' function were to be kept below the requisite magnitudes. The function $F(\psi)$ defined by equation 101 is more stable for small ψ but less economical to use than $f(\psi)$ for large ψ. N_t can be reduced by a factor of 3 if $F(\psi)$ were used to evaluate equation 98 when $\psi < 1^\circ$ while $f(\psi)$ were used for all other ψ. This calls for the use of data defined on a local coordinate system (ψ, α) for small ψ instead of the more universally applicable (ϕ_c, λ) system. Consequently, the definition of the former data set from the latter must precede local computations which is why it is preferable to restrict such conversions to as small a region as possible.

The effect of variations in Δg with position within the $n° \times n°$ square, on e_N and the consequent representation errors, are best studied by analyzing the statistical characteristics of Δg. The gravity anomaly as determined at any point on the Earth's surface is based on the following data.

1. Observed gravity.
2. Geocentric position of the gravity station.
3. The difference in potential ΔW with reference to the "geoid."

If Stokes' integral is to be solved by quadratures, it is relevant to investigate the errors which arise in the computed value of N_f due to the representation of a finite element of surface area by a single gravity anomaly.

A useful function for the study of e_N is the error of representation $E\{\Delta g\}_n$ for an $n° \times n°$ square (e.g., de Graaff Hunter 1935; Hirvonen 1956; Molodenskii et al 1962). More details of this important function and numerical magnitudes are given in section D of the Appendix. An empirical formula which describes the behaviour of this function is

$$E\{\Delta g\}_n = \pm C_1 \sqrt{N} \tag{104}$$

A value of C_1 which fits most modern data in regions where the topographical gradients are small is

$$C_1 \doteq 12 \quad \text{for} \quad 1/4° < n < 5°$$

for n in degrees and $E\{\Delta g\}_n$ in mgal. For $\psi < 1/4°$, a better definition of $E\{\Delta g\}_n$ is obtained if the relation

$$E\{\Delta g\}_n = \pm C_2 n \tag{105}$$

where $C_2 \doteq 3 \times 10$ under the same set of conditions as before.

The first problem to be looked into is the effect of representation errors on e_N if $E\{\Delta g\}_n$ is assumed to have random error characteristics. In such a case, any other determinations of the gravity anomaly field in a specified $n° \times n°$ area which is represented for computation purposes by an adopted value Δg_0, would deviate from Δg_0 exhibiting characteristics implicit in the normal distribution. If square sizes in excess of $n = 1°$ are excluded as these violate the prescribed linearity requirements of Stokes' function, as illustrated in table 1, it is interesting to verify whether variations in Δg over small squares are dominated by the gravity anomaly gradients $\partial \Delta g/\partial u_\alpha$, u_α forming an orthogonal and isometric angular coordinate system in the local horizon.

Let the smallest sub-division of relevance be an $m° \times m°$ square whose error of representation satisfies the relation

$$E\{\Delta g\}_m = o\{e^3 \Delta g\} = \pm o\{50 \ \mu gal\} \tag{106}$$

If equation 105 were true for very small n, m $\doteq$ 0.002°. The number of such regions in a larger n° × n° area would be N defined in equation 103. The gravity anomaly Δg and $F(\psi)$ in the larger n° × n° square could be represented by Taylor's series of the form

$$\Delta g = \Delta g_0 + \frac{u_\alpha^j}{j!} \left(\frac{\partial^j \Delta g}{\partial u_\alpha^i}\right)_{u_\alpha = 0} \tag{107}$$

and

$$F(\psi) = F(\psi_0) + \frac{u_\alpha^j}{j!} \left(\frac{\partial^j F(\psi)}{\partial u_\alpha^j}\right)_{u_\alpha = 0} \tag{108}$$

where the origin of the u_α coordinate system is at the center of the n° × n° square, the subscript o referring to values at this same point. On restricting the value of n to those square sizes where

$$\frac{\partial^j F(\psi)}{\partial u_\alpha^j} < o\{e^3 F(\psi)\} \quad \text{for all} \quad j \geq 2 \text{ (i.e., } n \leq 0.5°\text{)},$$

it is required to verify whether

$$\frac{\partial^j \Delta g}{\partial u_\alpha^j} < o\{e^3 \Delta g\} \quad \text{for all} \quad j \geq 2 \tag{109}$$

when n ≤ 0.5°. From equation A49 and 107

$$E\{\Delta g\}_n = \pm \frac{1}{N} \left[\sum_{i=1}^{N} \left[u_{\alpha i} \frac{\partial \Delta g}{\partial u_\alpha}\right]^2\right]^{1/2}$$

$$= \pm \left[\frac{1}{N}\left(\frac{\partial \Delta g}{\partial u_1}\right)^2 \sum_{i=1}^{N} u_{1i}^2 + \frac{2}{N} \frac{\partial \Delta g}{\partial u_1} \frac{\partial \Delta g}{\partial u_2} \sum_{i=1}^{N} u_{1i} u_{2i} + \frac{1}{N}\left(\frac{\partial \Delta g}{\partial u_2}\right)^2 \sum_{i=1}^{N} u_{2i}^2\right]^{1/2} \tag{110}$$

If the m° × m° grid were symmetrically distributed about the n° × n° square, it is easy to show that

$$\frac{1}{N}\sum_{i=1}^{N} u_{\alpha i}^2 = \frac{1}{n}\int_{-1/2\,n}^{1/2\,n} u_\alpha^2 \, d\,u_\alpha = \frac{1}{12}\,n^2 \qquad (111)$$

and

$$\frac{1}{N}\sum_{i=1}^{N} u_{1i}\, u_{2i} = 0$$

Hence equation 110 reduces to a relation of the form

$$E\,\{\Delta\, g\}_n = \pm\, C\, n$$

which agrees with the observed characteristics of $E\{\Delta g\}_n$ as described by equation 105 for small n. Practical experience however indicates that substantial deviations occur from this simplistic model especially in regions of rugged topography when the square size has to be reduced to an unacceptably small area to meet these specifications of linearity. It can be concluded that gravity anomaly variations are linear for square sizes under 0.3°, for purposes of statistical estimation, all departures having the characteristics of local noise. The contribution t of an n° × n° square to N_f and hence h_d, as obtained by the evaluation of Stokes' integral by quadratures, using N m° × m° squares on the basis of equations 98 and 100 is

$$t = K\, m^2 \sum_{i=1}^{N} \Delta\, g_i\; F\,(\psi_i)$$

The use of equations 107 and 108 gives

$$t = K\,m^2 \left[N \Delta g_0\, F(\psi_0) + F(\psi_0) \frac{\partial \Delta g}{\partial u_\alpha} \sum_{i=1}^{N} u_{\alpha i} \right.$$

$$+ \Delta g_0 \frac{\partial F(\psi)}{\partial \psi} \cos A'_\alpha \sum_{i=1}^{N} u_{\alpha i} + \frac{\partial F(\psi)}{\partial \psi} \left(\frac{\partial \Delta g}{\partial u_\alpha} \cos A'_\alpha \sum_{i=1}^{N} u^2_{\alpha i} \right.$$

$$\left. \left. + \frac{\partial \Delta g}{\partial u_\alpha} \sin A'_\alpha \sum_{i=1}^{N} u_{1i}\, u_{2i} \right) + \frac{1}{2} F(\psi_0) \frac{\partial^2 \Delta g}{\partial u^2_\alpha} \sum_{i=1}^{N} u^2_{\alpha i} \right] + o\,\{e^3\, t\}$$

$$= K\,n^2 \left[\Delta g_0\, F(\psi_0) + \frac{n^2}{12} \left[\frac{1}{2} F(\psi_0) \left(\frac{\partial^2 \Delta g}{\partial u_1^2} + \frac{\partial^2 \Delta g}{\partial u_2^2} \right) + \frac{\partial F(\psi)}{\partial \psi} \frac{\partial \Delta g}{\partial u_\alpha} \cos A'_\alpha \right] \right]$$

$$+ o\,\{e^3\, t\} \tag{112}$$

on using the results at 111, K being defined by equation 99, while A' has the same significance as at A 21. The magnitude of the contribution of the third term is governed by that of $\partial F(\psi)/\partial \psi$ which is two orders of magnitude smaller than the first for $n > 0.1°$, as can be seen from table 1. As the square mean $\overline{\Delta g_n}$ is given by

$$\overline{\Delta g_n} = \frac{1}{N} \sum_{i=1}^{N} \Delta g_i = \Delta g_0 + \frac{1}{N} \frac{\partial^j \Delta g}{\partial u^j_\alpha} \frac{1}{j\,!} \sum_{i=1}^{N} u^j_{\alpha i} \tag{113}$$

on using equation 107, it can be seen that the second term in equation 112 is also taken into consideration if $\overline{\Delta g_n}$ were adopted instead of Δg_0 when representing an $n° \times n°$ square for the evaluation of Stokes' integral by quadratures. The terms involving products of the type $(\partial F(\psi)/\partial u_\alpha) \times (\partial \Delta g/\partial u_\alpha)$ can be considered to have the characteristics of accidental errors only if no correlation existed on a large scale between variations in $F(\psi)$ and Δg. While such terms will make contributions of significance when t in equation 112 is computed from a single $n° \times n°$ square represented by $\overline{\Delta g_n}$ instead of N $m° \times m°$ squares, the error is unlikely to have a regionality in excess of 1°. The use of typical values for the case when $n = 0.1°$ indicates that the total contribution of this product term is less than 0.1 mgal. The figures in column 2 of table 2 show that such errors will not affect the final results to ±1 cm even if the signs of $\partial F(\psi)/\partial \psi$ and $\partial \Delta g/\partial u_\alpha$ were to hold the same sign over a $1° \times 1°$ area.

The above discussion may be summarized as follows: (a) The use of the mean value based on an evenly distributed sample gives a better representation than a single value when evaluating equation 112, the improvement being a function of the number of the sample and the moment of the distribution of gravity stations about the square center. (b) The nature of the gravity field is such that any residual error due to the use of a tenth degree grid instead of smaller subdivisions in non-mountainous regions has an effect less than 1 cm on the final value of N_f. Also see section D of the Appendix.

On adopting the basic square sizes specified in table 1, the remaining error characteristics can in the first instance be treated as random. The error e_{ta} in t due to the error of representation $E\{\Delta g\}_m$ of the N constituent $m° \times m°$ areas is given by the addition law as

$$e_{ta} = K m^2 \left(\sum_{i=1}^{N} [F(\psi_1)\, E\, \{\Delta g\}_m]^2 \right)^{1/2} = \pm o \left\{ K m^2\, E\, \{\Delta g\}_m \left(\sum_{i=1}^{N} F(\psi_i)^2 \right) \right\}^{1/2} \quad (114)$$

Over a limited n° × n° region $F(\psi)$ can be represented by the first two terms of a Taylor's series when

$$\sum_{i=1}^{N} F(\psi)^2 = N\, F(\psi_0)^2 + 2\, F(\psi_0)\, \frac{\partial F(\psi)}{\partial \psi} \sum_{i=1}^{N} d\, \psi_i + o \left\{ \left(\frac{\partial F(\psi)}{\partial \psi} \right)^2 \right\}$$

$$= N\, F(\psi_0)^2\, [1 + o\, \{10^{-3}\}] \quad (115)$$

as

$$\sum_{i=1}^{N} d\, \psi_i \to 0.$$

On using equation 103 and noting that $|F(\psi)| < 2.5$,

$$e_{ta} = \pm\, o\, \{K'\, m\, n\, E\, \{\Delta g\}_m\, (1 + o\, \{10^{-3}\})\} \quad (116)$$

where

$$K' = 4 \times 10^{-2}$$

The accumulated accidental error e_{Na} in N_f is given by

$$e_{Na} = \pm\, o \left\{ \left(\frac{360}{n} \times \frac{180}{n} \right)^{1/2} e_{ta} \right\} = \pm\, o\, \{K''\, m\, E\, \{\Delta g\}_m\} \quad (117)$$

where

$$K'' \doteq 10.$$

Column 3 of table 2 gives estimates of e_{Na} for various values of m in column 1 which represent the basic grid on which gravity stations are located to define the global gravity field for the evaluation of Stokes' integral by quadratures.

It is also necessary to estimate the effect of an error e_{ts} in evaluating t which retains its sign over an $\ell° \times \ell°$ area. For purposes of estimation, assume e_{ts} to retain its numerical magnitude and sign over the larger area. The error $e_{\ell s}$ in the larger block is obtained from equation 112 as

$$e_{\ell s} = \pm\, o \left\{ e_{ts} \frac{\ell^2}{n^2} \right\}$$

where

$$e_{ts} = \pm\, K'\, n^2\, e_{\Delta g},$$

$e_{\Delta g}$ being the systematic error in the gravity anomaly representing the $n° \times n$ square, K' having the same definition as in equation 116. The total contribution e_{Ns} to N_f is estimated as in the case of equation 117 to be

$$e_{Ns} = \pm\, o\, \{K''\, \ell\, e_{\Delta g}\} \tag{118}$$

Column 2 of table 2 gives estimates of $e_{\Delta g}$ for various values of ℓ, specified in column 1, which ensure that $e_{Ns} = o\{e^3 h_d\}$.

The following conclusions can be drawn concerning the evaluation of Stokes' integral by quadratures.

1. The most critical factor is the departures from linearity of $F(\psi)$, defined by equation 101. The use of $F(\psi)$ in practice is expensive as the (ψ, α) system of coordinates and not the (ϕ_c, λ) is used in computations, requiring the utilization of ring techniques which are less economic for computer use than the geographical square system. This is true even when use is restricted to those regions where $\psi < 3°$ and $f(\psi)$ is unstable. Computations with Stokes' function in such regions calls for a finer sub-division in representing the inner zone gravity field on the lines described in table 1 as $f(\psi) = o\{10^3\}$ for $\psi = o\{0.1°\}$ while it is of order 10^2 for $\psi = o\{1°\}$.

   If this were not done, the terms ignored in equation 115 could be as large as the magnitude of those considered. Further, K' in equation 116 could in such a case be 2 - 3 orders of magnitude larger. Thus the four tenth degree squares within 0.1° of the point of computation would contribute $\pm\, o\{0.3 \text{ cm}\}$ toward e_N while the 100 tenth degree squares within 0.5 degrees would give rise to a further $\pm\, o\{1 \text{ cm}\}$ due to departures from linearity (of order $e^2 f(\psi)$).

2. In view of the limited errors introduced into the result, it can be concluded that a global gravity field based on definition at corners of a $0.1° \times 0.1°$ may be adequate for the evaluation of Stokes' integral by quadratures to order $e^3 h_d$ ($\pm$ 5 cm) if correct computing procedures were adopted and the gravity anomalies were free from systematic errors over large extents as specified in column 2 of table 2.

3. It is desirable that a procedure similar to Rice's circular ring method (Rice 1952) be used to compute the inner zones when $F(\psi)$ should be used instead of $f(\psi)$ to circumvent the instability of the latter when $\psi < 1°$. This instability is <u>not</u> a consequence of variations in Δg and an adequate gravity field could be interpolated from the $0.1° \times 0.1°$ grid without introducing significant error in the final result for studies of sea surface topography where effects of very short wave lengths are of no concern.
4. The observation that interpolated values are not necessarily inferior to measured gravity anomalies was also made by Soviet geodesists (Molodenskii et al 1962, p. 172). The writer's own experience is that the extension of the gravity anomaly field represented by values on a uniform grid, to smaller subdivision in undisturbed regions, is stable without significant loss of accuracy (Mather 1967, p. 134). Thus if a $0.05° \times 0.05°$ grid were obtained by interpolation from a uniform $0.1° \times 0.1°$ grid on which $E\{\Delta g\}_{0.1°} = \pm 2.5$ mgal, then $E\{\Delta g\}_{0.05°}$ is held at this same value for the interpolated values, instead of the ± 1.5 mgal estimated from equation 105. Thus the use of a $0.2° \times 0.2°$ grid in lieu of the tenth degree grid as the fundamental basis of observations would result in $e_N = \pm\, o\{6 \text{ cm}\}$.
5. While considerable laxity can be tolerated in the accuracy with which a reading represents a basic (i.e., tenth degree) square, the effect of

systematic error which retain the same sign over considerable extents has to be carefully watched. Table 2 shows that errors of 0.1 mgal which hold their magnitude and sign over 500 km can affect the computed value in excess of the specified limits of error. This type of error can be due to one of three causes.

(a) Errors in the global gravity network controlling the gravity values used in the computations.

(b) Loss of accuracy in unifying the elevation datums in relation to a globally acceptable "geoid."

(c) Lack of precision in the geodetic coordinates used to compute the gravity anomaly as a consequence of regional datums not being related to the geocenter as described in section 2.2.

Thus IGSN 1971 can only be considered adequate in controlling the gravity fields in solutions to order e^3 if errors in defining individual station values in the net were uncorrelated and the interval between stations was not in excess of 200 km. Neither of these criteria are likely to be met. On the other hand there are no limitations to present day meteorology which would inhibit the establishment of a global net with sufficient precision. Absolute station accuracies could be held at $\pm 50\ \mu$gal as discussed earlier in this section using techniques similar to those used by Sakuma (1971).

The unification of the elevation datums is equivalent to defining the geoid to a degree which has not been achieved as yet if the order of accuracy implicit in this study is to be realized. The first stage in such a definition would be the adoption of a common epoch to which all so called "Mean Sea Level" datums are reduced. The principles underlying the establishment of such datums for a restricted region are well known and will not be discussed. The second stage calls for the definition of the sea surface topography and its departures with respect to a level surface which are stationary over long periods of time. The solution of the boundary value problem to order e^3 requires that errors of long wavelength in the definition of geopotential be kept to o {0.15 kgal m}. This could be achieved without difficulty if the ocean surface were an equipotential, on allowing for tidal and meteorological variations. Unfortunately, the comparison of tide gauge readings with the results of geodetic levelling have indicated the existence of stationary departures of the sea surface from an equipotential as reported in section 1, both in the United States and Australia. As current practice refers differences of geopotential to the sea surface instead of the equipotential, it becomes necessary to look into the effect such a procedure has on the computation of geoid heights with an accuracy of ±10 cm, which in turn calls for errors of less than ±15 cm in the definition of the geoid as a datum for elevations.

4.4 The Role of Satellite Solutions for the Gravity Field in Solutions of the Boundary Value Problem to Order e^3

The characteristics of the Earth's gravity field can be established by two different techniques.

(a) The measurement of gravity at discrete points at or near the Earths surface.

(b) The determination of the disturbing potential from the analysis of the orbital perturbations of near Earth satellites.

Solutions at (b) are interpreted in terms of spherical harmonic coefficients which can then be downward continued to the surface of the Earth with minimum mathematical complications. As Laplace's equation is satisfied at finite elevations exterior to the Earth's surface, the disturbing potential V_d' of the solid Earth satisfies equation 41 which can be written as

$$V_d' = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} \qquad n \neq 1 \tag{41}$$

where it is customary to express A_n in the form

$$A_n = k\ M\ a^n \sum_{m=0}^{n} C_{nm} \tag{119}$$

C_{nm} being defined by equation 25. The absence of the first degree harmonic places the center of the reference ellipsoid at the center of mass of the solid Earth.

The disturbing potential V_{ds} as used in the analysis of orbital perturbations, is defined as that which causes the geopotential to deviate from that of a sphere with the same mass as the Earth. A symmetrical mass distribution is also implied when referring perturbations to the model adopted for central force motion. It is this potential whose derivatives define Lagrange's equation of planetary motion (e.g., Kaula 1966, p. 29). As the gravitational effect of the atmosphere is estimated at less than 10 μgal at 30 km elevation, with the effect falling off rapidly with increase of elevation (IAG 1970, p. 72), it can be assumed that

$$V_d = V_d' + o\{e^3 V_d\}$$

at orbital elevations. The term of zero degree V_{d0} in V_d' has no effect on orbital perturbations though its numerical magnitude could have a scaling effect on the orbital parameters used in the evaluation of the coefficients at 119. A further difference between V_{ds} and V_d is due to the ellipsoidal reference model used in defining the latter as described in section 2.2 in contrast to the spherical model used in obtaining V_{ds}. If the effect on the gravitational potential is V_{de}, then

$$V_{ds} = V_d' - V_{d0} + V_{de} \tag{120}$$

On taking these factors into account when evaluating the coefficients $C_{\alpha\, nm}$ in equation 119, the height anomaly h_d at the surface of the Earth is given from equation 33, 41 and 119 as

$$h_d = \frac{k\,M}{R} \sum_{n=0}^{\infty} \left(\frac{a}{R}\right)^n \sum_{m=0}^{n} C_{nm} - \frac{W_0 - U_0}{\gamma} + \frac{V_a}{\gamma} + o\{e^3 h_d\},\ n \neq 1 \tag{121}$$

where R, V_a and γ refer to values at the relevant point at the surface of the Earth. The infinite series must by definition converge to the limit specified by equation 121. The evaluation of the coefficients $C_{\alpha\, nm}$ defined in equation 25, by the analysis of orbital perturbations, is dominated by two effects.

(1) The damping effect of the term $(a/R)^n$ as $R > a$ (e.g., see Mather 1971c, p. 67). Consequently the coefficients of degree n less than some limiting value n_s are well determined, n_s being a function of the orbital elevation of the satellite.

(2) The effect of resonance between the values of the set $\{n, m\}$ and the orbital period. This causes certain coefficients which by themselves, make no contribution of significance towards the representation of the Earth's gravitational field, to have marked effects on the perturbations of those orbits with sympathetic parameters. As a consequence, all orbits are sensitive to certain resonant harmonics whose identity can be predicted from the orbital elements (e.g. Wagner 1967).

At first glance it would appear that a very large number of satellites in a variety of orbits would afford a means for the complete determination of the Earth's gravitational field. The costs involved make such evaluation unlikely due to masking effects which make it difficult to separate some of the resonant terms unless adequate variations were available in the orbital inclination. Serious thought should be given to the role concepts of resonance should play in solutions of high resolution for the Earth's gravitational field <u>at the surface of the Earth</u> as it is most likely that only a limited number of satellites will be available for the task. These accurate determinations will suffer from a loss of resolution if not restricted to a limited interval of time. If such is the case, it may be preferable to treat higher degree resonant effects as sources of orbital perturbations rather than signals from the gravity field which could be meaningfully translated into representations at the surface of the Earth.

The gravity anomaly at the surface of the Earth is obtained from equations 41, 45 and 119 as

$$\Delta g = \frac{kM}{R^2} \sum_{\substack{n=0 \\ n \neq 1}}^{\infty} (n - 1 - 2c_{\phi}) \left(\frac{a}{R}\right)^n \sum_{m=0}^{n} C_{nm} + 2\frac{W_0 - U_0}{R} - \left(2\frac{V_a}{R} + \frac{\partial V_a}{\partial h}\right) + \frac{1}{2} g \zeta^2 + o\{e^3 \Delta g\}, \quad (122)$$

The comparison of the values of Δg computed from the $C_{\alpha\, nm}$ determined from the analysis of orbital perturbations with those established from surface gravity measurement on allowing for equation 120, provide an index of the success with which a truncated spherical harmonic series (i.e., $n < n_s$) can represent the gravity anomaly at the surface of the Earth. This could be extended further by incorporating those harmonics in the range $n_s < n < n_\ell$ from surface gravity to enhance the representation provided by the spherical harmonic series, thereby increasing the range of the power spectrum and reducing the residuals on comparison of deduced and observed values of the gravity anomaly at the surface of the Earth. Such concepts assume that those $C_{\alpha\, nm}$ for $n < n_s$ as determined from orbital perturbations were free from error as were the values of Δg at the surface of the Earth. It also has the advantage that errors in the framework controlling the gravity datum at the Earth's surface, which is established with difficulty, can be eliminated or at least minimized in the representation of the global field.

This procedure is not always strictly followed in common practice when general adjustment techniques are used to minimize residuals without holding any of the quantities fixed.

The variance of gravity anomalies at the surface of the Earth is approximately 1200 mgal^2. Solutions to (20, 20) absorb over 90% of the power inherent in the representation (Lerch et al 1972, p. A12). Thus

$$M\{(\Delta g_0 - \Delta g_s)^2\} = 100 \text{ mgal}^2, \tag{123}$$

where the subscripts o and s refer to terrestrial and satellite determined values respectively. The absorption of this balance 10% of the power spectrum is likely to require a great increase in the number of terms though some of this residual is due to deficiencies in the surface gravity data. From a study of the error of representation, given in section D of the appendix, a (20, 20) solution can be considered to be equivalent to a representation on an 0.5 degree grid only if the comparisons represented at 123 were with individual gravity values. This is not the case, as the surface gravity values were in the form of five degree area means. The conclusion which can be drawn is that the (20, 20) representation is equivalent to a global 5° × 5° coverage with 5 - 6 readings per square and zero moment of distribution about the square center.

The use of spherical harmonic representations of the gravity field to achieve the definition of the gravity field required in conclusion 2 to section 4.3 (i.e., a tenth degree grid) would require that the former absorbs all but 9 $mgal^2$ of the power spectrum on comparison with individual values. This will be equivalent to absorbing all but 0.1 $mgal^2$ of the power spectrum on comparison with adequately computed one degree area means, each based on 100 evenly spaced values with zero moment of distribution about the square center. The latter would involve analysis up to degree 180 (over 3×10^4 coefficients). It has yet to be established whether such refined determinations of the gravity field are possible by satellite to satellite tracking of low altitude satellites.

The requirements for a complete solution of the geodetic boundary value problem to order e^3 (i.e., ±5 cm in h_d) is a gravity field representation based on at least a tenth degree grid. This is equivalent to a spherical harmonic representation where $n = 1800$, involving $o\{3 \times 10^6\}$ terms, which is not significantly different from N_t in table 1. The use of such functions can therefore be justified in this case only if the amount of surface gravity information on the tenth degree grid were significantly low. It is unlikely that any favorable claim can be made at the present time regarding the achievement of this degree of resolution from the study of orbital perturbations. It would therefore appear that satellite determinations of the gravity field could well be inferior for the complete solution of the geodetic boundary value problem to ±5 cm if

(a) surface gravity data were available globally on a tenth degree grid; and

(b) systematic errors in the gravity anomalies were held to below $\pm 50\mu$ gal.

The low degree harmonics from orbital perturbations could however play a significant role in such solutions when (a) is true but not (b). The three major sources of systematic error in gravity anomalies which have long wavelength are given in note 5 to section 4.3. While (c) is likely to be resolved with minimal difficulty, systematic errors at (b) are complex primarily as a consequence of possible stationary departures of the sea surface from an equipotential. If the gravity anomalies have been corrected for effects at (b) and (c), any residual long wave discrepancies between surface gravity data based on adequate samples and the low degree harmonics obtained from the analysis of orbital perturbations and with the required precision, should provide an effective check on the systematic error propagation of the type at (a) in the note mentioned above.

The results obtained from the analysis of the orbital perturbations of satellites in near Earth orbits are unlikely therefore to provide the representation of the gravity field which is required for a complete solution of the geodetic boundary value problem to order e^3. The determination of the low degree harmonics in this representation with adequate precision will however be invaluable in resolving the systematic errors in the global gravity standardization network described in note 5 to section 4.3.

4.5 Departures of the Sea Surface Topography from an Equipotential Surface

Until recently, no attempt has been made to study the departure of the surface of the oceans from a level surface. The existence of such departures has been established on comparing the results of geodetic levelling with tide gauge readings. These departures which will be called stationary, in order that they could be differentiated from short term effects due to winds, other meteorological factors as well as the short period distortions on the geops due to tidal effects. The use of satellite altimeters provides a means of determining the instantaneous geocentric position of those features of the sea surface with wavelengths in excess of ℓ km. $\ell = 200$ for the proposed GEOS-C mission (Weiffenbach 1972, p. 1-1). The stationary departures can be obtained by allowing for the effect of tides and meteorological conditions, on differencing equivalent position vectors to the sea surface and the geoid.

As only features with wavelengths in excess of ℓ km are being studied, it is possible to use a truncated version of equation 121 to obtain the required definition of the geoid even to order $e^3 h_d$. Over oceanic regions, the telluroid coincides with the geoid and the elevation N of the latter above the ellipsoid is given by

$$N = h_d, \quad \text{if} \quad \Delta W = 0 \tag{124}$$

If the gravity field were represented by a global set of gravity anomalies, N could be obtained from the set of equations summarized in Section 4.1. Alternately, these anomalies could be analyzed for the equivalent harmonic coefficients using equation 122 and the values of N in oceanic areas obtained from equation 121. From the discussion in section 4.4, the representation should absorb all but 0.1 $mgal^2$ of the power spectrum on comparison with one degree square means compiled from 100 values spaced on a tenth degree grid, if wavelengths in excess of ℓ km in N is to be defined to e^3N. This should give an accuracy of ±10 cm in N on the basis of the results in table 2, which is a desirable goal in the definition of the sea topography (Williamstown Report 1969, 3-2).

Consider the use of equation 121. The harmonic coefficients could be obtained from surface gravity on controlling gravity standardization network errors with low degree harmonics determined from orbital perturbation analysis of adequate precision. In practice it is likely that the distribution of surface gravity information will continue to be non-uniform. It is therefore relevant to designate a desirable form in which the gravity data should be used in the analysis for harmonic coefficients. A global representation on a tenth degree (10 km) grid has an error of representation of approximately ±3 mgal, resulting in an accuracy of ±10 cm in N if the data is free from large scale systematic errors. A study of equation 116 indicates that if m = 0.1°, the precision required in the mean value for a n° × n° area to maintain the specified accuracy e_N in N is not $E\{\Delta g\}_n$ but $(E\{\Delta g\}_m \times m/n)$, all other things being equal. Thus the equivalent precision required from a 1° × 1° square mean is approximately ±0.3 mgal. Such a mean can be computed only if

(1) 100 values spaced on a tenth degree grid are used in its evaluation; and

(2) the moment of distribution of the gravity stations about the square center tends to zero.

This does not mean that each one degree square should contain 100 gravity readings on a tenth degree grid. It is well known that gravity anomaly values can be predicted under carefully controlled conditions such that the prediction error did not exceed the error of representation (e.g., Molodenskii et al 1962, p. 172; Mather 1967, p. 134). The exact technique to be used for this purpose is a matter for debate. In practice, the writer has found that practical and not theoretical considerations predominate in the choice of a particular method.

Any commonly used interpolation routine will give the desired accuracy provided

(a) sufficient data were available to avoid predictions based on readings which were not in the immediate vicinity of the point; and

(b) the correlation of gravity anomalies with elevation over limited regions were allowed for.

For example, an evenly space 50% coverage of a $1° \times 1°$ square (i.e., 50 readings) should give the required accuracy in the area mean if the latter were computed from 100 evenly spaced values with zero moment of distribution about the square center and the above requirements were met. Tests carried out for non-mountainous regions in Australia with considerable gravity variation, indicated that a 20% representation, again evenly spaced, could provide interpolated values whose error would be double that for $E\{\Delta g\}_{0.1}$ (ibid, p. 133). In such a case, $E\{\Delta g\}_1 = \pm\ o\ \{0.5 \text{ mgal}\}$. This figure falls off markedly if the moment of distribution about the square center did not approach zero.

The following conclusions can be drawn about the preparation of area means prior to harmonic analysis in regions which are incompletely represented by surface gravity data.

1. Values should be predicted from available observations represented on a tenth degree (10 km) grid using any reasonable interpolation routine or collocation techniques, and allowing for height correlation as well as the deviation of gravity station elevation from the mean elevation of the region it is intended to represent.

2. If a network of gravity stations were being planned, the stations should be cited such that the distance over which interpolations are made should be as small as practicable to avoid systematic effects.

3. The quality of the area mean is more dependent on the nature of distribution of the gravity stations about the square center, rather than the number of readings available. This is characterized by the moments M_α of the gravity station distribution defined by

$$M_\alpha = \sum_{i=1}^{N} d\,u_{\alpha i} \qquad (125)$$

where $du_{\alpha i}$ are the coordinate displacements of the i-th gravity station from the square center. More research is necessary into the role M_α should play in setting up observation equations for the determination of the harmonic coefficients.

The error $e_{\Delta g_n}$ in Δg as computed from equation 122 due to an error e_{C_n} in C_n is given by

$$e_{\Delta g_n} = \pm\, o\{\gamma(n - 1)\, e_{C_n}\},$$

where e_{C_n} is the r.m.s. of the sum of the variances of the 2n + 1 coefficients of degree n. The analysis of harmonic solutions of this type indicate that the magnitude of the average variance of coefficients of degree n are essentially constant (say $\bar{\sigma}^2$) for solutions up to degree 12 (Lerch et al. 1972, p. 21) while departures of individual variances from this mean, fall within a maximum amplitude σ_{max}. On assuming sinusoidal characteristics for the deviation ϵ_i of individual standard standard deviations from $\bar{\sigma}$, the total variance of terms of degree n is

$$\left.\begin{aligned} e^2_{C_n} &= o\left\{\sum_{i=1}^{2n+1} (\bar{\sigma} + \epsilon_i)^2 = o\left\{(2n + 1)\,\bar{\sigma}^2 + \sum_{i=1}^{2n+1} \epsilon_i^2\right\}\right. \\ &= o\left\{(2n + 1)\,(\sigma^2 + K\sigma^2_{max})\right\} = o\,\{(2n + 1)\, e^2_C\} \end{aligned}\right\} \qquad (126)$$

where e_C is a constant whose magnitude is approximately 2×10^{-8} for solutions obtained at the present time. The extension of these observations seem to indicate that $e_{\Delta g_n}$ is a function of n^2. On the other hand, the results in column 2 of table 2 indicate that larger errors could be tolerated in the higher degree harmonics without significantly worsening the results if equation 90 were used in the computation. The required accuracy for those of low degree is about 5 parts in 10^4 if each is treated as an isolated error source. It is difficult to estimate composite effects in solutions to very high degree in the absence of solution characteristics. It could be assumed that an adequate algorithm will result in the harmonic representation having error accumulation patterns similar to those of the original data, provided the latter were free from systematic error.

If surface gravity data were used to determine the geoidal slopes with wavelengths longer than 200 km, it would therefore be necessary to compute $1° \times 1°$ (100 km) area means from 100 evenly spaced values on a $0.1° \times 0.1°$ (10 km) grid in non-mountainous areas, such that the error of representation of the area mean is ±0.3 mgal. This would ensure ±10 cm accuracy in the computed result. The analysis of such a data set for the appropriate coefficients using equation 122, followed by the evaluation of h_d from equation 121, should give the required result. The existence of such a data set could also be used to give the same result through equations at section 4.1. In both cases it is extremely desirable to verify the correctness of the low degree harmonics against satellite determined values of adequate precision, to ensure that the results are free from systematic errors in the compilation of the global elevation and gravity datums.

Notes:

(1) It should be pointed out that it is quite valid to use the truncated spherical harmonic series in equation 121 for the evaluation of the characteristics of the geoid with wavelengths in excess of a certain minimum value,

provided such values in themselves are capable of meaningful interpretation. As this information is to be used in conjunction with altimeter data which can only evaluate similar characteristics of the sea surface, it is relevant to attempt the definitions of the long wave characteristics of geoid to ±10 cm, noting that such evaluations could deviate from the true stationary geoid over oceans by up to ±5 m.

(2) The development given above has only dealt with the techniques for the solution of the sea surface topography using determinations of the gravity field at the surface of the Earth. Satellite techniques which have been proposed for reaching similar goals (Williamstown Report 1969, 2-20 - 2-24) have not been considered as they fall beyond the scope of the present development. The equations in section 4.4 are of relevance however when formulating a solution to the problem in this case as well.

4.6 A Note on the Determination of Gravity Anomalies at the Surface of the Earth

The establishment of the gravity field at the surface of the Earth for determinations of sea surface topography with a resolution at the ±5 cm level does not require that individual gravity determinations are consequences of techniques achieving accuracies of better than ±50 μgal and equivalent station elevations to ±15 cm at each point. Instead what is required is the control of the propagation of systematic error due to those sources with long wavelength, to values below these exacting limits, on the basis of equation 118, as illustrated in column 2 of table 2.

The factors which influence such errors are the establishment of latitude on a geocentric datum, elevation and gravity such that these systematic error limits are not exceeded. The error e_ϕ in latitude is discussed in section 2.2. It is current practice to compute normal gravity from the value of ϕ referred to the local geodetic datum. If e_ϕ = 2 arcsec at ϕ = 45°, the resulting systematic error in $\Delta g = \pm 0$ {50 μgal}. It follows that the application of orientation vector corrections from any of the more recent satellite solutions for geocentric position prior to the computation of normal gravity, will ensure that this source of systematic error is eliminated.

The effect of elevation errors e_h in the gravity station elevations used in the computation of the gravity anomaly is not straightforward. Errors approaching ±50 μgal are obtained in Δg if $e_h = \pm$ o {15 cm}. Such a specification is at the noise level of internal errors in large first order regional geodetic level networks. As pointed out in section 4.3, the essential requirement is the control of systematic errors with long wavelength when establishing the global datum for elevations. This would call for a consideration of

(a) the time dependent variations in "Mean Sea Level"; and

(b) the stationary departures of the sea surface from the equipotential surface adopted as the geoid.

Techniques for the estimation of the former constitute areas of regular research in oceanography. The effects at (b) need evaluation only at those points on the coast which have been used to define the sea level datum for elevations. The geocentric position of a tide gauge on each of these reference datums could be established in the future by means of a connection to a suitable laser ranging station which forms part of a global network. The elevation of the sea surface can only be determined if the geoid height at this point were known. There is little choice but to iterate between improvements in the elevation datum and the determination of the geoid to obtain a solution of adequate accuracy, a procedure which could be quite expensive as there may be difficulties in making the solutions converge, as illustrated below.

Elevation errors of considerably larger magnitude can be tolerated in station elevations provided they are purely local in character. It should be noted that an error of 1 m in the elevation is approximately equivalent to 0.3 mgal in the gravity anomaly, which in turn can have an error of representation of ±3 mgal in the context of the global grid discussed earlier.

The use of accurately determined low degree harmonics of the Earth's gravitational field from the analysis of orbital perturbations for the verification of the global gravity standardization net will be successful only if the errors in the establishment of the global elevation datum have been satisfactorily resolved. The sea level datums in current use cannot be considered to be compatible for the purpose of solving the geodetic boundary value problem to order e^3, as no serious attempt has been made to define the term "geoid" on a global basis to ±15 cm. Elevations with respect to local determinations of the sea surface can be considered to be elevations above geoid only if

(1) corrections were applied for the epoch of determination; and

(2) the stationary departures of the sea surface from the equipotential were allowed for.

The latter is difficult to accomplish in advance of a geoid determination to ±15 cm unless all the tide gauges are linked by a single network of geodetic levelling. While such connections would be feasible for the American continents as a unit or Africa, Asia and Europe as a second entity, a global connection cannot be effected to achieve this end. If errors on this count averaged at ±1 m, causing effects in the gravity anomaly of approximately ±0.3 mgal with wavelengths of 1000 km, the accuracy of the computed value of N is estimated at ±15 cm. In such a case, the error in the determination of stationary departures of the sea surface from an equipotential can also be determined to ±15 cm, assuming that the geocentric positions of the tide gauges defining the elevation datum are established with this same order of accuracy either from laser ranging techniques or from satellite altimetry. The systematic errors in the gravity anomalies due to the revised height datum are almost an order of magnitude smaller and hence fall within the required limits of precision for a solution of the boundary value problem to order e^3.

If, on the other hand, the dominant stationary characteristics of the sea surface topography had twice the wavelength and magnitude as in the above case, the error in the computed value of N is estimated at ±60 cm in the initial iteration. Consequently two iterations are necessary to ensure the definition of the geoid to order $e^3 h_d$.

The use of collocation techniques in defining the unsurveyed portions of the Earth's gravity field is outside the scope of the present development. The accuracy of any predicted values are most likely to meet the criteria given in note 4 to section 4.3 if based on a minimum of four equidistant values, each pair of which subtends nearly equal angles at the point of prediction, and in regions where topographic variations are smooth. As pointed out in section 4.3, a network pre-planned in such a manner could be used to increase the gravity anomaly representation by a factor of 4: 1 in undisturbed regions without introducing significant error provided the gravity values at those points used to control the prediction, are substantially free from the sources of systematic error described above. Other criteria of significance are the following.

(i) Predictions should be restricted to regions where the behavior of the gravity field is sufficiently well known so that the error of prediction is no greater than $E\{\Delta g\}$, as discussed in section 4.3.

(ii) The prediction interval is small enough to permit the adequate representation of the correlation of gravity anomalies with elevation.

5. CONCLUSIONS

The above development has defined formulae for the height anomalies h_d at the surface of the Earth to order $e^3 h_d$ (±5 cm). h_d is equivalent to the elevation N of the geoid above the ellipsoid of reference in ocean regions. The solution obtained for the boundary value problem in geodesy includes Stokes' integral in circumstances where spherical harmonic expansions are resorted to only when physical validity exists for their use. The boundary value condition has been built around the potential of the solid Earth and oceans, excluding the atmosphere, to ensure the mathematical validity for the expression of the solution in terms of spherical harmonic expansions. Such a representation is desirable as it permits the ready incorporation of information regarding the Earth's gravitational field, as obtained from the analysis of the orbital perturbations of near Earth satellites, in representations at the surface of the Earth.

The solution referred to above exists only if the stratification of the atmosphere, assumed invariant with respect to the epoch in which the gravity field is determined, were known. A model has therefore to be defined for the Earth's atmosphere prior to effecting the solution, which is referred to the center of mass of the solid Earth. The latter can be related without difficulty to the geocenter if the mass distribution of the atmospheric model were known. This could be defined as a series of ellipsoidal shells (e.g., IAG 1970, p. 62). More correctly, the atmosphere can be considered to consist of such shells at altitudes greater than the maximum topographical elevation. At lower altitudes, the shells are not complete, the excluded portions being related to the topography of the Earth.

Contributions to the height anomaly h_d, as computed using the formulae listed in section 4.1, have three distinct orders of magnitude to $o\{e^3h_d\}$. The major term is obtained by the use of gravity anomalies at the surface of the Earth in Stokes' integral which gives over 90% of the total magnitude. The second is due to departures of the topography in the local area from a plane, contributing less than 10% of the total magnitude in regions of rugged terrain. The third set of terms is of order e^2h_d (±30 cm) and arise as a consequence of the ellipticity of the Earth, topographical gradients at the surface of the Earth and the consequences of the Stokesian assumptions.

No direct solution is possible, as pointed out in section C of the Appendix and an iterative procedure, described in section 4.2, has to be resorted to. The representation of the gravity field which would ensure adequate accuracy in the evaluation of Stokes' integral by quadratures can be estimated from those characteristics of gravity anomalies embodied in the error of representation $E\{\Delta g\}$. It is estimated that the definition of this field at the surface of the Earth by values on a tenth degree (10 km) grid in non-mountainous regions (97% of the globe) would ensure that the accuracy of the value computed for the Stokesian terms was of order e^3h_d, provided no systematic errors with long wavelength existed in the data. Regions of rugged topography and disturbed areas close to the point of computation should be represented by square sizes whose error of representation remains at ±3 mgal, if the accuracy of the final result is not to deteriorate by as much as a factor of 2.

It is therefore essential that the following criteria are met in defining the global gravity anomaly field.

(1) Stations comprising the global gravity standardization network should be established with absolute accuracies of ±50 μgal, and at intervals which are not much in excess of 1000 km

(2) The datum to which measurements of geopotential are referred (i.e., the "geoid") should be defined with an accuracy of ±15 cm. All "Mean Sea Level" datums should be reduced to the common epoch of the gravity measurements and the stationary departures of the sea surface from the equipotential corrected for prior to the computation of gravity anomalies.

(3) All values of normal gravity should be computed using the equivalent latitude on a geocentered ellipsoid, correct to ±2 arcsec, rather than values on a regional geodetic datum.

(4) Individual gravity stations should reflect the mean elevation of the region. It should be noted that comparatively large errors can be tolerated in individual gravity anomalies on the tenth degree grid provided they are random in character (i.e., purely local).

Gravity observations need not be made at every point on the tenth degree (10 km) grid in non-mountainous areas. It is common experience that interpolation techniques can give predicted values without increasing the error of representation under carefully controlled conditions. This factor should be taken into consideration when planning any large scale sampling of the global gravity field.

It is unlikely that numerical solutions aiming for an accuracy of ±10 cm in h_d can be achieved without significant roles being played by techniques from satellite geodesy. The first is in the establishment of geocentric position at tide gauges monitoring the sea surface and hence the datum for geopotential differences, and therefore gravity station elevations. This information would provide the link between the results of geodetic levelling and the geoid on a global basis, on using the iterative procedure described in section 4.5 if necessary. The method so described could prove ineffectual if the stationary departures of the sea surface from the geoid are characterized by very long wavelengths and amplitudes in excess of 2m.

The second role that could be played by techniques in satellite geodesy is the determination of the low degree harmonics of the gravity field from the analysis of orbital perturbations for the control of systematic errors with long wavelength in the gravity anomalies at the surface of the Earth. If the sources of error at (2) and (3) above are allowed for, such coefficients of adequate precision could control systematic effects in observed gravity due to standardization network errors, especially in ocean areas, where stations in such a network may be widely spaced. Consequently, the absolute error in the gravity control station, which should be less than $\pm 50\,\mu$gal, could be repeated in all gravity anomalies over a very large area.

The determination of features of the geoid with wavelength in excess of 200 km calls for the evaluation of equation 121 using a truncated harmonic series including all terms up to $n = 180$ (i.e., over 3×10^4 coefficients). Such determinations are meaningful in the context of quantifying the stationary departures of the sea surface from the geoid, as only wavelengths in excess of the figure given above can be resolved by altimetric techniques. The minimum surface gravity anomaly field necessary to obtain the resolution of such features to ± 10 cm is estimated to be a $1° \times 1°$ (100 km) grid which is represented by the area mean value computed from at least 100 equally spaced values with zero moment of distribution about the square center. The harmonic representation should be capable of absorbing all but 0.1 mgal^2 of the power spectrum of the gravity anomalies at the surface of the Earth as represented by correctly computed area mean values on the one degree grid.

Satellite geodesy, as distinct from satellite altimetry, still has an important role to play in defining stationary departures of the sea surface from the geoid, even if satellite-to-satellite tracking and drag-free satellites do not play a major role in the definition of the global gravity field. A world-wide system of tracking stations for laser ranging to satellites could provide both the resolution of the datum for the measurement of geopotential, as well as an accurate determination of the low degree harmonics of the Earth's gravity field.

2 of 2

There appear to be no long range obstacles that are likely to inhibit the definition of those characteristics of the geoid in ocean areas with wave lengths in excess of 200 km with a ±10 cm resolution. An evenly spaced sampling of the global gravity field on the lines described in section 4.3, and based on levelling and gravity control networks with suitably small systematic error characteristics, remains a necessary pre-requisite for a successful determination from surface gravity data. Information controlled in this manner and located on a grid where the station spacing could be as large as 20 km in non-mountainous and undisturbed regions, and when used in conjunction with prediction methods which took elevation correlations into account, is likely to provide the desired resolution.

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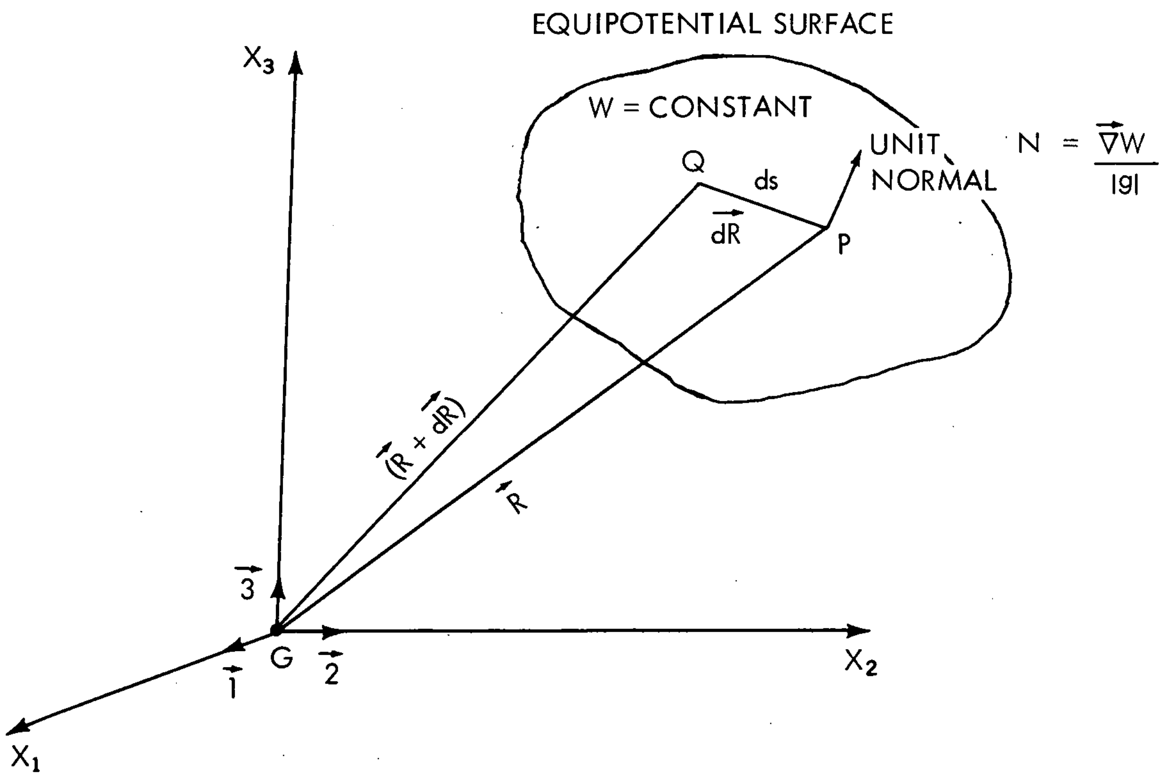

Figure 1. Gravity and Its Potential.

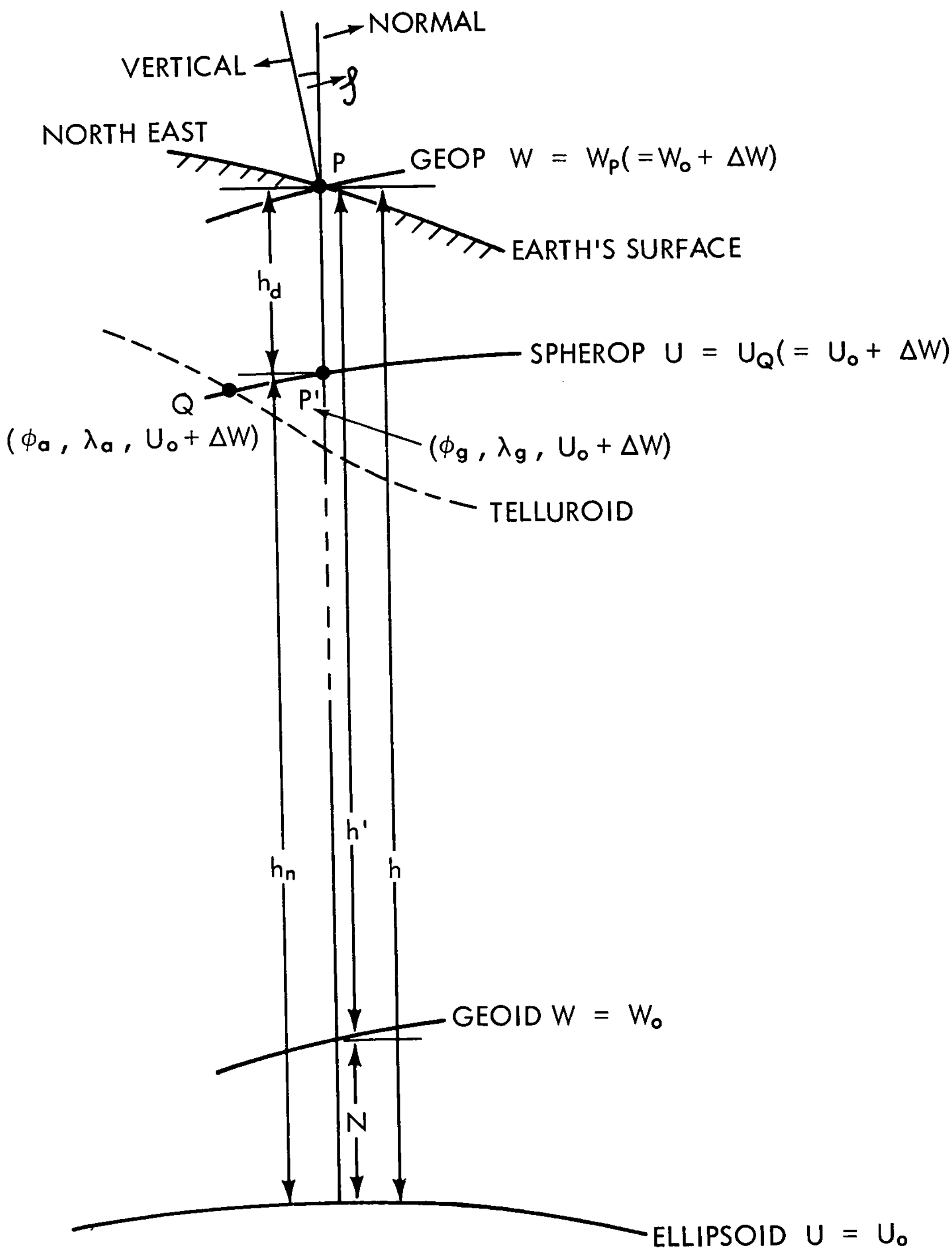

Figure 2. The Disturbing Potential at the Surface of the Earth.

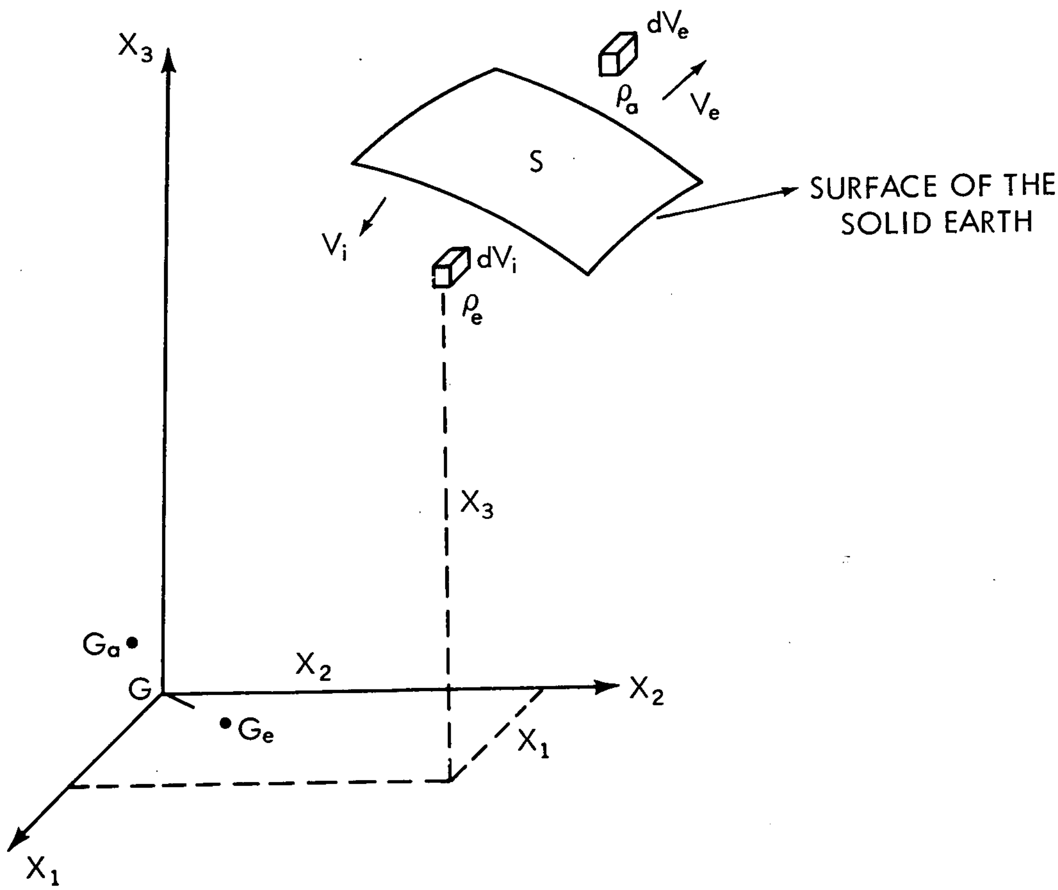

Figure 3. The First Order Inertia Tensor of the Solid Earth.

APPENDIX

A. Relations on the Ellipsoid

The relation between geocentric and geodetic coordinates are readily available in many texts on geometrical geodesy. These expressions do not, as a rule, consider the effects of the topography. Given an ellipsoid of revolution of equatorial radius a and flattening f, the geocentric latitude ϕ_{co} of a point P_0 on the reference ellipsoid, is related to its geodetic latitude by the formula

$$\delta\phi = \phi - \phi_{co} = \tan^{-1}\left[\frac{\tan\phi - \tan\phi_{co}}{1 + \tan\phi\tan\phi_{co}}\right]$$

As

$$\tan\phi_{co} = (1 - f)^2 \tan\phi, \qquad \delta\phi = \tan\delta\phi + o\{f^3\}.$$

Thus

$$\delta\phi = \frac{(1 - f)^{-2} - 1}{1 + 2f\sin^2\phi_c}\sin\phi_c\cos\phi_c$$

$$= f\sin 2\phi_c\left[1 + \frac{3}{2}f - 2f\sin^2\phi_c + o\{f^2\}\right] \qquad \text{(A-1)}$$

Further, if the elevation of the point P at the surface of the Earth, above P_0 is h, as illustrated in figure A-1 and if $PGP_0 = \delta\phi_c$, where G is the geocenter, assumed coincident with the center of the ellipsoid, it follows that

$$\frac{\sin\delta\phi_c}{h} = \frac{\sin\delta\phi}{R}$$

where R is the spherical coordinate of P on a geocentric spherical system of coordinates (R, ϕ_c, λ). On using equation A-1,

$$\delta\phi_c = \frac{h}{R} f \sin 2\phi_c + o\{f^3\} \tag{A-2}$$

Also,

$$R = R_0 \cos \delta\phi_c + h \cos (\delta\phi - \delta\phi_c)$$

$$= R_0 + h + o\{f^3 R\} \tag{A-3}$$

where R_0 is the distance to P_0 from the geocenter G. For most practical purposes,

$$R_0 = a\,[1 - f \sin^2 \phi_c + o\{f^2\}],$$

and the mean radius R_m of the Earth ellipsoid is given by

$$R_m = a \left[1 - \frac{1}{3} f + o\{f^2\}\right].$$

The combination of the above three relations gives

$$R_0 = R_m \left[1 + f \left(\frac{1}{3} - \sin^2 \phi_c\right) + o\{f^2\}\right] \tag{A-4}$$

and

$$R = R_m (1 + c_R) \tag{A-5}$$

where

$$c_R = \frac{h}{R_m} + f\left(\frac{1}{3} - \sin^2 \phi_c\right) + o\{f^2\} \tag{A-6}$$

The distance r between the surface element dS at $Q(R, \phi_c, \lambda)$ and the point of computation at $P(R_p, \phi_{cp}, \lambda_p)$ can be related to the angle subtended by the geocentric radii GP ($= R_p = R + \Delta R$) and GQ ($= R$) at G, as illustrated in figure A-2. As GP and GQ lie in the plane of the meridians through P and Q respectively, the angle $d\lambda$ between the meridian planes is given by

$$d\lambda = \lambda - \lambda_p \tag{A-7}$$

Thus

$$\psi = \cos^{-1} [\sin \phi_{cp} \sin \phi_c + \cos \phi_{cp} \cos \phi_c \cos d\lambda] \tag{A-8}$$

without approximation.

The Term x_3/r^2

The term x_3/r^2 in equation 38 is obtained from figure A-3 as

$$\frac{x_3}{r^2} = \frac{R \cos \delta - R_p \cos (\psi + \delta)}{r^2} \tag{A-9}$$

where

$$\delta = (\delta\phi - \delta\phi_c) \cos \alpha_\sigma$$

$$= f \sin 2\phi_c \cos \alpha_\sigma + o\{f^2\} \tag{A-10}$$

α_σ being the azimuth of P from the element of surface dS at Q. The distance r is given by

$$r^2 = R^2 + R_p^2 - 2R\,R_p \cos\psi = (R_p - R)^2 + 4R\,R_p \sin^2 \frac{1}{2}\psi$$

$$= (\Delta R)^2 + 4R_m^2 \sin^2 \frac{1}{2}\psi\,(1 + c_R + c_{Rp}),$$

where c_R is defined by equation A-6 and ΔR is given by

$$\Delta R = R_m\,(c_{Rp} - c_R) = R_m\, f(\sin^2 \phi_c - \sin^2 \phi_{cp}) + h_p - h + o\{f^2R\} \qquad \text{(A-11)}$$

On defining r_0 by the relation

$$r_0 = 2R_m \sin \frac{1}{2}\psi \qquad \text{(A-12)}$$

the expression for r^2 can be written as

$$r^2 = r_0^2\,(1 + c_r) \qquad \text{(A-13)}$$

where

$$c_r = c_{Rp} + c_R + \left(\frac{\Delta R}{r_0}\right)^2 + o\{f^2\} \qquad \text{(A-14)}$$

The expression for x_3/r^2 should be put in the form

$$\frac{x_3}{r^2} = \frac{1}{2R}\,(1 + \Phi) \qquad \text{(A-15)}$$

to facilitate the recovery of Stokes' integral from equation 38. The structure of Φ for purposes of numerical computations is dependent on the magnitude of ΔR and ψ. The third term in equation A14 could exceed unity when $h_p >> h$ for limiting values of ψ. To avoid loss of generality, it is preferable to retain closed expressions at this stage of the development. Equation 15 could be rewritten as

$$\Phi = \frac{2R\,x_3}{r^2} - 1 \tag{A-16}$$

The use of equations A-6, A-9 and A-13 in equation A-16 gives

$$\Phi = \frac{2R_m^2\,(1 + c_R)}{r_0^2\,(1 + c_r)}\,[(1 + c_R)\cos\delta - (1 + c_{Rp})\cos(\psi + \delta)] - 1$$

$$= \frac{2R_m^2}{r_0^2}(1 + c_r)^{-1}\left[2\sin^2\frac{1}{2}\psi + 2c_R + \delta\sin\psi - c_{Rp}\cos\psi + o\{f^2\}\right] - 1 \tag{A-17}$$

as $\delta = o\{f\}$.

Simplified working expressions for Φ are obtained by fixing maximum magnitudes for $(\Delta R/r_0)^2$. For example, if $[(h_p = h)/r_0]^2 = o\{f\}$, for small r_0, $c_r = o\{f\}$. Hence, equation A-17 can be written as

$$\Phi = \frac{2R_m^2}{r_0^2}\left[2c_R + \delta\sin\psi - c_{Rp}\cos\psi - 2c_r\sin^2\frac{1}{2}\psi - o\{f^2\}\right] \tag{A-18a}$$

Alternately, if $c_r = o\{10^{-1}\}$

$$\Phi = \frac{2R_m^2}{r_0^2}\left[(1 - c_r + c_r^2 + o\{f\})\,(2c_R + \delta\sin\psi - c_{Rp}\cos\psi + o\{f^2\}) + 2\sin^2\frac{1}{2}\psi\sum_{i=1}^{5}(-1)^i\,c_r^i + o\{f^2\}\right] \tag{A-18b}$$

It would be advisable to use equation A-17 for the evaluation of Φ when c_r takes larger values. In conclusion, Φ is usually a small quantity, except under limiting conditions when ψ and hence r is small, together with the elevation of P differing considerably from that of the surrounding topography. If terms of the order of the flattening were neglected, the contribution of this term arises only from elements with small ψ and great differences in $(h - h_p)$, when equation A-17 can be written as (Mather 1971b, p. 81)

$$\Phi = \frac{R_m}{r_0^2} [h - h_p - R_m (c_r - c_r^2 + o\{f\}) + o\{fh\}] \tag{A-19}$$

The expressions for Φ given at equations A-17 to A-19 can be programmed without problems to the required order of accuracy.

The Terms x_α/r^2

Let the angle between the line QP and the x_1x_2 plane be χ, as illustrated in figure A-3. Then PQG = $1/2\pi - (\chi - \delta)$ and QPG = $1/2\pi - (\psi - \chi + \delta)$. The application of sine formula to triangle PQG gives

$$\frac{r}{\sin\psi} = \frac{R_m (1 + c_R)}{\cos(\psi - \chi + \delta)} = \frac{R_m (1 + c_{Rp})}{\cos(\chi - \delta)} .$$

Thus

$$\cos(\psi - \chi + \delta)(1 + c_{Rp}) = \cos(\chi - \delta)(1 + c_R) \tag{A-20}$$

It follows from the use of equation A-20 on referral to figure A-3, that

$$\frac{x_\alpha}{r^2} = \frac{r \cos \chi \cos A'_\alpha}{r^2} = \frac{R \sin \psi}{r^2} \frac{\cos \chi}{\cos (\psi - \chi - \delta)} \cos A'_\alpha,$$

where

$$A'_1 = \alpha_\sigma \quad \text{and} \quad A'_2 = \frac{1}{2}\pi - \alpha_\sigma \tag{A-21}$$

α_σ being the azimuth of P from dS at Q. Equation A-20 also indicates that

$$\frac{\cos (\chi - \delta)}{\cos (\psi - \chi + \delta)} = 1 + \frac{\Delta R}{R_m} + o\{f^2\} \tag{A-22}$$

On ignoring terms of order f in equation A-22,

$$\cos \chi = \cos (\psi - \chi) + o\{f\}$$

or

$$\chi = \frac{1}{2}\psi + o\{f\},$$

provided ψ is not of the same order of magnitude as δ. Let

$$\frac{\cos \chi}{\cos (\psi - \chi + \delta)} = 1 + c_x \tag{A-23}$$

where c_x is usually a small quantity of order f except when $\psi = o\{\delta\}$. Also define χ by the equation

$$\chi = \frac{1}{2}\psi - \theta \tag{A-24}$$

The use of equations A-22 and A-23 gives

$$\cos\left(\frac{1}{2}\psi - \delta - \theta\right) = \cos\left(\frac{1}{2}\psi + \delta + \theta\right) \left[1 + \frac{\Delta R}{R_m} + o\{f^2\}\right]$$

The expansion and rearrangement of terms gives

$$\theta = \tan^{-1}\left[\frac{\cos\left(\frac{1}{2}\psi + \delta\right)\left[1 + \frac{\Delta R}{R_m}\right] - \cos\left(\frac{1}{2}\psi - \delta\right)}{\sin\left(\frac{1}{2}\psi + \delta\right)\left(1 + \frac{\Delta R}{R_m}\right) + \sin\left(\frac{1}{2}\psi - \delta\right)}\right]$$

$$= -\tan^{-1}\left[\frac{2\sin\delta\sin\frac{1}{2}\psi - \frac{\Delta R}{R_m}\cos\left(\frac{1}{2}\psi + \delta\right)}{2\cos\delta\sin\frac{1}{2}\psi + \frac{\Delta R}{R_m}\sin\left(\frac{1}{2}\psi + \delta\right)} + o\{f^2\}\right] \qquad \text{(A-25)}$$

If ψ is not a small angle, θ can be expressed as

$$\theta = \tan^{-1}\left[\frac{\frac{\Delta R}{R_m}\cos\frac{1}{2}\psi - 2\delta\sin\frac{1}{2}\psi + o\{f^2\}}{2\sin\frac{1}{2}\psi\left(1 + \frac{\Delta R}{2R_m} + o\{f^2\}\right)}\right]$$

or

$$\theta = \frac{1}{2}\frac{\Delta R}{R_m}\cot\frac{1}{2}\psi - \delta + o\{f^2\} \qquad \text{(A-26)}$$

The use of equations A-23 and A-24 gives

$$c_x = \frac{\cos\left(\frac{1}{2}\psi - \theta\right)}{\cos\left(\frac{1}{2}\psi + \delta + \theta\right)} - 1 \qquad \text{(A-27)}$$

For large ψ when $\theta = o\{f\}$,

$$c_x = \frac{\cos\frac{1}{2}\psi + \theta\sin\frac{1}{2}\psi}{\cos\frac{1}{2}\psi - (\theta+\delta)\sin\frac{1}{2}\psi} - 1$$

$$= 1 + \theta\tan\frac{1}{2}\psi + (\theta+\delta)\tan\frac{1}{2}\psi + o\{f^2\} - 1$$

$$= (2\theta + \delta)\tan\frac{1}{2}\psi + o\{f^2\} \qquad \text{(A-28)}$$

In conclusion, the use of these results in equation A-21 gives

$$\frac{x_\alpha}{r^2} = \frac{R\sin\psi}{r^2}(1 + c_x)\cos A'_\alpha \qquad \text{(A-29)}$$

where

$$c_x = \frac{\cos\left(\frac{1}{2}\psi - \theta\right)}{\cos\left(\frac{1}{2}\psi + \theta + \delta\right)} - 1,$$

and, for most practical purposes,

$$\theta = \frac{1}{2}\frac{\Delta R}{R_m}\cos\frac{1}{2}\psi - \delta + o\{f^2\} \qquad \text{(A-30)}$$

<u>The Horizontal Gradients of Potential</u>

The horizontal gradients of potential are obtained as follows. As V'_d is given by equation 33,

$$V'_d = W_0 - U_0 - V_a + \gamma h_d,$$

differentiation with respect to x gives

$$\frac{\partial V_d'}{\partial x_\alpha} \tan \beta_\alpha = -\frac{\partial V_a}{\partial x_\alpha} \tan \beta_\alpha + h_d \frac{\partial \gamma}{\partial x_\alpha} \tan \beta_\alpha + \gamma \frac{\partial h_d}{\partial x_\alpha} \tan \beta_\alpha \qquad \text{(A-31)}$$

on expressing the differential as it appears in equation 39. As changes with respect to x relate to the spherop $U = U_Q$, it follows that (e.g., Heiskanen and Moritz, 1967, p. 313)

$$\frac{\partial h_d}{\partial x_\alpha} = -\xi_\alpha,$$

where ξ_α are the components of the surface deflection of the vertical in the directions x_α.

The horizontal gradient of normal gravity is zero in the direction of the x_2 axis. The gradient along the x_1 axis is obtained from the formula for normal gravity (see entry against β_1 in section 1.2.1) as

$$\frac{\partial \gamma}{\partial x_1} = \gamma_e \beta_1 \sin 2\phi_c / R = o\{10^{-5} \text{ mgal cm}^{-1}\}.$$

thus the term

$$h_d \frac{\partial \gamma}{\partial x_1} \tan \beta_1 = o\{10^{-1} \tan \beta_1 \text{ mgal}\}$$

and cannot be ignored in either mountainous regions or areas where the height anomalies have significant magnitude.

The term $\partial V_a / \partial x_\alpha$ is purely a function of the topographical gradient, the change being approximately 20 μgal for a 250 m change in elevation (IAG 1970, p. 72). This is three orders of magnitude smaller than the change in Δg and is not of significance in the context of the other terms involved. Thus for the present development

$$\frac{\partial V_d'}{\partial x_\alpha} \tan \beta_\alpha = - \gamma \xi_\alpha \tan \beta_\alpha + h_d \frac{\partial \gamma}{\partial x_1} \tan \beta_1 \qquad \text{(A-32)}$$

the second term being two orders of magnitude smaller than the first.

B. Spherical Harmonic Expansions and the Geodetic Boundary Value Problem

B.1 The Product of Two Surface Harmonics

Let

$$S_{nm} = P_{nm} (\sin \phi_c) (C_{1_{nm}} \cos m\lambda + C_{2_{nm}} \sin m\lambda)$$

and

$$S'_{\ell k} = P_{\ell k} (\sin \phi_c) (C'_{1_{\ell k}} \cos k\lambda + C'_{2_{\ell k}} \sin k\lambda)$$

be two surface harmonics. The product

$$S_{nm} S'_{\ell k} = P_{nm} (\sin \phi_c) P_{\ell k} (\sin \phi_c) [C_{1_{nm}} C'_{1_{\ell k}} \cos m\lambda \cos k\lambda + C_{1_{nm}} C'_{2_{\ell k}} \cos m\lambda \sin k\lambda$$

$$+ C_{2_{nm}} C'_{1_{\ell k}} \sin m\lambda \cos k\lambda + C_{2_{nm}} C'_{2_{\ell k}} \sin m\lambda \sin k\lambda]$$

$$= P_{nm} (\sin \phi_c) P_{\ell k} (\sin \phi_c) \left[\frac{1}{2} \left(C_{1_{nm}} C'_{1_{\ell k}} - C_{2_{nm}} C'_{2_{\ell k}} \right) \cos (m + k) \lambda \right.$$

$$+ \frac{1}{2} \left(C'_{1_{\ell k}} C_{2_{nm}} + C_{1_{nm}} C'_{2_{\ell k}} \right) \sin (m + k) \lambda + \frac{1}{2} \left(C_{1_{nm}} C'_{1_{\ell k}} + C_{2_{nm}} C'_{2_{\ell k}} \right) \cos (m - k) \lambda$$

$$\left. + \frac{1}{2} \left(C'_{1_{\ell k}} C_{2_{nm}} - C_{1_{nm}} C'_{2_{\ell k}} \right) \sin (m - k) \lambda \right].$$

Further (e.g., Mather 1971c, p. 47), if $\sin \phi_c = \mu$,

$$P_{nm} (\mu) = \frac{1}{2^n} (1 - \mu^2)^{1/2\, m} \sum_{r=0}^{k'} (-1)^r \frac{(2n - 2r)!}{r!\,(n - m - 2r)!\,(n - r)!} \mu^{n-m-2r},$$

where $k' = 1/2\,(n - m)$ if $(n - m)$ is even or $1/2\,(n - m - 1)$ if $(n - m)$ is odd.

Hence

$$P_{nm} (\mu) P_{\ell k} (\mu) = \sum_{s=0}^{p} A_s \mu^s,$$

where $p = n + \ell$. It should therefore be possible to represent the above product by a relation of the form

$$P_{nm} (\mu) P_{\ell k} (\mu) = \sum_{i=0}^{p} \sum_{j=0}^{i} B_{ij} P_{ij} (\mu).$$

It would suffice for the purposes of the present study to draw the conclusion that the product $S_{nm} S'_{\ell k}$ can be fully represented by the set of surface harmonics

$$S_{nm}\, S'_k = \sum_{p=0}^{\infty} S''_p ,$$

where

$$S''_p = \sum_{q=0}^{p} P_{pq}\,(\sin\phi_c)\;[A_{pq}\cos q\lambda + B_{pq}\sin q\lambda].$$

B.2 The Orthogonal Property of Surface Harmonics over a Closed Surface

It is well known that two surface harmonics of different degrees satisfy the relation

$$\iint_S S_n\, S'_\ell\, dS = 0 \quad \text{if} \quad n \neq \ell.$$

This orthogonal property is usually derived in the case when S is a sphere, the derivation being a consequence of the surface harmonic constituting a spherical harmonic term (e.g., Jeffreys and Jeffreys 1962, p. 636). This property could be extended to cover any continuous closed surface, e.g., the physical surface of the Earth. It must be assumed that the surface harmonic expansion exists and provides unique definition at all points on the surface, the integration being taken over the element of solid angle between the appropriate limits defining the entire surface. This would imply that any set $\{\phi_c, \lambda\}$ would define a unique point on the surface. If $\sin\phi_c = \mu$,

$$S_{nm} = P_{nm}(\mu)\left[C_{1_{nm}}\cos m\lambda + C_{2_{nm}}\sin m\lambda\right]$$

and

$$S'_{\ell k} = P_{\ell k}(\mu)\left[C'_{1_{\ell k}} \cos k\lambda + C'_{2_{\ell k}} \sin k\lambda\right],$$

the surface integral

$$S_{nm} S'_k \, d\sigma = \int_{-1}^{1} \int_0^{2\pi} P_{nm}(\mu) P_{\ell k}(\mu)\left[C_{1_{nm}} C'_{1_{\ell k}} \cos m\lambda \cos k\lambda + C_{1_{nm}} C'_{2_{\ell k}} \cos m\lambda \sin k\lambda\right.$$

$$\left. + C_{2_{nm}} C'_{1_{\ell k}} \sin m\lambda \cos k\lambda + C_{2_{nm}} C'_{2_{\ell k}} \sin m\lambda \sin k\lambda\right] d\mu \, d\lambda$$

$$= 0 \quad \text{if} \quad m \neq k, \quad \text{as}$$

$$\int_0^{2\pi} \cos m\lambda \cos k\lambda \, d\lambda = \int_0^{2\pi} \sin m\lambda \sin k\lambda \, d\lambda = \int_0^{2\pi} \cos m\lambda \sin k\lambda \, d\lambda = 0.$$

If m = k, two terms remain, as

$$\int_0^{2\pi} \cos^2 m\lambda \, d\lambda = \int_0^{2\pi} \frac{1 + \cos 2m\lambda}{2} \, d\lambda = \int_0^{2\pi} \sin^2 m\lambda \, d\lambda$$

$$= \int_0^{2\pi} \frac{1 - \cos 2m\lambda}{2} \, d\lambda = \pi.$$

Therefore

$$\iint_\sigma S_{nm} S'_{\ell k} \, d\sigma = \left[C_{1_{nm}} C'_{1_{\ell m}} + C_{2_{nm}} C'_{2_{\ell m}}\right] \int_{-1}^{1} P_{nm}(\mu) P_{\ell m}(\mu) \, d\mu.$$

The integral can be evaluated by using the relation (e.g., Hobson 1965, p. 99)

$$P_{nm}(\mu) = (-1)^m \frac{(n+m)!}{(n-m)!} P_{n(-m)}(\mu),$$

and assuming that $n > \ell$, when

$$\int_{-1}^{1} P_{nm}(\mu) P_{\ell m}(\mu)\, d\mu = \frac{(-1)^m (n+m)!\,(1-\mu^2)^{(m-m)/2}}{2^{n+\ell}(n-m)!\,n!\,\ell!} \int_{-1}^{1} \frac{d^{n-m}}{d\mu^{n-m}}[(\mu^2-1)^n] \cdot \frac{d^{\ell+m}}{d\mu^{\ell+m}}[(\mu^2-1)^\ell]\, d\mu$$

On integrating the above equation (n − m) times by parts,

$$\int_{-1}^{1} P_{nm}(\mu) P_{\ell m}(\mu)\, d\mu = \frac{(-1)^n (n+m)!}{2^{n+\ell}\, n!\, \ell!\, (n-m)!} \int_{-1}^{1} (\mu^2-1)^n \frac{d^{\ell+n}}{d\mu^{\ell+n}}[(\mu^2-1)^\ell]\, d\mu$$

$$= 0 \quad \text{as} \quad n > \ell,$$

and the non-integrated product being zero on evaluation at the limits of integration, due to always having a factor $(\mu^2 - 1)$ (e.g., Mather 1971c, p. 51).

If $n < \ell$, then replace $P_{\ell m}(\mu)$ by $P_{\ell(-m)}(\mu)$ and proceed as before.

It can therefore be concluded that in all circumstances of integration over a closed surface on which a set of surface harmonics provide unique definition,

$$\iint_\sigma S_n S_\ell'\, d\sigma = 0, \quad n \neq \ell.$$

Thus the orthogonal properties of surface harmonics apply on integration over any closed surface which is single valued in the set $\{\phi_c, \lambda\}$.

B.3 Laplace's Equation at the Surface of the Earth

Attention has been drawn in note (i) to section 2.4, to the fact that the disturbing potential V_d, as defined by equation 5, does not satisfy Laplace's equation, but the function V_d', given by

$$V_d' = V_d - V_a,$$

where V_a is the potential due to the Earth's atmosphere, does so at all points exterior to the Earth's surface. S in equation 29 is the physical surface of the Earth, in the strictest sense.

The development in section 3 requires that V_d' be capable of representation by a set of spherical harmonics at all points on S. This cannot be claimed to be the case if S is the exact physical surface of the Earth. On the other hand, V_d' satisfies Laplace's equation at all points exterior to the Earth and right down to it. Thus V_d' can be expressed by a set of spherical harmonics at all points in space exterior to the Earth's surface and right down to but not <u>on</u> it, provided the reference ellipsoid is everywhere within the physical surface of the Earth.

Another important point is that physical manifestations of V_d' are not measured <u>at</u> the physical surface of the Earth but slightly exterior to it. Thus the surface S being defined in equation 29 is not the physical surface of the Earth itself, as a consequence of the nature of the observational data used to solve the problem, the former being measured slightly exterior to the physical surface of the Earth.

It can be concluded that physical validity exists for the representation of the scalar characteristics of the Earth's gravitational field by a spherical harmonic series), if based on the observations made "at" and exterior to the physical surface of the Earth. In the context of the present study which seeks resolution at the ±5 cm level, no significant errors occur if S is taken as a surface which is always slightly exterior to the physical surface of the Earth. Such a surface has the advantage that V_d' satisfies Laplace's equation at all points on it and therefore physical validity exists for adopting the representation

$$V_d' = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} + o\{f^2\}, \quad n \neq 1.$$

No common convention has been adopted for the definition of a "surface of measurement" for gravity determinations vis-a-vis the physical surface of the Earth. The accuracy requirements defined in section 4.3 for resolution of the geodetic boundary value problem at the ±5 cm level, call for the representation of the global gravity anomaly field so that systematic errors over large extents held to below ±50 μ gal. This in turn requires the establishment of the global gravity control network with individual station accuracy at this same level. The surface of measurement, which is only relevant when defining the global control net, can be deviated from by individual observation stations by ±10 cm without affecting solutions of the boundary value problem to order e^3.

C. A Non-Iterative Procedure for Evaluating the Gravitational Terms in the Solution of the Geodetic Boundary Value Problem

Section 4.2 shows that iterative techniques are necessary for solving the boundary value problem to order e^3. This is not an economic procedure and it is compelling to search for a non-iterative solution. The basic boundary condition is

$$V_{dp} = V_{ap} + \frac{1}{2\pi}\iint \frac{R^2}{r}\left(V_d'\left[\frac{x_\alpha \tan\beta_\alpha}{r^2} - \frac{x_3}{r^2}\right] + \frac{\partial V_d'}{\partial x_\alpha}\tan\beta_\alpha - \frac{\partial V_d'}{\partial h}\right) d\sigma \quad \text{(A-35)}$$

The first set of terms is obtained from equation A-29, while the second can be obtained from A-9 as

$$\frac{x_3}{r^2} = \frac{R}{r^2}\left(1 + c_{x_3}\right) \quad \text{(A-33)}$$

where

$$c_{x_3} = \delta \sin\psi - (1 + c_{Rp} - c_R)\cos\psi + o\,\{f^2\} \quad \text{(A-34)}$$

The third term, defined by equation A-32, cannot be satisfactorily included in the major gravitational term, even though it has significant magnitude. The last term is given by equation 34, the quantity h_d from equations 11 and 25 as

$$h_d = \frac{1}{\gamma}\left[V_d' + V_a - (W_0 - U_0)\right] \quad \text{(A-35)}$$

and equation 43 can be written as

$$\frac{\partial \gamma}{\partial h} = -\frac{2\gamma}{R}(1 + c_{\phi}) \tag{A-36}$$

where

$$c_{\phi} = f + m - 3 f \sin^2 \phi_c + o\{f^2\} \tag{A-37}$$

The combination of these relations gives

$$V_{dp} = V_a + I_A + I_B \tag{A-38}$$

where

$$I_A = \frac{1}{2\pi}\iint \frac{R^2}{r}\left[\frac{V_d'}{R} F(\psi, h) - \frac{\partial V_d'}{\partial h}\right] d\sigma \tag{A-39}$$

$$F \doteq F(\psi, h) = \left(\frac{R}{r}\right)^2 \left[\sin\psi (1 + c_x)\frac{dh}{dr} - \left(1 + c_{x_3}\right)\right] \tag{A-40}$$

and

$$I_B = \frac{1}{2\pi}\int\int \frac{R^2}{r}\left[-\gamma \xi_{\alpha} \tan\beta_{\alpha} + h_d \frac{\partial \gamma}{\partial x_1} \tan\beta_1\right] d\sigma \tag{A-41}$$

On adopting a spherical harmonic representation for V_d', the validity of which has been established in section B3,

$$V_d' = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}}, \quad n \neq 1$$

equation A-39 can be written as

$$I_A = \frac{1}{2\pi}\int\int \frac{R^2}{r} \sum_{n=0}^{\infty} (n + 1 + F) \frac{A_n}{R^{n+1}} \, d\sigma \quad n \neq 1 \tag{A-42}$$

The gravity anomaly is related to the disturbing potential through equations 13, A-35 and A-36 as

$$\Delta g = -\frac{\partial V_d'}{\partial h} - \frac{2 V_d'}{R}(1 + c_\phi) + c_{\Delta g} \tag{A-43}$$

where

$$c_{\Delta g} = \frac{1}{2} g \zeta^2 + 2 \frac{W_0 - U_0}{R} - 2 \frac{V_a}{R} - \frac{\partial V_a}{\partial h} + o\{f^2 \Delta g\} \tag{A-44}$$

Equation A-43 can be expressed in spherical harmonics as

$$\Delta g = \sum_{n=0}^{\infty} (n - 1 - 2 c_\phi) \frac{A_n}{R^{n+2}} + c_{\Delta g}, \quad n \neq 1 \tag{A-45}$$

The spherical harmonic function as evaluated at the surface of the Earth, can be expressed a set of surface harmonics for the reasons given in section 3. Let

$$\Delta g' = \sum_{n=0}^{\infty} G_n = \sum_{n=0}^{\infty} (n - 1 - 2 c_\phi) \frac{A_n}{R^{n+1}} \; : \; n \neq 1 = \Delta g - c_{\Delta g} \tag{A-46}$$

If conventional practice is to be followed, it will be necessary to prove that the replacement

$$A_n = \frac{R^{n+1}\, G_n}{n - 1 - 2\, c_\phi} \qquad n \neq 1 \tag{A-47}$$

is valid, in which case I_A can be written as

$$I_A = \frac{1}{2\pi} \iint \frac{R^2}{r} \sum_{n=0}^{\infty} \frac{n + 1 + F}{n - 1 - 2\, c_\phi}\, G_n \, d\,\sigma, \qquad n \neq 1 \tag{A-48}$$

R^2/r can be expressed as a set of surface harmonics for the reasons given in section B.1 of the Appendix as

$$\frac{R^2}{r} = R_m \sum_{n=0}^{\infty} S_n .$$

A non-iterative solution for the gravitational terms could only be obtained if I_A can be transformed into an expression of the type

$$I_A = \frac{R_m}{2\pi} \sum_{n=0}^{\infty} \frac{n + 1 + F}{n - 1 - 2\, c_\phi} \qquad S_n\, \Delta\, g_c \, d\,\sigma, \qquad n \neq 1.$$

This would be possible only if F and c_ϕ were unchanged on surface integration. As this is not the case, it would appear that it is not possible to solve the geodetic boundary value problem without resorting to iteration when evaluating the expressions containing the gravitational terms.

D. The Error of Representation of Gravity Anomalies at the Surface of the Earth

The error of representation $E\{\Delta g\}_{nm}$ for an n° × m° area at the surface of the Earth is given by

$$(E\{\Delta g\}_{nm})^2 = \sum_{i=1}^{N} \frac{(\Delta g_i - \overline{\Delta g})^2}{N} \tag{A-49}$$

where the Δg_i are individual determinations of the gravity anomaly at N points within the area, and $\overline{\Delta g}$ is given by

$$\overline{\Delta g} = \frac{1}{N} \sum_{i=0}^{N} \Delta g_i \tag{A-50}$$

For a meaningful estimation, $E\{\Delta g\}_{nm}$ must be the mean of several such evaluations. Further, the gravity stations must be evenly distributed about the region center with N being very large. Estimates of the error of representation for various square sizes by several researchers are given in table A-1. Linear units have been converted to equivalent angular values using 10 km = 0.09 degrees. The figures given in table A-1 are heavily, if not totally biased toward continental areas and with two exceptions, to regions where topographical gradients are small. It should be noted that no correlation is implied between the value of $E\{\Delta g\}$ and elevation. Thus $E\{\Delta g\}$ for an elevated plateau should have a magnitude similar to that for a coastal plain. In rugged mountainous terrain, $E\{\Delta g\}$ can be 3 to 10 times as great, especially for smaller regions. This should not however preclude the use of $E\{\Delta g\}$ to represent the statistical characteristics of the global gravity anomaly field for error estimation purposes. For example, in such cases, $E\{\Delta g\}_{0.1}$ can be as much as 10 times greater than the value (±3 mgal) in flat areas, the effect being confined to 3% of the Earth's surface area where rugged topography occurs. This would increase e_{ta} in equation 116 by a factor of 10 while e_{NA} in equation 117 will be twice as large.

Table A-1

Error of Representation for Free Air Anomalies

Source	Square Size n = m (degrees)	Latitude (degrees)	$E\{\Delta g\}_n$ mgal	Region
	0	0.0		
MO	0.05	0	1.5	USSR
H	0.1	0	2.8	Finland
MO	0.1	0	2.8	USSR
H	0.2	30(1)	5.4	Global
MO	0.2	0	4.4	USSR
MO	0.3	0	7.0	USSR (Plains)
MO	0.3	0	10	Urals
MO	0.3	0	25	Caucasus Mountains, USSR
H	0.5	30(1)	9.0	Global
M	0.5	30*	10.1	Australia
MO	0.6	0	10.1	USSR
H	1	30(1)	12.7	Global
H & M	1	45	12.4	
M	1	30*	13.5	Australia
MO	1.1	0	13.3	USSR
MO	1.6	0	16.0	USSR
H	2	30(1)	17.6	Global
H&M	2	45	20.8	
M	2	30*	17.7	Australia
MO	2.2	0	16.3	USSR
H	5	30(1)	23.1	Global
H&M	5	45	27.6	
H	10	30(1)	26.6	Global
H&M	10	45	29.3	

Key: -1. Source H = Hirvonen 1956, p. 3.
M = Mather 1967, p. 131
H&M= Heiskanen & Moritz 1967, p. 279
MO = Molodenskii et al 1962, p. 172

Col. 3: (1) = based on global sample
* = mean latitude for region of studies
0 = converted from data for squares with equidistant sides

Table 1

Range of ψ for Linear Variations in $f(\psi)$ and $F(\psi)$ to Order e^3

N = Number of contributions t as at equation 100 to the quadratures evaluation of N_f

Square size (in degrees) n	$f(\psi)$ Range of ψ (in degrees)	$f(\psi)$ N ($\times 10^5$)	$F(\psi)$ Range of ψ (in degrees)	$F(\psi)$ N ($\times 10^5$)
0.001	$\psi > 0.07$	10.0	$\psi > 0.0$	–
0.005	$\psi > 0.5$	0.6	$\psi > 0.0$	–
0.01	$\psi > 0.8$	3.2	$\psi > 0.0$	0.0
0.05	$\psi > 3$	0.4	$\psi > 0.2$	0.0
0.1	$\psi > 6$	0.1	$\psi > 0.4$	0.2
0.2	$\psi > 13$	3.4	$\psi > 2$	16.2
0.5	$\psi > 60$	2.3	–	–
	N_t	20.0		16.4

Table 2

Estimates of Systematic and Random Error Effects on the

Computation of Stokes' Integral

n (in degrees)	Maximum tolerable Systematic Error in Δg Over Range n ($\pm$ mgal)	Error in N_f due to $E\{\Delta g\}_n$ ($\pm$ cm)
0.01	50	0.03
0.1	5	3
0.5	1	50
1.0	0.5	120
5.0	0.1	1400

FIGURE CAPTIONS

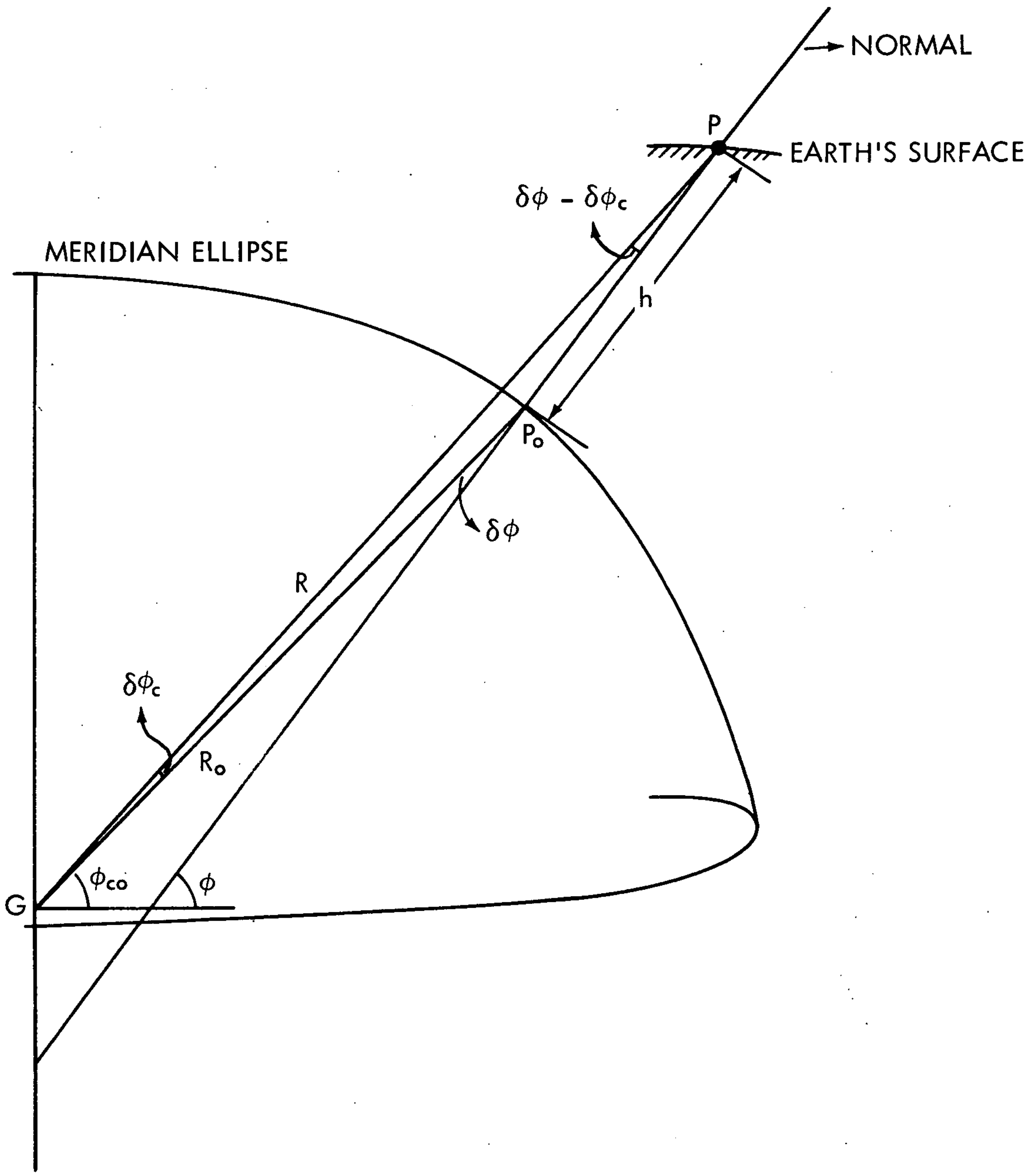

Figure A-1. The Meridian Ellipse and the Topography.

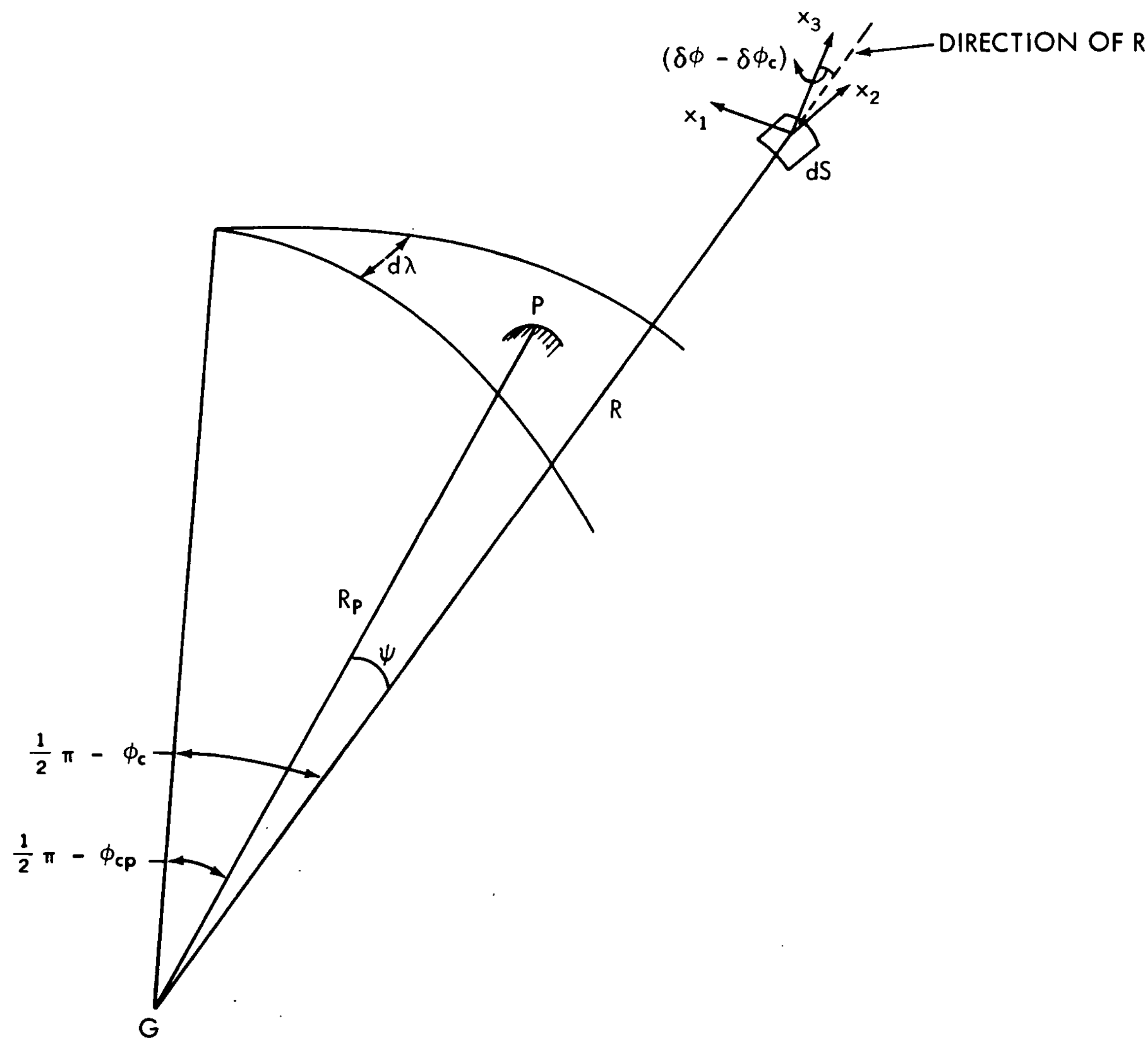

Figure A-2. The Spherical and Ellipsoidal Co-ordinates.

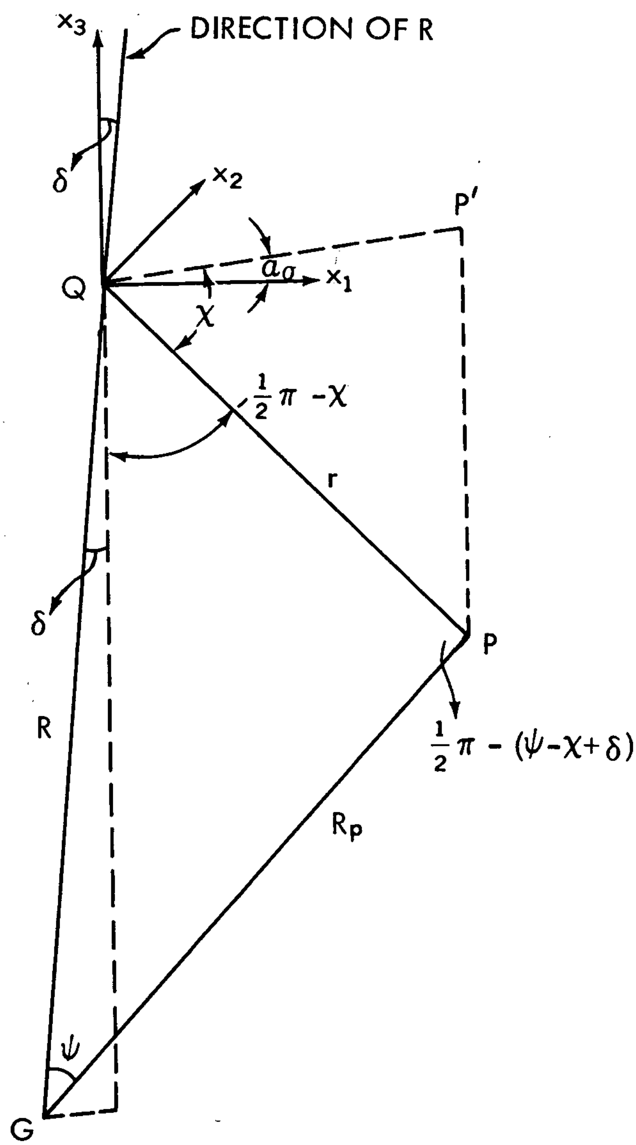

Figure A-3. The x_i Cartesian System in the Local Laplacian Triad and Geocentric Spherical Co-ordinates.

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