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EFFECT OF BOTTOM TOPOGRAPHY, EDDY DIFFUSIVITY, AND WIND VARIATION ON CIRCULATION IN A TWO-LAYER STRATIFIED LAKE

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16. Abstract

The steady-state, wind-driven circulation is calculated in a stratified lake composed of two layers having uniform but unequal densities and eddy diffusivities. The position of the thermocline and the velocities in both layers are calculated from an asymptotic solution of the shallow lake equations when the Ekman number in the epilimnion (upper layer) is of order one but the ratio of hypolimnion (lower layer) to epilimnion eddy diffusivities is much less than one. Large differences in the thermocline shape and the velocities occur between the solution for uniform wind stress and the one for unit order wind stress gradients. For the latter solution the hypolimnion eddy diffusivity magnitude and the bottom topography have a large and important effect.

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SUMMARY

The steady-state, wind-driven circulation is calculated in a stratified lake composed of two layers having uniform but unequal densities and eddy diffusivities. The position of the thermocline and the three dimensional velocities in both layers are calculated from an asymptotic solution of the shallow lake equations when the Ekman number in the epilimnion (upper layer) is of order one but the ratio of hypolimnion (lower layer) to epilimnion eddy diffusivities is much less than one. This analysis differs from previous ones in that the upper and bottom layers are coupled since no assumption is made about the hypolimnion velocities being negligible.

The solutions are very dependent on the wind conditions. For a uniform wind stress of moderate strength, the "zeroth order" horizontal pressure gradients in the hypolimnion are zero and the only hypolimnion velocities are in a thin boundary layer adjacent to the thermocline. For this case, the bottom topography and value of the hypolimnion eddy diffusivity only affect the solution to a small degree. However, for a wind stress with unit order gradient, small hypolimnion horizontal pressure gradients do occur producing significant hypolimnion geostrophic (inviscid) velocities. For this case, the value of the hypolimnion eddy diffusivity has a large effect on the thermocline shape. In addition, the shape of the lake bottom also becomes important. With unit order wind stress gradients, large differences in thermocline shape and horizontal velocities occur between the asymptotic solution which assumes small hypolimnion eddy diffusivity and previous solutions which uncouple the two layers by assuming the hypolimnion velocities are zero (hypolimnion eddy diffusivity very large).

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INTRODUCTION

Observations of stratified lakes suggest that during steady wind periods they may be modeled by considering the lake to be made up of two homogeneous layers each with different densities and eddy diffusivities with the interface between the two layers being located at the thermocline. This two-layer model is analyzed here. A more complete discussion of the applicability of the two-layer model is given in reference 1.

Welander (ref. 2) and many others have studied a two-layer model for the oceans which included the Ekman dynamics assumption that the shear stress at the thermocline and the ocean bottom are proportional to the geostrophic velocities in either the hypolimnion or epilimnion. Others such as Hamblin (ref. 3) have used this approach in the Great Lakes. In fresh water lakes, the thickness of the friction layer generated by the wind is of the order of or greater than the average thickness of the epilimnion; and there is no real geostrophic velocity in the epilimnion to which the shear stress at the thermocline can be made proportional. Therefore, Ekman dynamics cannot be used. In this analysis, an extension of the shallow lake equations originally derived by Welander (ref. 4) and shown by Gedney and Lick (ref. 5) to yield good quantitative results for Lake Erie during uniform temperature conditions is used.

In their two-layer analyses, Welander (ref. 2) and Hamblin (ref. 3) initially uncoupled the bottom layer from the top layer by assuming that the motion in the bottom layer can be neglected (quasi-compensation assumption). After finding the position of the thermocline with this assumption they then allow small motions in the bottom layer. This small motion is calculated using the thermocline position and shear stress calculated from the upper layer solution.

As will be shown later in this report, the quasi-compensation assumption does not appear to be valid for a two-layer lake subjected to a spatially variable wind stress. We therefore make no assumption pertaining to the hypolimnion velocity magnitude. In order to solve the governing equations we assume that the ratio of hypolimnion to epilimnion eddy diffusivity (ν_{M2}/ν_{M1}) is small and perform an asymptotic expansion in that parameter. The solution is valid for any value of the vertical Ekman number (E_{v1}) in the epilimnion. For the normal case in a large lake, E_{v1} in the epilimnion is of order 1. For moderate winds, the value of ν_{M2} which makes $\nu_{M2}/\nu_{M1} << 1$ is only 6 to 12 times that of the molecular kinematic viscosity. According to Hutchinson (ref. 6) and Sundaran, Esterbrook, Piech, and Rudinger (ref. 7) this is the correct magnitude for ν_{M2}. Larger hypolimnion eddy diffusivities than those considered here have been investigated numerically by Gedney, Lick, and Molls (ref. 1) and Lee and Liggett (ref. 8).

In order to determine the error induced by making the quasi-compensation assumption, the results of the shallow lake equations with the quasi-compensation assumption included are given and compared with the asymptotic results of the complete two-layer equations.

SYMBOLS

A_1	coefficient in eq. (7)
A_2	coefficient defined by eq. (8)
a_1	$\omega\alpha_1\xi$
a_2	$\omega\alpha_2(\xi + h)$
B, C	coefficients in equations (32) and (33)
D_2	coefficients in eqs. (9) and (10)
D_{2_ξ}	$\partial D_2/\partial\xi$
d_1	epilimnion friction depth
d_2	hypolimnion friction depth
E_{v1}	epilimnion Ekman number, $\nu_{M1}/f_c L^2$
f	arbitrary function
f_c	Coriolis parameter
f_1	unit order depth function
f_2	unit order wind stress curl function
G	coefficient in equations (32) and (33)
g	acceleration of gravity
H	coefficient in equations (32) and (33)
h	nondimensional lake depth
h_m	$\partial h/\partial m$
h_t	$\partial h/\partial t$
K	coefficient in eqs. (7) and (8)
L	reference length of lake
M_{T1}	$M_{x1} + iM_{y1}$, epilimnion volume transport
M_{T2}	$M_{x2} + iM_{y2}$, epilimnion volume transport
m	coordinate normal to boundary or curve

m_x, m_y	x- and y-components of $\hat{m}$
$\hat{m}$	unit normal vector to boundary
$\frac{\partial}{\partial n}$	$\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}$
$\frac{\partial}{\partial n^*}$	$\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$
p	nondimensional pressure
s	integer exponent
t	coordinate tangential to boundary or curve
U_w	wind velocity
u, v, w	dimensionless velocity, respectively, in x-, y-, z-directions
x, y, z	dimensionless Cartesian coordinates
α_1	$2\pi\xi_{ref}/d_1$
α_2	$2\pi\xi_{ref}/d_2$
Γ_1	$u_1 + iv_1$
Γ_2	$u_2 + iv_2$
δ	$d_2/(d_1\rho_r)$
ϵ	boundary layer stretching parameter
ζ	nondimensional surface displacement
η	$\zeta + \xi$
ν_{M1}	epilimnion eddy diffusivity of momentum
ν_{M2}	hypolimnion eddy diffusivity of momentum
ξ	nondimensional thermocline depth
ρ	density of water
ρ_a	density of air
ρ_r	ρ_1/ρ_2
$\Delta\rho$	$(\rho_2 - \rho_1)/\rho_1$
τ^w	$\tau_x^w + i\tau_y^w$, dimensionless wind shear stress $(\bar{\tau}^w/\tau_{ref}^w)$
ω	$(1 + i)/2$

Subscripts:

b	value at lake boundary
c	value at lake center
ref	dimensional reference quantity
x	x-component
y	y-component
0	value at $x = 0$, $y = 0$
1	epilimnion
2	hypolimnion

Superscripts:

(i)	imaginary part
(r)	real part
(0), (1), (2)	zeroth, first, second term in asymptotic expansion
$-$	dimensional component
$\sim$	unit order function
+	stretched coordinate

BASIC EQUATIONS AND BOUNDARY CONDITIONS

In the present analysis, the lake is considered to be composed of two layers of different density as shown in figure 1. In each layer, the basic approximations are that the water density is constant, the vertical eddy viscosity is independent of position, the pressure is hydrostatic, and the lateral friction and nonlinear acceleration terms can be neglected. Effects due to the Earth's curvature and to the variation in Coriolis force with position are neglected since the length scales of lakes are much less than the radius of the Earth. The explanation for the density and eddy diffusivity being considered constant in each layer is given in reference 1. Gedney and Lick (ref. 5) have shown the other assumptions to be good approximations for the Great Lakes.

The analysis performed here will assume a steady wind. As is well known, the location of the thermocline in a stratified lake being acted on by a steady wind will slowly become deeper. The rate of deepening has been measured in Cayuga Lake (see ref. 7) to be in the neighborhood of 15 to 30 centimeters per day. This rate is of such magnitude that the time derivative inertia terms in the momentum equations can be neglected and the problem is then "quasi-steady." With the steady wind restriction plus the other assumptions stated, the nondimensional continuity and momentum equations applicable to each layer shown in figure 1 are as shown in reference 1:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{1}$$

$$\frac{\partial u_2}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial w_2}{\partial z} = 0 \tag{2}$$

$$\frac{1}{\alpha_1^2} \frac{\partial^2 \Gamma_1}{\partial z^2} - \frac{i}{2} \Gamma_1 = \frac{1}{2\pi} \frac{\partial \zeta}{\partial n} = \frac{1}{2\pi} \frac{\partial p_1}{\partial n} \tag{3}$$

$$\frac{1}{\alpha_2^2} \frac{\partial^2 \Gamma_2}{\partial z^2} - \frac{i}{2} \Gamma_2 = \frac{\rho_r}{2\pi} \left(\frac{\partial \xi}{\partial n} + \frac{\partial \zeta}{\partial n} \right) = \frac{\rho_r}{2\pi} \frac{\partial p_2}{\partial n} \tag{4}$$

The Cartesian coordinate system used (see fig. 1) has x increasing eastward, y northward, and z vertically upwards with the corresponding velocities being u_j, v_j, and w_j where $j = 1$ indicates the epilimnion and $j = 2$ indicates the hypolimnion. In these equations $\Gamma_1 = u_1 + iv_1$, $\Gamma_2 = u_2 + iv_2$, ζ is the lake surface, ξ the thermocline position, p_j $(j = 1, 2)$ the pressure, $\partial/\partial n = \partial/\partial x + i(\partial/\partial y)$, $\alpha_1 = 2\pi\xi_{ref}/d_1$, $\alpha_2 = 2\pi\xi_{ref}/d_2$, and $\rho_r = \rho_1/\rho_2$ where d_j and ρ_j $(j = 1, 2)$ are the friction depth and water density in either the epilimnion or hypolimnion. In forming these nondimensional equations, we have used the following relations:

$$x = \frac{\bar{x}}{L} \qquad \zeta = \frac{\bar{\zeta}}{\zeta_{ref}}$$

$$y = \frac{\bar{y}}{L} \qquad u = \frac{\bar{u}}{u_{ref}}$$

$$z = \frac{\bar{z}}{\xi_{ref}} \qquad v = \frac{\bar{v}}{u_{ref}}$$

$$\xi = \frac{\bar{\xi}}{\xi_{ref}} \qquad w = \frac{\bar{w}}{w_{ref}}$$

where

$$\zeta_{ref} = \frac{L\tau^{w}_{ref}}{\rho_1 d_1 g} \qquad d_1 = \pi\sqrt{\frac{2\nu_{M1}}{f_c}}$$

$$\xi_{ref} = \frac{\zeta_{ref}}{\Delta\rho} \qquad d_2 = \pi\sqrt{\frac{2\nu_{M2}}{f_c}}$$

$$u_{ref} = \frac{\pi\tau^{w}_{ref}}{\rho_1 d_1 f_c} \qquad \Delta\rho = \frac{\rho_2 - \rho_1}{\rho_1}$$

$$w_{ref} = \frac{\xi_{ref}}{L} U_{ref}$$

and the overbar indicates the dimensional quantity. Here τ^{w}_{ref} is the reference wind stress, g the acceleration due to gravity, f_c the Coriolis parameter, ν_{Mj} (j = 1, 2) the vertical momentum eddy diffusivity, and L the reference length of the basin.

The system of equations (1) to (4) must be solved subject to the following boundary conditions:

$$\left.\begin{array}{l}
\tau^{w} = \tau^{w}_{x} + i\tau^{w}_{y} = \dfrac{1}{\alpha_1}\dfrac{\partial\Gamma_1}{\partial z} \quad \text{at } z = 0 \\[2ex]
\Gamma_2 = w_2 = 0 \quad \text{at } z = -h(x,y) \\[2ex]
\left.\begin{array}{l}
\Gamma_1 = \Gamma_2 \\[2ex]
\dfrac{\partial\Gamma_1}{\partial z} = \dfrac{d_2^2}{d_1^2}\dfrac{1}{\rho_r}\dfrac{\partial\Gamma_2}{\partial z}
\end{array}\right\} \quad \text{at } z = \xi(x,y)
\end{array}\right\} \tag{5}$$

$$
\left.\begin{aligned}
& w_1 = 0 \quad \text{at } z = 0 \\
& \left.\begin{aligned}
w_1 &= u_1 \frac{\partial \xi}{\partial x} + v_1 \frac{\partial \xi}{\partial y} \\
w_2 &= u_2 \frac{\partial \xi}{\partial x} + v_2 \frac{\partial \xi}{\partial y}
\end{aligned}\right\} \quad \text{at } z = \xi(x, y)
\end{aligned}\right\} \tag{6}
$$

where h is the depth of the lake and τ_x^w and τ_y^w are, respectively, the x- and y-components of wind shear stress. The boundary conditions on w_1 and w_2 at $z = \xi(x, y)$ are such that there is no flow normal to the thermocline. For strong enough winds the thermocline may intersect the surface of the water. This case is not considered here. The details associated with the derivation of equations (1) to (6) can be found in Gedney, Lick, and Molls (ref. 1).

We have said nothing about the average thickness of the epilimnion which is determined by the thermal problem. In this report, the z value of the thermocline will be specified at one point on the boundary so that the average thickness of the epilimnion is consistent with observations.

GOVERNING EQUATIONS FOR THE HORIZONTAL VELOCITIES

Equations (3) and (4) subject to the boundary conditions (5) can be solved for the horizontal velocities in the hypolimnion and epilimnion as functions of the surface displacement ζ, the thermocline position ξ, the lake depth h, and the wind stress τ^w. The resulting solutions are

$$
\Gamma_1 = A_1 e^{\omega \alpha_1 (\xi - z)} + A_1 e^{\omega \alpha_1 (\xi + z)} + \frac{\tau^w}{\omega} e^{\omega \alpha_1 z} + \frac{i}{\pi} \frac{\partial \zeta}{\partial n} \qquad \xi \leq z \leq 0 \tag{7}
$$

$$
\Gamma_2 = \left(A_2 - \frac{i\rho_r}{\pi} \frac{\partial \eta}{\partial n}\right) e^{\omega \alpha_2 (h+z)} - A_2 e^{-\omega \alpha_2 (h+z)} + \frac{i\rho_r}{\pi} \frac{\partial \eta}{\partial n} \qquad -h \leq z \leq \xi \tag{8}
$$

where

$$
A_1 = \frac{i\rho_r \delta}{\pi K} \frac{\partial \eta}{\partial n}\left(1 + e^{-2a_2} - 2e^{-a_2}\right) - \frac{\delta i}{\pi K}\left(\frac{\partial \zeta}{\partial n}\right)\left(1 + e^{-2a_2}\right) + \frac{\tau^w}{\omega} \frac{e^{a_1}}{K}\left[\left(1 - e^{-2a_2}\right) - \delta\left(1 + e^{-2a_2}\right)\right]
$$

$$A_2 = \frac{2\tau^w}{\omega}\frac{e^{a_1}e^{-a_2}}{K} - \frac{i\rho_r}{\pi K}\left[\left(e^{2a_1} - 1\right)\left(1 - e^{-a_2}\right) - \delta\left(e^{2a_1} + 1\right)\right]\frac{\partial\eta}{\partial n} - \frac{i}{\pi K}e^{-a_2}\left(e^{2a_1} - 1\right)\left(\frac{\partial\zeta}{\partial n}\right)$$

$$a_1 = \omega\alpha_1\xi$$

$$a_2 = \omega\alpha_2(h + \xi)$$

$$\omega = \frac{1 + i}{2}$$

$$\delta = \frac{d_1}{d_1\rho_r}$$

$$\frac{\partial\eta}{\partial n} = \frac{\partial\zeta}{\partial n} + \frac{\partial\xi}{\partial n}$$

$$K = \delta\left(1 + e^{-2a_2}\right)\left(1 + e^{2a_1}\right) - \left(1 - e^{-2a_2}\right)\left(e^{2a_1} - 1\right)$$

The w_1 and w_2 velocities can be determined by differentiating equations (7) and (8), substituting in equations (1) and (2), and then integrating.

ASYMPTOTIC EXPANSION OF TWO-LAYER LAKE EQUATIONS

As explained in the INTRODUCTION we consider here the case when $\nu_{M2}/\nu_{M1} << 1$ so that $\delta = (1/\rho_r)\sqrt{\nu_{M2}/\nu_{M1}} << 1 (\rho_r = \rho_1/\rho_2 \cong 1.0)$. We then perform an asymptotic expansion which is to be valid as $\delta \rightarrow 0$. The asymptotic expansion which we perform is valid as $\delta \rightarrow 0$ provided that two conditions are met. The first condition is that

$$e^{-a_2} = e^{-2\pi\omega\rho_r(\bar{h}+\xi)/\delta d_1} < \delta^3$$

In the limit as $\delta \rightarrow 0$ this requires that $\bar{h} + \bar{\xi} \neq 0$ or that the thermocline does not intersect the lake bottom (note in the analysis $\bar{h}$ is always positive and $\bar{\xi}$ is always negative). The second condition is that $\left(e^{2a_1} + 1\right)\Big/\left(e^{2a_1} - 1\right)$ is of order unity ($a_1 = \omega\alpha_1\xi$), which requires that the thermocline be below the surface of the lake by a certain amount (i.e., $\bar{\xi} < 0$). The bounds on $\bar{h}$ and $\bar{\xi}$ which ensure that these two conditions are met for the cases to be calculated will be discussed subsequently.

The cases to be calculated will be for -

(1) A moderate wind of 5.2 m/s ($\nu_{M1} = 16.8$ cm^2/sec)

(2) $1/576 \leq \nu_{M2}/\nu_{M1} \leq 1/144$ (ν_{M2} is therefore a few times larger than the molecular diffusivity)

(3) $\rho_2 - \rho_1 = 0.002203$ g/cm^3

(4) Lake length of 96 km

(5) Wind shear stress $\tau^w_{ref} \cong 1$ dyne/cm^2

For these conditions the values of δ, α_2, and α_1 are

$$\frac{1}{24} \leq \delta \leq \frac{1}{12}$$

$$84 \leq \alpha_2 \leq 168$$

$$\alpha_1 = 8.4$$

With this range of δ and α_2, $e^{-a_2} < \delta^3$ providing the lake depth is approximately 10 percent deeper than the value of $|\xi|$. The value of ξ_{ref} is approximately 30 meters. For $\alpha_1 = 8.4$, the quotient $\left(e^{2a_1} + 1\right)\Big/\left(e^{2a_1} - 1\right)$ is of order unity provided the minimum values of $|\bar{\xi}|$ are of the order of 3 to 4 meters.

With these restrictions understood we now proceed with the asymptotic expansion. First,

$$\frac{1}{K} = \frac{-1}{\left(e^{2a_1} - 1\right)}\left[1 + \delta\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right) + \delta^2\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)^2 + \ldots\right]$$

The expansions for A_1 and A_2 are

$$A_1 = \frac{-\tau_w e^{a_1}}{\omega\left(e^{2a_1} - 1\right)}\left[1 + \frac{2\delta}{\left(e^{2a_1} - 1\right)} - \delta^2\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right) + \delta^2\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)^2\right]$$

$$+ \frac{\delta}{\left(e^{2a_1} - 1\right)}\left[1 + \delta\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)\right]\left(- \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} + \frac{i}{\pi}\frac{\partial\zeta}{\partial n}\right) + O(\delta^3)$$

$$A_2 = e^{-a_2}\left[1 + \delta\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)\right]\left[\frac{-2\tau_w e^{a_1}}{\omega\left(e^{2a_1} - 1\right)} - \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} + \frac{i}{\pi}\frac{\partial\zeta}{\partial n}\right] + \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} + O(\delta^5)$$

When the expansion for A_2 is substituted into equation (8), some cancellation takes place to give the more compact form

$$\Gamma_2 = \left[\frac{-2\tau_w e^{a_1}}{\omega\left(e^{2a_1} - 1\right)} - \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} + \frac{i}{\pi}\frac{\partial\zeta}{\partial n}\right] \times \left[1 + \delta\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)\right] e^{\omega\alpha_2(z-\xi)} - \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} e^{-\omega\alpha_2(h+z)}$$

$$+ \frac{i\rho_r}{\pi}\frac{\partial\eta}{\partial n} + O(\delta^2) \qquad \text{(8a)}$$

When A_1 is substituted into equation (7), no such simplification for Γ_1 results.

The velocity equations (7) and (8a) can be integrated vertically to give the volume transport equations

$$M_{T1} = M_{x1} + iM_{y1} = \int_{\xi}^{0} \alpha_1 \Gamma_1 \, dz = -2i\tau^w - \frac{i\alpha_1}{\pi} \frac{\partial \zeta}{\partial n} \xi$$

$$- \delta \left[1 + \delta \left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1} \right) \right] \left(- \frac{2\omega\rho_r}{\pi} \frac{\partial \eta}{\partial n} + \frac{2\omega}{\pi} \frac{\partial \zeta}{\partial n} + \tau^w D_2 \right) + O(\delta^3) \qquad (9)$$

$$M_{T2} = M_{x2} + iM_{y2} = \int_{-h}^{\xi} \frac{\alpha_1 \Gamma_2}{\rho_r} \, dz = \frac{i\alpha_1}{\pi} \frac{\partial \eta}{\partial n} (h + \xi) - \frac{2\omega\delta\rho_r}{\pi} \frac{\partial \eta}{\partial n}$$

$$+ \delta \left[1 + \delta \left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1} \right) \right] \left(\tau^w D_2 - \frac{2\omega\rho_r}{\pi} \frac{\partial \eta}{\partial n} + \frac{2\omega}{\pi} \frac{\partial \zeta}{\partial n} \right) + O(\delta^3) \qquad (10)$$

Integrating equations (1) and (2) in the z-direction and applying equation (6) give

$$\text{Real}\left(\frac{\partial M_{T1}}{\partial n^*} \right) = 0 \qquad (11)$$

$$\text{Real}\left(\frac{\partial M_{T2}}{\partial n^*} \right) = 0 \qquad (12)$$

where

$$\frac{\partial}{\partial n^*} = \frac{\partial}{\partial x} - i\left(\frac{\partial}{\partial y} \right)$$

and

$$D_2(\xi) = \left[\frac{4ie^{a_1}}{\left(e^{2a_1} - 1 \right)} \right]$$

Terms of order δ^3 have been neglected in equations (9) and (10). The quantity M_{Tj} $(j = 1, 2)$ is the total volume transport in either the epilimnion or hypolimnion, and

M_{xj} and M_{yj} are, respectively, the volume transport components in the x- and y-directions.

Equations (9) to (12) can be combined to give the governing equations for ζ and ξ. This can be most readily accomplished by forming

$$\text{Real}\left(\frac{\partial M_{T1}}{\partial n^*} + \frac{\partial M_{T2}}{\partial n^*}\right) = 0$$

which results in

$$\delta\rho_r \nabla^2\eta + \alpha_1\left(\frac{\partial \eta}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial \eta}{\partial x}\frac{\partial h}{\partial y}\right) = 2\pi\left(\frac{\partial \tau_y^w}{\partial x} - \frac{\partial \tau_x^w}{\partial y}\right) + O(\delta^3) \tag{13}$$

Combining equations (10) and (12) results in the second governing equation

$$\frac{\delta\rho_r}{\pi}\nabla^2\eta - \text{Real}\left(\frac{i\alpha_1}{\pi}\left(\frac{\partial \eta}{\partial n}\right)\left(\frac{\partial h}{\partial n^*} + \frac{\partial \xi}{\partial n^*}\right) + \delta\frac{\partial}{\partial n^*}\left\{\left[1 + \delta\left(\frac{e^{2a_1} + 1}{e^{2a_1} - 1}\right)\right]\left[\tau^w D_2(\xi) - \frac{2\omega\rho_r}{\pi}\frac{\partial \eta}{\partial n} + \frac{2\omega}{\pi}\frac{\partial \zeta}{\partial n}\right]\right\}\right) = 0 \tag{14}$$

Equations (13) and (14) which neglect terms of order δ^3 are to be solved for the dependent variables subject to the boundary condition that the volume transport normal to the boundary is zero. This boundary condition is then

$$M_{x1}m_x + M_{y1}m_y = 0 \tag{15a}$$

$$M_{x2}m_x + M_{y2}m_y = 0 \tag{15b}$$

where m_x and m_y are the x- and y-components of unit normal $\hat{m}$ to the boundary.

SOLUTION OF GOVERNING EQUATIONS

The solution of the governing equations (13) to (15) will be performed using an asymptotic expansion procedure. To develop the asymptotic expansion, we substitute

$$\left.\begin{aligned}
\eta &= \eta^{(0)} + \delta\eta^{(1)} + \delta^2\eta^{(2)} + \ldots \\
\xi &= \xi^{(0)} + \delta\xi^{(1)} + \delta^2\xi^{(2)} + \ldots \\
\zeta &= \zeta^{(0)} + \delta\zeta^{(1)} + \delta^2\zeta^{(2)} + \ldots \\
p_2 &= p_2^{(0)} + \delta p_2^{(1)} + \delta^2 p_2^{(2)} + \ldots
\end{aligned}\right\} \tag{16}$$

into equations (13) and (14) to give

$$\delta\rho_r\nabla^2\eta^{(0)} + \delta^2\rho_r\nabla^2\eta^{(1)} + \alpha_1\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial h}{\partial y}\right) + \delta\,\alpha_1\left(\frac{\partial\eta^{(1)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(1)}}{\partial x}\frac{\partial h}{\partial y}\right)$$

$$+ \delta^2\alpha_1\left(\frac{\partial\eta^{(2)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(2)}}{\partial x}\frac{\partial h}{\partial y}\right) = 2\pi\left(\frac{\partial\tau_y^W}{\partial x} - \frac{\partial\tau_x^W}{\partial y}\right) + O(\delta^3) \tag{13a}$$

$$\frac{\delta\rho_r}{\pi}\nabla^2\eta^{(0)} + \frac{\delta^2\rho_r}{\pi}\nabla^2\eta^{(1)} + \frac{\alpha_1}{\pi}\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial h}{\partial y}\right) + \frac{\alpha_1\delta}{\pi}\left(\frac{\partial\eta^{(1)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(1)}}{\partial x}\frac{\partial h}{\partial y}\right)$$

$$+ \frac{\alpha_1}{\pi}\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y}\right) + \frac{\alpha_1\delta}{\pi}\left(\frac{\partial\eta^{(1)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(1)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y}\right)$$

$$+ \frac{\alpha_1\delta}{\pi}\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(1)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(1)}}{\partial y}\right) - \delta\mathrm{Real}\left\{\frac{\partial}{\partial n^*}\left[\tau^W D_2(\xi^{(0)})\right.\right.$$

$$\left.\left. - \frac{2\omega}{\pi}\rho_r\frac{\partial\eta^{(0)}}{\partial n} + \frac{2\omega}{\pi}\frac{\partial\zeta^{(0)}}{\partial n}\right]\right\} - \delta^2\mathrm{Real}\left\{\frac{\partial}{\partial n^*}\left[-\frac{2\omega}{\pi}\rho_r\frac{\partial\eta^{(1)}}{\partial n}\right.\right.$$

$$\left.\left. + \frac{2\omega}{\pi}\frac{\partial\zeta^{(1)}}{\partial n}\right]\right\} + O(\delta^2)(\text{terms involving first derivatives})$$

$$+ O(\delta^3) = 0 \tag{14a}$$

Similarly, the expansion form of the boundary conditions can be obtained by combining equations (9), (10), (15), and (16).

Case 1 - Uniform Wind and Constant Depth

Consider the asymptotic solution for a constant depth square basin which is acted on by a uniform wind over the entire water surface. As determined from equation (13a), the zeroth order term (the term which is of order δ) from differential equation (13) is

$$\nabla^2 \eta^{(0)} = 0 \tag{17a}$$

The boundary conditions for $\eta^{(0)}$ are determined from the zeroth order terms of equation (15b) and are

$$\left.\begin{aligned} \frac{\partial \eta^{(0)}}{\partial y} &= 0 \qquad \text{at } x = 0, 1 \\ \frac{\partial \eta^{(0)}}{\partial x} &= 0 \qquad \text{at } y = 0, 1 \end{aligned}\right\} \tag{17b}$$

and, therefore, $\eta^{(0)}$ = constant on the entire boundary.

As is well known, the solution of the boundary value problem represented by equations (17) is $\eta^{(0)}$ = constant. Therefore, to zeroth order, the horizontal pressure gradient (see eq. (4)),

$$\frac{\partial p_2^{(0)}}{\partial n} = \frac{\partial \eta^{(0)}}{\partial n} = \frac{\partial \xi^{(0)}}{\partial n} + \frac{\partial \zeta^{(0)}}{\partial n}$$

in the hypolimnion is zero. This does not mean the velocities in the entire hypolimnion are zero. There is a shear stress transfer at the thermocline which imparts motion to the hypolimnion water in a thin boundary layer adjacent to the thermocline. Below this boundary layer, the zeroth order velocities in the hypolimnion are zero. This can be shown from equation (8a) using a typical value for α_2 of the order of 168.

It is possible to obtain $\xi^{(0)}$ along the boundary from the zeroth order component of the equation (15a) boundary conditions. This results in

$$\left.\begin{aligned} 2\tau_y^w - \frac{\alpha_1}{\pi}\frac{\partial \xi^{(0)}}{\partial y}\xi^{(0)} &= 0 \qquad \text{at } x = 0, 1 \\ 2\tau_x^w - \frac{\alpha_1}{\pi}\frac{\partial \xi^{(0)}}{\partial x}\xi^{(0)} &= 0 \qquad \text{at } y = 0, 1 \end{aligned}\right\} \tag{18a}$$

where we have used the relations

$$\frac{\partial \xi^{(0)}}{\partial x} = -\frac{\partial \zeta^{(0)}}{\partial x}$$

and

$$\frac{\partial \xi^{(0)}}{\partial y} = -\frac{\partial \zeta^{(0)}}{\partial y}$$

These equations can be directly integrated for the value of $\xi^{(0)}$ along the boundary to give

$$\left.\begin{aligned} \xi^{(0)} &= -\sqrt{\tau_y^w \frac{4\pi y}{\alpha_1} + \left|\xi_0^{(0)}\right|^2} \qquad \text{at } x = 0 \\ \xi^{(0)} &= -\sqrt{\tau_x^w \frac{4\pi x}{\alpha_1} + \left|\xi_0^{(0)}\right|^2} \qquad \text{at } y = 0 \\ \xi^{(0)} &= -\sqrt{\left(y\tau_y^w + \tau_x^w\right)\frac{4\pi}{\alpha_1} + \left|\xi_0^{(0)}\right|^2} \qquad \text{at } x = 1 \\ \xi^{(0)} &= -\sqrt{\left(x\tau_x^w + \tau_y^w\right)\frac{4\pi}{\alpha_1} + \left|\xi_0^{(0)}\right|^2} \qquad \text{at } y = 1 \end{aligned}\right\} \tag{18b}$$

where $\xi_0^{(0)}$ is the value of $\xi^{(0)}$ at $x = 0$ and $y = 0$.

Since $\eta^{(0)}$ is a constant, the zeroth order component of equation (14a) yields nothing. The second-order terms of equation (14a) give a governing equation for $\xi^{(0)}$ but it involves $\eta^{(1)}$. Therefore, to calculate $\xi^{(0)}$ we must also solve for $\eta^{(1)}$. The governing equation for $\eta^{(1)}$ is obtained from the first-order terms of equation (13a). These $\eta^{(1)}$ and $\xi^{(0)}$ governing equations are

$$\nabla^2 \eta^{(1)} = 0 \tag{19}$$

$$\nabla^2 \xi^{(0)} + \alpha_1 \left(\frac{\partial \eta^{(1)}}{\partial y} \frac{\partial \xi^{(0)}}{\partial x} - \frac{\partial \eta^{(1)}}{\partial x} \frac{\partial \xi^{(0)}}{\partial y} \right) + \pi \left\{ -\tau_x^w \left[D_2^{(r)}(\xi^{(0)}) \frac{\partial \xi^{(0)}}{\partial x} + D_2^{(i)}(\xi^{(0)}) \frac{\partial \xi^{(0)}}{\partial y} \right] \right.$$

$$\left. + \tau_y^w \left[D_{2\xi}^{(i)}(\xi^{(0)}) \frac{\partial \xi^{(0)}}{\partial x} - D_{2\xi}^{(r)}(\xi^{(0)}) \frac{\partial \xi^{(0)}}{\partial y} \right] \right\} = 0 \tag{20}$$

where $D_{2\xi} = \partial D_2 / 2\xi$ and the superscripts r and i indicate, respectively, the real and imaginary parts. In deriving equation (20), we have used the fact that

$$\frac{\partial \eta^{(0)}}{\partial n} = \frac{\partial \xi^{(0)}}{\partial n} + \frac{\partial \zeta^{(0)}}{\partial n} = 0$$

These coupled equations are solved using the $\xi^{(0)}$ boundary conditions specified by equation (18) and the following $\eta^{(1)}$ boundary conditions derived from the first-order terms from equation (15b):

$$\mathrm{Real}\left[\tau^w D_2(\xi^{(0)})\right] + \frac{1}{\pi}\left(\frac{\partial \xi^{(0)}}{\partial y} - \frac{\partial \xi^{(0)}}{\partial x} \right) - \frac{\alpha_1}{\pi} \frac{\partial \eta^{(1)}}{\partial y} (h + \xi^{(0)}) = 0 \qquad \text{at } x = 0, 1 \tag{21a}$$

$$\mathcal{I}m\left[\tau^w D_2(\xi^{(0)})\right] - \frac{1}{\pi}\left(\frac{\partial \xi^{(0)}}{\partial x} + \frac{\partial \xi^{(0)}}{\partial y} \right) + \frac{\alpha_1}{\pi} \frac{\partial \eta^{(1)}}{\partial x} (h + \xi^{(0)}) = 0 \qquad \text{at } y = 0, 1 \tag{21b}$$

The solution for $\xi^{(0)}$ and $\eta^{(1)}$ were obtained numerically using finite differences. The numerical method used was Newton's method with point underrelaxation.

Case 2 - Uniform Wind With Variable Depth

As has been just discussed for case 1, a uniform wind acting on the water surface of a constant depth basin produces no zeroth order flow in the hypolimnion except for a thin boundary layer adjacent to the thermocline. Implicit in the assumptions of this whole analysis is that this thin hypolimnion boundary layer does not contact the lake bottom. As a result, there are no flow velocities in the region of the lake bottom which could be affected by variations in depth. Therefore, variations in bottom depth should not appreciably effect the zeroth order solution for the uniform wind case. This conclusion has been verified by reference 1 where the complete two-layer lake equations have been solved numerically for a lake whose depth is constant along the shore but is variable in the interior.

In this section the asymptotic solution $\eta^{(0)}$ for a variable depth lake will be obtained to again demonstrate this conclusion. Here we consider a lake whose depth is described by

$$h = h_b + 16.0(h_c - h_b)(x - x^2)(y - y^2) \qquad 0 \le x \le 1,\ 0 \le y \le 1 \tag{22}$$

where h_b is the depth at the lake boundary which is taken to be a constant and h_c is the maximum lake depth which is taken to occur at the center of the lake. Note that the h derivatives tangential to the boundary are directly proportional to the distance from the boundary. At a small distance Δ from the boundary, the tangential depth derivatives are of order Δ.

The zeroth order terms from the governing equation (13a) is

$$\frac{\partial \eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial \eta^{(0)}}{\partial x}\frac{\partial h}{\partial y} = 0 \tag{23a}$$

where we assume at this point in the analysis that no $\eta^{(0)}$ boundary layer exists which would make the $\nabla^2\eta^{(0)}$ term in equation (13a) or order one. The general solution of equation (23a) is

$$\eta^{(0)} = f\left[h(x, y)\right] \tag{23b}$$

where f is an arbitrary function of the depth $h(x, y)$. The boundary conditions on $\eta^{(0)}$ can be shown from the expansion of equation (15b) to be $\eta^{(0)}$ = constant. Since by equation (22) h is constant on the boundary, equation (23b) satisfies the boundary condition for $\eta^{(0)}$.

In order to determine the form of f we investigate the zeroth order term of equation (14a) which is

$$\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial h}{\partial y} + \frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y} = 0 \tag{23c}$$

Substituting equation (23b) into this equation results in

$$\frac{\partial h}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial\xi^{(0)}}{\partial y} = 0 \tag{23d}$$

which has the general solution

$$\xi^{(0)} = f\left[h(x, y)\right] \tag{23e}$$

where f is again some arbitrary function. The boundary conditions for $\xi^{(0)}$ are determined from the zeroth order term of (15a) and are the same as equation (18b). Since h is constant on the boundary, no function f will be able to satisfy these boundary conditions.

A possibility exists that there is a $\xi^{(0)}$ boundary layer so that equation (23d) does not apply in the boundary region. To determine the boundary layer equations, it is necessary to introduce the stretched coordinates

$$\left.\begin{aligned} m^{+} &= \frac{x}{\epsilon(\delta)}, \quad t^{+} = y \qquad \text{along } x = 0, 1 \\ m^{+} &= \frac{y}{\epsilon(\delta)}, \quad t^{+} = x \qquad \text{along } y = 0, 1 \end{aligned}\right\} \tag{24}$$

in equation (14a). The resulting zeroth order boundary layer equations are then

$$\frac{\delta}{\epsilon^2}\frac{\partial^2\xi^{(0)}}{\partial m^{+2}} + \alpha_1\frac{\partial f}{\partial h}\left[\frac{(-1)^{S+1}}{\epsilon}\frac{\partial\xi^{(0)}}{\partial m^{+}} h_t + (-1)^S\frac{\partial\xi^{(0)}}{\partial t^{+}} h_m\right] = 0 \tag{25a}$$

where this equation applies along $x = 0, 1$ when $S = 1$ and along $y = 0, 1$ when $S = 2$ and where h_m and h_t are, respectively, the normal and tangential partial derivatives of h. All derivative terms except for h_t and $\partial\xi^{(0)}/\partial t^{+}$ in this equation have been scaled such that they are of order one. Equation (22) shows that in the boundary layer region

$$h_t = \frac{\partial h}{\partial t^+} = \epsilon f_1(m^+, t^+) \tag{25b}$$

where f_1 is of order one along the entire boundary. In order to estimate the magnitude of $\partial\xi^{(0)}/\partial t^+$ in the boundary layer we must consider the boundary conditions (eq. 18(b)) and the interior solution (eq. (23(e)). The boundary conditions show that $\partial\xi^{(0)}/\partial t^+$ will be of order one along $y = 0, 1$ but zero along $x = 0, 1$. The interior solution (eq. (23(e)) shows that near the edge of the possible boundary layer

$$\frac{\partial \xi^{(0)}}{\partial t^+} = \frac{df}{dh} \frac{\partial h}{\partial t^+} = \epsilon \frac{df}{dh} f_1(m^+, t^+) \tag{25c}$$

where ϵ is the thickness of the boundary layer. A Taylor series expansion for $\xi^{(0)}$ using equations (18b) and (25c) will show that $\partial\xi^{(0)}/\partial t^+$ in the $x = 0, 1$ boundary layers will be of $O(\epsilon)$. In the $y = 0, 1$ boundary layers $\partial\xi^{(0)}/\partial t^+$ is of $O(1)$.

When equation (25b) is substituted into equation (25a) and the estimate is used for $\partial\xi^{(0)}/\partial t^+$, the boundary layer equation along $x = 0, 1$ becomes

$$\frac{\delta}{\epsilon^2} \frac{\partial^2 \xi^{(0)}}{\partial m^{+2}} + \alpha_1 \frac{\partial f}{\partial h} \frac{\partial \xi^{(0)}}{\partial m^+} f_1(m^+, t^+) + O(\epsilon) = 0$$

When the principle of least degeneracy is used, the proper scale is chosen as $\epsilon = \delta^{1/2}$ and the final form of the $x = 0, 1$ boundary layer equation became

$$\frac{\partial^2 \xi^{(0)}}{\partial m^{+2}} + \alpha_1 \frac{\partial f}{\partial h} \frac{\partial \xi^{(0)}}{\partial m^+} f_1(m^+, t^+) + O(\delta^{1/2}) = 0$$

The boundary layer resulting from this equation is an increasing or decreasing exponential depending on the sign of $\partial h/\partial t^+ = \epsilon f_1(m^+, t^+)$. Only exponential decreasing functions can be allowed. Because $\partial h/\partial t^+ = \epsilon f_1$ changes sign about $y = 0.5$, it will only be possible to have a boundary layer along one-half of the $x = 0, 1$ boundaries. With equations (22) and (23e), the values of $\xi^{(0)}$ outside the boundary layer will be symmetric about $y = 0.5$. But this means that a $\xi^{(0)}$ boundary layer will be needed along the entire x boundary. We conclude from this that no $\xi^{(0)}$ boundary layer is possible along the $x = 0, 1$ boundaries which will permit the $\xi^{(0)}$ boundary conditions to be satisfied. As a result, no $\xi^{(0)}$ solution exists when we assume a nonconstant $\eta^{(0)}$ form for equation (23b). The only other possibility is that $\eta^{(0)}$ is a constant.

Before finally concluding that $\eta^{(0)}$ is constant for this variable depth basin case, we must admit the possibility of an $\eta^{(0)}$ boundary layer that would make the $\nabla^2\eta^{(0)}$ term in equation (13a) of order one. If this is done, the same arguments just used to show that no $\xi^{(0)}$ boundary layer existed will show that no $\eta^{(0)}$ boundary layer can exist. Therefore, we conclude that $\eta^{(0)}$ is a constant and that the zeroth order horizontal pressure gradient in the hypolimnion $\left(\partial p_2^{(0)}/\partial n = \partial\eta^{(0)}/\partial n\right)$ is zero whether or not the lake depth is variable.

Case 3 - Wind Stress Gradients of Order δ With Constant Depth

The zeroth order terms from the governing equation (13a) when the lake depth is constant but the wind shear stress curl $\left(\partial\tau_y^w/\partial x - \partial\tau_x^w/\partial y\right)$ is of order δ is

$$\nabla^2\eta^{(0)} = \frac{2\pi}{\rho_r} f_2\left(\frac{\partial\tau_y^w}{\partial x}, \frac{\partial\tau_x^w}{\partial y}\right) \tag{26}$$

where

$$f_2\left(\frac{\partial\tau_y^w}{\partial x}, \frac{\partial\tau_x^w}{\partial y}\right) = \frac{1}{\delta}\left(\frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y}\right)$$

is of order one. The boundary conditions for $\eta^{(0)}$ can be derived from equation (15b) and again $\eta^{(0)}$ is a constant on the boundary. This boundary value problem for $\eta^{(0)}$ has a nonconstant solution which will result in hypolimnion pressure gradients $\left(\partial p_2^{(0)}/\partial n = \partial\eta^{(0)}/\partial n\right)$.

The governing differential equation for $\xi^{(0)}$ is obtained from the zeroth order component of equation (14a). This equation for $\xi^{(0)}$ is singular and on the interior region of the basin is

$$\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y} = 0 \tag{27a}$$

In order to satisfy the boundary conditions, a "boundary layer" must exist along portions of the boundaries. To determine the boundary layer equations, it is necessary as in the previous case to introduce stretched coordinates given by equations (24). The resulting zeroth order boundary layer equations are then

$$\frac{\delta}{\epsilon^2}\frac{\partial^2\xi^{(0)}}{\partial m^{+2}} + \alpha_1\left[(-1)^{s+1}\eta_m^{(0)}\frac{\partial\xi^{(0)}}{\partial t^+} + (-1)^s\frac{1}{\epsilon}\eta_t^{(0)}\frac{\partial\xi^{(0)}}{\partial m^+}\right] = 0$$

where this equation applies along $y = 0, 1$ when $s = 1$ and along $x = 0, 1$ when $s = 2$ and where $\eta_m^{(0)}$ and $\eta_t^{(0)}$ are, respectively, the normal and tangential partial derivatives of $\eta^{(0)}$ given by the solution of equation (26). Because of the boundary condition on $\eta^{(0)}$ (i.e., $\eta^{(0)}$ = constant on the boundary), $\eta_t^{(0)}$ within a possible boundary layer thickness ϵ can be shown by a Taylor series expansion to be $O(\epsilon)$. We denote here $\eta_t^{(0)} = \epsilon\widetilde{\eta_t^{(0)}}$ where $\widetilde{\eta_t^{(0)}}$ is of order one. Using the principle of least degeneracy the proper value of ϵ is then $\epsilon = \delta^{1/2}$ and the boundary layer equation becomes

$$\frac{\partial^2\xi^{(0)}}{\partial m^{+2}} + \alpha_1\left[(-1)^{s+1}\eta_m^{(0)}\frac{\partial\xi^{(0)}}{\partial t^+} + (-1)^s\widetilde{\eta_t^{(0)}}\frac{\partial\xi^{(0)}}{\partial m^+}\right] = 0 \tag{27b}$$

Instead of obtaining the interior and boundary layer solutions separately, we obtain the complete $\eta^{(0)}$ solution from the combined equation

$$\delta\nabla^2\xi^{(0)} + \alpha_1\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y}\right) = 0 \tag{28}$$

The boundary conditions for $\xi^{(0)}$ are the same as equation (18). The solution of equation (28) is easily obtained numerically using successive overrelaxation.

Case 4 - Wind Stress Gradients With Variable Lake Depth

The results to be given subsequently for case 3 where the basin depth is constant and the wind stress curl is of order δ show that significant geostrophic velocities occur in the hypolimnion due to the finite values of $\partial p_2^{(0)}/\partial n = \partial\eta^{(0)}/\partial n$. With significant hypolimnion velocities, variations in lake depth should be very important. Consider again a square lake whose depth is described by equation (22) which is acted on by a wind with a stress curl $\left(\partial\tau_x^W/\partial y - \partial\tau_y^W/\partial x\right)$ of order δ (case 4a) and order 1 (case 4b). At the lake boundary, normal derivatives of the wind stress are nonzero but tangential derivatives are assumed zero.

Case 4a - wind stress curl of order δ. - The zeroth order governing equation for $\eta^{(0)}$ and $\xi^{(0)}$ for this case are the same as case 2. The solution is therefore $\eta^{(0)}$ = constant. The variation in lake depth is then enough to eliminate the hypolimnetic horizontal pressure gradient $(\partial\eta^{(0)}/\partial n)$ which was created by the δ wind stress curl when the depth was constant. The effect of variation of lake depth is therefore very strong.

Case 4b - wind stress curl of order one. - For this case the zeroth order term of the governing equation (13a) is

$$\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial h}{\partial y} = \frac{2\pi}{\alpha_1}\left(\frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y}\right) \tag{29}$$

The boundary condition on $\eta^{(0)}$ from equation (15a) is $\eta^{(0)}$ = constant. At the boundaries, equation (22) shows that the tangential depth derivative h_t is zero and the normal depth derivative h_m is finite. Using this fact in the nonhomogeneous equation (29) shows that the tangential $\eta^{(0)}$ derivative at the lake boundary is nonzero. As a result, equation (29) can not satisfy the $\eta^{(0)}$ = constant boundary conditions. We must therefore consider an $\eta^{(0)}$ boundary layer which will be compatible with a solution of equation (29) in the interior of the lake.

To determine the boundary layer we introduce stretched coordinates given by equation (24) into equation (13a) which gives

$$\frac{\delta}{\epsilon^2}\rho_r\frac{\partial^2\eta^{(0)}}{\partial m^{+2}} + \alpha_1\left[\frac{(-1)^{s+1}}{\epsilon}\frac{\partial\eta^{(0)}}{\partial m^+}h_t + (-1)^s\frac{\partial\eta^{(0)}}{\partial t^+}h_m\right] = 2\pi\left(\frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y}\right)$$

where this equation applies along $y = 0, 1$ when $s = 1$ and along $x = 0, 1$ when $s = 2$ and where h_t and h_m are, respectively, the normal and tangential partial derivatives of h with respect to the boundary. Now equation (22) shows that

$$h_t = \epsilon f_1(m^+, t^+)$$

where f_1 is of order one. Again using the principle of least degeneracy gives the proper scaling as $\epsilon = \delta^{1/2}$. The boundary layer equation is then

$$\rho_r\frac{\partial^2\eta^{(0)}}{\partial m^{+2}} + \alpha_1\left[(-1)^{s+1}\frac{\partial\eta^{(0)}}{\partial m^+}f_1(m^+, t^+) + (-1)^s\frac{\partial\eta^{(0)}}{\partial t^+}h_m(m^+, t^+)\right] = 2\pi\left(\frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y}\right)$$

Instead of obtaining the interior and boundary layer solutions separately, we choose to solve the combined equation

$$\delta \rho_r \nabla^2 \eta^{(0)} + \alpha_1 \left(\frac{\partial \eta^{(0)}}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial \eta^{(0)}}{\partial x} \frac{\partial h}{\partial y} \right) = 2\pi \left(\frac{\partial \tau_y^w}{\partial x} - \frac{\partial \tau_x^w}{\partial y} \right) \tag{30}$$

over the entire region. The boundary conditions on $\eta^{(0)}$ are shown from the expansion of equation (15b) to be $\eta^{(0)}$ = constant.

The solution of equation (29a) will result in $\eta^{(0)} = \eta^{(0)}(x, y)$ with a boundary layer of order $\delta^{1/2}$. The governing equation for $\xi^{(0)}$ is obtained from equation (14a), and the boundary value problem is easily seen to be singular. In the interior region,

$$\left(\frac{\partial \eta^{(0)}}{\partial y} \frac{\partial h}{\partial x} - \frac{\partial \eta^{(0)}}{\partial x} \frac{\partial h}{\partial y} \right) + \left(\frac{\partial \eta^{(0)}}{\partial y} \frac{\partial \xi^{(0)}}{\partial x} - \frac{\partial \eta^{(0)}}{\partial x} \frac{\partial \xi^{(0)}}{\partial y} \right) = 0$$

The equation valid in the boundary layer can also be readily derived. Introducing the equation (24) stretched coordinates into equation (14a) results in

$$(2\rho_r - 1)\widetilde{\eta_{mm}^{(0)}} + \alpha_1 \left[\epsilon (-1)^{s+1} \eta_m^{(0)} f_1(m^+, t^+) + (-1)^s \eta_t^{(0)} h_m \right]$$
$$+ \alpha_1 \left[(-1)^{s+1} \eta_m^{(0)} \frac{\partial \xi^{(0)}}{\partial t^+} + \frac{(-1)^s}{\epsilon} \eta_t^{(0)} \frac{\partial \xi^{(0)}}{\partial m^+} \right] + \frac{\delta}{\epsilon^2} \frac{\partial^2 \xi^{(0)}}{\partial m^{+2}} = 0$$

where $\widetilde{\eta_{mm}^{(0)}}$ is the second partial derivative of $\eta^{(0)}$ with respect to m and is of order one or less. From the $\eta^{(0)}$ solution determined from equation (29a), $\eta_t^{(0)}$ (within a distance $\delta^{1/2}$ from the boundary) was found to be $\leq \delta^{1/2} \widetilde{\eta_t^{(0)}}$ where $\widetilde{\eta_t^{(0)}}$ is of order one (this same behavior could also have been found by expanding $\eta^{(0)}$ in a Taylor series about the boundary). Two possible $\xi^{(0)}$ boundary layers can exist depending on the magnitude of $\eta_m^{(0)}$ in the boundary region. In some boundary regions an $\eta^{(0)}$ boundary layers exists so that $\eta_m^{(0)} = \widetilde{\eta_m^{(0)}}/\delta^{1/2}$; in other boundary regions $\eta_m^{(0)} = \widetilde{\eta_m^{(0)}}$ ($\widetilde{\eta_m^{(0)}}$ is of order 1). Where an $\eta^{(0)}$ boundary layer exists, the $\xi^{(0)}$ boundary layer is of order $\delta^{3/4}$ ($\epsilon = \delta^{3/4}$) and the appropriate boundary layer equation is

$$\frac{\partial^2 \xi^{(0)}}{\partial m^{+2}} + \alpha_1 (-1)^{s+1} \widetilde{\eta_m^{(0)}} \frac{\partial \xi^{(0)}}{\partial t^+} = 0$$

In regions where no $\eta^{(0)}$ boundary layer exists, the $\xi^{(0)}$ boundary layer is of order $\delta^{1/2}$ ($\epsilon = \delta^{1/2}$) and the appropriate boundary layer equation is

$$\frac{\partial^2 \xi^{(0)}}{\partial m^{+2}} + (2\rho_r - 1)\widetilde{\eta^{(0)}_{mm}} + \alpha_1\left[(-1)^{s+1}\widetilde{\eta^{(0)}_m}\,\frac{\partial \xi^{(0)}}{\partial t} + (-1)^s\widetilde{\eta^{(0)}_t}\,\frac{\partial \xi^{(0)}}{\partial m^+}\right] = 0$$

However, as with the solution for $\eta^{(0)}$, we do not choose to obtain the interior solution and boundary layer solution separately but to solve the combined equation

$$\delta\nabla^2\xi^{(0)} + \delta(2\rho_r - 1)\nabla^2\eta^{(0)} + \alpha_1\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial h}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial h}{\partial y}\right) + \alpha_1\left(\frac{\partial\eta^{(0)}}{\partial y}\frac{\partial\xi^{(0)}}{\partial x} - \frac{\partial\eta^{(0)}}{\partial x}\frac{\partial\xi^{(0)}}{\partial y}\right) = 0 \tag{31}$$

Again the boundary conditions on $\xi^{(0)}$ are the same as for equation (18).

SOLUTION WITH ZERO VELOCITY IN THE HYPOLIMNION

As discussed in the INTRODUCTION, the "quasi-compensation" assumption is commonly made in two-layer lake analyses to find the thermocline position. The quasi-compensation assumption is that the horizontal pressure gradient and the velocity to be zero in the hypolimnion. Taking the hypolimnion velocity equal to zero is equivalent to taking the hypolimnion eddy diffusivity (ν_{M2}) to be infinite. The zero hypolimnion velocity case is considered here so as to be able to contrast its results with the results for the more reasonable case where the hypolimnion eddy diffusivity is small.

The advantage of using the zero hypolimnion velocity assumption is that it results in a single governing equation for the thermocline position ξ. If the boundary condition $w_1 = \Gamma_1 = 0$ at $z = \xi$ is used at the thermocline, the following governing equation and boundary conditions for ξ can readily be derived using equations (1), (3), (11), and (15a):

$$G\nabla^2\xi + \frac{\partial G}{\partial \xi}\left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right] = +2\pi\left[-B\left(\frac{\partial\tau_x^w}{\partial x} + \frac{\partial\tau_y^w}{\partial y}\right) + C\left(\frac{\partial\tau_y^w}{\partial x} - \frac{\partial\tau_x^w}{\partial y}\right)\right.$$

$$\left. - \left(\frac{\partial B}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial C}{\partial\xi}\frac{\partial\xi}{\partial y}\right)\tau_y^w + \left(\frac{\partial C}{\partial\xi}\frac{\partial\xi}{\partial x} - \frac{\partial B}{\partial\xi}\frac{\partial\xi}{\partial y}\right)\tau_y^w\right] \tag{32}$$

$$\left.\begin{aligned} G\frac{\partial\xi}{\partial x} - H\frac{\partial\xi}{\partial y} &= -2\pi\left(B\tau_x^w - C\tau_y^w\right) \quad \text{at } x = 0, 1 \\ H\frac{\partial\xi}{\partial x} + G\frac{\partial\xi}{\partial y} &= -2\pi\left(C\tau_x^w + B\tau_y^w\right) \quad \text{at } y = 0, 1 \end{aligned}\right\} \tag{33}$$

where G, H, B, and C are given by

$$G = + \frac{\sin(\alpha_1 \xi) - \sinh(\alpha_1 \xi)}{\cos(\alpha_1 \xi) + \cosh(\alpha_1 \xi)}$$

$$H = - \frac{\sin(\alpha_1 \xi) + \sinh(\alpha_1 \xi)}{\cos(\alpha_1 \xi) + \cosh(\alpha_1 \xi)} + \alpha_1 \xi$$

$$B = \frac{2 \sin(\alpha_1 \xi/2) \sinh(\alpha_1 \xi/2)}{\cos(\alpha_1 \xi) + \cosh(\alpha_1 \xi)}$$

$$C = \frac{2 \cos(\alpha_1 \xi/2) \cosh(\alpha_1 \xi/2)}{\cos(\alpha_1 \xi) + \cosh(\alpha_1 \xi)} - 1$$

The derivation of these equations may also be obtained directly from the results given by Welander (ref. 4) for a shallow homogeneous lake. When $\overline{h}$ is replaced by $-\overline{\xi}$ and $\partial\overline{\zeta}/\partial\overline{x}$ by $-\partial\overline{\xi}/\partial\overline{x}$, etc, Welander's equations (11) and (13) lead to equations (32) and (33) after they are nondimensionalized in the manner given herein. It should be noted that equations (32) and (33) are not part of an asymptotic expansion but include terms of all order. Equations (32) and (33) can be solved numerically using successive underrelaxation with Newton's method incorporated. At each iteration the equations are linearized about the ξ values determined at the previous iteration.

RESULTS

The governing equations derived in the previous sections were solved for the position of the thermocline ξ, the position of the lake surface ζ, and the hypolimnion and epilimnion horizontal velocities u and v. All results are presented for a square basin with the following parameter values:

Basin length, L, km	96.6
Epilimnion temperature, $^{\circ}C$	22
Hypolimnion temperature, $^{\circ}C$	4
Density difference, $\Delta\rho$	0.002203
Density ratio, ρ_r	0.9977969
Epilimnion eddy viscosity, ν_{M1}, cm^2/sec	16.8
Nominal wind velocity, $\overline{U}_w$, m/sec	5.2

The ν_{M1} value corresponds to that generated by a wind velocity of 5.2 meters per second. Small variations in wind velocity about the nominal value will be allowed. The wind shear stress will be determined by

$$\tau_x^w = -(0.00273)(\rho_a)(\overline{U}_w)^2$$

$$\tau_y^w = 0$$

where ρ_a is the density of the air and the 0.00273 wind drag coefficient is that proposed by Wilson (ref. 9).

The wind drag coefficient and ν_{M1} values used here are the same as the ones used by Gedney and Lick (ref. 5) for current calculations in Lake Erie during the uniform water temperature period (winter period). With these values agreement between the Lake Erie calculations and measurements were very satisfactory. The epilimnion in our two-layer model is considered to be of uniform temperature and for the cases calculated here will have an average thickness of approximately 20 meters. Since 20 meters is also near the average depth of Lake Erie, the ν_{M1} value should be very similar to that used in the Lake Erie calculations.

In obtaining all the thermocline solution the boundary conditions (18b) were adjusted to give the desired mean thermocline depth.

Case 1 - 5.2 Meter per Second Uniform Wind and Constant Depth

Zeroth order asymptotic solution. - The zeroth order solution is governed by equations (17) to (21) and gives the pressure gradient in the hypolimnion as zero; that is,

$$\frac{\partial p_2^{(0)}}{\partial n} = \frac{\partial \xi^{(0)}}{\partial n} + \frac{\partial \zeta^{(0)}}{\partial n} = 0$$

Therefore, there is no zeroth order geostrophic inviscid current in the hypolimnion. There are, however, hypolimnion velocities created by the shear stress at the thermocline. The zeroth order thermocline $(\overline{\xi}^{(0)})$ contours are shown in figure 2(a). The thermocline is at a shallow depth (4.6 m) at the upwind end of the lake and a large depth (≅30 m) at the downwind end. There is very little tilting of the thermocline in the cross wind direction and, as determined by the equation (18b), there is no tilting at the upwind and downwind boundaries of the lake. The dimensional form of equation (18b) for the thermocline boundary values is

$$\bar{\xi}^{(0)} = -\sqrt{\frac{2\bar{\tau}_x^W \bar{x}}{g(\rho_2 - \rho_1)} + \left|\bar{\xi}_0^{(0)}\right|^2} \quad \text{at } \bar{y} = 0, 1$$

$$\bar{\xi}^{(0)} = -\left|\bar{\xi}_0^{(0)}\right| \quad \text{at } \bar{x} = 0$$

$$\bar{\xi}^{(0)} = -\sqrt{\frac{2\bar{\tau}_x^W L}{g(\rho_2 - \rho_1)} + \left|\bar{\xi}_0^{(0)}\right|^2} \quad \text{at } \bar{x} = 1$$

where the overbar indicates dimensional quantities. It is important to note that the boundary values of the zeroth order solution for the thermocline do not depend on (1) the eddy diffusivity values of (2) the depth of the lake. Furthermore, the length to width ratio of the lake has no influence on the $\xi^{(0)}$ boundary values. This zeroth order solution, of course, is only valid as long as $d_2/d_1 = \sqrt{\nu_{M2}/\nu_{M1}} << 1$ and the lake is deeper than the thermocline by the amount d_2, which is the friction depth in the hypolimnion.

In figures 2, the horizontal velocities are given at the lake surface and at depths of 6.7, 15, and 20 meters. These velocity results have been determined from equations (7) and (8) assuming $d_2 = 1.5$ meters. The dashed line included on some of the velocity plots is the intersection of the thermocline with the horizontal plane at the particular depth. The velocity patterns down to 10 meters are similar to those which occur in a homogeneous lake. The Coriolis force causes the deflection of the surface velocities to the right of the wind and a clockwise rotation of the current vector with depth. The effect of the epilimnion thickness being smaller at the upwind end of the lake is easily discerned from the figures. At depths closer to the thermocline, the epilimnion velocities rotate even more clockwise until they are in a southerly cross wind direction. As we go deeper to a region below the thermocline the velocities become very small in magnitude. At a distance approximately equal to d_2 below the thermocline the velocities are essentially zero.

Effect of increasing the hypolimnion eddy diffusivity. - The effect of increasing the eddy diffusivity on the thermocline position is shown by considering solutions for values of $\delta\left(= 1/\rho_r \left(\sqrt{\nu_{M2}/\nu_{M1}}\right)\right)$ of 0.0, 0.5 and ∞ for the uniform wind and constant depth condition. The $\delta = 0$ case is obtained from the zeroth order solution governed by equations (19) and (21). The results for the $\delta = 0.5$ case are obtained from the numerical solution of the complete two-layer lake equations by Gedney, Lick, and Molls (ref. 1).

The results for the $\delta = \infty$ case are obtained from equations (32) and (33) which assume zero hypolimnion velocity. As previously discussed in the INTRODUCTION, the zero hypolimnion velocity assumption has been used by others (refs. 2 and 3) to solve for the position of the thermocline since it uncouples the lower layer from the top layer.

The thermocline positions for $\delta = 0,\ 0.5$, and ∞ are shown in figure 3. In each case the minimum depth of the thermocline is approximately 8.4 meters and occurs along the right boundary. As δ is increased (ν_{M2} increased) the thermocline tilt increases though not dramatically. For uniform wind conditions, the $\delta = \infty$ case, which assumes zero hypolimnion velocities, does not differ greatly from the more realistic $\delta < 1$ case.

Case 2 - Uniform Wind With Variable Depth

As has been shown by the case 1 asymptotic results with $\delta << 1$, a uniform wind acting on the water surface of a constant depth basin produces no zeroth order flow in the hypolimnion except for a thin boundary layer adjacent to the thermocline. The zeroth order horizontal pressure gradient for case 1 is zero. From these results we expect that a variation in lake depth should produce no or a negligible effect when the winds are uniform. This was shown in the analysis section to be true. The zeroth order horizontal pressure gradient in the hypolimnion is zero for a uniform wind with variable depth.

Case 3 - Wind Stress Gradients of Order δ

In this section we examine the effect of a variable wind stress of the form

$$\left.\begin{aligned} \tau_x^w &= -0.914 - 8.0\,\delta(x - x^2)(y - y^2) \qquad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \\ \tau_y^w &= 0.0 \end{aligned}\right\} \tag{34}$$

acting on the surface of a constant depth basin. This τ_x^w has the largest wind stress at the center of the lake which often occurs in actual lakes. (Note that $x = 1,\ y = 1$ corresponds to $\bar{x} = 96.6$ km, $\bar{y} = 96.6$ km.) This wind stress also creates a maximum wind stress curl $\left|\partial\tau_x^y/\partial y\right|$ of 2δ at $x = 0.5$ and $y = 0.0$ and 1.0. The governing equations for this case are equations (26) and (28).

The thermocline depth contours are shown in figure 4(a) for $\delta = 1/24$. These results show that the thermocline shape is very sensitive to spacial variations in the wind stress. Whereas the thermocline generally has very little variation in the cross wind direction when the wind stress is constant, a wind stress curl of order δ creates a large variation.

The $\bar{\xi}^{(0)}$ boundary layer of width $\delta^{1/2}L$ predicted by equation (27) occurs along several portions of the boundary as shown in figure 4(a). The "outer" solution governed by equation (27a) extends throughout the rest of the lake other than these boundary layer regions.

Shown in figure 4(b) are significant horizontal velocities below the thermocline at a depth of 32 meters. These velocities occur below the viscous boundary layer adjacent to the thermocline and therefore are geostrophic (inviscid) velocities created by the nonzero horizontal pressure gradient in the hypolimnion. The governing equation for the geostrophic velocity can be obtained from equation (4) by dropping the viscous terms and is

$$\frac{\partial p_2^{(0)}}{\partial n} = \frac{\partial \eta^{(0)}}{\partial n} = \frac{\partial \xi^{(0)}}{\partial n} + \frac{\partial \zeta^{(0)}}{\partial n} = \frac{-\pi i}{\rho_r}\left[u_2^{(0)} + iv_2^{(0)}\right]$$

The geostrophic velocity occurs at a right angle to the direction of the horizontal pressure gradient $\nabla p_2^{(0)} = \nabla \eta^{(0)}$. As a result, the geostrophic velocities are tangent to the $\eta^{(0)}$ contours shown in figure 4(c). The geostrophic flow in the region of the lake bottom is, of course, brought to rest in a velocity boundary layer at the lake bottom.

With significant geostrophic flow in the hypolimnion, we expect a depth variation to be very important. As shown in the analysis section, the inclusion of a variable depth with an order δ wind stress curl eliminates the hypolimnion horizontal pressure gradient (i.e., $\eta^{(0)}$ is constant) and therefore the hypolimnion geostrophic velocity.

The influence of bottom topography is therefore very strong. We next consider the important case where the wind stress gradients are of order one over a basin of variable depth.

Case 4 - Wind Stress Gradients of Order One

In this section we consider the two-layer solution for a variable wind stress of the form

$$\left.\begin{aligned} \tau_x^w &= -0.914 - 2.0(x - x^2)(y - y^2) \qquad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \\ \tau_y^w &= 0 \end{aligned}\right\} \qquad (35)$$

acting on the surface of a variable depth basin of the form

$$\left.\begin{aligned} \frac{\partial h}{\partial x} &= 2.0(1 - 2x)(y - y^2) \\ \frac{\partial h}{\partial y} &= 2.0(1 - 2y)(x - x^2) \end{aligned}\right\} \quad \begin{aligned} 0 \le x \le 1 \\ 0 \le y \le 1 \end{aligned}$$

(Note that $x = 1$, $y = 1$ corresponds to $\bar{x} = 96.6$, $\bar{y} = 96.6$ km.) The wind stress creates a maximum wind stress curl $(\partial \tau_x^W/\partial y)$ of -0.5 at $x = 0.5$ and $y = 0.0$ and 1.0. The asymptotic expansion solution to this case is governed by equations (30) and (31).

The thermocline depth contours are shown in figure 5(a) for a $\delta = 1/24$. The $\bar{\xi}^{(0)}$ solution contains boundary layers along $\bar{y} = 0$, $\bar{y} = 96.6$, and $\bar{x} = 0$ boundaries. These boundary layer regions are more clearly shown in figure 5(b) which is a plot of the thermocline depth along the $\bar{x} = 48.3$ km ($x = 0.5$) station. Figure 5(b) shows the thermocline to have a large shape variation in the cross wind direction.

The horizontal velocities at the surface and at 6.7, 20.0, and 35.0 meters are shown in figures 5(c) to (f). The figures show the effect of the shape of the thermocline on the velocities. The thermocline shape produces the clockwise gyre shown in figure 5(e). Below the thermocline at a depth of 35.0 meters geostrophic velocities of order 1.0 to 5.0 centimeters per second occur. As explained in the previous section (case 3), these geostrophic (inviscid) velocities are created by and are perpendicular to the horizontal pressure gradient $\nabla p_2^{(0)} = \nabla \eta^{(0)}$. The geostrophic velocities are therefore tangent to the $\eta^{(0)}$ contours shown in figure 5(g). The high geostrophic velocities near the boundaries are created by the $\eta^{(0)}$ boundary layers shown in figure 5(g).

In contrast to the uniform wind stress solution, the unit order wind stress gradient solution is very dependent on the value of the hypolimnion eddy diffusivity. Figure 5(b) shows the shape of the thermocline at $\bar{x} = 48.3$ kilometers ($x = 0.5$) for the same applied wind stress for the cases where $\delta = 1/24$, 1/12, and ∞. The $\delta = \infty$ case corresponds to the solution governed by equations (32) and (33) where the velocities on the entire hypolimnion are assumed zero. The thermocline contour plots for the $\delta = 1/12$ and $\delta = \infty$ cases are shown, respectively, in figures 5(h) and (i). In all three cases the minimum thermocline depth of 6.9 meters occurs along the right boundary. As the hypolimnion eddy diffusivity (ν_{M2}) increases (δ increases), the cross wind variation in the thermocline depth can be seen to become much less.

CONCLUDING REMARKS

The steady-state, wind-driven circulation has been calculated in a stratified lake composed of two layers having uniform but unequal densities and eddy diffusivities. The solution is obtained from the governing equations assuming the ratio of hypolimnion (lower layer) to epilimnion (upper layer) eddy diffusivities (ν_{M2}/ν_{M1}) is much less than one. A value of this ratio much less than one is generally recognized as being true for lakes during the late summer stratification period.

The results indicate that the thermocline shape is strongly dependent on (1) spatial variation of the wind (on the order of 1 m/sec) over the lake, (2) the variation in lake depth, and (3) the magnitude of the hypolimnion eddy diffusivity.

When the wind magnitude varies spatially over a lake, significant velocities occur in the entire hypolimnion. Previous two-layer lake models have calculated the thermocline position by making an interim assumption that the velocity in the entire hypolimnion is zero (this is equivalent to assuming the hypolimnion eddy diffusivity is infinite). The thermocline shape calculated in this manner is found to differ by a large amount from the asymptotic solution which assumes a small hypolimnion eddy diffusivity.

Lewis Research Center,
National Aeronautics and Space Administration,
Cleveland, Ohio, January 24, 1973,
160-75.

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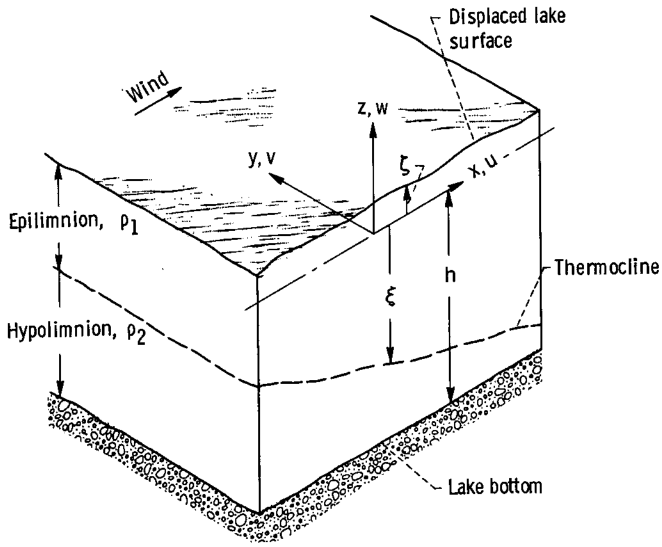
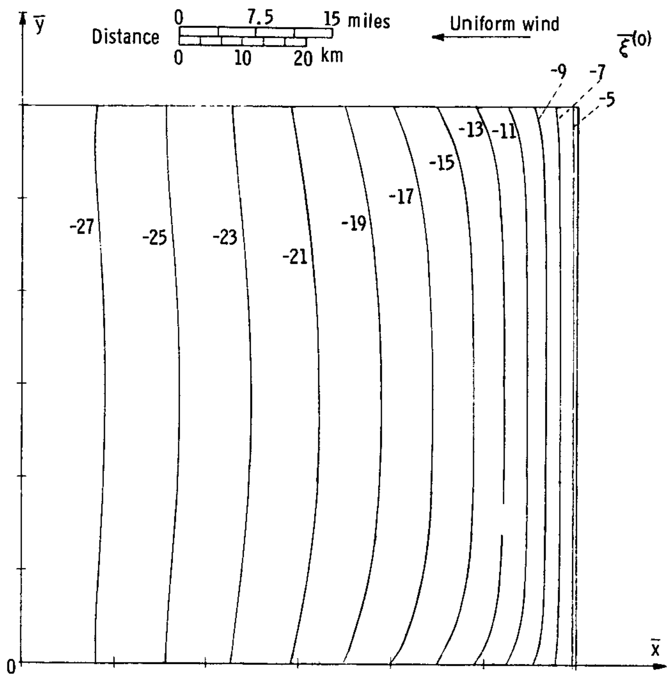
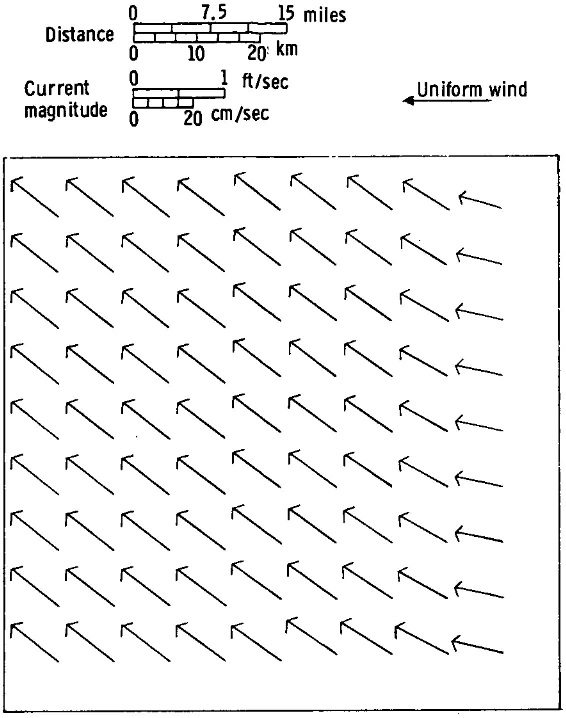


Figure 1. - Cartesian coordinates for stratified lake.

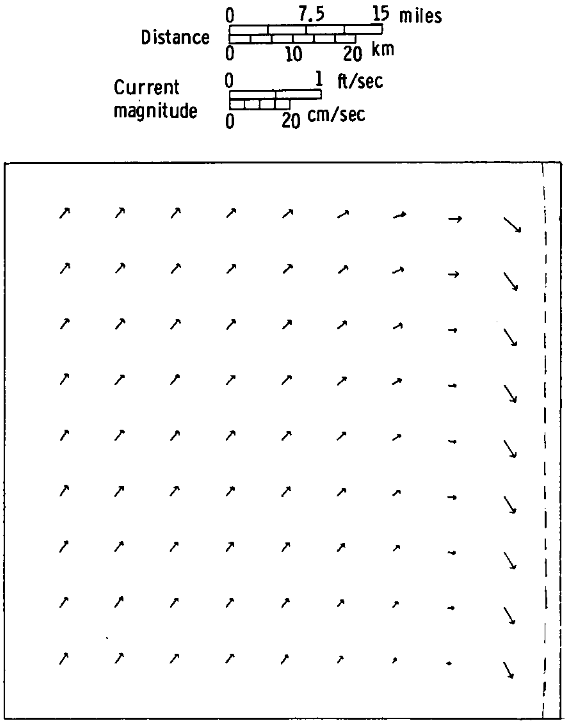


(a) Thermocline depth contours, $\bar{\xi}^{(o)}$ (m).

Figure 2. - Zeroth order two-layer lake solution with uniform wind and constant depth (δ = 1/24).

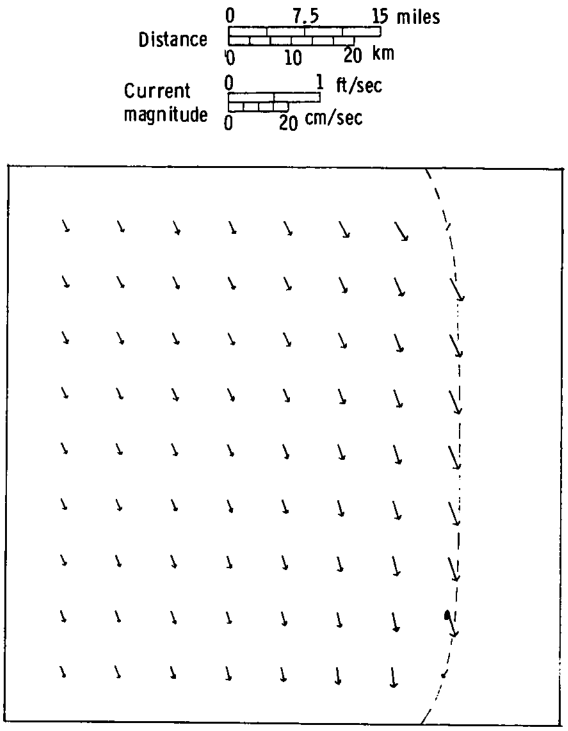


(b) Surface velocities.

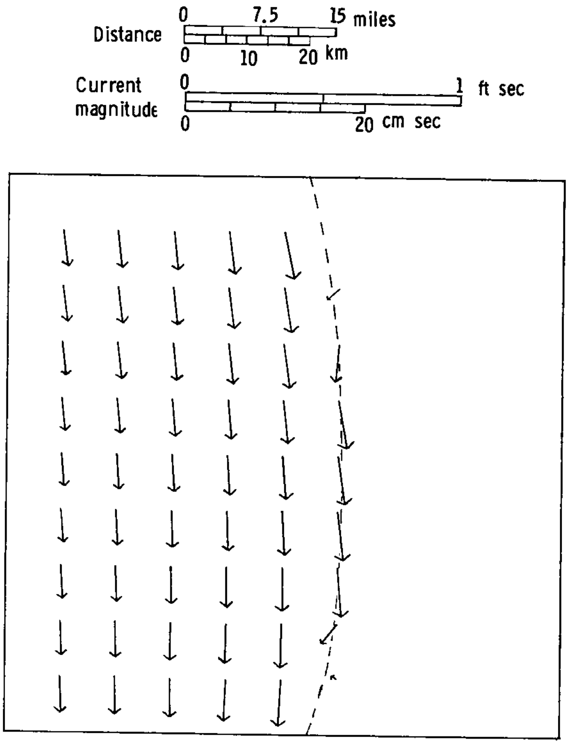


(c) Horizontal velocities at 6.7 meters.

Figure 2. - Continued.

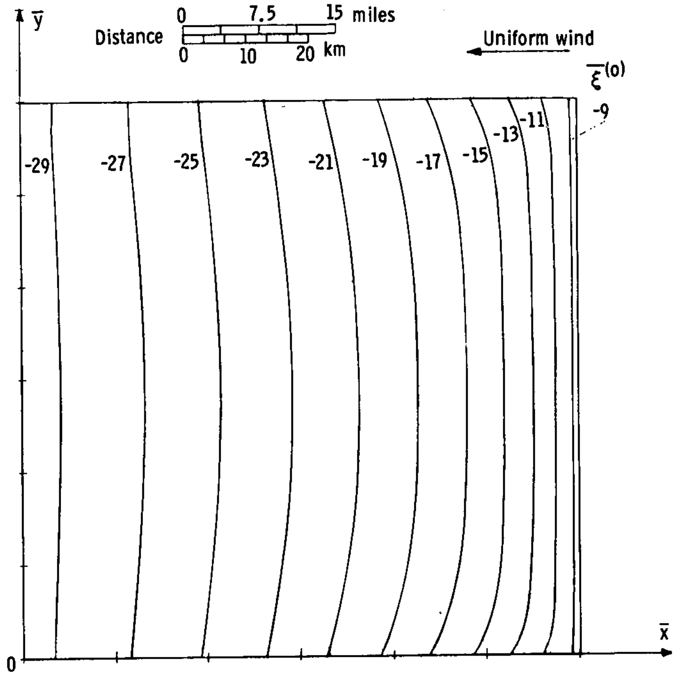


(d) Horizontal velocities at 15 meters.

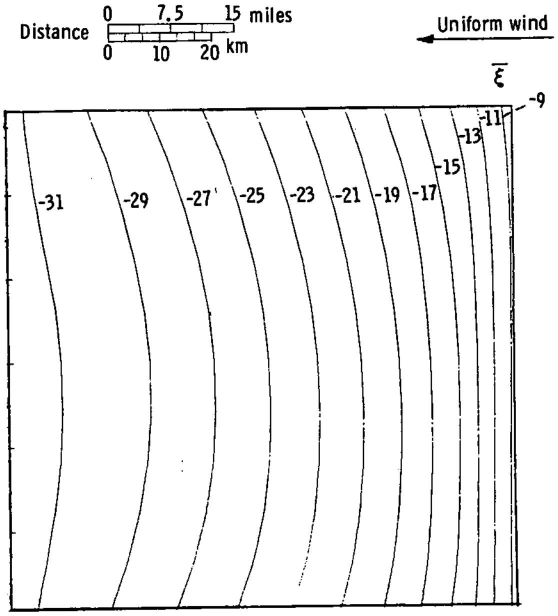


(e) Horizontal velocities at 20 meters.

Figure 2. - Concluded.

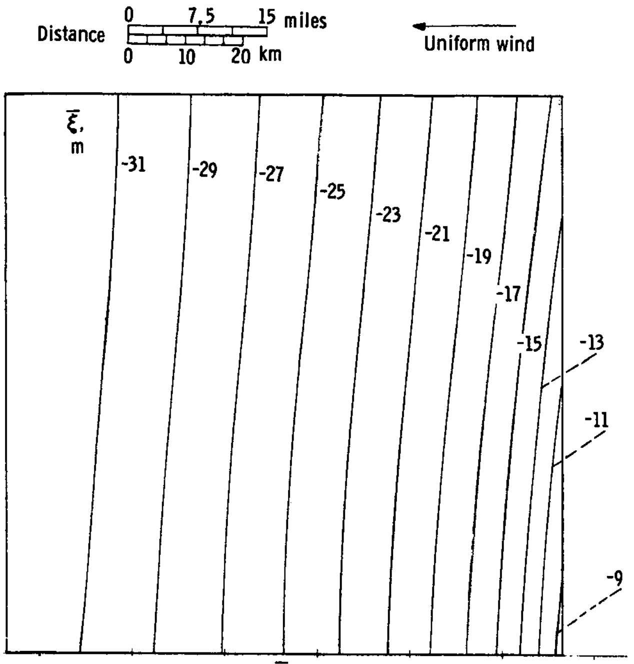


(a) Thermocline depth contours, $\bar{\xi}^{(0)}$ (m); predicted by asymptotic two-layer equations (20) and (21) ($\delta = 0$).



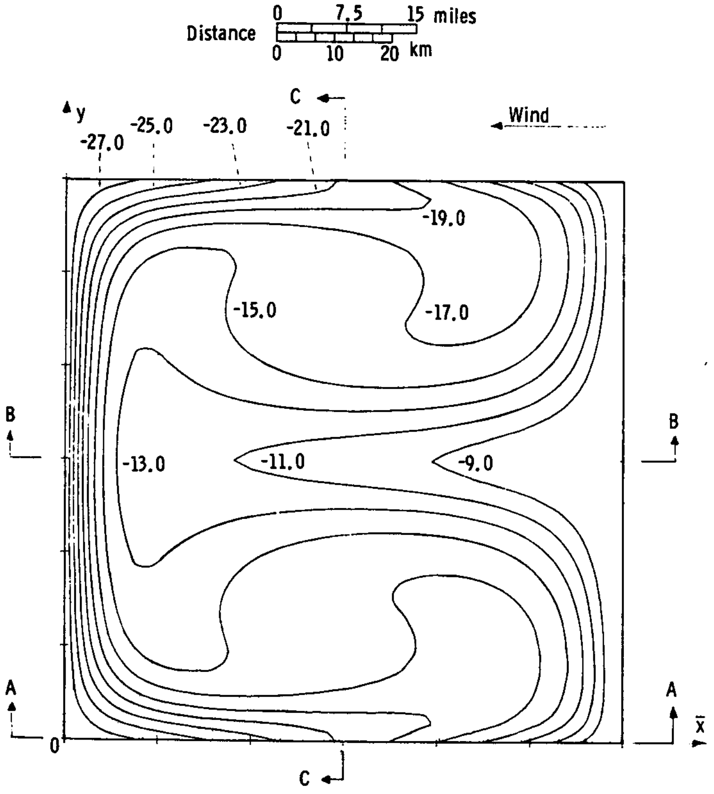
(b) Thermocline depth contours, $\bar{\xi}$ (m); predicted by complete two-layer equations ($\delta = 0.5$; ref. 1).

Figure 3. - Comparison of two-layer lake solutions for $\delta = 0.0$, 0.5, and ∞.



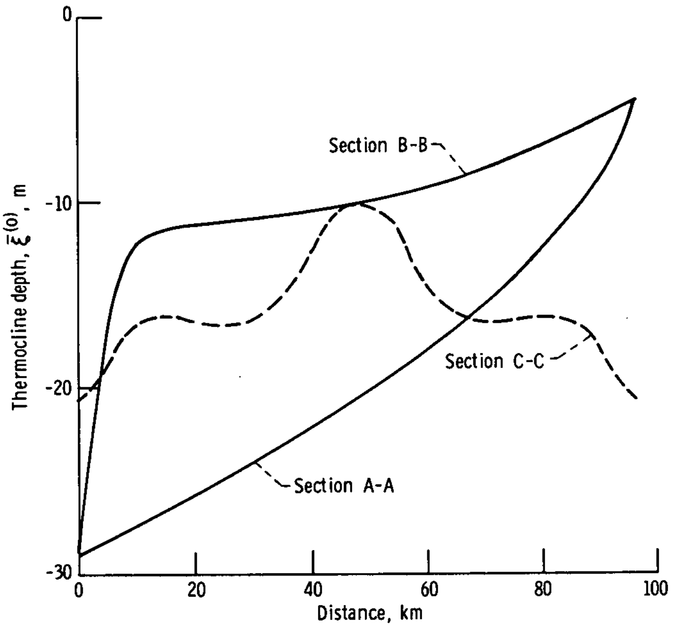
(c) Thermocline depth contours $\bar{\xi}$ (m) when hypolimnion velocities are zero ($\delta = \infty$).

Figure 3. - Concluded.



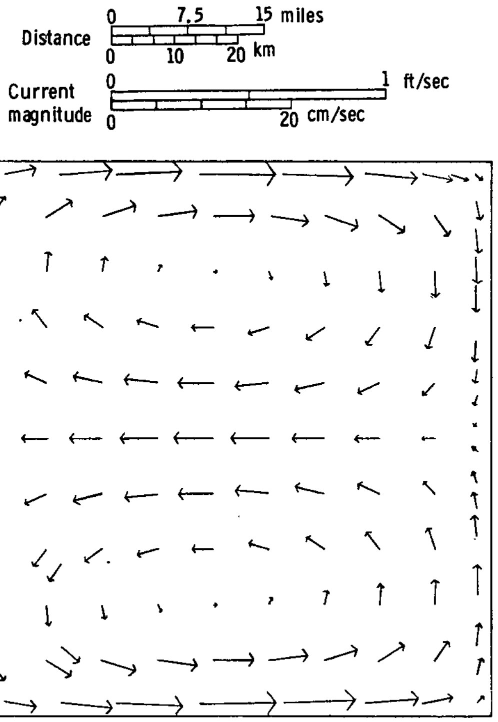
(a) Thermocline depth contours, $\xi^{(0)}$ (m).

Figure 4. - Constant depth two-layer lake solution with wind stress gradients of order δ(δ = 1/24). See equation (34) for form of wind stress.



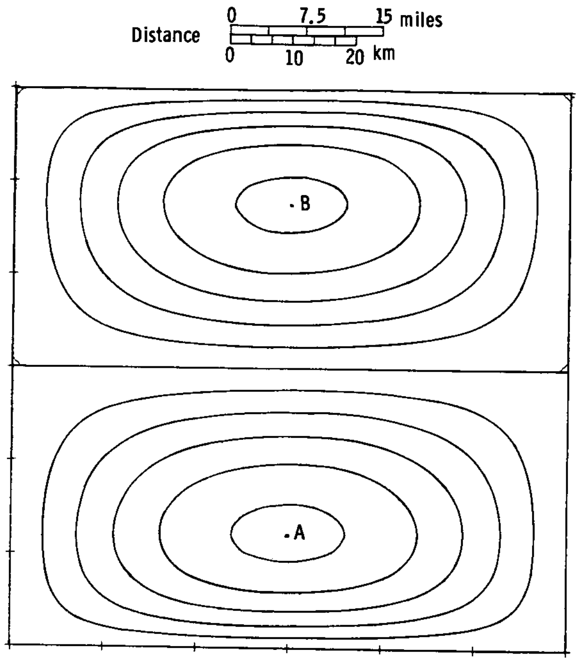
(a) Concluded.

Figure 4. - Continued.



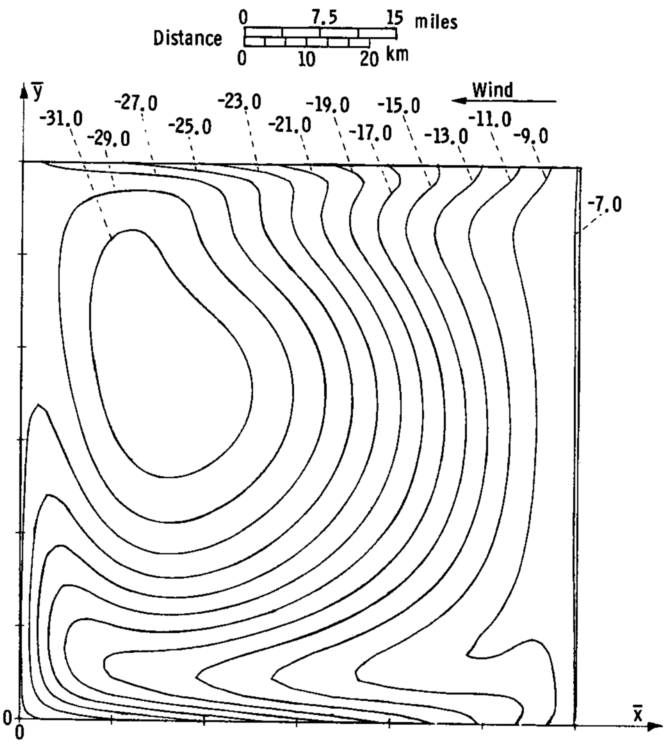
(b) Horizontal velocities at 32-meter depth.

Figure 4. - Continued.



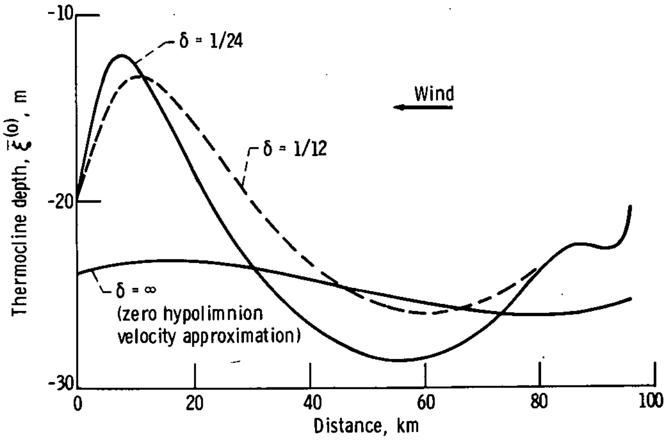
(c) $\eta^{(0)}$ Contours. Difference between each $\eta^{(0)}$ contour is $\pi \times 10^{-2}$; $\eta^{(0)}$ increases from point A to point B ($\delta = 1/24$).

Figure 4. - Concluded.

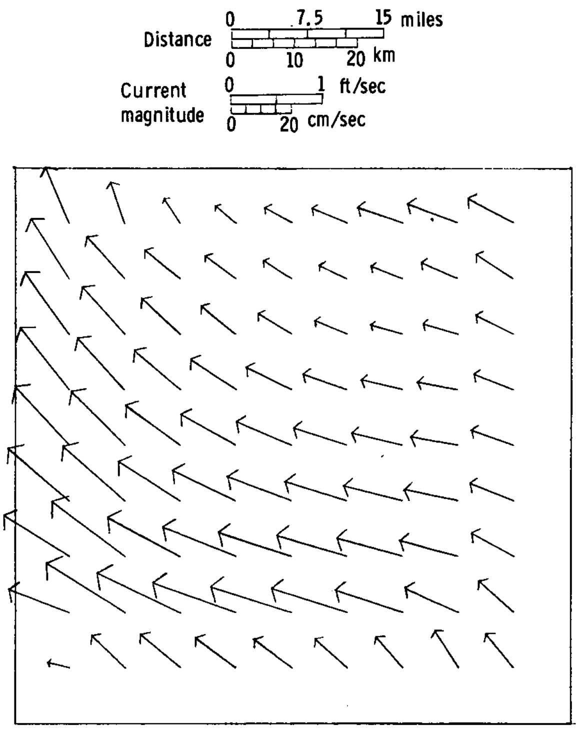


(a) Thermocline depth contours, $\overline{\xi}^{(0)}$ (m); $\delta = 1/24$ (from eqs. (30) and (31)).

Figure 5. - Variable depth two-layer lake solution with wind stress curl of order one. See equation (35) for form of wind stress.

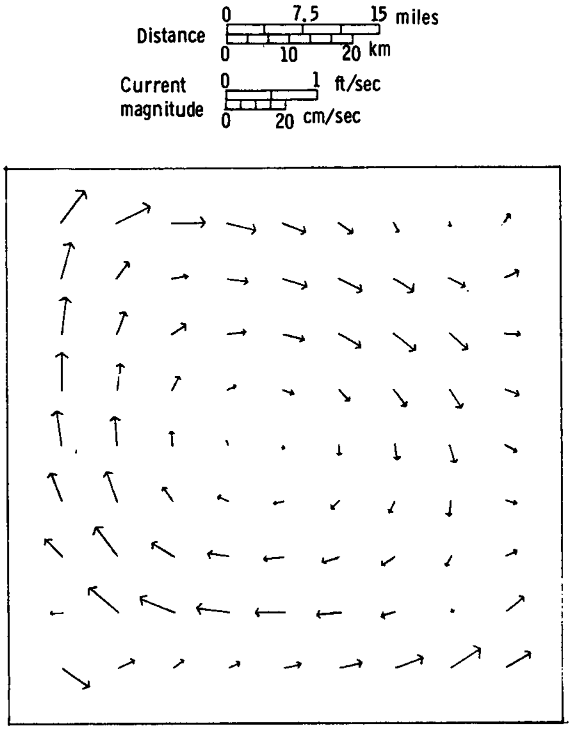


(b) Thermocline profile at $\bar{x}$ = 48.3 kilometers.

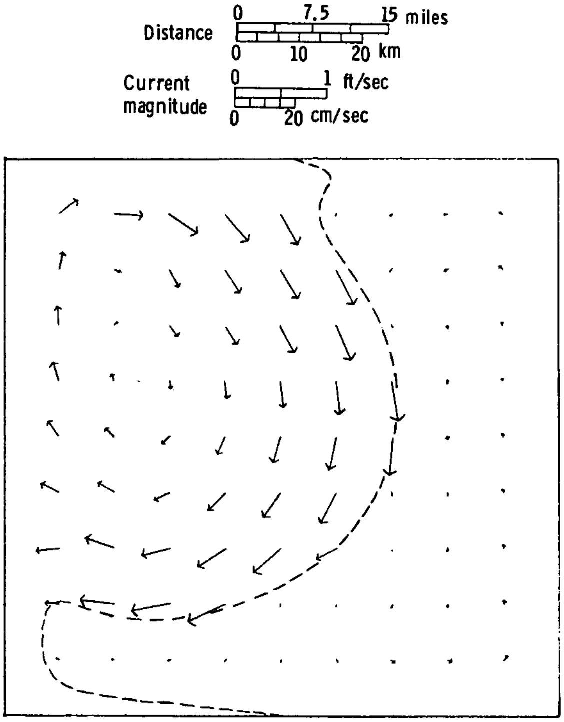


(c) Horizontal surface velocity (δ = 1/24).

Figure 5. - Continued.

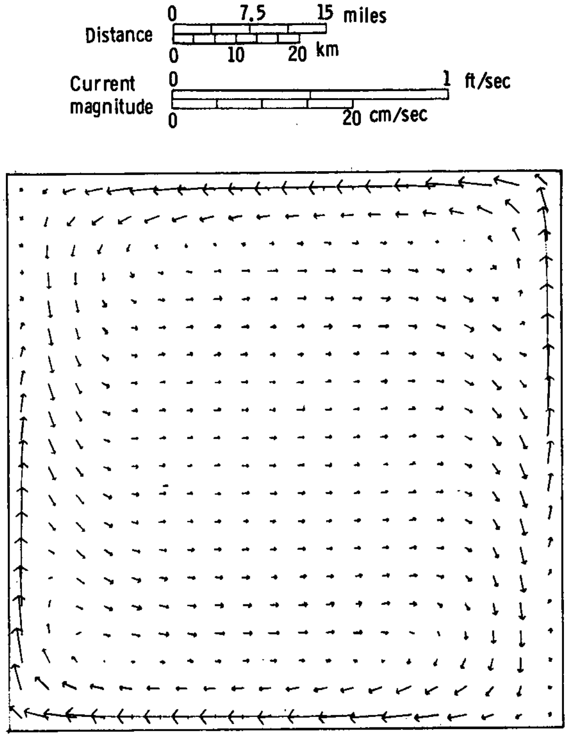


(d) Horizontal velocity at 6.7-meter depth (δ = 1/24).

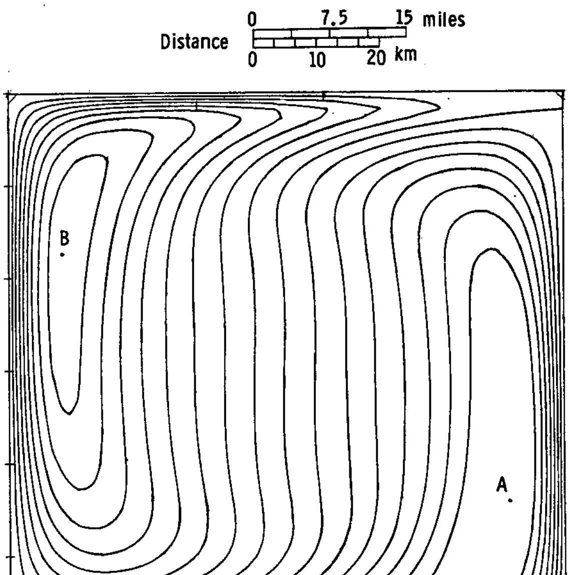


(e) Horizontal velocities at 20-meter depth (δ = 1/24).

Figure 5. - Continued.

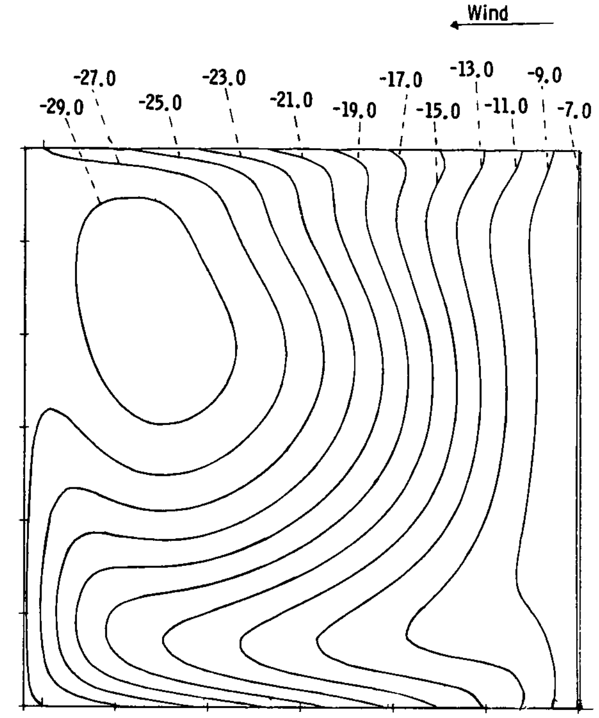


(f) Horizontal velocities at 35.0-meter depth ($\delta = 1/24$).

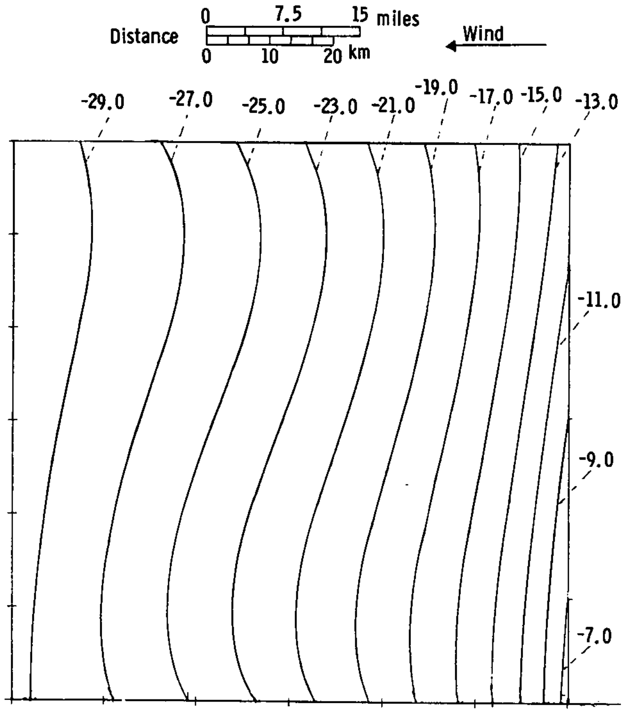


(g) $\eta^{(0)}$ Contours. Difference between each $\eta^{(0)}$ contour is $2\pi x 10^{-3}$; $\eta^{(0)}$ increases from point A to point B ($\delta = 1/24$).

Figure 5. - Continued.



(h) Thermocline depth contours, $\bar{\xi}^{(o)}$ (m); $\delta = 1/12$ (from eqs. (30) and (31)).



(i) Thermocline depth contours, $\bar{\xi}$ (m); $\delta = \infty$ (from eqs. (32) and (33)).

Figure 5. - Concluded.

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