FIXED POINT THEOREMS AND DISSIPATIVE PROCESSES

by

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I. Introduction

Suppose \( X \) is a Banach space, \( T: X \to X \) is a continuous mapping. The map \( T \) is said to be dissipative if there is a bounded set \( B \) in \( X \) such that for any \( x \in X \), there is an integer \( N = N(x) \) with the property that \( T^n x \in B \) for \( n \geq N(x) \). In his study of ordinary differential equations in \( n \)-dimensional Euclidean space (which were \( \omega \)-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with \( T x \) representing the solution of the differential equation at time \( \omega \) which started at \( x \) at time zero. The basic problem is to give information about the limiting behavior of orbits of \( T \) and to discuss the existence of fixed points of \( T \). Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of \( T \) has a fixed point and he characterized the maximal compact invariant set of \( T \). Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of \( T \) has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space \( X \) arising in retarded functional differential equations, and \( T \) completely continuous, Jones [9] and Yoshizawa [16] showed that \( T \) has a fixed point by using Browder's theorem. For an arbitrary Banach space \( X \) and \( T \) completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently,
Billotti and LaSalle [1] have obtained the same result with $T$ completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when $T$ is condensing on balls in $X$; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subseteq X$ and $\alpha$ is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of $T$. More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators $T$ which includes $\alpha$-contractions or $k$-set contractions; that is, there is a constant $k$, $0 < k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subseteq X$. They have characterized the maximal compact invariant set of $T$, shown that it is asymptotically stable, and proved that some iterate of $T$ has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to $\omega$-periodic retarded functional differential equations, the hypothesis that $T$ is completely continuous implies that the period $\omega$ in the equation is greater than or equal to the delay $r$ in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral
functional differential equations, the operator $T$ is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an $\alpha$-contraction. However, the above theory for this case implies only that some iterate of $T$ has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on $T$ which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for $T$ condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The $\varepsilon$-neighborhood of a set $K \subset X$ will be denoted by $B_{\varepsilon}(K)$, the closure by $\text{Cl}(K)$ and the convex closure by $\overline{\text{co}}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set $K$ in $X$ (see [3]).

Suppose $T$ is a continuous map $T : X \to X$. The map $T$ is said to be weak condensing if for any bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map $T$ is said to be a weak $\alpha$-contraction if there is a constant $0 < k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$.

If $T$ takes bounded sets into bounded sets, then a weak $\alpha$-contraction is an $\alpha$-contraction. The map $T^{n_0}$ is said to be weak completely continuous if there is an integer $n_0$ such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^n x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If $T$ is weak completely continuous it is weak condensing.
If $T$ is completely continuous then $T$ is weak completely continuous.

The map $T$ is said to be asymptotically smooth if for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subset X$ attracts a set $H \subset X$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $T^n H \subset B_\varepsilon(K)$ for $n > N(H, \varepsilon)$. We say $K$ attracts compact sets of $X$ if $K$ attracts each compact set $H \subset X$. We say $K$ attracts neighborhoods of compact sets of $X$ if for any compact set $H \subset X$, there is a neighborhood $H_0$ of $H$ such that $K$ attracts $H_0$.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n > N(x)$. If $B$ satisfies the property that for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n A \subset B$ for $n > N(A)$, then $T$ is said to be compact dissipative.

If $B$ satisfies the property that for any $x \in X$, there is an open neighborhood $O_x$ and an integer $N(x)$ such that $T^n O_x \subset B$, $n > N(x)$, then $T$ is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

Lemma 1. a) (Hale, LaSalle, Slemrod [7]). If $T$ is continuous, local dissipative and asymptotically smooths, then there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$. 


b) (Billotti and LaSalle [1]) If $T$ is continuous, point dissipative and $T^{p_0}$ is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood $H_0$ of $H$ and an integer $N(H)$ such that $\bigcup_{j \geq 0} T^j H_0$ is bounded and $T^j H_0 \subset K$ for $n \geq N(H)$. In particular, $T$ is local dissipative and $T$ asymptotically smooths.

**Lemma 2.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of $X$, then

a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of $H$, such that $\bigcup_{n} \geq 0 T^n H_1$ is bounded;

b) $\bigcup_{j \geq 0} T^j B$ is precompact if $B$ is compact.

**Proof:**

a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H, \mathcal{C})$ is the number occurring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where $T$ is bounded. Define $\Omega_N = B_\varepsilon(K)$, $\Gamma_N = \Omega_N$, $\Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

b) The set $A = \bigcup_{j \geq 0} T^j B$ is bounded. Since $T^j(B)$ is compact for any $j$ we have $\alpha(A) = \alpha(\bigcup_{j \geq n} T^j(B))$ for any $n$. But given $\varepsilon > 0$, if $n \geq n_1(B, \varepsilon)$, we have $\bigcup_{j \geq n} T^j B \subset B_\varepsilon(B)$ and thus $\alpha(A) \leq 2 \varepsilon$. Thus $\alpha(A) = 0$ and $A$ is compact. This proves the Lemma.
The following result was proved in [7] if we use Lemmas 1 and 2.

**Theorem 1.** If \( T: X \to X \) is continuous and there is a compact set \( K \subset X \) which attracts neighborhoods of compact sets of \( X \), then \( J = \cap_{j \geq 0} T^j(K) \) is independent of the sets \( K \) satisfying the above property, \( J \) is the maximal compact invariant of \( T \) and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood \( U \) of \( K \) and an integer \( n \) such that \( T^n(U) \subset U \). Thus, if \( T \) possesses the fixed point property, then some iterate of \( T \) has a fixed point (see [7]).

Regarding fixed points of \( T \), it is known (see [9], [14], [5], [8], [1]) that \( T \) completely continuous and point dissipative implies \( T \) has a fixed point. Below, we give some weaker conditions which assert that \( T \) has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

**Theorem 2.**

a) If \( T: X \to X \) is continuous, weak condensing and compact dissipative, then there is a compact invariant set \( K \) which attracts compact sets of \( X \) and \( T \) is local dissipative.

b) If \( T \) is weak condensing and point dissipative then there is a compact invariant set \( K \) that attracts points of \( X \).

**Proof:**

a) It is an easy matter to prove the following fact: If \( H \) is a compact set such that \( T: H \to H \), then the set \( A = \cap T^n(H) \) is compact, non empty, \( T(A) = A \) and \( T^n(H) \) tends to \( A \) in the Hausdorff metric.
Now, for any compact set $L$ of $X$, let $L_1 = \bigcup_{j \geq 0} T^j(L)$. Since $L_1$ is bounded, $L_1 = L \cup T(L_1)$ and $T$ is weak condensing, it follows that $\alpha(L_1) = 0$ and thus $H = \text{Cl}(L_1)$ is compact. Also $T(H) \subset H$. Let $A_L = \bigcap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subset X$ such that $A_L \subset B$ for each compact set $L$.

Since $T(\bigcup A_L) = \bigcup A_L$, where the union is taken over all compact sets $L \subset X$, it follows that the set $K = \text{Cl}(\bigcup A_L)$ is compact, $T(K) \subset K$, and $K$ attracts compact sets of $X$.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if $T$ is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

**Lemma 3.** If $T$ is a weak $\alpha$-contraction, then $T$ asymptotically smooths.

**Proof:** If $B$ is a bounded set, then $B^* = \text{Cl}(\bigcup A_x)$, where $A_x$ is constructed as above for the elements $x \in B$ such that $T^n x \in B$, for any $n \geq 0$.

**Corollary.** If $T$ is a weak $\alpha$-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

3. **Fixed point theorems.** In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious:
Lemma 4. If $A$ is a compact set of $X$ and $F \subset X$ contains a sequence $\{x_n\}$ such that $d(x_n,A) \to 0$ as $n \to \infty$, then $A \cap \overline{F} \neq \emptyset$.

Theorem 3. Suppose $K \subset B \subset S \subset X$ are convex subsets with $K$ compact, $S$ closed, bounded, and $B$ open in $S$. If $T: S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, and $K$ attracts points of $B$, then there is a convex, closed bounded subset $A$ of $S$ such that

$$A = \overline{\co} \left( \bigcup_{j \geq 1} T^j(B \cap A) \right), A \cap K \neq \emptyset.$$

Proof: Let $\mathcal{F}$ be the set of convex, closed, bounded subsets $L$ of $S$ such that $T^j(B \cap L) \subset L$ for $j \geq 1$ and $L \cap K \neq \emptyset$. The family $\mathcal{F}$ is not empty because $S \in \mathcal{F}$. If $L \in \mathcal{F}$, let $L_1 = \overline{\co} \left( \bigcup_{j \geq 1} T^j(B \cap L) \right)$. By Lemma 4, $L_1 \cap K \neq \emptyset$. Also, $L_1$ is convex, closed, and contained in $S$. Since $L \in \mathcal{F}$, we have $L \supset L_1$ and $L_1 \supset T^j(B \cap L) \supset T^j(B \cap L_1)$ for all $j \geq 1$. Thus, $L_1 \in \mathcal{F}$. It follows that a minimal element $A$ of $\mathcal{F}$ will satisfy the conditions of the theorem.

To prove such a minimal element exists, let $(L_\alpha)_{\alpha \in I}$ be a totally ordered family of sets in $\mathcal{F}$. The set $L = \bigcap_{\alpha \in I} L_\alpha$ is closed, convex and contained in $S$. Also, $T^j(B \cap L) \subset T^j(B \cap L_\alpha) \subset L_\alpha$ for any $\alpha \in I$ and $j \geq 1$. Thus, $T^j(B \cap L) \subset L$ for $j \geq 1$. If $J$ is any finite subset of $I$, we have $K \cap (\bigcap_{\alpha \in J} L_\alpha) \neq \emptyset$ and, from compactness, it follows that $K \cap (\bigcap_{\alpha \in I} L_\alpha) \neq \emptyset$. Thus, $L \in \mathcal{F}$ and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:
Theorem 4. The set $A$ of Theorem 3 is compact if and only if there is a compact set $Q = Q(B)$ such that $Q \cap B \neq \emptyset$ and $T^j(Q \cap B) \subseteq Q$ for all $j \geq 0$.

Lemma 5. (Horn [8]). Let $S_0 \subseteq S_1 \subseteq S_2$ be convex subsets of a Banach space $X$ with $S_0, S_2$ compact and $S_1$ open in $S_2$. Let $T: S_2 \to X$ be a continuous mapping such that for some integer $m > 0$, $T^j(S_1) \subseteq S_2$, $0 \leq j \leq m-1$, $T^j(S_1) \subseteq S_0$, $m \leq j \leq 2m-1$. Then $T$ has a fixed point.

Theorem 5. Suppose $K \subseteq B \subseteq S \subseteq X$ are convex subsets with $K$ compact, $S$ closed bounded and $B$ open in $S$. If $T: S \to X$ is continuous, $T^jB \subseteq S$, $j \geq 0$, $K$ attracts compact sets of $B$ and the set $A$ of Theorem 3 is compact, then $T$ has a fixed point.

Proof: Since $K$ is compact and convex, the set $B$ can be taken as $S \cap B_{\varepsilon}(K)$ for some $\varepsilon > 0$. Let $Q$ be as in Theorem 4, $S_0 = \text{Cl}(B_{\varepsilon/2}(K)) \cap Q$, $S_1 = B_{\varepsilon}(K) \cap Q$ and $S_2 = S \cap Q$. Then $S_0 \subseteq S_1 \subseteq S_2$, $S_0, S_2$ compact and $S_1$ is open in $S_2$. Also, $T^j(S_1) \subseteq S_2$, $0 \leq j \leq n_1(K,\varepsilon)$ and $T^j(S_1) \subseteq S_0$ for $j \geq n_1(K,\varepsilon)$ for some integer $n_1(K,\varepsilon)$. An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map $T$ which will ensure that the set $A$ in Theorem 5 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is
Theorem 6. If $T$ is weak condensing, then the set $A$ in Theorem 5 is compact.

Proof: If $\bar{A} = \bigcup_{j \geq 1} T^j(B \cap A)$, then $\bar{X} = T(B \cap A) \cup T(\bar{X})$ and $\alpha(A) = \alpha(\bar{X}) = \max(\alpha(T(B \cap A)), \alpha(T(\bar{X})))$. Since $\alpha(T(\bar{X})) < \alpha(\bar{X})$ if $\alpha(\bar{X}) > 0$, it follows that $\alpha(\bar{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then $\alpha(A) = \alpha(\bar{A}) < \alpha(B \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(B \cap A) = 0$. However, this implies $\alpha(A) = 0$ and $A$ is compact, proving the theorem.

Corollary 1. If the sets $K, B, S$ in Theorem 5 exist, if $K$ attracts the compact sets of $B$ and $T$ is weak condensing, then $T$ has a fixed point.

Proof. This is immediate from Theorems 5 and 6.
Corollary 2. If T: X → X is continuous, point-wise dissipative and T is weak completely continuous, then T has a fixed point.

Proof: This is immediate from Lemma 1b) and Corollary 1.

Corollary 3. If T is a weak α-contraction and there are sets K, B, S as in Corollary 1, then T has a fixed point.

Corollary 4. If T is weak condensing and compact dissipative, then T has a fixed point.

Proof: From Theorem 2a), T is a local dissipative system. Thus co K has an open convex neighborhood B with bounded orbit. The result now follows from Theorems 2, 5, 6.

For α-contractions, this result is contained in [13].

Corollary 5. If T₀ is weak completely continuous, T is weak condensing and point dissipative, then T has a fixed point.

Proof: This follows from Lemma 1b) and Corollary 4.
Lemma 6. If \( S: X \to X \) is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, \( \| \cdot \|_1 \), in \( X \) such that \( \| S \|_1 < 1 \).

Proof. Define \( \| x \|_1 = |x| + |Sx| + \ldots + |S^n x| + \ldots \).

The assumption on the spectrum implies there is an \( 0 < r < 1 \) such that \( |S^n| < r^n \) if \( n \) is sufficiently large. Thus, there is a constant \( K \) such that \( |x| \leq \| x \|_1 \leq K|x| \). Also, for \( x \neq 0 \)

\[
\frac{|Sx|}{\|x\|_1} = 1 - \left[ 1 + \frac{|Sx|}{\|x\|_1} + \frac{|S^2 x|}{\|x\|_1} + \ldots \right]^{-1} \leq 1 - \frac{1}{K}.
\]

The lemma is proved.

Corollary 6. If \( T \) is compact dissipative, \( T = S + U \), where \( S \) is linear and continuous with spectrum contained in the open unit ball and \( T(\Omega) \) bounded implies \( \text{Cl}(U(\Omega)) \) compact for any \( \Omega \subset X \), then \( T \) has a fixed point. If, in addition, \( S^0 \) is completely continuous and \( T \) is only point dissipative, then \( T \) has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that \( T^0 \) is \( S^0 \) plus a completely continuous operator.
The next result generalizes an asymptotic fixed point theorem of Browder [2].

**Theorem 7.** Suppose \( S_0, S_1, S_2 \) are subsets of a Banach space, \( S_0, S_2 \) convex, closed, \( S_1 \) open, \( S_2 \) bounded, \( S_0 \subset S_1 \subset S_2 \). Assume \( T : S_2 \to X \) is condensing in the following sense: if \( \Omega, T(\Omega) \) are contained in \( S_2 \) and \( \alpha(\Omega) > 0 \), then \( \alpha(T(\Omega)) < \alpha(\Omega) \). Assume also that \( T \) satisfies: for any compact set \( H \subset S_1 \), \( T^j(H) \subset S_2 \), \( j \geq 0 \), and there is a number \( N(H) \) such that \( T^j(H) \subset S_0 \) for \( j \geq N(H) \). Then \( T \) has a fixed point.

**Proof:** Following the proof of Theorem 2, there is a compact set \( K \) which attracts the compact sets of \( S_1 \). Since \( K \subset S_0 \), it follows that \( \overline{\text{co}} \ K \subset S_0 \). Let \( B \) be a closed, convex neighborhood of \( \overline{\text{co}} \ K \), \( B \subset S_1 \). Theorems 4 and 5 complete the proof.

4. **Dissipative flows.**

Let \( \{T(t), t \geq 0\} \) be a flow in a Banach space \( X \). A point \( x \in X \) is said to be an equilibrium point if \( T(t)x_0 = x_0 \) for any \( t \geq 0 \). We also say that a compact set \( J \) attracts a compact set \( H \) if, for any \( \varepsilon > 0 \), there is a \( t^*(H,\varepsilon) \) such that \( T(t)H \subset B_\varepsilon(J) \) for \( t \geq t^*(H,\varepsilon) \). A set \( Q \subset X \) is said to be **boundedly compact** if \( P \cap Q \) is compact for any closed bounded set \( P \subset X \).
Theorem 8. If \( \{ T(t), t \geq 0 \} \) satisfies:

\( a_1 \) there is a compact set \( J \) that attracts the compact sets of \( J_0 \);

\( a_2 \) there is a number \( w > 0 \) and a family of boundedly compact (in particular compact) sets \( Q(\tau), 0 < \tau < w \) such that \( Q(\tau) \cap J_0 \neq \emptyset \) and \( T(k\tau)Q(\tau) \cap Q(\tau), 0 < \tau \leq w \), \( k \) positive integer;

then there is an equilibrium point.

**Proof:** Take the sequence \( w_n = \frac{w}{n} \). From Theorems 2, 4, 5, it follows that for each \( n \), there is an \( x_n \) satisfying: \( T(w_n)x_n = x_n \). Since \( w_n > 0 \), \( x_n \) is in \( J \). Changing the notation if necessary, we may assume that \( x_n \) converges to \( x_o \). Let \( k_n(t) \) be the integer defined by:

\[ k_n(t)w_n < t < (k_n(t) + 1)w_n \]

Then, \( T(k_n(t)w_n)x_n = x_n \) and so:

\[ |T(t)x_o - x_o| \leq |T(t)x_o - T(k_n(t)w_n)x_o| + |T(k_n(t)w_n)x_o - T(k_n(t)w_n)x_n| + |x_n - x_o| \]

Since \( k_n(t)w_n \) tends to \( t \) as \( n \to \infty \), the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4, 5 and 8, we have:

**Corollary 7:** If \( \{ T(t), t \geq 0 \} \) is weak condensing for any \( t > 0 \) and satisfies \( a_1 \), then there is an equilibrium point.
Corollary 8. If \( \{T(t), t \geq 0\} \) is a weak \( \alpha \)-contraction and satisfies \( \alpha \), then there is an equilibrium point.

A flow \( \{T(t), t \geq 0\} \) is said to be local dissipative if there is a bounded set \( B \) such that for any point \( x \in X \) there is a neighborhood \( O_x \) of \( x \) and a \( t(x) \) such that \( T(t)O_x \subseteq B \) for \( t \geq t(x) \).

A flow \( \{T(t), t \geq 0\} \) is compact (point) dissipative if there is a bounded set \( B \) such that for any compact set \( H \) (any point \( x \)) there is a \( t(H) \) (\( t(x) \)) such that \( T(t)H \subseteq B \) (\( T(t)x \in B \)) for \( t \geq t(H) \) (\( t \geq t(x) \)).

Lemma 7.

a). If \( T(w) \) is weak condensing for some \( w > 0 \) and is compact (point) dissipative with compact attractor \( K \), then \( \{T(t), t \geq 0\} \) is compact (point) dissipative with attractor \( J = \bigcup_{0 \leq t \leq w} T(t)K \).

b). If there is a compact set \( K \) such that \( T(w)K \subseteq K \) and \( K \) attracts neighborhoods of points, then the set \( J \) above attracts neighborhoods of points relative to the flow \( \{T(t), t \geq 0\} \).

c). If \( \{T(t) \geq 0\} \) is weak condensing for some \( \omega > 0 \) and compact dissipative, it is local dissipative.

Proof: For any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( T(t)B_\delta(K) \subseteq B_\varepsilon(J) \), \( 0 \leq t \leq w \). Since \( T(w)K \subseteq K \), parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).
Corollary 9. If for some \( w > 0 \), \( T(w) \) is weak completely continuous and point dissipative then \( \{T(t), t \geq 0\} \) is local dissipative.

Proof: The set \( K \) in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

Theorem 9. If \( \{T(t), t > 0\} \) is weak condensing for \( t > 0 \), then the following assertions hold:

a). If \( \{T(t), t \geq 0\} \) is compact dissipative, then there is an equilibrium point;

b). If \( \{T(t), t \geq 0\} \) is point dissipative and \( T(w) \) is weak completely continuous for some \( w > 0 \), then there is an equilibrium point.

Corollary 10. If \( \{T(t) = S(t) + U(t), t \geq 0\} \) then the following assertions hold:

a). If \( \{U(t), t \geq 0\} \) is compact dissipative, \( S(t) \) is linear with spectrum contained inside the unit ball for \( t > 0 \) and \( U(t) \) is weak completely continuous, then there is an equilibrium point.

b). If \( S(w) \) is completely continuous for some \( w > 0 \) and \( \{T(t), t \geq 0\} \) is point dissipative, there exists an equilibrium point.
5. **Functional differential equations.**

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let $r \geq 0$ be a given real number, $E^n$ be an $n$-dimensional linear vector space with norm $|\cdot|$, $C([a,b],E^n)$ be the space of continuous functions from $[a,b]$ to $E^n$ with the uniform topology and let $C = C([-r,0],E^n)$. For $\varphi \in C$, $|\varphi| = \sup_{-r < \theta < 0} |\varphi(\theta)|$. For any $x \in C([-r,A),E^n)$, $A \geq 0$, let $x_t \in C$, $t \in [0,A]$, be defined by $x_t(\theta) = x(t+\theta), -r \leq \theta \leq 0$. Suppose $D : R \times C \rightarrow E^n$ is a continuous linear operator $D(\varphi) = \varphi(0) - g(t,\varphi)$,

$$g(t,\varphi) = \int_{-r}^0 [d\mu(t,\theta)]\varphi(\theta)$$

(1)

$$|\int_{-s}^0 [d\mu(t,\theta)]\varphi(\theta)| \leq \gamma(s)|\varphi|,$$

for $s \geq 0$, $\varphi \in C$ where $\mu$ is an $n \times n$ matrix function of bounded variation, $\gamma$ is continuous and nondecreasing on $[0,r]$, $\gamma(0) = 0$. If $f : R \times C \rightarrow E^n$ is continuous, then a NFDE is a relation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t).$$

A solution $x = x(\sigma)$ through $\varphi$ at time $\sigma$ is a continuous function defined on $[\sigma-r,\sigma+A)$, $A > 0$, such that $x_\sigma = \varphi$, $D(t,x_t)$ is continuously differentiable on $(\sigma,\sigma+A)$ and (2) is satisfied on $(\sigma,\sigma+A)$. We assume
We assume a solution $x(\varphi)$ of (2) through any $\varphi \in C$ exists on $[\sigma-r, \infty)$, is unique and $x(\varphi)(t)$ depends continuously on $(\varphi, t) \in C \times [\sigma-r, \infty)$. In the following, we let $T_D(t, \sigma) : C \to C$, $t \geq 0$, be the continuous linear operator defined by $T_D(t, \sigma)\varphi = y_t(\varphi)$, $t \geq \sigma$, where $y = y(\varphi)$ is the solution of

\begin{equation}
\frac{d}{dt} D(t, y_t) = 0, \ y_0 = \varphi.
\end{equation}

If $D$ is $\omega$-periodic in $t$, $C_D = \{ \varphi \in C : D(0, \varphi) = 0 \}$, then $C_D$ is a Banach space with the topology of $C$, $T_D(\omega, 0) : C_D \to C_D$, and $T_D(n\omega, 0) = T_D^N(\omega, 0)$.

The operator $D$ is said to be uniformly stable if there exist constants $K \geq 1$, $\alpha > 0$, such that

\begin{equation}
|T_D(t, \sigma)\varphi| \leq Ke^{-\alpha(t-\sigma)}|\varphi|, \ \varphi \in C_D, \ t \geq \sigma.
\end{equation}

Notice the operator $D\varphi = \varphi(0)$ corresponding to retarded functional differential equations is always stable.

Remark. The conclusion of the main theorem below is valid under the weaker hypothesis that $D(\varphi) = D_0(\varphi) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta$ where $D_0$ is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that $D$ uniformly stable implies there exists an $n \times n$ matrix function $B(t)$ defined and of bounded variation on $[-r, \infty)$, continuous from the left, $B(t) = 0$, $-r \leq t \leq 0$, and a constant $M_1$ such that
and, for any continuous function $h: [0, \infty) \to \mathbb{R}^n$, the solution of the problem

$$D(t, x_t) = D(0, \varphi) + \int_0^t h(s) ds, \quad x_0 = \varphi$$

is given by

$$x_t = T_D(t) \varphi - \int_0^t B(t-s) h(s) ds.$$  

Furthermore, there exist $n$ functions $\varphi_1, \ldots, \varphi_n$ in $C$ such that $D(0, \varphi) = I$, the identity, where $\varphi = (\varphi_1, \ldots, \varphi_n)$.

Let $\psi: C \to C_D$ be the continuous linear operator defined by $\psi(\varphi) = \varphi - \Phi D(\varphi)$.

**Lemma 8.** If $D$ is uniformly stable and $f$ maps bounded sets of $\mathbb{R} \times C$ into bounded sets of $\mathbb{R}^n$, then there is a family of continuous transformations $T_1(t): C \to C$, $t \geq 0$ which are weak completely continuous and

$$T(t, 0) \varphi = T(t) \varphi = x_t(\varphi) = T_D(t) \psi(\varphi) + T_1(t) \varphi$$

If $D \varphi = \varphi(0)$, then $T(t)$ is weak completely continuous for $t \geq r$.
Proof: Equation (2) with initial value $x_0 = \varphi$ is equivalent to

$$D(x_t) = D(\varphi) + \int_0^t f(s, x_s) ds, \quad t \geq 0, \quad x_0 = \varphi,$$

which from (7) is equivalent to

$$T(t) = T(t)\psi(\varphi) + T(t)\psi_D(\varphi) - \int_0^t B(t-s) f(s, x_s) ds = T(t)\psi(\varphi) + T(t).$$

It is now an easy matter to verify the assertions in the theorem.

Since the condition that $D$ is uniformly stable implies the linear operator $S(\omega) = T_D(\omega)\psi$ has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

**Theorem 4.** If there exists an $\omega > 0$ such that $f(t+\omega, \varphi) = f(t, \varphi)$ for all $\varphi \in C$, $f$ takes bounded sets of $\mathbb{R} \times C \to E^n$ and system (2) is compact dissipative, then there is an $\omega$-periodic solution of (2). If $f$ satisfies the same hypotheses and is independent of $t$, then there is a constant function $c$ in $C$ such that $f(c) = 0$; that is, an equilibrium point of (2). If $D(\varphi) = \varphi(0)$, then the same conclusions are true for point dissipative.
References


FIXED POINT THEOREMS AND DISSIPATIVE PROCESSES

by

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1. Introduction

Suppose $X$ is a Banach space, $T: X \to X$ is a continuous mapping. The map $T$ is said to be dissipative if there is a bounded set $B$ in $X$ such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n \geq N(x)$. In his study of ordinary differential equations in n-dimensional Euclidean space (which were $\omega$-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with $T x$ representing the solution of the differential equation at time $\omega$ which started at $x$ at time zero. The basic problem is to give information about the limiting behavior of orbits of $T$ and to discuss the existence of fixed points of $T$. Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of $T$ has a fixed point and he characterized the maximal compact invariant set of $T$. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of $T$ has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space $X$ arising in retarded functional differential equations, and $T$ completely continuous, Jones [9] and Yoshizawa [16] showed that $T$ has a fixed point by using Browder's theorem. For an arbitrary Banach space $X$ and $T$ completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently,
Billotti and LaSalle [1] have obtained the same result with $T$ completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when $T$ is condensing on balls in $X$; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subseteq X$ and $\alpha$ is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of $T$.

More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators $T$ which includes $\alpha$-contractions or $k$-set contractions; that is, there is a constant $k$, $0 < k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subseteq X$. They have characterized the maximal compact invariant set of $T$, shown that it is asymptotically stable, and proved that some iterate of $T$ has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to $\omega$-periodic retarded functional differential equations, the hypothesis that $T$ is completely continuous implies that the period $\omega$ in the equation is greater than or equal to the delay $r$ in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral
functional differential equations, the operator $T$ is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an $\alpha$-contraction. However, the above theory for this case implies only that some iterate of $T$ has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on $T$ which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for $T$ condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The $\varepsilon$-neighborhood of a set $K \subset X$ will be denoted by $B_\varepsilon(K)$, the closure by $\text{Cl}(K)$ and the convex closure by $\text{co}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set $K$ in $X$ (see [3]). Suppose $T$ is a continuous map $T: X \to X$. The map $T$ is said to be weak condensing if for any bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map $T$ is said to be a weak $\alpha$-contraction if there is a constant $0 < k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$. If $T$ takes bounded sets into bounded sets, then a weak $\alpha$-contraction is an $\alpha$-contraction. The map $T^{n_0}$ is said to be weak completely continuous if there is an integer $n_0$ such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^n x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If $T$ is weak completely continuous it is weak condensing.
If \( T \) is completely continuous then \( T \) is weak completely continuous.

The map \( T \) is said to be \underline{asymptotically smooth} if for any bounded set \( B \subset X \), there is a compact set \( B^* \subset X \) such that for any \( \varepsilon > 0 \), there is an integer \( n_0(\varepsilon, B) \) with the property that \( T^n x \in B \) for \( n \geq 0 \) implies \( T^n x \in B_\varepsilon(B^*) \) for \( n \geq n_0(\varepsilon, B) \).

For a given continuous map \( T: X \to X \), we say a set \( K \subset X \) \underline{attracts} a set \( H \subset X \) if for any \( \varepsilon > 0 \), there is an integer \( N(H, \varepsilon) \) such that \( T^n(H) \subset B_\varepsilon(K) \) for \( n \geq N(H, \varepsilon) \). We say \( K \) \underline{attracts compact sets of} \( X \) if \( K \) attracts each compact set \( H \subset X \). We say \( K \) \underline{attracts neighborhoods of compact sets of} \( X \) if for any compact set \( H \subset X \), there is a neighborhood \( H^*_0 \) of \( H \) such that \( K \) attracts \( H^*_0 \).

A continuous map \( T: X \to X \) is said to be \underline{point dissipative} if there is a bounded set \( B \subset X \) with the property that, for any \( x \in X \), there is an integer \( N(x) \) such that \( T^n x \in B \) for \( n \geq N(x) \). If \( B \) satisfies the property that for any compact set \( A \subset X \), there is an integer \( N(A) \) such that \( T^n(A) \subset B \) for \( n \geq N(A) \), then \( T \) is said to be \underline{compact dissipative}.

If \( B \) satisfies the property that for any \( x \in X \), there is an open neighborhood \( O_x \) and an integer \( N(x) \) such that \( T^n O_x \subset B, n \geq N(x) \), then \( T \) is said to be \underline{local dissipative}. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

\textbf{Lemma 1. a) (Hale, LaSalle, Slemrod [7]).} If \( T \) is continuous, local dissipative and asymptotically smooths, then there is a compact set \( K \subset X \) which attracts neighborhoods of compact sets of \( X \).
5.

b) (Billotti and LaSalle [1]) If $T$ is continuous, point dissipative and $T^0$ is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood $H_0$ of $H$ and an integer $N(H)$ such that $U_j \geq 0 \ T^j H_0$ is bounded and $T^n H_0 \subset K$ for $n \geq N(H)$. In particular, $T$ is local dissipative and $T$ asymptotically smooths.

Lemma 2. If $T: X \rightarrow X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of $X$, then

a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of $H$, such that $\bigcup_{n \geq 0} T^n H_1$ is bounded;

b) $\bigcup_{j \geq 0} T^j B$ is precompact if $B$ is compact.

Proof: a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H, \varepsilon)$ is the number occurring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where $T$ is bounded. Define $\Omega_N = B_{\varepsilon}(K), \quad \Gamma_N = \Omega_N, \quad \Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

b) The set $A = \bigcup_{j \geq 0} T^j B$ is bounded. Since $T^j(B)$ is compact for any $j$, we have $\alpha(A) = \alpha(\bigcup_{j \geq 0} T^j(B))$ for any $n$. But given $\varepsilon > 0$, if $n \geq n_1(B, \varepsilon)$, we have $\bigcup_{j \geq n} T^j B \subset B_\varepsilon(B)$ and thus $\alpha(A) \leq 2\varepsilon$. Thus $\alpha(A) = 0$ and $A$ is compact. This proves the Lemma.
The following result was proved in [7] if we use Lemmas 1 and 2.

**Theorem 1.** If $\mathbf{T} : \mathbf{X} \to \mathbf{X}$ is continuous and there is a compact set $\mathbf{K} \subset \mathbf{X}$ which attracts neighborhoods of compact sets of $\mathbf{X}$, then $J = \cap_j > 0 \mathbf{T}^j(\mathbf{K})$ is independent of the sets $\mathbf{K}$ satisfying the above property, $J$ is the maximal compact invariant of $\mathbf{T}$ and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood $\mathbf{U}$ of $\mathbf{K}$ and an integer $n$ such that $\mathbf{T}^n(\mathbf{U}) \subset \mathbf{U}$. Thus, if $\mathbf{T}$ possesses the fixed point property, then some iterate of $\mathbf{T}$ has a fixed point (see [7]).

Regarding fixed points of $\mathbf{T}$, it is known (see [9], [14], [5], [8], [1]) that $\mathbf{T}$ completely continuous and point dissipative implies $\mathbf{T}$ has a fixed point. Below, we give some weaker conditions which assert that $\mathbf{T}$ has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

**Theorem 2.** a) If $\mathbf{T} : \mathbf{X} \to \mathbf{X}$ is continuous, weak condensing and compact dissipative, then there is a compact invariant set $\mathbf{K}$ which attracts compact sets of $\mathbf{X}$ and $\mathbf{T}$ is local dissipative.

b) If $\mathbf{T}$ is weak condensing and point dissipative then there is a compact invariant set $\mathbf{K}$ that attracts points of $\mathbf{X}$.

**Proof:** a) It is an easy matter to prove the following fact: If $\mathbf{H}$ is a compact set such that $\mathbf{T} : \mathbf{H} \to \mathbf{H}$, then the set $A = \cap \mathbf{T}^n(\mathbf{H})$ is compact, non empty, $\mathbf{T}(A) = A$ and $\mathbf{T}^n(\mathbf{H})$ tends to $A$ in the Hausdorff metric.
Now, for any compact set $L$ of $X$, let $L^1 = \bigcup_{j \geq 0} T^j(L)$. Since $L^1$ is bounded, $L^1 = L \cup T(L^1)$ and $T$ is weak condensing, it follows that $\alpha(L^1) = 0$ and thus $H = \text{Cl}(L^1)$ is compact. Also $T(H) \subseteq H$. Let $A_L = \cap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subseteq X$ such that $A_L \subseteq B$ for each compact set $L$.

Since $T(UA_L) = UA_L$, where the union is taken over all compact sets $L \subseteq X$, it follows that the set $K = \text{Cl}(UA_L)$ is compact, $T(K) \subseteq K$, and $K$ attracts compact sets of $X$.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if $T$ is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

**Lemma 3.** If $T$ is a weak $\alpha$-contraction, then $T$ asymptotically smooths.

**Proof:** If $B$ is a bounded set, then $B^* = \text{Cl}(UA_x)$, where $A_x$ is constructed as above for the elements $x \in B$ such that $T^n x \in B$, for any $n \geq 0$.

**Corollary.** If $T$ is a weak $\alpha$-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

3. **Fixed point theorems.** In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious:
Lemma 4. If \( A \) is a compact set of \( X \) and \( F \subseteq X \) contains a sequence \( \{x_n\} \) such that \( d(x_n, A) \to 0 \) as \( n \to \infty \), then \( A \cap F \neq \emptyset \).

Theorem 3. Suppose \( K \subseteq B \subseteq S \subseteq X \) are convex subsets with \( K \) compact, \( S \) closed, bounded, and \( B \) open in \( S \). If \( T: S \to X \) is continuous, \( T^j B \subseteq S \), \( j \geq 0 \), and \( K \) attracts points of \( B \), then there is a convex, closed bounded subset \( A \) of \( S \) such that

\[
A = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap A) \right], \quad A \cap K \neq \emptyset.
\]

Proof: Let \( \mathcal{F} \) be the set of convex, closed, bounded subsets \( L \) of \( S \) such that \( T^j(B \cap L) \subseteq L \) for \( j \geq 1 \) and \( L \cap K \neq \emptyset \). The family \( \mathcal{F} \) is not empty because \( S \in \mathcal{F} \). If \( L \in \mathcal{F} \), let \( L_1 = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap L) \right] \). By Lemma 4, \( L_1 \cap K \neq \emptyset \). Also, \( L_1 \) is convex, closed, and contained in \( S \).

Since \( L \in \mathcal{F} \), we have \( L \supseteq L_1 \) and \( L_1 \supseteq T^j(B \cap L) \supseteq T^j(B \cap L_1) \) for all \( j \geq 1 \). Thus, \( L_1 \in \mathcal{F} \). It follows that a minimal element \( A \) of \( \mathcal{F} \) will satisfy the conditions of the theorem.

To prove such a minimal element exists, let \( (L_\alpha)_{\alpha \in I} \) be a totally ordered family of sets in \( \mathcal{F} \). The set \( L = \bigcap_{\alpha \in I} L_\alpha \) is closed, convex and contained in \( S \). Also, \( T^j(B \cap L) \subseteq T^j(B \cap L_\alpha) \subseteq L_\alpha \) for any \( \alpha \in I \) and \( j \geq 1 \). Thus, \( T^j(B \cap L) \subseteq L \) for \( j \geq 1 \). If \( J \) is any finite subset of \( I \), we have \( K \cap (\bigcap_{\alpha \in J} L_\alpha) \neq \emptyset \) and, from compactness, it follows that \( K \cap (\bigcap_{\alpha \in I} L_\alpha) \neq \emptyset \). Thus, \( L \in \mathcal{F} \) and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:
Theorem 4. The set $A$ of Theorem 3 is compact if and only if there is a compact set $Q = Q(B)$ such that $Q \cap B \neq \emptyset$ and $T^j(Q \cap B) \subseteq Q$ for all $j \geq 0$.

Lemma 5. (Horn [8]). Let $S_0 \subseteq S_1 \subseteq S_2$ be convex subsets of a Banach space $X$ with $S_0, S_2$ compact and $S_1$ open in $S_2$. Let $T : S_2 \to X$ be a continuous mapping such that for some integer $m > 0$, $T^j(S_1) \subseteq S_2$, $0 \leq j \leq m - 1$, $T^j(S_1) \subseteq S_0$, $m \leq j \leq 2m - 1$. Then $T$ has a fixed point.

Theorem 5. Suppose $K \subseteq B \subseteq S \subseteq X$ are convex subsets with $K$ compact, $S$ closed bounded and $B$ open in $S$. If $T : S \to X$ is continuous, $T^j(B) \subseteq S$, $j \geq 0$, $K$ attracts compact sets of $B$ and the set $A$ of Theorem 3 is compact, then $T$ has a fixed point.

Proof: Since $K$ is compact and convex, the set $B$ can be taken as $S \cap B_e(K)$ for some $e > 0$. Let $Q$ be as in Theorem 4, $S_0 = \text{Cl}(B_e/2(K)) \cap Q$, $S_1 = B_e(K) \cap Q$ and $S_2 = S \cap Q$. Then $S_0 \subseteq S_1 \subseteq S_2$, $S_0, S_2$ compact and $S_1$ is open in $S_2$. Also, $T^j(S_1) \subseteq S_2$, $0 \leq j \leq n_1(K,e)$ and $T^j(S_1) \subseteq S_0$ for $j \geq n_1(K,e)$ for some integer $n_1(K,e)$. An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map $T$ which will ensure that the set $A$ in Theorem 3 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is
Theorem 6. If $T$ is weak condensing, then the set $A$ in Theorem 5 is compact.

Proof: If $\overline{A} = \bigcup_{j \geq 1} T^j(B \cap A)$, then $\overline{A} = T(B \cap A) \cup T(\overline{A})$ and $\alpha(A) = \alpha(\overline{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\overline{A})))$. Since $\alpha(T(\overline{A})) < \alpha(\overline{A})$ if $\alpha(\overline{A}) > 0$, it follows that $\alpha(\overline{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then $\alpha(A) = \alpha(\overline{A}) < \alpha(B \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(B \cap A) = 0$. However, this implies $\alpha(A) = 0$ and $A$ is compact, proving the theorem.

Corollary 1. If the sets $K, B, S$ in Theorem 5 exist, if $K$ attracts the compact sets of $B$ and $T$ is weak condensing, then $T$ has a fixed point.

Proof. This is immediate from Theorems 5 and 6.
Corollary 2. If $T: X \to X$ is continuous, point-wise dissipative and $T$ is weak completely continuous, then $T$ has a fixed point.

Proof: This is immediate from Lemma 1b) and Corollary 1.

Corollary 3. If $T$ is a weak $\alpha$-contraction and there are sets $K, B, S$ as in Corollary 1, then $T$ has a fixed point.

Corollary 4. If $T$ is weak condensing and compact dissipative, then $T$ has a fixed point.

Proof: From Theorem 2a), $T$ is a local dissipative system. Thus $\overline{co K}$ has an open convex neighborhood $B$ with bounded orbit. The result now follows from Theorems 2, 5, 6.

For $\alpha$-contractions, this result is contained in [13].

Corollary 5. If $T^n$ is weak completely continuous, $T$ is weak condensing and point dissipative, then $T$ has a fixed point.

Proof: This follows from Lemma 1b) and Corollary 4.
Lemma 6. If $S: X \to X$ is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, $\| \cdot \|_1$, in $X$ such that $\| S \|_1 < 1$.

Proof. Define $\| x \|_1 = \| x \| + \| Sx \| + \ldots + \| S^n x \| + \ldots$.

The assumption on the spectrum implies there is an $0 < r < 1$ such that $\| S^n \| < r^n$ if $n$ is sufficiently large. Thus, there is a constant $k$ such that $\| x \| \leq \| x \|_1 \leq k \| x \|$. Also, for $x \neq 0$

$$\frac{\| Sx \|_1}{\| x \|_1} = 1 - \left[ 1 + \frac{\| Sx \|}{\| x \|} + \frac{\| S^2 x \|}{\| x \|} + \ldots \right]^{-1} \leq 1 - \frac{1}{k}.$$

The lemma is proved.

Corollary 6. If $T$ is compact dissipative, $T = S + U$, where $S$ is linear and continuous with spectrum contained in the open unit ball and $T(\Omega)$ bounded implies $Cl(U(\Omega))$ compact for any $\Omega \subseteq X$, then $T$ has a fixed point. If, in addition, $S_0^n$ is completely continuous and $T$ is only point dissipative, then $T$ has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that $T_0^n$ is $S_0^n$ plus a completely continuous operator.
The next result generalizes an asymptotic fixed point theorem of Browder [2].

**Theorem 7.** Suppose $S_0, S_1, S_2$ are subsets of a Banach space, $S_0, S_2$ convex, closed, $S_1$ open, $S_2$ bounded, $S_0 \subseteq S_1 \subseteq S_2$. Assume $T: S_2 \to X$ is condensing in the following sense: if $\Omega, T(\Omega)$ are contained in $S_2$ and $\alpha(\Omega) > 0$, then $\alpha(T(\Omega)) < \alpha(\Omega)$. Assume also that $T$ satisfies: for any compact set $H \subseteq S_1$, $T^j(H) \subseteq S_2$, $j \geq 0$, and there is a number $N(H)$ such that $T^j(H) \subseteq S_0$ for $j \geq N(H)$. Then $T$ has a fixed point.

**Proof:** Following the proof of Theorem 2, there is a compact set $K$ which attracts the compact sets of $S_1$. Since $K \subseteq S_0$, it follows that $\overline{co} K \subseteq S_0$. Let $B$ be a closed, convex neighborhood of $\overline{co} K$, $B \subseteq S_1$. Theorems 4 and 5 complete the proof.

4. **Dissipative flows.**

Let $\{T(t), t \geq 0\}$ be a flow in a Banach space $X$. A point $x \in X$ is said to be an equilibrium point if $T(t)x_0 = x_0$ for any $t \geq 0$. We also say that a compact set $J$ attracts a compact set $H$ if, for any $\varepsilon > 0$, there is a $t^\ast(H, \varepsilon)$ such that $T(t)H \subseteq B_\varepsilon(J)$ for $t \geq t^\ast(H, \varepsilon)$. A set $Q \subseteq X$ is said to be **boundedly compact** if $P \cap Q$ is compact for any closed bounded set $P \subseteq X$. 
Theorem 8. If \( \{T(t), t \geq 0\} \) satisfies:

1) there is a compact set \( J \) that attracts the compact sets of \( J_0 \);

2) there is a number \( w > 0 \) and a family of boundedly compact (in particular compact) sets \( Q(\tau), 0 < \tau < w \) such that \( Q(\tau) \cap J_0 \neq \emptyset \) and \( T(k\tau)Q(\tau) \cap Q(\tau), 0 < \tau \leq w \), \( k \) positive integer;

then there is an equilibrium point.

Proof: Take the sequence \( w_n = w/n \). From Theorems 2, 4, 5, it follows that for each \( n \), there is an \( x_n \) satisfying: \( T(w_n)x_n = x_n \). Since \( w_n > 0 \), \( x_n \) is in \( J \).

Changing the notation if necessary, we may assume that \( x_n \) converges to \( x_0 \). Let \( k_n(t) \) be the integer defined by:

\[ k_n(t)w_n \leq t < (k_n(t) + 1)w_n \]

Then, \( T(k_n(t)w_n)x_n = x_n \) and so:

\[ |T(t)x_0 - x_0| \leq |T(t)x_0 - T(k_n(t)w_n)x_n| + |T(k_n(t)w_n)x_n - T(k_n(t)w_n)x_n| + |x_n - x_0| \]

Since \( k_n(t)w_n \) tends to \( t \) as \( n \to \infty \), the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4, 5 and 8, we have:

Corollary 7: If \( \{T(t), t \geq 0\} \) is weak condensing for any \( t > 0 \) and satisfies \( \alpha_1 \), then there is an equilibrium point.
Corollary 8. If \( \{ T(t), t \geq 0 \} \) is a weak \( \alpha \) - contraction and satisfies \( \alpha_1 \), then there is an equilibrium point.

A flow \( \{ T(t), t \geq 0 \} \) is said to be local dissipative if there is a bounded set \( B \) such that for any point \( x \in X \) there is a neighborhood \( O_x \) of \( x \) and a \( t(x) \) such that \( T(t)O_x \subseteq B \) for \( t \geq t(x) \).

A flow \( \{ T(t), t \geq 0 \} \) is compact (point) dissipative if there is a bounded set \( B \) such that for any compact set \( H \) (any point \( x \)) there is a \( t(H) \) (\( t(x) \)) such that \( T(t)H \subseteq B \) (\( T(t)x \in B \)) for \( t \geq t(H) \) (\( t \geq t(x) \)).

Lemma 7:

a). If \( T(w) \) is weak condensing for some \( w > 0 \) and is compact (point) dissipative with compact attractor \( K \), \( T(w)K \subseteq K \), then \( \{ T(t), t \geq 0 \} \) is compact (point) dissipative with attractor \( J = \bigcup_0 \leq t \leq w T(t)K \).

b). If there is a compact set \( K \) such that \( T(w)K \subseteq K \) and \( K \) attracts neighborhoods of points, then the set \( J \) above attracts neighborhoods of points relative to the flow \( \{ T(t), t \geq 0 \} \).

c). If \( \{ T(t) \geq 0 \} \) is weak condensing for some \( w > 0 \) and compact dissipative, it is local dissipative.

Proof: For any \( \varepsilon > 0 \), there is a \( s > 0 \) such that \( T(t)B_\varepsilon(K) \subseteq B_\varepsilon(J), 0 \leq t \leq w \). Since \( T(w)K \subseteq K \), parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).
Corollary 9. If for some \( w > 0 \), \( T(w) \) is weak completely continuous and point dissipative then \{T(t), t \geq 0\} is local dissipative.

Proof: The set \( K \) in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

Theorem 2. If \{T(t), t \geq 0\} is weak condensing for \( t > 0 \), then the following assertions hold:

a). If \{T(t), t \geq 0\} is compact dissipative, then there is an equilibrium point;

b). If \{T(t), t \geq 0\} is point dissipative and \( T(w) \) is weak completely continuous for some \( w > 0 \), then there is an equilibrium point.

Corollary 10. If \{T(t) = S(t) + U(t), t \geq 0\} then the following assertions hold:

a). If \{T(t), t \geq 0\} is compact dissipative, \( S(t) \) is linear with spectrum contained inside the unit ball for \( t > 0 \) and \( U(t) \) is weak completely continuous, then there is an equilibrium point.

b). If \( S(w) \) is completely continuous for some \( w > 0 \) and \{T(t), t \geq 0\} is point dissipative, there exists an equilibrium point.
5. Functional differential equations.

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let $r \geq 0$ be a given real number, $E^n$ be an $n$-dimensional linear vector space with norm $| \cdot |$, $C([a,b], E^n)$ be the space of continuous functions from $[a,b]$ to $E^n$ with the uniform topology and let $C = C([a,A], E^n)$. For $\varphi \in C$, $|\varphi| = \sup_{-r < \theta < 0} |\varphi(\theta)|$. For any $x \in C([{-r},A], E^n)$, $A \geq 0$, let $x_t \in C$, $t \in [0,A]$, be defined by $x_t(\theta) = x(t + \theta)$, $-r < \theta < 0$. Suppose $D: \mathbb{R} \times C \rightarrow E^n$ is a continuous linear operator $D\varphi = \varphi(0) - g(t, \varphi)$,

$$
g(t, \varphi) = \int_{-r}^{0} [d\mu(t, \theta)]\varphi(\theta)
$$

(1)

$$
|\int_{-s}^{0} [d\mu(t, \theta)]\varphi(\theta)| \leq r(s)|\varphi|,
$$

for $s \geq 0$, $\varphi \in C$ where $\mu$ is an $n \times n$ matrix function of bounded variation, $r$ is continuous and nondecreasing on $[0,r]$, $r(0) = 0$. If $f: \mathbb{R} \times C \rightarrow E^n$ is continuous, then a NFDE is a relation

$$
\frac{d}{dt} D(t, x_t) = f(t, x_t).
$$

(2)

A solution $x = x(\sigma)$ through $\sigma$ at time $\sigma$ is a continuous function defined on $[\sigma-r, \sigma+A]$, $A > 0$, such that $x_\sigma = \varphi$, $D(t, x_t)$ is continuously differentiable on $(\sigma, \sigma+A)$ and (2) is satisfied on $(\sigma, \sigma+A)$. We assume
We assume a solution $x(\varphi)$ of (2) through any $\varphi \in C$ exists on $[\sigma-r, \infty)$, is unique and $x(\varphi)(t)$ depends continuously on $(\varphi, t) \in C \times [\sigma-r, \infty)$.

In the following, we let $T_D(t, \sigma): C \to C$, $t \geq 0$, be the continuous linear operator defined by $T_D(t, \sigma)\varphi = y_t(\varphi)$, $t \geq \sigma$, where $y = y(\varphi)$ is the solution of

$$\frac{d}{dt} D(t, y_t) = 0, \ y_\sigma = \varphi.$$ \hspace{1cm} (3)

If $D$ is $\omega$-periodic in $t$, $C_D = \{ \varphi \in C: D(0, \varphi) = 0 \}$, then $C_D$ is a Banach space with the topology of $C$, $T_D(\omega, 0): C_D \to C_D$, and $T_D(n\omega, 0) = T_D(\omega, 0)$.

The operator $D$ is said to be uniformly stable if there exist constants $K \geq 1$, $\alpha > 0$, such that

$$|T_D(t, \sigma)\varphi| \leq Ke^{-\alpha(t-\sigma)}|\varphi|, \ \varphi \in C_D, \ t \geq \sigma.$$ \hspace{1cm} (4)

Notice the operator $D\varphi = \varphi(0)$ corresponding to retarded functional differential equations is always stable.

Remark. The conclusion of the main theorem below is valid under the weaker hypothesis that $D(\varphi) = D_0(\varphi) + \int_0^\sigma A(\theta)\varphi(\theta)d\theta$ where $D_0$ is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that $D$ uniformly stable implies there exists an $n \times n$ matrix function $B(t)$ defined and of bounded variation on $[-r, \infty)$, continuous from the left, $B(t) = 0$, $-r \leq t \leq 0$, and a constant $M_1$ such that
(5) \(|T_D(t)\varphi| \leq M_1|\varphi|, \quad t \geq 0, \quad \varphi \in C, \quad \sup_{t \geq -r} E(t) \leq M_1,

and, for any continuous function \(h: [0, \infty) \rightarrow \mathbb{R}^n\), the solution of the problem

(6) \[D(t, x_t) = D(0, x_0) + \int_0^t h(s) ds, \quad x_0 = \varphi\]

is given by

(7) \[x_t = T_D(t)\varphi - \int_0^t B_t s \cdot h(s) ds\,.

Furthermore, there exist \(n\) functions \(\varphi_1, \ldots, \varphi_n\) in \(C\) such that \(D(0, \varphi) = I\), the identity, where \(\varphi = (\varphi_1, \ldots, \varphi_n)\).

Let \(\Psi: C \rightarrow C_D\) be the continuous linear operator defined by \(\Psi(\varphi) = \varphi - \varphi D(\varphi)\).

Lemma 8. If \(D\) is uniformly stable and \(f\) maps bounded sets of \(\mathbb{R}^n\) into bounded sets of \(\mathbb{R}^n\), then there is a family of continuous transformations \(T_1(t): C \rightarrow C\), \(t \geq 0\) which are weak completely continuous and

\[T(t, 0)\varphi = T(t)\varphi = x_t(\varphi) = T_D(t)\psi(\varphi) + T_1(t)\psi(\varphi)\]

If \(D\varphi = \varphi(0)\), then \(T(t)\) is weak completely continuous for \(t \geq r\).
Proof: Equation (2) with initial value \( x_0 = \varphi \) is equivalent to

\[
D(x_t) = D(\varphi) + \int_0^t f(s,x_s)ds, \quad t \geq 0, \quad x_0 = \varphi,
\]

which from (7) is equivalent to

\[
T(t) = x_t = T_D(t)\varphi + T_D(t)D(\varphi) - \int_0^t B(t-s)f(s,x_s)ds = T_D(t)\varphi + I(t).
\]

It is now an easy matter to verify the assertions in the theorem.

Since the condition that \( D \) is uniformly stable implies the linear operator \( S(\omega) = T_D(\omega) \) has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

**Theorem 4.** If there exists an \( \omega > 0 \) such that \( f(t+\omega,\varphi) = f(t,\varphi) \) for all \( \varphi \in C \), \( f \) takes bounded sets of \( R \times C \rightarrow \mathbb{R}^n \) and system (2) is compact dissipative, then there is an \( \omega \)-periodic solution of (2). If \( f \) satisfies the same hypotheses and is independent of \( t \), then there is a constant function \( c \) in \( C \) such that \( f(c) = 0 \); that is, an equilibrium point of (2). If \( D(\varphi) = \varphi(0) \), then the same conclusions are true for point dissipative.
References


