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FIXED POINT THEOREMS AND DISSIPATIVE PROCESSES

by

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1. Introduction

Suppose X is a Banach space, $T: X \to X$ is a continuous mapping. The map T is said to be dissipative if there is a bounded set B in X such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n \geq N(x)$. In his study of ordinary differential equations in n-dimensional Euclidean space (which were ω-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with Tx representing the solution of the differential equation at time ω which started at x at time zero. The basic problem is to give information about the limiting behavior of orbits of T and to discuss the existence of fixed points of T. Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of T has a fixed point and he characterized the maximal compact invariant set of T. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of T has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space X arising in retarded functional differential equations, and T completely continuous, Jones [9] and Yoshizawa [16] showed that T has a fixed point by using Browder's theorem. For an arbitrary Banach space X and T completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently, Billotti and LaSalle [1] have obtained the same result with T completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when T is condensing on balls in X; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subset X$ and α is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of T. More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators T which includes α-contractions or k-set contractions; that is, there is a constant k, $0 \leq k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subset X$. They have characterized the maximal compact invariant set of T, shown that it is asymptotically stable, and proved that some iterate of T has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to ω-periodic retarded functional differential equations, the hypothesis that T is completely continuous implies that the period ω in the equation is greater than or equal to the delay r in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral functional differential equations, the operator T is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an α-contraction. However, the above theory for this case implies only that some iterate of T has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on T which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for T condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The ε-neighborhood of a set $K \subset X$ will be denoted by $B_\varepsilon(K)$, the closure by $Cl(K)$ and the convex closure by $\overline{co}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set K in X (see [3]). Suppose T is a continuous map $T: X \to X$. The map T is said to be weak condensing if for any bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map T is said to be a weak α-contraction if there is a constant $k, 0 \leq k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$. If T takes bounded sets into bounded sets, then a weak α-contraction is an α-contraction. The map T^{n_0} is said to be weak completely continuous if there is an integer n_0 such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^n x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If T is weak completely continuous it is weak condensing.

If T is completely continuous then T is weak completely continuous. The map T is said to be asymptotically smooth if for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subset X$ attracts a set $H \subset X$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $T^n(H) \subset B_\varepsilon(K)$ for $n \geq N(H, \varepsilon)$. We say K attracts compact sets of X if K attracts each compact set $H \subset X$. We say K attracts neighborhoods of compact sets of X if for any compact set $H \subset X$, there is a neighborhood H_0 of H such that K attracts H_0.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$. If B satisfies the property that for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n(A) \subset B$ for $n \geq N(A)$, then T is said to be compact dissipative. If B satisfies the property that for any $x \in X$, there is an open neighborhood O_x and an integer $N(x)$ such that $T^n O_x \subset B$, $n \geq N(x)$, then T is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

Lemma 1. a) (Hale, LaSalle, Slemrod [7]). If T is continuous, local dissipative and asymptotically smooths, then there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of X.

b) (Billotti and LaSalle [1]) If T is continuous, point dissipative and T^{n_0} is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood H_0 of H and an integer $N(H)$ such that $\cup_{j \geq 0} T^j H_0$ is bounded and $T^n H_0 \subset K$ for $n \geq N(H)$. In particular, T is local dissipative and T asymptotically smooths.

<u>Lemma 2</u>. If $T: X \to X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of X, then

a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of H, such that $\cup_{n \geq 0} T^n H_1$ is bounded;

b) $\cup_{j \geq 0} T^j B$ is precompact if B is compact.

<u>Proof</u>: a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H,\varepsilon)$ is the number occuring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where T is bounded. Define $\Omega_N = B_\varepsilon(K)$, $\Gamma_N = \Omega_N$, $\Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

b) The set $A = \cup_{j \geq 0} T^j B$ is bounded. Since $T^j(B)$ is compact for any j we have $\alpha(A) = \alpha(\cup_{j \geq n} T^j(B))$ for any n. But given $\varepsilon > 0$, if $n \geq n_1(B,\varepsilon)$, we have $\cup_{j \geq n} T^j B \subset B_\varepsilon(B)$ and thus $\alpha(A) \leq 2\varepsilon$. Thus $\alpha(A) = 0$ and A is compact. This proves the Lemma.

The following result was proved in [7] if we use Lemmas 1 and 2.

Theorem 1. If $T: X \to X$ is continuous and there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of X, then $J = \cap_{j \geq 0} T^j(K)$ is independent of the sets K satisfying the above property, J is the maximal compact invariant of T and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood U of K and an integer n such that $T^n(U) \subset U$. Thus, if T possesses the fixed point property, then some iterate of T has a fixed point (see [7]).

Regarding fixed points of T, it is known (see [9], [14], [5], [8], [1]) that T completely continuous and point dissipative implies T has a fixed point. Below, we give some weaker conditions which assert that T has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

Theorem 2. a) If $T:X \to X$ is continuous, weak condensing and compact dissipative, then there is a compact invariant set K which attracts compact sets of X and T is local dissipative.

b) If T is weak condensing and point dissipative then there is a compact invariant set K that attracts points of X.

Proof: a) It is an easy matter to prove the following fact: If H is a compact set such that $T:H \to H$, then the set $A = \cap T^n(H)$ is compact, non empty, $T(A) = A$ and $T^n(H)$ tends to A in the Hausdorff metric.

Now, for any compact set L of X, let $L_1 = \cup_{j\geq 0} T^j(L)$. Since L_1 is bounded, $L_1 = L \cup T(L_1)$ and T is weak condensing, it follows that $\alpha(L_1) = 0$ and thus $H = Cl(L_1)$ is compact. Also $T(H) \subset H$. Let $A_L = \cap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subset X$ such that $A_L \subset B$ for each compact set L. Since $T(\cup A_L) = \cup A_L$, where the union is taken over all compact sets $L \subset X$, it follows that the set $K = Cl(\cup A_L)$ is compact, $T(K) \subset K$, and K attracts compact sets of X.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if T is is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

Lemma 3. If T is a weak α - contraction, then T asymptotically smooths.

Proof: If B is a bounded set, then $B^* = Cl(\cup A_x)$, where A_x is constructed as above for the elements $x \in B$ such that $T^n x \in B$, for any $n \geq 0$.

Corollary. If T is a weak α-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

3. Fixed point theorems. In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious :

Lemma 4. If A is a compact set of X and $F \subset X$ contains a sequence $\{x_n\}$ such that $d(x_n, A) \to 0$ as $n \to \infty$, then $A \cap \overline{F} \neq \Phi$.

Theorem 3. Suppose $K \subset B \subset S \subset X$ are convex subsets with K compact, S closed, bounded, and B open in S. If $T\colon S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, and K attracts points of B, then there is a convex, closed bounded subset A of S such that

$$A = \overline{co}\,[\cup_{j \geq 1} T^j(B \cap A)],\ A \cap K \neq \Phi .$$

Proof: Let $\mathscr{F}$ be the set of convex, closed, bounded subsets L of S such that $T^j(B \cap L) \subset L$ for $j \geq 1$ and $L \cap K \neq \Phi$. The family $\mathscr{F}$ is not empty because $S \in \mathscr{F}$. If $L \in \mathscr{F}$, let $L_1 = \overline{co}\,[\cup_{j \geq 1} T^j(B \cap L)]$. By Lemma 4, $L_1 \cap K \neq \Phi$. Also, L_1 is convex, closed, and contained in S. Since $L \in \mathscr{F}$, we have $L \supset L_1$ and $L_1 \supset T^j(B \cap L) \supset T^j(B \cap L_1)$ for all $j \geq 1$. Thus, $L_1 \in \mathscr{F}$. It follows that a minimal element A of $\mathscr{F}$ will satisfy the conditions of the theorem.

To prove such a minimal element exists, let $(L_\alpha)_{\alpha \in I}$ be a totally ordered family of sets in $\mathscr{F}$. The set $L = \cap_{\alpha \in I}$ is closed, convex and contained in S. Also, $T^j(B \cap L) \subset T^j(B \cap L_\alpha) \subset L_\alpha$ for any $\alpha \in I$ and $j \geq 1$. Thus, $T^j(B \cap L) \subset L$ for $j \geq 1$. If J is any finite subset of I, we have $K \cap (\cap_{\alpha \in J} L_\alpha) \neq \Phi$ and, from compactness, it follows that $K \cap (\cap_{\alpha \in I} L_\alpha) \neq \Phi$. Thus, $L \in \mathscr{F}$ and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:

Theorem 4. The set A of Theorem 3 is compact if and only if there is a compact set $Q = Q(B)$ such that $Q \cap B \neq \phi$ and $T^j(Q \cap B) \subset Q$ for all $j \geq 0$.

Lemma 5. (Horn [8]). Let $S_o \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_o, S_2 compact and S_1 open in S_2. Let $T: S_2 \to X$ be a continuous mapping such that for some integer $m > 0$, $T^j(S_1) \subset S_2$, $0 \leq j \leq m-1$, $T^j(S_1) \subset S_o$, $m \leq j \leq 2m - 1$. Then T has a fixed point.

Theorem 5. Suppose $K \subset B \subset S \subset X$ are convex subsets with K compact, S closed bounded and B open in S. If $T: S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, K attracts compact sets of B and the set A of Theorem 3 is compact, then T has a fixed point.

Proof: Since K is compact and convex, the set B can be taken as $S \cap \mathscr{B}_\varepsilon(K)$ for some $\varepsilon > 0$. Let Q be as in Theorem 4, $S_o = \mathrm{Cl}(\mathscr{B}_{\varepsilon/2}(K)) \cap Q$, $S_1 = \mathscr{B}_\varepsilon(K) \cap Q$ and $S_2 = S \cap Q$. Then $S_o \subset S_1 \subset S_2$, S_o, S_2 compact and S_1 is open in S_2. Also, $T^j(S_1) \subset S_2$, $0 \leq j \leq n_1(K,\varepsilon)$ and $T^j(S_1) \subset S_o$ for $j \geq n_1(K,\varepsilon)$ for some integer $n_1(K,\varepsilon)$. An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map T which will ensure that the set A in Theorem 3 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is

Theorem 6. If T is weak condensing, then the set A in Theorem 5 is compact.

Proof: If $\tilde{A} = \cup_{j \geq 1} T^j(B \cap A)$, then $\tilde{A} = T(B \cap A) \cup T(\tilde{A})$ and $\alpha(A) = \alpha(\tilde{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\tilde{A}))$. Since $\alpha(T(\tilde{A})) < \alpha(\tilde{A})$ if $\alpha(\tilde{A}) > 0$, it follows that $\alpha(\tilde{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then $\alpha(A) = \alpha(\tilde{A}) < \alpha(B \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(B \cap A) = 0$. However, this implies $\alpha(A) = 0$ and A is compact, proving the theorem.

Corollary 1. If the sets K, B, S in Theorem 5 exist, if K attracts the compact sets of B and T is weak condensing, then T has a fixed point.

Proof. This is immediate from Theorems 5 and 6.

Corollary 2. If $T: X \to X$ is continuous, pointwise dissipative and T is weak completely continuous, then T has a fixed point.

Proof: This is immediate from Lemma 1b) and Corollary 1.

Corollary 3. If T is a weak α - contraction and there are sets K,B,S as in Corollary 1, then T has a fixed point.

Corollary 4. If T is weak condensing and compact dissipative, then T has a fixed point.

Proof: From Theorem 2a), T is a local dissipative system. Thus $\overline{co}\, K$ has an open convex neighborhood B with bounded orbit. The result now follows from Theorems 2, 5, 6.

For α-contractions, this result is contained in [13].

Corollary 5. If T^{n_0} is weak completely continuous, T is weak condensing and point dissipative, then T has a fixed point.

Proof: This follows from Lemma 1b) and Corollary 4.

Lemma 6. If $S: X \to X$ is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, $|\cdot|_1$, in X such that $|S|_1 < 1$.

Proof. Define $|x|_1 = |x| + |Sx| + \dots + |S^n x| + \dots$. The assumption on the spectrum implies there is an $0 \le r < 1$ such that $|S^n| < r^n$ if n is sufficiently large. Thus, there is a constant K such that $|x| \le |x|_1 \le K|x|$. Also, for $x \neq 0$

$$\frac{|Sx|_1}{|x|_1} = 1 - \left[1 + \frac{|Sx|}{|x|} + \frac{|S^2 x|}{|x|} + \cdots\right]^{-1} \le 1 - \frac{1}{K} .$$

The lemma is proved.

Corollary 6. If T is compact dissipative, $T = S + U$, where S is linear and continuous with spectrum contained in the open unit ball and $T(\Omega)$ bounded implies $Cl(U(\Omega))$ compact for any $\Omega \subset X$, then T has a fixed point. If, in addition, S^{n_0} is completely continuous and T is only point dissipative, then T has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that T^{n_0} is S^{n_0} plus a completely continuous operator.

The next result generalizes an asymptotic fixed point theorem of Browder [2].

Theorem 7. Suppose S_o, S_1, S_2 are subsets of a Banach space, S_o, S_2 convex, closed, S_1 open, S_2 bounded, $S_o \subset S_1 \subset S_2$. Assume $T: S_2 \to X$ is condensing in the following sense: if Ω, $T(\Omega)$ are contained in S_2 and $\alpha(\Omega) > 0$, then $\alpha(T(\Omega)) < \alpha(\Omega)$. Assume also that T satisfies: for any compact set $H \subset S_1$, $T^j(H) \subset S_2$, $j \geq 0$, and there is a number $N(H)$ such that $T^j(H) \subset S_o$ for $j \geq N(H)$. Then T has a fixed point.

Proof: Following the proof of Theorem 2, there is a compact set K which attracts the compact sets of S_1. Since $K \subset S_o$, it follows that $\overline{co}\, K \subset S_o$. Let B be a closed, convex neighborhood of $\overline{co}\, K$, $B \subset S_1$. Theorems 4 and 5 complete the proof.

4. Dissipative flows.

Let $\{T(t),\ t \geq 0\}$ be a flow in a Banach space X. A point $x \in X$ is said to be an equilibrium point if $T(t)x_o = x_o$ for any $t \geq 0$. We also say that a compact set J attracts a compact set H if, for any $\varepsilon > 0$, there is a $t^*(H,\varepsilon)$ such that $T(t)H \subset B_\varepsilon(J)$ for $t \geq t^*(H,\varepsilon)$. A set $Q \subset X$ is said to be boundedly compact if $P \cap Q$ is compact for any closed bounded set $P \subset X$.

Theorem 8. If $\{T(t), t \geq 0\}$ satisfies:

$\alpha_1)$ there is a compact set J that attracts the compact sets of J_o;

$\alpha_2)$ there is a number $w > 0$ and a family of boundedly compact (in particular compact) sets $Q(\tau)$, $0 < \tau < w$ such that $Q(\tau) \cap J_o \neq \phi$ and $T(k\tau)Q(\tau) \cap Q(\tau)$, $0 < \tau \leq w$, k positive integer;

then there is an equilibrium point.

Proof: Take the sequence $w_n = w/n$. From Theorems 2, 4, 5, it follows that for each n, there is an x_n satisfying: $T(w_n)x_n = x_n$. Since $w_n > 0$, x_n is in J. Changing the notation if necessary, we may assume that x_n converges to x_o. Let $k_n(t)$ be the integer defined by: $k_n(t)w_n \leq t < (k_n(t) + 1)w_n$. Then, $T(k_n(t)w_n)x_n = x_n$ and so: $|T(t)x_o - x_o| \leq |T(t)x_o - T(k_n(t)w_n)x_o| + |T(k_n(t)w_n)x_o - T(k_n(t)w_n)x_n| + |x_n - x_o|$. Since $k_n(t)w_n$ tends to t as $n \to \infty$, the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4,5 and 8, we have:

Corollary 7: If $\{T(t), t \geq 0\}$ is weak condensing for any $t > 0$ and satisfies $\alpha_1)$, then there is an equilibrium point.

Corollary 8. If $\{T(t), t \geq 0\}$ is a weak α - contraction and satisfies $\alpha_1)$, then there is an equilibrium point.

A flow $\{T(t), t \geq 0\}$ is said to be local dissipative if there is a bounded set B such that for any point $x \in X$ there is a neighborhood O_x of x and a $t(x)$ such that $T(t)O_x \subset B$ for $t \geq t(x)$.

A flow $\{T(t), t \geq 0\}$ is compact (point) dissipative if there is a bounded set B such that for any compact set H (any point x) there is a $t(H)$ $(t(x))$ such that $T(t)H \subset B$ $(T(t)x \in B)$ for $t \geq t(H)$ $(t \geq t(x))$.

Lemma 7.

a). If $T(w)$ is weak condensing for some $w > 0$ and is compact (point) dissipative with compact attractor K, $T(w)K \subset K$, then $\{T(t), t \geq 0\}$ is compact (point) dissipative with attractor $J = \cup_{0 \leq t \leq w} T(t)K$.

b). If there is a compact set K such that $T(w)K \subset K$ and K attracts neighborhoods of points, then the set J above attracts neighborhoods of points relative to the flow $\{T(t), t \geq 0\}$.

c). If $\{T(t) \geq 0\}$ is weak condensing for some $\omega > 0$ and compact dissipative, it is local dissipative.

Proof: For any $\varepsilon > 0$, there is a $\delta > 0$ such that $T(t)B_\delta(K) \subset B_\varepsilon(J)$, $0 \leq t \leq w$. Since $T(w)K \subset K$, parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).

<u>Corollary 9</u>. If for some $w > 0$, $T(w)$ is weak completely continuous and point dissipative then $\{T(t), t \geq 0\}$ is local dissipative.

<u>Proof</u>: The set K in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

<u>Theorem 9</u>. If $\{T(t), t \geq 0\}$ is weak condensing for $t > 0$, then the following assertions hold:

a). If $\{T(t), t \geq 0\}$ is compact dissipative, then there is an equilibrium point;

b). If $\{T(t), t \geq 0\}$ is point dissipative and $T(w)$ is weak completely continuous for some $w > 0$, then there is an equilibrium point.

<u>Corollary 10</u>. If $\{T(t) = S(t) + U(t), t \geq 0\}$ then the following assertions hold:

a). If $\{T(t), t \geq 0\}$ is compact dissipative, $S(t)$ is linear with spectrum contained inside the unit ball for $t > 0$ and $U(t)$ is weak completely continuous, then there is an equilibrium point.

b). If $S(w)$ is completely continuous for some $w > 0$ and $\{T(t), t \geq 0\}$ is point dissipative, there exists an equilibrium point.

5. Functional differential equations.

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let $r \geq 0$ be a given real number, E^n be an n-dimensional linear vector space with norm $|\cdot|$, $C([a,b],E^n)$ be the space of continuous functions from $[a,b]$ to E^n with the uniform topology and let $C = C([-r,0],E^n)$. For $\varphi \in C$, $|\varphi| = \sup_{-r\leq\theta\leq 0}|\varphi(\theta)|$. For any $x \in C([-r,A),E^n)$, $A \geq 0$, let $x_t \in C$, $t \in [0,A]$, be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Suppose $D\colon R \times C \to E^n$ is a continuous linear operator $D\varphi = \varphi(0)-g(t,\varphi)$,

$$
\begin{aligned}
&g(t,\varphi) = \int_{-r}^{0} [d\mu(t,\theta)]\varphi(\theta) \\
(1)\qquad &\left|\int_{-s^+}^{0}[d\mu(t,\theta)]\varphi(\theta)\right| \leq \gamma(s)|\varphi|,
\end{aligned}
$$

for $s \geq 0$, $\varphi \in C$ where μ is an $n \times n$ matrix function of bounded variation, γ is continuous and nondecreasing on $[0,r]$, $\gamma(0) = 0$. If $f\colon R \times C \to E^n$ is continuous, then a NFDE is a relation

$$(2)\qquad \frac{d}{dt} D(t,x_t) = f(t,x_t).$$

A solution $x = x(\varphi)$ through φ at time σ is a continuous function defined on $[\sigma-r,\sigma+A)$, $A > 0$, such that $x_\sigma = \varphi$, $D(t,x_t)$ is continuously differentiable on $(\sigma,\sigma+A)$ and (2) is satisfied on $(\sigma,\sigma+A)$. We assume

We assume a solution $x(\varphi)$ of (2) through any $\varphi \in C$ exists on $[\sigma-r,\infty)$, is unique and $x(\varphi)(t)$ depends continuously on $(\varphi,t) \in C \times [\sigma-r,\infty)$.

In the following, we let $T_D(t,\sigma)\colon C \to C$, $t \geq 0$, be the continuous linear operator defined by $T_D(t,\sigma)\varphi = y_t(\varphi)$, $t \geq \sigma$, where $y = y(\varphi)$ is the solution of

$$\frac{d}{dt}D(t,y_t) = 0, \quad y_\sigma = \varphi. \tag{3}$$

If D is ω-periodic in t, $C_D = \{\varphi \in C\colon D(0,\varphi) = 0\}$, then C_D is a Banach space with the topology of C, $T_D(w,0)\colon C_D \to C_D$, and $T_D(nw,0) = T_D^n(w,0)$.

The operator D is said to be <u>uniformly stable</u> if there exist constants $K \geq 1$, $\alpha > 0$, such that

$$|T_D(t,\sigma)\varphi| \leq Ke^{-\alpha(t-\sigma)}|\varphi|, \quad \varphi \in C_D, \quad t \geq \sigma. \tag{4}$$

Notice the operator $D\varphi = \varphi(0)$ corresponding to retarded functional differential equations is always stable.

<u>Remark.</u> The conclusion of the main theorem below is valid under the weaker hypothesis that $D(\varphi) = D_0(\varphi) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta$ where D_0 is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that D uniformly stable implies there exists an $n \times n$ matrix function $B(t)$ defined and of bounded variation on $[-r,\infty)$, continuous from the left, $B(t) = 0$, $-r \leq t \leq 0$, and a constant M_1 such that

$$|T_D(t)\varphi| \leq M_1|\varphi|, \quad t \geq 0, \quad \varphi \in C, \quad \sup_{t \geq -r} B(t) \leq M_1, \tag{5}$$

and, for any continuous function $h: [0,\infty) \to E^n$, the solution of the problem

$$D(t,x_t) = D(0,\varphi) + \int_0^t h(s)\,ds, \quad x_0 = \varphi \tag{6}$$

is given by

$$x_t = T_D(t)\varphi - \int_0^t B_{t-s}h(s)\,ds\ . \tag{7}$$

Furthermore, there exist n functions $\varphi_1,\ldots,\varphi_n$ in C such that $D(0,\Phi) = I$, the identity, where $\Phi = (\varphi_1,\ldots,\varphi_n)$.

Let $\psi: C \to C_D$ be the continuous linear operator defined by $\psi(\varphi) = \varphi - \Phi D(\varphi)$.

<u>Lemma</u> 8. If D is uniformly stable and f maps bounded sets of $R \times C$ into bounded sets of E^n, then there is a family of continuous transformations $T_1(t): C \to C$, $t \geq 0$ which are weak completely continuous and

$$T(t,0)\varphi = T(t)\varphi \overset{\text{def}}{=} x_t(\varphi) = T_D(t)\psi(\varphi) + T_1(t)\varphi$$

If $D\varphi = \varphi(0)$, then $T(t)$ is weak completely continuous for $t \geq r$.

Proof: Equation (2) with initial value $x_0 = \varphi$ is equivalent to

$$D(x_t) = D(\varphi) + \int_0^t f(s,x_s)\,ds, \quad t \geq 0, \; x_0 = \varphi,$$

which from (7) is equivalent to

$$T(t)\varphi \overset{\text{def}}{=} x_t = T_D(t)\psi(\varphi) + T_D(t)\Phi D(\varphi) - \int_0^t B^D_{t-s} f(s,x_s)\,ds \overset{\text{def}}{=} T_D(t)\psi(\varphi) + T_1(t).$$

It is now an easy matter to verify the assertions in the theorem.

Since the condition that D is uniformly stable implies the linear operator $S(\omega) = T_D(\omega)\psi$ has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

Theorem 4. If there exists an $\omega > 0$ such that $f(t+\omega,\varphi) = f(t,\varphi)$ for all $\varphi \in C$, f takes bounded sets of $R \times C \to E^n$ and system (2) is compact dissipative, then there is an ω-periodic solution of (2). If f satisfies the same hypotheses and is independent of t, then there is a constant function c in C such that $f(c) = 0$; that is, an equilibrium point of (2). If $D(\varphi) = \varphi(0)$, then the same conclusions are true for point dissipative.

References

[1] Billotti, J. E. and J. P. LaSalle, Periodic dissipative processes. Bull. American Math. Soc., 6(1971), 1082-1089.

[2] Browder, F. E., On a generalization of the Schauder fixed point theorem. Duke Math. J., 26(1959), 291-303.

[3] Darbo, G., Punti uniti in trasformazione a codominio non compatto. Rend. Sem. Mat. Univ. Padova 24(1955), 84-92.

[4] Gerstein, V. M., On the theory of dissipative differential equations in a Banach space. Funk. Ana. i Priloz. 4(1970), 99-100.

[5] Gerstein, V. M. and M. A. Krasnoselskii, Structure of the set of solutions of dissipative equations. Dokl. Akad. Nauk. SSSR 183 (1968), 267-269.

[6] Hale, J. K. and M. A. Cruz, Asymptotic behavior of neutral functional differential equations. Arch. Rat. Mech. Ana. 5(1969), 331-353.

[7] Hale, J. K., LaSalle, J. P. and M. Slemrod, Theory of a general class of dissipative processes: J. Math. Ana. Appl.

[8] Horn, W. A., Some fixed point theorems for compact mappings and flows on a Banach space. Trans. Amer. Math. Soc. 149 (1970), 391-404.

[9] Jones, G. S., Stability and asymptotic fixed point theory. Proc. Nat. Acad. Sci. USA, 53 (1965), 1262-1264.

[10] Jones, G. S., Fixed point theorems for asymptotically compact mappings. Univ. of Maryland Report BN-503(1967).

[11] LaSalle, J. P., A study of synchronous asymptotic stability. Annals Math. 65 (1957), 571-581.

[12] Levinson, N., Transformation theory of nonlinear differential equations of second order. Annals. Math. 45 (1944), 723-737.

[13] Nussbaum, R., Some asymptotic fixed point theorems. Trans. of Amer. Math. Soc. To appear.

[14] Pliss, V. A., Nonlocal Problems of the Theory of Oscillations. Academic Press, New York, 1966 (translation of 1964 Russian edition).

[15] Reissig, R., Sansone, G. and R. Conti, Nichtlineare Differential Gleichungen Höherer Ordnung, Edizioni Cremonese, Roma, 1969.

[16] Yoshizawa, T., Stability Theory by Liapunov's Second Method. Math. Soc. of Japan, Tokyo, 1966.

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FIXED POINT THEOREMS AND DISSIPATIVE PROCESSES

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Fixed Point Theorems and Dissipative Processes

by

Jack K. Hale and Orlando Lopes

1. Introduction

Suppose X is a Banach space, $T: X \to X$ is a continuous mapping. The map T is said to be dissipative if there is a bounded set B in X such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n \geq N(x)$. In his study of ordinary differential equations in n-dimensional Euclidean space (which were ω-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with Tx representing the solution of the differential equation at time ω which started at x at time zero. The basic problem is to give information about the limiting behavior of orbits of T and to discuss the existence of fixed points of T. Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of T has a fixed point and he characterized the maximal compact invariant set of T. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of T has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space X arising in retarded functional differential equations, and T completely continuous, Jones [9] and Yoshizawa [16] showed that T has a fixed point by using Browder's theorem. For an arbitrary Banach space X and T completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently, Billotti and LaSalle [1] have obtained the same result with T completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when T is condensing on balls in X; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subset X$ and α is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of T. More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators T which includes α-contractions or k-set contractions; that is, there is a constant k, $0 \leq k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subset X$. They have characterized the maximal compact invariant set of T, shown that it is asymptotically stable, and proved that some iterate of T has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to ω-periodic retarded functional differential equations, the hypothesis that T is completely continuous implies that the period ω in the equation is greater than or equal to the delay r in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral functional differential equations, the operator T is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an α-contraction. However, the above theory for this case implies only that some iterate of T has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on T which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for T condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The ε-neighborhood of a set $K \subset X$ will be denoted by $B_\varepsilon(K)$, the closure by $Cl(K)$ and the convex closure by $\overline{co}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set K in X (see [3]). Suppose T is a continuous map $T: X \to X$. The map T is said to be <u>weak condensing</u> if for any bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map T is said to be a <u>weak α-contraction</u> if there is a constant $k, 0 \leq k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$. If T takes bounded sets into bounded sets, then a weak α-contraction is an α-contraction. The map T^{n_0} is said to be <u>weak completely continuous</u> if there is an integer n_0 such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^n x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If T is weak completely continuous it is weak condensing.

If T is completely continuous then T is weak completely continuous. The map T is said to be asymptotically smooth if for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subset X$ attracts a set $H \subset X$ if for any $\varepsilon > 0$, there is an integer $N(H,\varepsilon)$ such that $T^n(H) \subset B_\varepsilon(K)$ for $n \geq N(H,\varepsilon)$. We say K attracts compact sets of X if K attracts each compact set $H \subset X$. We say K attracts neighborhoods of compact sets of X if for any compact set $H \subset X$, there is a neighborhood H_0 of H such that K attracts H_0.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$. If B satisfies the property that for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n(A) \subset B$ for $n \geq N(A)$, then T is said to be compact dissipative. If B satisfies the property that for any $x \in X$, there is an open neighborhood O_x and an integer $N(x)$ such that $T^n O_x \subset B$, $n \geq N(x)$, then T is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

Lemma 1. a) (Hale, LaSalle, Slemrod [7]). If T is continuous, local dissipative and asymptotically smooths, then there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of X.

b) (Billotti and LaSalle [1]) If T is continuous, point dissipative and T^{n_0} is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood H_0 of H and an integer $N(H)$ such that $\cup_{j \geq 0} T^j H_0$ is bounded and $T^n H_0 \subset K$ for $n \geq N(H)$. In particular, T is local dissipative and T asymptotically smooths.

Lemma 2. If $T: X \to X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of X, then

a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of H, such that $\cup_{n \geq 0} T^n H_1$ is bounded;

b) $\cup_{j \geq 0} T^j B$ is precompact if B is compact.

Proof: a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H,\varepsilon)$ is the number occuring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where T is bounded. Define $\Omega_N = B_\varepsilon(K)$, $\Gamma_N = \Omega_N$, $\Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

b) The set $A = \cup_{j \geq 0} T^j B$ is bounded. Since $T^j(B)$ is compact for any j we have $\alpha(A) = \alpha(\cup_{j \geq n} T^j(B))$ for any n. But given $\varepsilon > 0$, if $n \geq n_1(B,\varepsilon)$, we have $\cup_{j \geq n} T^j B \subset B_\varepsilon(B)$ and thus $\alpha(A) \leq 2\varepsilon$. Thus $\alpha(A) = 0$ and A is compact. This proves the Lemma.

The following result was proved in [7] if we use Lemmas 1 and 2.

Theorem 1. If $T: X \to X$ is continuous and there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of X, then $J = \cap_{j \geq 0} T^j(K)$ is independent of the sets K satisfying the above property, J is the maximal compact invariant of T and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood U of K and an integer n such that $T^n(U) \subset U$. Thus, if T possesses the fixed point property, then some iterate of T has a fixed point (see [7]).

Regarding fixed points of T, it is known (see [9], [14], [5], [8], [1]) that T completely continuous and point dissipative implies T has a fixed point. Below, we give some weaker conditions which assert that T has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

Theorem 2. a) If $T:X \to X$ is continuous, weak condensing and compact dissipative, then there is a compact invariant set K which attracts compact sets of X and T is local dissipative.

b) If T is weak condensing and point dissipative then there is a compact invariant set K that attracts points of X.

Proof: a) It is an easy matter to prove the following fact: If H is a compact set such that $T:H \to H$, then the set $A = \cap T^n(H)$ is compact, non empty, $T(A) = A$ and $T^n(H)$ tends to A in the Hausdorff metric.

Now, for any compact set L of X, let $L_1 = \cup_{j\geq 0} T^j(L)$. Since L_1 is bounded, $L_1 = L \cup T(L_1)$ and T is weak condensing, it follows that $\alpha(L_1) = 0$ and thus $H = Cl(L_1)$ is compact. Also $T(H) \subset H$. Let $A_L = \cap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subset X$ such that $A_L \subset B$ for each compact set L. Since $T(\cup A_L) = \cup A_L$, where the union is taken over all compact sets $L \subset X$, it follows that the set $K = Cl(\cup A_L)$ is compact, $T(K) \subset K$, and K attracts compact sets of X.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if T is is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

<u>Lemma 3</u>. If T is a weak α - contraction, then T asymptotically smooths.

<u>Proof</u>: If B is a bounded set, then $B^* = Cl(\cup A_x)$, where A_x is constructed as above for the elements $x \in B$ such that $T^n x \in B$, for any $n \geq 0$.

<u>Corollary</u>. If T is a weak α-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

<u>3. Fixed point theorems</u>. In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious :

Lemma 4. If A is a compact set of X and $F \subset X$ contains a sequence $\{x_n\}$ such that $d(x_n, A) \to 0$ as $n \to \infty$, then $A \cap \overline{F} \neq \Phi$.

Theorem 3. Suppose $K \subset B \subset S \subset X$ are convex subsets with K compact, S closed, bounded, and B open in S. If $T: S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, and K attracts points of B, then there is a convex, closed bounded subset A of S such that

$$A = \overline{co}\,[\cup_{j \geq 1} T^j(B \cap A)],\ A \cap K \neq \Phi .$$

Proof: Let $\mathscr{F}$ be the set of convex, closed, bounded subsets L of S such that $T^j(B \cap L) \subset L$ for $j \geq 1$ and $L \cap K \neq \Phi$. The family $\mathscr{F}$ is not empty because $S \in \mathscr{F}$. If $L \in \mathscr{F}$, let $L_1 = \overline{co}\,[\cup_{j \geq 1} T^j(B \cap L)]$. By Lemma 4, $L_1 \cap K \neq \Phi$. Also, L_1 is convex, closed, and contained in S. Since $L \in \mathscr{F}$, we have $L \supset L_1$ and $L_1 \supset T^j(B \cap L) \supset T^j(B \cap L_1)$ for all $j \geq 1$. Thus, $L_1 \in \mathscr{F}$. It follows that a minimal element A of $\mathscr{F}$ will satisfy the conditions of the theorem.

To prove such a minimal element exists, let $(L_\alpha)_{\alpha \in I}$ be a totally ordered family of sets in $\mathscr{F}$. The set $L = \cap_{\alpha \in I}$ is closed, convex and contained in S. Also, $T^j(B \cap L) \subset T^j(B \cap L_\alpha) \subset L_\alpha$ for any $\alpha \in I$ and $j \geq 1$. Thus, $T^j(B \cap L) \subset L$ for $j \geq 1$. If J is any finite subset of I, we have $K \cap (\cap_{\alpha \in J} L_\alpha) \neq \Phi$ and, from compactness, it follows that $K \cap (\cap_{\alpha \in I} L_\alpha) \neq \Phi$. Thus, $L \in \mathscr{F}$ and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:

<u>Theorem 4</u>. The set A of Theorem 3 is compact if and only if there is a compact set $Q = Q(B)$ such that $Q \cap B \neq \Phi$ and $T^j(Q \cap B) \subset Q$ for all $j \geq 0$.

<u>Lemma 5</u>. (Horn [8]). Let $S_o \subset S_1 \subset S_2$ be convex subsets of a Banach space X with S_o, S_2 compact and S_1 open in S_2. Let $T\colon S_2 \to X$ be a continuous mapping such that for some integer $m > 0$, $T^j(S_1) \subset S_2$, $0 \leq j \leq m-1$, $T^j(S_1) \subset S_o$, $m \leq j \leq 2m-1$. Then T has a fixed point.

<u>Theorem 5</u>. Suppose $K \subset B \subset S \subset X$ are convex subsets with K compact, S closed bounded and B open in S. If $T\colon S \to X$ is continuous, $T^j B \subset S$, $j \geq 0$, K attracts compact sets of B and the set A of Theorem 3 is compact, then T has a fixed point.

<u>Proof</u>: Since K is compact and convex, the set B can be taken as $S \cap \mathscr{B}_\varepsilon(K)$ for some $\varepsilon > 0$. Let Q be as in Theorem 4, $S_o = Cl(\mathscr{B}_{\varepsilon/2}(K)) \cap Q$, $S_1 = \mathscr{B}_\varepsilon(K) \cap Q$ and $S_2 = S \cap Q$. Then $S_o \subset S_1 \subset S_2$, S_o, S_2 compact and S_1 is open in S_2. Also, $T^j(S_1) \subset S_2$, $0 \leq j \leq n_1(K,\varepsilon)$ and $T^j(S_1) \subset S_o$ for $j \geq n_1(K,\varepsilon)$ for some integer $n_1(K,\varepsilon)$. An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map T which will ensure that the set A in Theorem 3 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is

Theorem 6. If T is weak condensing, then the set A in Theorem 5 is compact.

Proof: If $\tilde{A} = \cup_{j \geq 1} T^j(B \cap A)$, then $\tilde{A} = T(B \cap A) \cup T(\tilde{A})$ and $\alpha(A) = \alpha(\tilde{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\tilde{A}))$. Since $\alpha(T(\tilde{A})) < \alpha(\tilde{A})$ if $\alpha(\tilde{A}) > 0$, it follows that $\alpha(\tilde{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then $\alpha(A) = \alpha(\tilde{A}) < \alpha(B \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(B \cap A) = 0$. However, this implies $\alpha(A) = 0$ and A is compact, proving the theorem.

Corollary 1. If the sets K, B, S in Theorem 5 exist, if K attracts the compact sets of B and T is weak condensing, then T has a fixed point.

Proof. This is immediate from Theorems 5 and 6.

<u>Corollary 2</u>. If $T: X \to X$ is continuous, pointwise dissipative and T is weak completely continuous, then T has a fixed point.

<u>Proof</u>: This is immediate from Lemma 1b) and Corollary 1.

<u>Corollary 3</u>. If T is a weak α - contraction and there are sets K,B,S as in Corollary 1, then T has a fixed point.

<u>Corollary 4</u>. If T is weak condensing and compact dissipative, then T has a fixed point.

<u>Proof</u>: From Theorem 2a), T is a local dissipative system. Thus $\overline{co}\,K$ has an open convex neighborhood B with bounded orbit. The result now follows from Theorems 2, 5, 6.

For α-contractions, this result is contained in [13].

<u>Corollary 5</u>. If T^{n_0} is weak completely continuous, T is weak condensing and point dissipative, then T has a fixed point.

<u>Proof</u>: This follows from Lemma 1b) and Corollary 4.

Lemma 6. If $S: X \to X$ is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, $|\cdot|_1$, in X such that $|S|_1 < 1$.

Proof. Define $|x|_1 = |x| + |Sx| + \ldots + |S^n x| + \ldots$. The assumption on the spectrum implies there is an $0 \le r < 1$ such that $|S^n| < r^n$ if n is sufficiently large. Thus, there is a constant K such that $|x| \le |x|_1 \le K|x|$. Also, for $x \neq 0$

$$\frac{|Sx|_1}{|x|_1} = 1 - \left[1 + \frac{|Sx|}{|x|} + \frac{|S^2x|}{|x|} + \cdots\right]^{-1} \le 1 - \frac{1}{K} .$$

The lemma is proved.

Corollary 6. If T is compact dissipative, $T = S + U$, where S is linear and continuous with spectrum contained in the open unit ball and $T(\Omega)$ bounded implies $Cl(U(\Omega))$ compact for any $\Omega \subset X$, then T has a fixed point. If, in addition, S^{n_0} is completely continuous and T is only point dissipative, then T has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that T^{n_0} is S^{n_0} plus a completely continuous operator.

The next result generalizes an asymptotic fixed point theorem of Browder [2].

Theorem 7. Suppose S_o, S_1, S_2 are subsets of a Banach space, S_o, S_2 convex, closed, S_1 open, S_2 bounded, $S_o \subset S_1 \subset S_2$. Assume $T: S_2 \to X$ is condensing in the following sense: if Ω, $T(\Omega)$ are contained in S_2 and $\alpha(\Omega) > 0$, then $\alpha(T(\Omega)) < \alpha(\Omega)$. Assume also that T satisfies: for any compact set $H \subset S_1$, $T^j(H) \subset S_2$, $j \geq 0$, and there is a number $N(H)$ such that $T^j(H) \subset S_o$ for $j \geq N(H)$. Then T has a fixed point.

Proof: Following the proof of Theorem 2, there is a compact set K which attracts the compact sets of S_1 . Since $K \subset S_o$, it follows that $\overline{co}\, K \subset S_o$. Let B be a closed, convex neighborhood of $\overline{co}\, K$, $B \subset S_1$. Theorems 4 and 5 complete the proof.

4. Dissipative flows.

Let $\{T(t), t \geq 0\}$ be a flow in a Banach space X . A point $x \in X$ is said to be an equilibrium point if $T(t)x_o = x_o$ for any $t \geq 0$. We also say that a compact set J attracts a compact set H if, for any $\varepsilon > 0$, there is a $t^*(H,\varepsilon)$ such that $T(t)H \subset B_\varepsilon(J)$ for $t \geq t^*(H,\varepsilon)$. A set $Q \subset X$ is said to be boundedly compact if $P \cap Q$ is compact for any closed bounded set $P \subset X$.

Theorem 8. If $\{T(t),\ t \geq 0\}$ satisfies:

α_1) there is a compact set J that attracts the compact sets of J_o;

α_2) there is a number $w > 0$ and a family of boundedly compact (in particular compact) sets $Q(\tau)$, $0 < \tau < w$ such that $Q(\tau) \cap J_o \neq \phi$ and $T(k\tau)Q(\tau) \cap Q(\tau)$, $0 < \tau \leq w$, k positive integer;

then there is an equilibrium point.

Proof: Take the sequence $w_n = w/n$. From Theorems 2, 4, 5, it follows that for each n, there is an x_n satisfying: $T(w_n)x_n = x_n$. Since $w_n > 0$, x_n is in J. Changing the notation if necessary, we may assume that x_n converges to x_o. Let $k_n(t)$ be the integer defined by: $k_n(t)w_n \leq t < (k_n(t) + 1)w_n$. Then, $T(k_n(t)w_n)x_n = x_n$ and so: $|T(t)x_o - x_o| \leq |T(t)x_o - T(k_n(t)w_n)x_o| + |T(k_n(t)w_n)x_o - T(k_n(t)w_n)x_n| + |x_n - x_o|$. Since $k_n(t)w_n$ tends to t as $n \to \infty$, the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4,5 and 8, we have:

Corollary 7: If $\{T(t),\ t \geq 0\}$ is weak condensing for any $t > 0$ and satisfies α_1), then there is an equilibrium point.

<u>Corollary 8</u>. If $\{T(t), t \geq 0\}$ is a weak α - contraction and satisfies α_1), then there is an equilibrium point.

A flow $\{T(t), t \geq 0\}$ is said to be local dissipative if there is a bounded set B such that for any point $x \in X$ there is a neighborhood 0_x of x and a $t(x)$ such that $T(t)0_x \subset B$ for $t \geq t(x)$.

A flow $\{T(t), t \geq 0\}$ is compact (point) dissipative if there is a bounded set B such that for any compact set H (any point x) there is a $t(H)$ $(t(x))$ such that $T(t)H \subset B$ $(T(t)x \in B)$ for $t \geq t(H)$ $(t \geq t(x))$.

<u>Lemma 7</u>.

a). If $T(w)$ is weak condensing for some $w > 0$ and is compact (point) dissipative with compact attractor K, $T(w)K \subset K$, then $\{T(t), t \geq 0\}$ is compact (point) dissipative with attractor $J = \cup_{0 \leq t \leq w} T(t)K$.

b). If there is a compact set K such that $T(w)K \subset K$ and K attracts neighborhoods of points, then the set J above attracts neighborhoods of points relative to the flow $\{T(t), t \geq 0\}$.

c). If $\{T(t) \geq 0\}$ is weak condensing for some $\omega > 0$ and compact dissipative, it is local dissipative.

<u>Proof</u>: For any $\varepsilon > 0$, there is a $\delta > 0$ such that $T(t)B_\delta(K) \subset B_\varepsilon(J)$, $0 \leq t \leq w$. Since $T(w)K \subset K$, parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).

Corollary 9. If for some $w > 0$, $T(w)$ is weak completely continuous and point dissipative then $\{T(t), t \geq 0\}$ is local dissipative.

Proof: The set K in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

Theorem 9. If $\{T(t), t \geq 0\}$ is weak condensing for $t > 0$, then the following assertions hold:

a). If $\{T(t), t \geq 0\}$ is compact dissipative, then there is an equilibrium point;

b). If $\{T(t), t \geq 0\}$ is point dissipative and $T(w)$ is weak completely continuous for some $w > 0$, then there is an equilibrium point.

Corollary 10. If $\{T(t) = S(t) + U(t), t \geq 0\}$ then the following assertions hold:

a). If $\{T(t), t \geq 0\}$ is compact dissipative, $S(t)$ is linear with spectrum contained inside the unit ball for $t > 0$ and $U(t)$ is weak completely continuous, then there is an equilibrium point.

b). If $S(w)$ is completely continuous for some $w > 0$ and $\{T(t), t \geq 0\}$ is point dissipative, there exists an equilibrium point.

5. Functional differential equations.

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let $r \geq 0$ be a given real number, E^n be an n-dimensional linear vector space with norm $|\cdot|$, $C([a,b],E^n)$ be the space of continuous functions from $[a,b]$ to E^n with the uniform topology and let $C = C([-r,0],E^n)$. For $\varphi \in C$, $|\varphi| = \sup_{-r\leq\theta\leq 0}|\varphi(\theta)|$. For any $x \in C([-r,A),E^n)$, $A \geq 0$, let $x_t \in C$, $t \in [0,A]$, be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. Suppose $D: R \times C \to E^n$ is a continuous linear operator $D\varphi = \varphi(0)-g(t,\varphi)$,

$$g(t,\varphi) = \int_{-r}^{0} [d\mu(t,\theta)]\varphi(\theta) \tag{1}$$

$$\left|\int_{-s^+}^{0}[d\mu(t,\theta)]\varphi(\theta)\right| \leq \gamma(s)|\varphi|,$$

for $s \geq 0$, $\varphi \in C$ where μ is an $n \times n$ matrix function of bounded variation, γ is continuous and nondecreasing on $[0,r]$, $\gamma(0) = 0$. If $f: R \times C \to E^n$ is continuous, then a NFDE is a relation

$$\frac{d}{dt} D(t,x_t) = f(t,x_t). \tag{2}$$

A solution $x = x(\varphi)$ through φ at time σ is a continuous function defined on $[\sigma-r,\sigma+A)$, $A > 0$, such that $x_\sigma = \varphi$, $D(t,x_t)$ is continuously differentiable on $(\sigma,\sigma+A)$ and (2) is satisfied on $(\sigma,\sigma+A)$. We assume

We assume a solution $x(\varphi)$ of (2) through any $\varphi \in C$ exists on $[\sigma-r,\infty)$, is unique and $x(\varphi)(t)$ depends continuously on $(\varphi,t) \in C \times [\sigma-r,\infty)$.

In the following, we let $T_D(t,\sigma): C \to C$, $t \geq 0$, be the continuous linear operator defined by $T_D(t,\sigma)\varphi = y_t(\varphi)$, $t \geq \sigma$, where $y = y(\varphi)$ is the solution of

$$(3) \qquad \frac{d}{dt} D(t,y_t) = 0, \quad y_\sigma = \varphi.$$

If D is ω-periodic in t, $C_D = \{\varphi \in C: D(0,\varphi) = 0\}$, then C_D is a Banach space with the topology of C, $T_D(w,0): C_D \to C_D$, and $T_D(nw,0) = T_D^n(w,0)$.

The operator D is said to be <u>uniformly stable</u> if there exist constants $K \geq 1$, $\alpha > 0$, such that

$$(4) \qquad |T_D(t,\sigma)\varphi| \leq Ke^{-\alpha(t-\sigma)}|\varphi|, \quad \varphi \in C_D, \quad t \geq \sigma.$$

Notice the operator $D\varphi = \varphi(0)$ corresponding to retarded functional differential equations is always stable.

<u>Remark.</u> The conclusion of the main theorem below is valid under the weaker hypothesis that $D(\varphi) = D_0(\varphi) + \int_{-r}^{0} A(\theta)\varphi(\theta)d\theta$ where D_0 is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that D uniformly stable implies there exists an $n \times n$ matrix function $B(t)$ defined and of bounded variation on $[-r,\infty)$, continuous from the left, $B(t) = 0$, $-r \leq t \leq 0$, and a constant M_1 such that

$$(5) \qquad |T_D(t)\varphi| \le M_1|\varphi|, \; t \ge 0, \; \varphi \in C, \; \sup_{t \ge -r} B(t) \le M_1,$$

and, for any continuous function $h: [0,\infty) \to E^n$, the solution of the problem

$$(6) \qquad D(t,x_t) = D(0,\varphi) + \int_0^t h(s)\,ds, \quad x_0 = \varphi$$

is given by

$$(7) \qquad x_t = T_D(t)\varphi - \int_0^t B_{t-s}h(s)\,ds \; .$$

Furthermore, there exist n functions $\varphi_1,\ldots,\varphi_n$ in C such that $D(0,\Phi) = I$, the identity, where $\Phi = (\varphi_1,\ldots,\varphi_n)$.

Let $\psi: C \to C_D$ be the continuous linear operator defined by $\psi(\varphi) = \varphi - \Phi D(\varphi)$.

<u>Lemma 8</u>. If D is uniformly stable and f maps bounded sets of $R\times C$ into bounded sets of E^n, then there is a family of continuous transformations $T_1(t): C \to C$, $t \ge 0$ which are weak completely continuous and

$$T(t,0)\varphi = T(t)\varphi \overset{\text{def}}{=} x_t(\varphi) = T_D(t)\psi(\varphi) + T_1(t)\varphi$$

If $D\varphi = \varphi(0)$, then $T(t)$ is weak completely continuous for $t \ge r$.

Proof: Equation (2) with initial value $x_0 = \varphi$ is equivalent to

$$D(x_t) = D(\varphi) + \int_0^t f(s,x_s)ds, \quad t \geq 0, \; x_0 = \varphi,$$

which from (7) is equivalent to

$$T(t)\varphi \overset{\text{def}}{=} x_t = T_D(t)\psi(\varphi) + T_D(t)\Phi D(\varphi) - \int_0^t B^D_{t-s} f(s,x_s)ds \overset{\text{def}}{=} T_D(t)\psi(\varphi) + T_1(t).$$

It is now an easy matter to verify the assertions in the theorem.

Since the condition that D is uniformly stable implies the linear operator $S(\omega) = T_D(\omega)\psi$ has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

Theorem 4. If there exists an $\omega > 0$ such that $f(t+\omega,\varphi) = f(t,\varphi)$ for all $\varphi \in C$, f takes bounded sets of $R \times C \to E^n$ and system (2) is compact dissipative, then there is an ω-periodic solution of (2). If f satisfies the same hypotheses and is independent of t, then there is a constant function c in C such that $f(c) = 0$; that is, an equilibrium point of (2). If $D(\varphi) = \varphi(0)$, then the same conclusions are true for point dissipative.

References

[1] Billotti, J. E. and J. P. LaSalle, Periodic dissipative processes. Bull. American Math. Soc., 6(1971), 1082-1089.

[2] Browder, F. E., On a generalization of the Schauder fixed point theorem. Duke Math. J., 26(1959), 291-303.

[3] Darbo, G., Punti uniti in trasformazione a codominio non compatto. Rend. Sem. Mat. Univ. Padova 24(1955), 84-92.

[4] Gerstein, V. M., On the theory of dissipative differential equations in a Banach space. Funk. Ana. i Priloz. 4(1970), 99-100.

[5] Gerstein, V. M. and M. A. Krasnoselskii, Structure of the set of solutions of dissipative equations. Dokl. Akad. Nauk. SSSR 183 (1968), 267-269.

[6] Hale, J. K. and M. A. Cruz, Asymptotic behavior of neutral functional differential equations. Arch. Rat. Mech. Ana. 5(1969), 331-353.

[7] Hale, J. K., LaSalle, J. P. and M. Slemrod, Theory of a general class of dissipative processes. J. Math. Ana. Appl.

[8] Horn, W. A., Some fixed point theorems for compact mappings and flows on a Banach space. Trans. Amer. Math. Soc. 149 (1970), 391-404.

[9] Jones, G. S., Stability and asymptotic fixed point theory. Proc. Nat. Acad. Sci. USA, 53 (1965), 1262-1264.

[10] Jones, G. S., Fixed point theorems for asymptotically compact mappings. Univ. of Maryland Report BN-503(1967).

[11] LaSalle, J. P., A study of synchronous asymptotic stability. Annals Math. 65 (1957), 571-581.

[12] Levinson, N., Transformation theory of nonlinear differential equations of second order. Annals. Math. 45 (1944), 723-737.

[13] Nussbaum, R., Some asymptotic fixed point theorems. Trans. of Amer. Math. Soc. To appear.

[14] Pliss, V. A., Nonlocal Problems of the Theory of Oscillations. Academic Press, New York, 1966 (translation of 1964 Russian edition).

[15] Reissig, R., Sansone, G. and R. Conti, Nichtlineare Differential Gleichungen Höherer Ordnung, Edizioni Cremonese, Roma, 1969.

[16] Yoshizawa, T., Stability Theory by Liapunov's Second Method. Math. Soc. of Japan, Tokyo, 1966.