FIXED POINT THEOREMS AND DISSIPATIVE PROCESSES

by

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1. Introduction

Suppose $X$ is a Banach space, $T: X \to X$ is a continuous mapping. The map $T$ is said to be dissipative if there is a bounded set $B$ in $X$ such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n > N(x)$. In his study of ordinary differential equations in $n$-dimensional Euclidean space (which were $\omega$-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with $T x$ representing the solution of the differential equation at time $\omega$ which started at $x$ at time zero. The basic problem is to give information about the limiting behavior of orbits of $T$ and to discuss the existence of fixed points of $T$. Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of $T$ has a fixed point and he characterized the maximal compact invariant set of $T$. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of $T$ has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space $X$ arising in retarded functional differential equations, and $T$ completely continuous, Jones [9] and Yoshizawa [16] showed that $T$ has a fixed point by using Browder's theorem. For an arbitrary Banach space $X$ and $T$ completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently,
Billotti and LaSalle [1] have obtained the same result with $T$ completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when $T$ is condensing on balls in $X$; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subset X$ and $\alpha$ is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of $T$.

More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators $T$ which includes $\alpha$-contractions or $k$-set contractions; that is, there is a constant $k$, $0 \leq k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subset X$. They have characterized the maximal compact invariant set of $T$, shown that it is asymptotically stable, and proved that some iterate of $T$ has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to $\omega$-periodic retarded functional differential equations, the hypothesis that $T$ is completely continuous implies that the period $\omega$ in the equation is greater than or equal to the delay $r$ in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral
functional differential equations, the operator $T$ is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an $\alpha$-contraction. However, the above theory for this case implies only that some iterate of $T$ has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on $T$ which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for $T$ condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The $\varepsilon$-neighborhood of a set $K \subset X$ will be denoted by $B_\varepsilon(K)$, the closure by $\overline{\text{Cl}}(K)$ and the convex closure by $\overline{\text{co}}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set $K$ in $X$ (see [3]). Suppose $T$ is a continuous map $T: X \to X$. The map $T$ is said to be weak condensing if for any bounded $K \subset X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map $T$ is said to be a weak $\alpha$-contraction if there is a constant $k, 0 \leq k < 1$, such that for any bounded set $K \subset X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k \alpha(K)$. If $T$ takes bounded sets into bounded sets, then a weak $\alpha$-contraction is an $\alpha$-contraction. The map $T^{n_0}$ is said to be weak completely continuous if there is an integer $n_0$ such that for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^n x \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If $T$ is weak completely continuous it is weak condensing.
If $T$ is completely continuous then $T$ is weak completely continuous. The map $T$ is said to be asymptotically smooth if for any bounded set $B \subset X$, there is a compact set $B^* \subset X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subset X$ attracts a set $H \subset X$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $T^n(H) \subset B_\varepsilon(K)$ for $n \geq N(H, \varepsilon)$. We say $K$ attracts compact sets of $X$ if $K$ attracts each compact set $H \subset X$. We say $K$ attracts neighborhoods of compact sets of $X$ if for any compact set $H \subset X$, there is a neighborhood $H_0$ of $H$ such that $K$ attracts $H_0$.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subset X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$. If $B$ satisfies the property that for any compact set $A \subset X$, there is an integer $N(A)$ such that $T^n(A) \subset B$ for $n \geq N(A)$, then $T$ is said to be compact dissipative. If $B$ satisfies the property that for any $x \in X$, there is an open neighborhood $0_x$ and an integer $N(x)$ such that $T^n 0_x \subset B$, $n \geq N(x)$, then $T$ is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

**Lemma 1.** a) [Hale, LaSalle, Slemrod [7]]. If $T$ is continuous, local dissipative and asymptotically smooths, then there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$. 
b) (Billotti and LaSalle [1]) If \( T \) is continuous, point dissipative and \( T^{p_0} \) is weak completely continuous, then there is a compact set \( K \subset X \) such that for any compact set \( H \subset X \), there is an open neighborhood \( H_0 \) of \( H \) and an integer \( N(H) \) such that \( \bigcup_{j \geq 0} T^j H_0 \) is bounded and \( T^N H_0 \subset K \) for \( n \geq N(H) \). In particular, \( T \) is local dissipative and \( T \) asymptotically smooths.

Lemma 2. If \( T : X \to X \) is continuous and there is a compact set \( K \subset X \) that attracts neighborhoods of compact sets of \( X \), then

a) there is a neighborhood \( H_1 \subset H_0 \), the above neighborhood of \( H \), such that \( \bigcup_{n \geq 0} T^n H_1 \) is bounded;

b) \( \bigcup_{j \geq 0} T^j B \) is precompact if \( B \) is compact.

Proof:

a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If \( H \subset X \) is compact and \( N = n_1(H, \varepsilon) \) is the number occurring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets \( H, T(H), \ldots, T^{N-1}(H) \). Let \( \Omega_0, \ldots, \Omega_{N-1} \) be corresponding neighborhoods where \( T \) is bounded. Define \( \Omega_N = B_{\varepsilon}(K), \Gamma_N = \Omega_N, \Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i \). The set \( H_1 = \Gamma_0 \) satisfies the required property.

b) The set \( A = \bigcup_{j \geq 0} T^j B \) is bounded. Since \( T^j(B) \) is compact for any \( j \) we have \( \alpha(A) = \alpha(\bigcup_{j \geq n} T^j(B)) \) for any \( n \). But given \( \varepsilon > 0 \), if \( n \geq n_1(B, \varepsilon) \), we have \( \bigcup_{j \geq n} T^j B \subset B_{\varepsilon}(B) \) and thus \( \alpha(A) \leq 2\varepsilon \). Thus \( \alpha(A) = 0 \) and \( A \) is compact. This proves the Lemma.
The following result was proved in [7] if we use Lemmas 1 and 2.

**Theorem 1.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$, then $J = \cap_{j \geq 0} T^j(K)$ is independent of the sets $K$ satisfying the above property, $J$ is the maximal compact invariant of $T$ and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood $U$ of $K$ and an integer $n$ such that $T^n(U) \subset U$. Thus, if $T$ possesses the fixed point property, then some iterate of $T$ has a fixed point (see [7]).

Regarding fixed points of $T$, it is known (see [9], [14], [5], [8], [1]) that $T$ completely continuous and point dissipative implies $T$ has a fixed point. Below, we give some weaker conditions which assert that $T$ has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

**Theorem 2.** a) If $T: X \to X$ is continuous, weak condensing and compact dissipative, then there is a compact invariant set $K$ which attracts compact sets of $X$ and $T$ is local dissipative.

b) If $T$ is weak condensing and point dissipative then there is a compact invariant set $K$ that attracts points of $X$.

**Proof:** a) It is an easy matter to prove the following fact: If $H$ is a compact set such that $T: H \to H$, then the set $A = \cap T^n(H)$ is compact, non empty, $T(A) = A$ and $T^n(H)$ tends to $A$ in the Hausdorff metric.
Now, for any compact set $L$ of $X$, let $L_1 = \bigcup_{j \geq 0} T^j(L)$. Since $L_1$ is bounded, $L_1 = L \cup T(L_1)$ and $T$ is weak condensing, it follows that $\alpha(L_1) = 0$ and thus $H = \text{Cl}(L_1)$ is compact. Also $T(H) \subseteq H$. Let $A_L = \bigcap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subseteq X$ such that $A_L \subseteq B$ for each compact set $L$.

Since $T(UA_L) = UA_L$, where the union is taken over all compact sets $L \subseteq X$, it follows that the set $K = \text{Cl}(UA_L)$ is compact, $T(K) \subseteq K$, and $K$ attracts compact sets of $X$.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if $T$ is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

**Lemma 3.** If $T$ is a weak $\alpha$-contraction, then $T$ asymptotically smooths.

**Proof:** If $B$ is a bounded set, then $B^* = \text{Cl}(UA_x)$, where $A_x$ is constructed as above for the elements $x \in B$ such that $T^n x \in B$, for any $n \geq 0$.

**Corollary.** If $T$ is a weak $\alpha$-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

**3. Fixed point theorems.** In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious:
Lemma 4. If A is a compact set of X and \( F \subset X \) contains a sequence \( \{x_n\} \) such that \( d(x_n, A) \to 0 \) as \( n \to \infty \), then \( A \cap F \neq \emptyset \).

Theorem 3. Suppose \( K \subset B \subset S \subset X \) are convex subsets with \( K \) compact, \( S \) closed, bounded, and \( B \) open in \( S \). If \( T : S \to X \) is continuous, \( T^j B \subset S \), \( j \geq 0 \), and \( K \) attracts points of \( B \), then there is a convex, closed bounded subset \( A \) of \( S \) such that

\[
A = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap A) \right], \quad A \cap K \neq \emptyset.
\]

Proof: Let \( \mathcal{F} \) be the set of convex, closed, bounded subsets \( L \) of \( S \) such that \( T^j(B \cap L) \subset L \) for \( j \geq 1 \) and \( L \cap K \neq \emptyset \). The family \( \mathcal{F} \) is not empty because \( S \in \mathcal{F} \). If \( L \in \mathcal{F} \), let \( L_1 = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap L) \right] \). By Lemma 4, \( L_1 \cap K \neq \emptyset \). Also, \( L_1 \) is convex, closed, and contained in \( S \). Since \( L \in \mathcal{F} \), we have \( L \supset L_1 \) and \( L_1 \supset T^j(B \cap L) \subset T^j(B \cap L_1) \) for all \( j \geq 1 \). Thus, \( L_1 \in \mathcal{F} \). It follows that a minimal element \( A \) of \( \mathcal{F} \) will satisfy the conditions of the theorem.

To prove such a minimal element exists, let \((L_\alpha)_{\alpha \in I}\) be a totally ordered family of sets in \( \mathcal{F} \). The set \( L = \bigcap_{\alpha \in I} L_\alpha \) is closed, convex and contained in \( S \). Also, \( T^j(B \cap L) \subset T^j(B \cap L_\alpha) \subset L_\alpha \) for any \( \alpha \in I \) and \( j \geq 1 \). Thus, \( T^j(B \cap L) \subset L \) for \( j \geq 1 \). If \( J \) is any finite subset of \( I \), we have \( K \cap (\bigcap_{\alpha \in J} L_\alpha) \neq \emptyset \) and, from compactness, it follows that \( K \cap (\bigcap_{\alpha \in I} L_\alpha) \neq \emptyset \). Thus, \( L \in \mathcal{F} \) and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:
Theorem 4. The set \( A \) of Theorem 3 is compact if and only if there is a compact set \( Q = Q(B) \) such that \( Q \cap B \neq \phi \) and \( T^j(Q \cap B) \subseteq Q \) for all \( j \geq 0 \).

Lemma 5. (Horn [8]). Let \( S_0 \subseteq S_1 \subseteq S_2 \) be convex subsets of a Banach space \( X \) with \( S_0, S_2 \) compact and \( S_1 \) open in \( S_2 \). Let \( T : S_2 \to X \) be a continuous mapping such that for some integer \( m > 0 \), \( T^j(S_1) \subseteq S_2 \), \( 0 \leq j \leq m-1 \), \( T^j(S_1) \subseteq S_0 \), \( m \leq j \leq 2m-1 \). Then \( T \) has a fixed point.

Theorem 5. Suppose \( K \subseteq B \subseteq S \subseteq X \) are convex subsets with \( K \) compact, \( S \) closed bounded and \( B \) open in \( S \). If \( T : S \to X \) is continuous, \( T^j B \subseteq S \), \( j \geq 0 \), \( K \) attracts compact sets of \( B \) and the set \( A \) of Theorem 3 is compact, then \( T \) has a fixed point.

Proof: Since \( K \) is compact and convex, the set \( B \) can be taken as \( S \cap B_{\epsilon}(K) \) for some \( \epsilon > 0 \). Let \( Q \) be as in Theorem 4, \( S_0 = Cl(B_{\epsilon/2}(K)) \cap Q \), \( S_1 = B_{\epsilon}(K) \cap Q \) and \( S_2 = S \cap Q \). Then \( S_0 \subseteq S_1 \subseteq S_2 \), \( S_0, S_2 \) compact and \( S_1 \) is open in \( S_2 \). Also, \( T^j(S_1) \subseteq S_2 \), \( 0 \leq j \leq n_1(K, \epsilon) \) and \( T^j(S_1) \subseteq S_0 \) for \( j \geq n_1(K, \epsilon) \) for some integer \( n_1(K, \epsilon) \). An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map \( T \) which will ensure that the set \( A \) in Theorem 5 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is
Theorem 6. If $T$ is weak condensing, then the set $A$ in Theorem 5 is compact.

Proof: If $\tilde{A} = \bigcup_{j \geq 1} T^j(B \cap A)$, then $\tilde{A} = T(B \cap A) \cup T(\tilde{A})$ and $\alpha(A) = \alpha(\tilde{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\tilde{A})))$. Since $\alpha(T(\tilde{A})) < \alpha(\tilde{A})$ if $\alpha(\tilde{A}) > 0$, it follows that $\alpha(\tilde{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(\tilde{A} \cap A) > 0$, then $\alpha(A) = \alpha(\tilde{A}) < \alpha(\tilde{A} \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(\tilde{A} \cap A) = 0$. However, this implies $\alpha(A) = 0$ and $A$ is compact, proving the theorem.

Corollary 1. If the sets $K,B,S$ in Theorem 5 exist, if $K$ attracts the compact sets of $B$ and $T$ is weak condensing, then $T$ has a fixed point.

Proof. This is immediate from Theorems 5 and 6.
Corollary 2. If $T: X \to X$ is continuous, point-wise dissipative and $T$ is weak completely continuous, then $T$ has a fixed point.

Proof: This is immediate from Lemma 1b) and Corollary 1.

Corollary 3. If $T$ is a weak $\alpha$-contraction and there are sets $K, B, S$ as in Corollary 1, then $T$ has a fixed point.

Corollary 4. If $T$ is weak condensing and compact dissipative, then $T$ has a fixed point.

Proof: From Theorem 2a), $T$ is a local dissipative system. Thus $\overline{co} K$ has an open convex neighborhood $B$ with bounded orbit. The result now follows from Theorems 2, 5, 6.

For $\alpha$-contractions, this result is contained in [13].

Corollary 5. If $T^n$ is weak completely continuous, $T$ is weak condensing and point dissipative, then $T$ has a fixed point.

Proof: This follows from Lemma 1b) and Corollary 4.
12.

Lemma 6. If \( S : X \to X \) is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, \( \| \cdot \|_1 \), in \( X \) such that \( \| S \|_1 < 1 \).

Proof. Define \( \| x \|_1 = \| x \| + \| Sx \| + \ldots + \| S^n x \| + \ldots \). The assumption on the spectrum implies there is an \( 0 < r < 1 \) such that \( \| S^n \| < r^n \) if \( n \) is sufficiently large. Thus, there is a constant \( K \) such that \( \| x \| \leq \| x \|_1 \leq K \| x \| \). Also, for \( x \neq 0 \)

\[
\frac{\| Sx \|_1}{\| x \|_1} = 1 - \left[ 1 + \frac{\| Sx \|}{\| x \|} + \frac{\| S^2 x \|}{\| x \|} + \ldots \right]^{-1} \leq 1 - \frac{1}{K}.
\]

The lemma is proved.

Corollary 6. If \( T \) is compact dissipative, \( T = S + U \), where \( S \) is linear and continuous with spectrum contained in the open unit ball and \( T(\Omega) \) bounded implies \( Cl(U(\Omega)) \) compact for any \( \Omega \subset X \), then \( T \) has a fixed point. If, in addition, \( S^\infty \) is completely continuous and \( T \) is only point dissipative, then \( T \) has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that \( T^\infty \) is \( S^\infty \) plus a completely continuous operator.
The next result generalizes an asymptotic fixed point theorem of Browder [2].

**Theorem 7.** Suppose $S_0, S_1, S_2$ are subsets of a Banach space, $S_0, S_2$ convex, closed, $S_1$ open, $S_2$ bounded, $S_0 \subset S_1 \subset S_2$. Assume $T: S_2 \to X$ is condensing in the following sense: if $\Omega, T(\Omega)$ are contained in $S_2$ and $\alpha(\Omega) > 0$, then $\alpha(T(\Omega)) < \alpha(\Omega)$. Assume also that $T$ satisfies: for any compact set $H \subset S_1$, $T^j(H) \subset S_2$, $j \geq 0$, and there is a number $N(H)$ such that $T^j(H) \subset S_0$ for $j \geq N(H)$. Then $T$ has a fixed point.

**Proof:** Following the proof of Theorem 2, there is a compact set $K$ which attracts the compact sets of $S_1$. Since $K \subset S_0$, it follows that $\overline{co} K \subset S_0$. Let $B$ be a closed, convex neighborhood of $\overline{co} K$, $B \subset S_1$. Theorems 4 and 5 complete the proof.

4. **Dissipative flows.**

Let $\{T(t), t \geq 0\}$ be a flow in a Banach space $X$. A point $x \in X$ is said to be an equilibrium point if $T(t)x_0 = x_0$ for any $t \geq 0$. We also say that a compact set $J$ attracts a compact set $H$ if, for any $\varepsilon > 0$, there is a $t^*(H, \varepsilon)$ such that $T(t)H \subset B_{\varepsilon}(J)$ for $t \geq t^*(H, \varepsilon)$. A set $Q \subset X$ is said to be **boundedly compact** if $P \cap Q$ is compact for any closed bounded set $P \subset X$. 
Theorem 8. If \( \{T(t), t \geq 0\} \) satisfies:

\( \alpha_1 \): there is a compact set \( J \) that attracts the compact sets of \( J_0 \);

\( \alpha_2 \): there is a number \( w > 0 \) and a family of boundedly compact (in particular compact) sets \( Q(t), 0 < t < w \) such that \( Q(t) \cap J_0 \neq \emptyset \) and \( T(k\tau)Q(t) \cap Q(t), 0 < t \leq w \), \( k \) positive integer; then there is an equilibrium point.

**Proof:** Take the sequence \( w_n = w/n \). From Theorems 2, 4, 5, it follows that for each \( n \), there is an \( x_n \) satisfying: \( T(w_n)x_n = x_n \). Since \( w_n > 0 \), \( x_n \) is in \( J \).

Changing the notation if necessary, we may assume that \( x_n \) converges to \( x_0 \). Let \( k_n(t) \) be the integer defined by:

\[ k_n(t)w_n < t < (k_n(t) + 1)w_n \]

Then, \( T(k_n(t)w_n)x_n = x_n \) and so:

\[ |T(t)x_0 - x_0| \leq |T(t)x_0 - T(k_n(t)w_n)x_0| + |T(k_n(t)w_n)x_0 - T(k_n(t)w_n)x_n| + |x_n - x_o| \]

Since \( k_n(t)w_n \) tends to \( t \) as \( n \to \infty \), the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4, 5 and 8, we have:

**Corollary 7:** If \( \{T(t), t \geq 0\} \) is weak condensing for any \( t > 0 \) and satisfies \( \alpha_1 \), then there is an equilibrium point.
Corollary 8. If \( \{T(t), t \geq 0\} \) is a weak \( \alpha \)-contraction and satisfies \( \alpha_1 \), then there is an equilibrium point.

A flow \( \{T(t), t \geq 0\} \) is said to be local dissipative if there is a bounded set \( B \) such that for any point \( x \in X \) there is a neighborhood \( O_x \) of \( x \) and a \( t(x) \) such that \( T(t)O_x \subset B \) for \( t \geq t(x) \).

A flow \( \{T(t), t \geq 0\} \) is compact (point) dissipative if there is a bounded set \( B \) such that for any compact set \( H \) (any point \( x \)) there is a \( t(H) \) (\( t(x) \)) such that \( T(t)H \subset B \) (\( T(t)x \in B \)) for \( t \geq t(H) \) (\( t \geq t(x) \)).

Lemma 7.

a). If \( T(w) \) is weak condensing for some \( w > 0 \) and is compact (point) dissipative with compact attractor \( K \), \( T(w)K \subset K \), then \( \{T(t), t \geq 0\} \) is compact (point) dissipative with attractor \( J = \bigcup_{0 \leq t \leq w} T(t)K \).

b). If there is a compact set \( K \) such that \( T(w)K \subset K \) and \( K \) attracts neighborhoods of points, then the set \( J \) above attracts neighborhoods of points relative to the flow \( \{T(t), t \geq 0\} \).

c). If \( \{T(t) \geq 0\} \) is weak condensing for some \( \omega > 0 \) and compact dissipative, it is local dissipative.

Proof: For any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( T(t)B_\varepsilon(K) \subset B_\varepsilon(J) \), \( 0 \leq t \leq w \). Since \( T(w)K \subset K \), parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).
Corollary 9. If for some \( w > 0 \), \( T(w) \) is weak completely continuous and point dissipative then \( \{ T(t), t \geq 0 \} \) is local dissipative.

Proof: The set \( K \) in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

Theorem 9. If \( \{ T(t), t \geq 0 \} \) is weak condensing for \( t > 0 \), then the following assertions hold:

a). If \( \{ T(t), t \geq 0 \} \) is compact dissipative, then there is an equilibrium point;

b). If \( \{ T(t), t \geq 0 \} \) is point dissipative and \( T(w) \) is weak completely continuous for some \( w > 0 \), then there is an equilibrium point.

Corollary 10. If \( \{ T(t) = S(t) + U(t), t \geq 0 \} \) then the following assertions hold:

a). If \( \{ U(t), t \geq 0 \} \) is compact dissipative, \( S(t) \) is linear with spectrum contained inside the unit ball for \( t > 0 \) and \( U(t) \) is weak completely continuous, then there is an equilibrium point.

b). If \( S(w) \) is completely continuous for some \( w > 0 \) and \( \{ T(t), t \geq 0 \} \) is point dissipative, there exists an equilibrium point.
5. Functional differential equations.

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let \( r > 0 \) be a given real number, \( E^n \) be an \( n \)-dimensional linear vector space with norm \( \| \cdot \| \), \( C([a, b], E^n) \) be the space of continuous functions from \([a, b]\) to \( E^n \) with the uniform topology and let \( C = C([0, b], E^n) \). For \( \varphi \in C \), \( \| \varphi \| = \sup_{0 < \theta \leq b} |\varphi(\theta)| \). For any \( \mu \in C([0, b], E^n) \), \( A > 0 \), let \( x_t \in C \), \( t \in [0, A] \), be defined by \( x_t(\theta) = x(t+\theta), -r < \theta < 0 \). Suppose \( D: \mathbb{R} \times C \to E^n \) is a continuous linear operator \( D\varphi = \varphi(0) - g(t, \varphi) \),

\[
g(t, \varphi) = \int_{r}^{0} [d\mu(t, \theta)] \varphi(\theta)
\]

(1)

\[
\left| \int_{-s}^{0} [d\mu(t, \theta)] \varphi(\theta) \right| \leq \gamma(s) \| \varphi \|
\]

for \( s > 0 \), \( \varphi \in C \) where \( \mu \) is an \( n \times n \) matrix function of bounded variation, \( \gamma \) is continuous and nondecreasing on \([0, r]\), \( \gamma(0) = 0 \). If \( f: \mathbb{R} \times C \to E^n \) is continuous, then a NFDE is a relation

(2)

\[
\frac{d}{dt} D(t, x_t) = f(t, x_t).
\]

A solution \( x = x(\varphi) \) through \( \varphi \) at time \( \sigma \) is a continuous function defined on \([\sigma-r, \sigma+A]\), \( A > 0 \), such that \( x_\sigma = \varphi \), \( D(t, x_t) \) is continuously differentiable on \((\sigma, \sigma+A)\) and (2) is satisfied on \((\sigma, \sigma+A)\). We assume
We assume a solution \( x(\varphi) \) of (2) through any \( \varphi \in C \) exists on \([\sigma-r, \infty)\), is unique and \( x(\varphi)(t) \) depends continuously on \((\varphi, t) \in C \times [\sigma-r, \infty)\).

In the following, we let \( T_D(t, \sigma): C \rightarrow C, t \geq 0 \), be the continuous linear operator defined by \( T_D(t, \sigma) \varphi = y_t(\varphi), t \geq \sigma \), where \( y = y(\varphi) \) is the solution of

\[
\frac{d}{dt}y^*_t(y_t) = 0, \quad y_0 = \varphi.
\]

If \( D \) is \( \omega \)-periodic in \( t \), \( C_D = \{ \varphi \in C: D(0, \varphi) = 0 \} \), then \( C_D \) is a Banach space with the topology of \( C \), \( T_D(\varphi, 0): C_D \rightarrow C_D \), and \( T_D(n\varphi, 0) = T_D^n(\varphi, 0) \).

The operator \( D \) is said to be uniformly stable if there exist constants \( K \geq 1, \alpha > 0 \), such that

\[
|T_D(t, \sigma)\varphi| \leq K e^{-\alpha(t-\sigma)}|\varphi|, \quad \varphi \in C_D, \quad t \geq \sigma.
\]

Notice the operator \( D\varphi = \varphi(0) \) corresponding to retarded functional differential equations is always stable.

Remark. The conclusion of the main theorem below is valid under the weaker hypothesis that \( D(\varphi) = D_0(\varphi) + \int_{-r}^{0} A(\theta) \varphi(\theta) d\theta \) where \( D_0 \) is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that \( D \) uniformly stable implies there exists an \( n \times n \) matrix function \( B(t) \) defined and of bounded variation on \([-r, \infty)\), continuous from the left, \( B(t) = 0, -r \leq t \leq 0 \), and a constant \( M_1 \) such that
\( |T_D(t)\phi| \leq M_1|\phi|, \quad t \geq 0, \quad \phi \in C, \quad \sup_{t \geq r} B(t) \leq M_1, \)

and, for any continuous function \( h: [0, \infty) \to \mathbb{E}^n \), the solution of the problem

\[
D(t, x_t) = D(O, \phi) + \int_0^t h(s)ds, \quad x_0 = \phi
\]

is given by

\[
x_t = T_D(t)\phi - \int_0^t B_{t-s}h(s)ds.
\]

Furthermore, there exist \( n \) functions \( \phi_1, \ldots, \phi_n \) in \( C \) such that

\[
D(O, \phi) = I, \quad \text{the identity, where } \phi = (\phi_1, \ldots, \phi_n).
\]

Let \( \psi: C \to C_D \) be the continuous linear operator defined by \( \psi(\phi) = \phi - \Phi D(\phi) \).

**Lemma 8.** If \( D \) is uniformly stable and \( f \) maps bounded sets of \( \mathbb{R} \times C \) into bounded sets of \( \mathbb{E}^n \), then there is a family of continuous transformations \( T_1(t): C \to C, \quad t \geq 0 \) which are weak completely continuous and

\[
def T(t, 0)\phi = T(t)\phi = x_t(\phi) = T_D(t)\psi(\phi) + T_1(t)\phi
\]

If \( D\phi = \phi(0) \), then \( T(t) \) is weak completely continuous for \( t \geq r \).
Proof: Equation (2) with initial value $x_0 = \varphi$ is equivalent to

$$D(x_t) = D(\varphi) + \int_0^t f(s,x_s)ds, \quad t \geq 0, \quad x_0 = \varphi,$$

which from (7) is equivalent to

$$T(t)\varphi = x_t = T_D(t)\psi(\varphi) + T_D(t)T_D(\varphi) - \int_0^t T_D(t-s)f(s,x_s)ds = T_D(t)\psi(\varphi) + T_1(t).$$

It is now an easy matter to verify the assertions in the theorem.

Since the condition that $D$ is uniformly stable implies the linear operator $S(\omega) = T_D(\omega)\psi$ has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

Theorem 4. If there exists an $\omega > 0$ such that $f(t+\omega,\varphi) = f(t,\varphi)$ for all $\varphi \in C$, $f$ takes bounded sets of $R \times C \to E^n$ and system (2) is compact dissipative, then there is an $\omega$-periodic solution of (2). If $f$ satisfies the same hypotheses and is independent of $t$, then there is a constant function $c$ in $C$ such that $f(c) = 0$; that is, an equilibrium point of (2). If $D(\varphi) = \varphi(0)$, then the same conclusions are true for point dissipative.
References


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Fixed Point Theorems and Dissipative Processes

by

Jack K. Hale and Orlando Lopes

1. Introduction

Suppose $X$ is a Banach space, $T : X \to X$ is a continuous mapping. The map $T$ is said to be dissipative if there is a bounded set $B$ in $X$ such that for any $x \in X$, there is an integer $N = N(x)$ with the property that $T^n x \in B$ for $n \geq N(x)$. In his study of ordinary differential equations in $n$-dimensional Euclidean space (which were ω-periodic in time), Levinson [12] in 1944 initiated the study of dissipative systems with $T x$ representing the solution of the differential equation at time $\omega$ which started at $x$ at time zero. The basic problem is to give information about the limiting behavior of orbits of $T$ and to discuss the existence of fixed points of $T$. Since 1944, a tremendous literature has accumulated on this subject and the reader may consult LaSalle [11], Pliss [14], Reissig, Sansone and Conti [15] and Yoshizawa [16] for references. Levinson [12] showed that some iterate of $T$ has a fixed point and he characterized the maximal compact invariant set of $T$. Later (see Pliss [14]), it was shown that the maximal compact invariant set was globally asymptotically stable and that some power of $T$ has a fixed point, the latter being proved by applying the fixed point theorem of Brouwer.

For the special case of the Banach space $X$ arising in retarded functional differential equations, and $T$ completely continuous, Jones [9] and Yoshizawa [16] showed that $T$ has a fixed point by using Browder's theorem. For an arbitrary Banach space $X$ and $T$ completely continuous, the same result was obtained by Horn [8] and by Gerstein and Krasnoselskii [5] with applications to parabolic partial differential equations. Recently,
Billotti and LaSalle [1] have obtained the same result with $T$ completely continuous. They have in addition characterized the maximal compact invariant set and proved that it is globally asymptotically stable.

Gerstein [4] has considered the case when $T$ is condensing on balls in $X$; in particular, if $\alpha(TB) < \alpha(B)$ for any ball $B \subset X$ and $\alpha$ is the measure of noncompactness introduced by Kuratowski (see Darbo [3]). Gerstein showed there is a maximal compact invariant set and a few other properties, but said nothing about fixed points of $T$. More recently, Hale, LaSalle and Slemrod [7] have considered a slightly stronger definition of dissipative and a class of operators $T$ which includes $\alpha$-contractions or $k$-set contractions; that is, there is a constant $k$, $0 < k < 1$, such that $\alpha(TB) \leq k\alpha(B)$ for any bounded $B \subset X$. They have characterized the maximal compact invariant set of $T$, shown that it is asymptotically stable, and proved that some iterate of $T$ has a fixed point.

There are a number of deficiencies in the above theories, two of which are the following: First, in the applications to $\omega$-periodic retarded functional differential equations, the hypothesis that $T$ is completely continuous implies that the period $\omega$ in the equation is greater than or equal to the delay $r$ in the differential system. In particular, this implies the above theory can not be employed to show the existence of an equilibrium point for an autonomous equation by taking a sequence of periods approaching zero. However, with the available knowledge on asymptotic fixed point theory (see for example, Jones [10]), the retarded equations can be handled directly for any $\omega > 0$. Secondly, in neutral
functional differential equations, the operator $T$ is not even completely continuous when $\omega \geq r$ and the most that can be obtained is a special form of an $\alpha$-contraction. However, the above theory for this case implies only that some iterate of $T$ has a fixed point.

It is the purpose of this paper to consider the same type of operators as considered by Hale, LaSalle and Slemrod [7] and to impose an additional condition on $T$ which will ensure that it has a fixed point. At first glance, this latter condition looks very strange, but it will be shown that the condition is always satisfied for $T$ condensing and local dissipative. Applications are given to a class of neutral functional differential equations.

2. Dissipative systems.

The $\varepsilon$-neighborhood of a set $K \subseteq X$ will be denoted by $B_\varepsilon(K)$, the closure by $\overline{Cl}(K)$ and the convex closure by $\overline{co}(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set $K$ in $X$ (see [3]). Suppose $T$ is a continuous map $T: X \to X$. The map $T$ is said to be weak condensing if for any bounded $K \subseteq X$ for which $\alpha(K) > 0$ and $T(K)$ is bounded it follows that $\alpha(T(K)) < \alpha(K)$. The map $T$ is said to be a weak $\alpha$-contraction if there is a constant $0 < k < 1$, such that for any bounded set $K \subseteq X$ for which $T(K)$ is bounded, it follows that $\alpha(T(K)) \leq k\alpha(K)$. If $T$ takes bounded sets into bounded sets, then a weak $\alpha$-contraction is an $\alpha$-contraction. The map $T^{n_0}$ is said to be weak completely continuous if there is an integer $n_0$ such that for any bounded set $B \subseteq X$, there is a compact set $B^* \subseteq X$ with the property that, for any integer $N \geq n_0$ and any $x \in X$ with $T^nx \in B$ for $0 \leq n \leq N$, it follows that $T^n x \in B^*$ for $n_0 \leq n \leq N$. If $T$ is weak completely continuous it is weak condensing.
If $T$ is completely continuous then $T$ is weak completely continuous. The map $T$ is said to be asymptotically smooth if for any bounded set $B \subseteq X$, there is a compact set $B^* \subseteq X$ such that for any $\varepsilon > 0$, there is an integer $n_0(\varepsilon, B)$ with the property that $T^n x \in B$ for $n \geq 0$ implies $T^n x \in B_\varepsilon(B^*)$ for $n \geq n_0(\varepsilon, B)$.

For a given continuous map $T: X \to X$, we say a set $K \subseteq X$ attracts a set $H \subseteq X$ if for any $\varepsilon > 0$, there is an integer $N(H, \varepsilon)$ such that $T^n(H) \subseteq B_\varepsilon(K)$ for $n \geq N(H, \varepsilon)$. We say $K$ attracts compact sets of $X$ if $K$ attracts each compact set $H \subseteq X$. We say $K$ attracts neighborhoods of compact sets of $X$ if for any compact set $H \subseteq X$, there is a neighborhood $H_0$ of $H$ such that $K$ attracts $H_0$.

A continuous map $T: X \to X$ is said to be point dissipative if there is a bounded set $B \subseteq X$ with the property that, for any $x \in X$, there is an integer $N(x)$ such that $T^n x \in B$ for $n \geq N(x)$. If $B$ satisfies the property that for any compact set $A \subseteq X$, there is an integer $N(A)$ such that $T^n(A) \subseteq B$ for $n \geq N(A)$, then $T$ is said to be compact dissipative. If $B$ satisfies the property that for any $x \in X$, there is an open neighborhood $0_x$ and an integer $N(x)$ such that $T^n 0_x \subseteq B$, $n \geq N(x)$, then $T$ is said to be local dissipative. Obviously, local dissipative implies compact dissipative implies point dissipative.

We now give a few relations among the above concepts.

**Lemma 1.** a) (Hale, LaSalle, Slemrod [7]). If $T$ is continuous, local dissipative and asymptotically smooth, then there is a compact set $K \subseteq X$ which attracts neighborhoods of compact sets of $X$. 

b) (Billotti and LaSalle [1]) If $T$ is continuous, point dissipative and $T^0$ is weak completely continuous, then there is a compact set $K \subset X$ such that for any compact set $H \subset X$, there is an open neighborhood $H_0$ of $H$ and an integer $N(H)$ such that $U_j \geq 0 \cdot T^j H_0$ is bounded and $T^n H_0 \subset K$ for $n \geq N(H)$. In particular, $T$ is local dissipative and $T$ asymptotically smooths.

**Lemma 2.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ that attracts neighborhoods of compact sets of $X$, then

a) there is a neighborhood $H_1 \subset H_0$, the above neighborhood of $H$, such that $U_n \geq 0 \cdot T^n H_1$ is bounded;

b) $U_j \geq 0 \cdot T^j B$ is precompact if $B$ is compact.

**Proof:**

a) First of all observe that a continuous function is bounded in some neighborhood of a compact set. If $H \subset X$ is compact and $N = n_1(H, \varepsilon)$ is the number occurring in the definition of the concept of attracts neighborhoods of compact sets, consider the sets $H, T(H), \ldots, T^{N-1}(H)$. Let $\Omega_0, \ldots, \Omega_{N-1}$ be corresponding neighborhoods where $T$ is bounded. Define $\Omega_N = B_0(K), \Gamma_N = \Omega_N, \Gamma_i = T^{-1}(\Omega_{i+1}) \cap \Omega_i$. The set $H_1 = \Gamma_0$ satisfies the required property.

b) The set $A = U_j \geq 0 \cdot T^j B$ is bounded. Since $T^j(B)$ is compact for any $j$ we have $\alpha(A) = \alpha(U_j \geq n \cdot T^j(B))$ for any $n$. But given $\varepsilon > 0$, if $n \geq n_1(B, \varepsilon)$, we have $U_j \geq n \cdot T^j B \subset B_\varepsilon(B)$ and thus $\alpha(A) \leq 2\varepsilon$. Thus $\alpha(A) = 0$ and $A$ is compact. This proves the Lemma.
The following result was proved in [7] if we use Lemmas 1 and 2.

**Theorem 1.** If $T: X \to X$ is continuous and there is a compact set $K \subset X$ which attracts neighborhoods of compact sets of $X$, then $J = \bigcap_j \bigcup_l T^j(K)$ is independent of the sets $K$ satisfying the above property, $J$ is the maximal compact invariant of $T$ and is globally asymptotically stable.

The hypotheses of Lemma 2 also imply there is a closed, bounded convex neighborhood $U$ of $K$ and an integer $n$ such that $T^n(U) \subset U$. Thus, if $T$ possesses the fixed point property, then some iterate of $T$ has a fixed point (see [7]).

Regarding fixed points of $T$, it is known (see [9], [14], [5], [8], [1]) that $T$ completely continuous and point dissipative implies $T$ has a fixed point. Below, we give some weaker conditions which assert that $T$ has a fixed point, but before beginning this discussion, there is one other interesting result regarding condensing maps which was stated without proof by Gerstein [4] for point dissipative systems.

**Theorem 2.**

a) If $T: X \to X$ is continuous, weak condensing and compact dissipative, then there is a compact invariant set $K$ which attracts compact sets of $X$ and $T$ is local dissipative.

b) If $T$ is weak condensing and point dissipative then there is a compact invariant set $K$ that attracts points of $X$.

**Proof:** a) It is an easy matter to prove the following fact: If $H$ is a compact set such that $T: H \to H$, then the set $A = \bigcap T^n(H)$ is compact, non empty, $T(A) = A$ and $T^n(H)$ tends to $A$ in the Hausdorff metric.
Now, for any compact set $L$ of $X$, let $L_1 = \bigcup_{j \geq 0} T^j(L)$. Since $L_1$ is bounded, $L_1 = L \cup T(L_1)$ and $T$ is weak condensing, it follows that $\alpha(L_1) = 0$ and thus $H = \text{Cl}(L_1)$ is compact. Also $T(H) \subseteq H$. Let $A_L = \bigcap_{n \geq 0} T^n(H)$. But, by hypothesis, there is a closed bounded set $B \subseteq X$ such that $A_L \subseteq B$ for each compact set $L$.

Since $T(\bigcup A_L) = \bigcup A_L$, where the union is taken over all compact sets $L \subseteq X$, it follows that the set $K = \text{Cl}(\bigcup A_L)$ is compact, $T(K) \subseteq K$, and $K$ attracts compact sets of $X$.

Nussbaum [13] has shown that if a non-empty invariant set attracts compact sets then it attracts neighborhoods of points and so if $T$ is weak condensing and compact dissipative it is local dissipative. This proves a) and the proof of b) is the same.

With a slight change in the argument above, we can prove the following:

**Lemma 3.** If $T$ is a weak $\alpha$-contraction, then $T$ asymptotically smooths.

**Proof:** If $B$ is a bounded set, then $B^* = \text{Cl}(\bigcup A_x)$, where $A_x$ is constructed as above for the elements $x \in B$ such that $T^nx \in B$, for any $n \geq 0$.

**Corollary.** If $T$ is a weak $\alpha$-contraction and local dissipative, then there is a compact invariant set that attracts neighborhoods of compact sets.

3. **Fixed point theorems.** In this section, we prove some fixed point theorems which have applications to the dissipative systems of the previous section. We need the following obvious:
Lemma 4. If $A$ is a compact set of $X$ and $F \subset X$ contains a sequence $\{x_n\}$ such that $d(x_n, A) \to 0$ as $n \to \infty$, then $A \cap \overline{F} \neq \emptyset$.

Theorem 3. Suppose $K \subset B \subset S \subset X$ are convex subsets with $K$ compact, $S$ closed, bounded, and $B$ open in $S$. If $T: S \to X$ is continuous, $T^j(B) \subset S$, $j \geq 0$, and $K$ attracts points of $B$, then there is a convex, closed bounded subset $A$ of $S$ such that

$$A = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap A) \right], A \cap K \neq \emptyset.$$

Proof: Let $\mathcal{F}$ be the set of convex, closed, bounded subsets $L$ of $S$ such that $T^j(B \cap L) \subset L$ for $j \geq 1$ and $L \cap K \neq \emptyset$. The family $\mathcal{F}$ is not empty because $S \in \mathcal{F}$. If $L \in \mathcal{F}$, let $L_1 = \overline{\text{co}} \left[ \bigcup_{j \geq 1} T^j(B \cap L) \right]$. By Lemma 4, $L_1 \cap K \neq \emptyset$. Also, $L_1$ is convex, closed, and contained in $S$. Since $L \in \mathcal{F}$, we have $L \supset L_1$ and $L_1 \supset T^j(B \cap L) \supset T^j(B \cap L_1)$ for all $j \geq 1$. Thus, $L_1 \in \mathcal{F}$. It follows that a minimal element $A$ of $\mathcal{F}$ will satisfy the conditions of the theorem.

To prove such a minimal element exists, let $(L_\alpha)_{\alpha \in I}$ be a totally ordered family of sets in $\mathcal{F}$. The set $L = \bigcap_{\alpha \in I} L_\alpha$ is closed, convex and contained in $S$. Also, $T^j(B \cap L) \subset T^j(B \cap L_\alpha) \subset L_\alpha$ for any $\alpha \in I$ and $j \geq 1$. Thus, $T^j(B \cap L) \subset L$ for $j \geq 1$. If $J$ is any finite subset of $I$, we have $K \cap \bigcap_{\alpha \in J} L_\alpha \neq \emptyset$ and, from compactness, it follows that $K \cap \bigcap_{\alpha \in I} L_\alpha \neq \emptyset$. Thus, $L \in \mathcal{F}$ and Zorn's lemma yields the conclusion of the theorem.

The same proof as given in Theorem 3 also proves the following:
Theorem 4. The set \( A \) of Theorem 3 is compact if and only if there is a compact set \( Q = Q(B) \) such that \( Q \cap B \neq \emptyset \) and \( T^j(Q \cap B) \subseteq Q \) for all \( j \geq 0 \).

Lemma 5. (Horn [8]). Let \( S_0 \subseteq S_1 \subseteq S_2 \) be convex subsets of a Banach space \( X \) with \( S_0, S_2 \) compact and \( S_1 \) open in \( S_2 \). Let \( T : S_2 \to X \) be a continuous mapping such that for some integer \( m > 0 \), \( T^j(S_1) \subseteq S_2 \), \( 0 \leq j \leq m-1 \), \( T^j(S_1) \subseteq S_0 \), \( m \leq j \leq 2m - 1 \). Then \( T \) has a fixed point.

Theorem 5. Suppose \( K \subseteq B \subseteq S \subseteq X \) are convex subsets with \( K \) compact, \( S \) closed bounded and \( B \) open in \( S \). If \( T : S \to X \) is continuous, \( T^jB \subseteq S \), \( j \geq 0 \), \( K \) attracts compact sets of \( B \) and the set \( A \) of Theorem 3 is compact, then \( T \) has a fixed point.

Proof: Since \( K \) is compact and convex, the set \( B \) can be taken as \( S \cap \mathcal{B}_\varepsilon(K) \) for some \( \varepsilon > 0 \). Let \( Q \) be as in Theorem 4, \( S_0 = \text{Cl}(\mathcal{B}_{\varepsilon/2}(K)) \cap Q \), \( S_1 = \mathcal{B}_\varepsilon(K) \cap Q \) and \( S_2 = S \cap Q \). Then \( S_0 \subseteq S_1 \subseteq S_2 \), \( S_0 \), \( S_2 \) compact and \( S_1 \) is open in \( S_2 \). Also, \( T^j(S_1) \subseteq S_2 \), \( 0 \leq j \leq n_1(K,\varepsilon) \) and \( T^j(S_1) \subseteq S_0 \) for \( j \geq n_1(K,\varepsilon) \) for some integer \( n_1(K,\varepsilon) \). An application of Lemma 4 completes the proof of the theorem.

It is clear that Theorem 5 is equivalent to the theorem of Horn.

Any additional conditions on the map \( T \) which will ensure that the set \( A \) in Theorem 3 is compact will yield a fixed point theorem using Theorem 5. One result in this direction is
Theorem 6. If $T$ is weak condensing, then the set $A$ in Theorem 5 is compact.

Proof: If $\widetilde{A} = \bigcup_{j \geq 1} T^j(B \cap A)$, then $\widetilde{A} = T(B \cap A) \cup T(\widetilde{A})$ and $\alpha(A) = \alpha(\widetilde{A}) = \max(\alpha(T(B \cap A)), \alpha(T(\widetilde{A})))$. Since $\alpha(T(\widetilde{A})) < \alpha(\widetilde{A})$ if $\alpha(\widetilde{A}) > 0$, it follows that $\alpha(\widetilde{A}) = \alpha(T(B \cap A))$. Thus, if $\alpha(B \cap A) > 0$, then $\alpha(A) = \alpha(\widetilde{A}) < \alpha(B \cap A) \leq \alpha(A)$ and this is a contradiction. Thus, $\alpha(B \cap A) = 0$. However, this implies $\alpha(A) = 0$ and $A$ is compact, proving the theorem.

Corollary 1. If the sets $K, B, S$ in Theorem 5 exist, if $K$ attracts the compact sets of $B$ and $T$ is weak condensing, then $T$ has a fixed point.

Proof. This is immediate from Theorems 5 and 6.
Corollary 2. If $T: X \to X$ is continuous, point-wise dissipative and $T$ is weak completely continuous, then $T$ has a fixed point.

Proof: This is immediate from Lemma 1b) and Corollary 1.

Corollary 3. If $T$ is a weak $\alpha$-contraction and there are sets $K, B, S$ as in Corollary 1, then $T$ has a fixed point.

Corollary 4. If $T$ is weak condensing and compact dissipative, then $T$ has a fixed point.

Proof: From Theorem 2a), $T$ is a local dissipative system. Thus $\text{co} \ K$ has an open convex neighborhood $B$ with bounded orbit. The result now follows from Theorems 2, 5, 6.

For $\alpha$-contractions, this result is contained in [13].

Corollary 5. If $T^n$ is weak completely continuous, $T$ is weak condensing and point dissipative, then $T$ has a fixed point.

Proof: This follows from Lemma 1b) and Corollary 4.
Lemma 6. If $S: X \to X$ is a bounded linear operator with spectrum contained in the open unit ball, then there is an equivalent norm, $\| \cdot \|_1$, in $X$ such that $\| S \|_1 < 1$.

Proof. Define $\| x \|_1 = \| x \| + \| Sx \| + \ldots + \| S^n x \| + \ldots$.

The assumption on the spectrum implies there is an $0 < r < 1$ such that $\| S^n \| < r^n$ if $n$ is sufficiently large. Thus, there is a constant $K$ such that $\| x \| \leq \| x \|_1 \leq K \| x \|$. Also, for $x \neq 0$

$$\frac{\| Sx \|_1}{\| x \|_1} = 1 - \left[ 1 + \frac{\| Sx \|}{\| x \|} + \frac{\| S^2 x \|}{\| x \|} + \ldots \right]^{-1} \leq 1 - \frac{1}{K}.$$

The lemma is proved.

Corollary 6. If $T$ is compact dissipative, $T = S + U$, where $S$ is linear and continuous with spectrum contained in the open unit ball and $T(\Omega)$ bounded implies $\text{Cl}(U(\Omega))$ compact for any $\Omega \subseteq X$, then $T$ has a fixed point. If, in addition, $S^{n_0}$ is completely continuous and $T$ is only point dissipative, then $T$ has a fixed point.

Proof: The first statement is immediate from Corollary 4 and Lemma 6. The second follows from Corollary 5 and the observation that $T^{n_0}$ is $S^{n_0}$ plus a completely continuous operator.
The next result generalizes an asymptotic fixed point theorem of Browder [2].

**Theorem 7.** Suppose $S_o, S_1, S_2$ are subsets of a Banach space, $S_o, S_2$ convex, closed, $S_1$ open, $S_2$ bounded, $S_o \subseteq S_1 \subseteq S_2$. Assume $T: S_2 \to X$ is condensing in the following sense: if $\Omega, T(\Omega)$ are contained in $S_2$ and $\alpha(\Omega) > 0$, then $\alpha(T(\Omega)) < \alpha(\Omega)$. Assume also that $T$ satisfies: for any compact set $H \subseteq S_1$, $T_j(H) \subseteq S_2$, $j > 0$, and there is a number $N(H)$ such that $T_j(H) \subseteq S_o$ for $j > N(H)$. Then $T$ has a fixed point.

**Proof:** Following the proof of Theorem 2, there is a compact set $K$ which attracts the compact sets of $S_1$. Since $K \subseteq S_o$, it follows that $\overline{co} K \subseteq S_o$. Let $B$ be a closed, convex neighborhood of $\overline{co} K, B \subseteq S_1$. Theorems 4 and 5 complete the proof.

4. **Dissipative flows.**

Let $\{T(t), t \geq 0\}$ be a flow in a Banach space $X$. A point $x \in X$ is said to be an equilibrium point if $T(t)x_0 = x_0$ for any $t \geq 0$. We also say that a compact set $J$ attracts a compact set $H$ if, for any $\varepsilon > 0$, there is a $t^*(H,\varepsilon)$ such that $T(t)H \subseteq B_{\varepsilon}(J)$ for $t \geq t^*(H,\varepsilon)$. A set $Q \subseteq X$ is said to be **boundedly compact** if $P \cap Q$ is compact for any closed bounded set $P \subseteq X$. 
Theorem 8. If \( \{T(t), t \geq 0\} \) satisfies:

1. there is a compact set \( J \) that attracts the compact sets of \( J_o \);

2. there is a number \( w > 0 \) and a family of boundedly compact (in particular compact) sets \( Q(\tau), 0 < \tau < w \) such that \( Q(\tau) \cap J_o \neq \emptyset \) and \( T(k\tau)Q(\tau) \cap Q(\tau), 0 < \tau \leq w \), \( k \) positive integer;

then there is an equilibrium point.

Proof: Take the sequence \( w_n = w/n \). From Theorems 2, 4, 5, it follows that for each \( n \), there is an \( x_n \) satisfying: \( T(w_n)x_n = x_n \). Since \( w_n > 0 \), \( x_n \) is in \( J \).

Changing the notation if necessary, we may assume that \( x_n \) converges to \( x_0 \). Let \( k_n(t) \) be the integer defined by:

\[
k_n(t)w_n < t < (k_n(t) + 1)w_n
\]

Then, \( T(k_n(t)w_n)x_n = x_n \) and so:

\[
|T(t)x_o - x_o| \leq |T(t)x_o - T(k_n(t)w_n)x_n| + |T(k_n(t)w_n)x_n - T(k_n(t)w_n)x_n| + |x_n - x_o|
\]

Since \( k_n(t)w_n \) tends to \( t \) as \( n \to \infty \), the right hand side of the above expression goes to zero and this proves the theorem.

As an application of Theorems 4, 5 and 8, we have:

Corollary 7: If \( \{T(t), t \geq 0\} \) is weak condensing for any \( t > 0 \) and satisfies \( \alpha_1 \), then there is an equilibrium point.
Corollary 8. If \( \{T(t), t \geq 0\} \) is a weak \( \alpha \) - contraction and satisfies \( \alpha_1 \), then there is an equilibrium point.

A flow \( \{T(t), t \geq 0\} \) is said to be local dissipative if there is a bounded set \( B \) such that for any point \( x \in X \) there is a neighborhood \( O_x \) of \( x \) and a \( t(x) \) such that \( T(t)O_x \subset B \) for \( t \geq t(x) \).

A flow \( \{T(t), t \geq 0\} \) is compact (point) dissipative if there is a bounded set \( B \) such that for any compact set \( H \) (any point \( x \)) there is a \( t(H) \) (\( t(x) \)) such that \( T(t)H \subset B \) (\( T(t)x \in B \)) for \( t \geq t(H) \) (\( t \geq t(x) \)).

Lemma 7.

a). If \( T(w) \) is weak condensing for some \( w > 0 \) and is compact (point) dissipative with compact attractor \( K \), \( T(w)K \subset K \), then \( \{T(t), t \geq 0\} \) is compact (point) dissipative with attractor \( J = \bigcup_0^w T(t)K \).

b). If there is a compact set \( K \) such that \( T(w)K \subset K \) and \( K \) attracts neighborhoods of points, then the set \( J \) above attracts neighborhoods of points relative to the flow \( \{T(t), t \geq 0\} \).

c). If \( \{T(t) \geq 0\} \) is weak condensing for some \( \omega > 0 \) and compact dissipative, it is local dissipative.

Proof: For any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( T(t)B_\delta(K) \subset B_\varepsilon(J), 0 \leq t \leq \omega \). Since \( T(w)K \subset K \), parts a) and b) are proved. Part c) follows from Theorem 2, and parts a) and b).
Corollary 9. If for some \( w > 0 \), \( T(w) \) is weak completely continuous and point dissipative then \( \{T(t), t \geq 0\} \) is local dissipative.

Proof: The set \( K \) in Lemma 1(b) may be chosen to satisfy the hypothesis of Lemma 7(b). From the previous theorem, we can state the following:

Theorem 9. If \( \{T(t), t \geq 0\} \) is weak condensing for \( t > 0 \), then the following assertions hold:

a). If \( \{T(t), t \geq 0\} \) is compact dissipative, then there is an equilibrium point;

b). If \( \{T(t), t \geq 0\} \) is point dissipative and \( T(w) \) is weak completely continuous for some \( w > 0 \), then there is an equilibrium point.

Corollary 10. If \( \{T(t) = S(t) + U(t), t \geq 0\} \) then the following assertions hold:

a). If \( \{Q(t), t \geq 0\} \) is compact dissipative, \( S(t) \) is linear with spectrum contained inside the unit ball for \( t > 0 \) and \( U(t) \) is weak completely continuous, then there is an equilibrium point.

b). If \( S(w) \) is completely continuous for some \( w > 0 \) and \( \{T(t), t \geq 0\} \) is point dissipative, there exists an equilibrium point.
5. Functional differential equations.

As an application of the previous results, we consider a special class of neutral functional differential equations which are periodic in time. Let \( r > 0 \) be a given real number, \( E^n \) be an \( n \)-dimensional linear vector space with norm \( \| \cdot \| \), \( C([a,b],E^n) \) be the space of continuous functions from \([a,b]\) to \( E^n \) with the uniform topology and let \( C = C([a,b],E^n) \). For \( \varphi \in C \), \( |\varphi| = \sup_{-r < \theta < 0} |\varphi(\theta)| \). For any \( x \in C([a,b],E^n) \), \( A > 0 \), let \( x_t(\theta) = x(t+\theta), -r < \theta < 0 \). Suppose \( D: \mathbb{R} \times C \rightarrow E^n \) is a continuous linear operator \( D\varphi = \varphi(0) - g(t,\varphi) \),

\[
\begin{align*}
g(t,\varphi) &= \int_{-r}^{0} [d\mu(t,\theta)]\varphi(\theta) \\
|\int_{-s}^{0} [d\mu(t,\theta)]\varphi(\theta)| &\leq \gamma(s)|\varphi|,
\end{align*}
\]

for \( s \geq 0 \), \( \varphi \in C \) where \( \mu \) is an \( n \times n \) matrix function of bounded variation, \( \gamma \) is continuous and nondecreasing on \([0,r]\), \( \gamma(0) = 0 \). If \( f: \mathbb{R} \times C \rightarrow E^n \) is continuous, then a NFDE is a relation

\[
\frac{d}{dt} D(t,x_t) = f(t,x_t).
\]

A solution \( x = x(\varphi) \) through \( \varphi \) at time \( \sigma \) is a continuous function defined on \([\sigma-r,\sigma+A)\), \( A > 0 \), such that \( x_\sigma = \varphi \), \( D(t,x_t) \) is continuously differentiable on \((\sigma,\sigma+A)\) and (2) is satisfied on \((\sigma,\sigma+A)\). We assume
We assume a solution $x(\varphi)$ of (2) through any $\varphi \in C$ exists on $[\sigma-r, \infty)$, is unique and $x(\varphi)(t)$ depends continuously on $(\varphi, t) \in C \times [\sigma-r, \infty)$.

In the following, we let $T_D(t, \sigma): C \to C$, $t \geq 0$, be the continuous linear operator defined by $T_D(t, \sigma)\varphi = y_t(\varphi)$, $t \geq \sigma$, where $y = y(\varphi)$ is the solution of

$$
\frac{d}{dt} y(t) = 0, \quad y_\sigma = \varphi.
$$

If $D$ is $\omega$-periodic in $t$, $C_D = \{ \varphi \in C : D(0, \varphi) = 0 \}$, then $C_D$ is a Banach space with the topology of $C$, $T_D(w, 0): C_D \to C_D$, and $T_D(nw, 0) = T_D(w, 0)$.

The operator $D$ is said to be uniformly stable if there exist constants $K > 1$, $\alpha > 0$, such that

$$
|T_D(t, \sigma)\varphi| \leq Ke^{-\alpha(t-\sigma)}|\varphi|, \quad \varphi \in C_D, \quad t \geq \sigma.
$$

Notice the operator $D\varphi = \varphi(0)$ corresponding to retarded functional differential equations is always stable.

Remark. The conclusion of the main theorem below is valid under the weaker hypothesis that $D(\varphi) = D_0(\varphi) + \int_{-r}^0 A(\theta)\varphi(\theta)d\theta$ where $D_0$ is stable. For simplicity in notation, we do not consider this more general case.

We need some results from Hale and Cruz [6]. It is shown in [6] that $D$ uniformly stable implies there exists an $n \times n$ matrix function $B(t)$ defined and of bounded variation on $[-r, \infty)$, continuous from the left, $B(t) = 0$, $-r \leq t \leq 0$, and a constant $M_+$ such that
and, for any continuous function \( h: [0, \infty) \to \mathbb{R}^n \), the solution of the problem

\[
D(t, x_t) = D(0, x_0) + \int_0^t h(s) \, ds, \quad x_0 = \varphi
\]

is given by

\[
x_t = T_D(t) \varphi - \int_0^t B_{t-s} h(s) \, ds.
\]

Furthermore, there exist \( n \) functions \( \varphi_1, \ldots, \varphi_n \) in \( C \) such that \( D(0, \varphi) = I \), the identity, where \( \varphi = (\varphi_1, \ldots, \varphi_n) \).

Let \( \Psi: C \to C_D \) be the continuous linear operator defined by \( \Psi(\varphi) = \varphi - \varphi D(\varphi) \).

**Lemma 8.** If \( D \) is uniformly stable and \( f \) maps bounded sets of \( \mathbb{R} \times C \) into bounded sets of \( \mathbb{R}^n \), then there is a family of continuous transformations \( T_1(t): C \to C \), \( t \geq 0 \) which are weak completely continuous and

\[
T(t, 0) \varphi = T(t) \varphi = x_t(\varphi) = T_D(t) \Psi(\varphi) + T_1(t) \varphi
\]

If \( D \varphi = \varphi(0) \), then \( T(t) \) is weak completely continuous for \( t \geq r \).
Proof: Equation (2) with initial value \( x_0 = \phi \) is equivalent to

\[
D(x_t) = D(\phi) + \int_0^t f(s, x_s) ds, \quad t \geq 0, \quad x_0 = \phi,
\]

which from (7) is equivalent to

\[
T(t) \psi = x_t = T_D(t) \psi(\phi) + T_D(t) \phi D(\phi) - \int_0^t B_t f(s, x_s) ds = T_D(t) \psi(\phi) + T_1(t).
\]

It is now an easy matter to verify the assertions in the theorem.

Since the condition that \( D \) is uniformly stable implies the linear operator \( S(\omega) = T_D(\omega) \psi \) has spectrum contained inside the unit ball, Corollary 6, Lemma 1b) and Corollary 10 imply

**Theorem 4.** If there exists an \( \omega > 0 \) such that \( f(t+\omega, \varphi) = f(t, \varphi) \) for all \( \varphi \in C \), \( f \) takes bounded sets of \( R \times C \to E^n \) and system (2) is compact dissipative, then there is an \( \omega \)-periodic solution of (2). If \( f \) satisfies the same hypotheses and is independent of \( t \), then there is a constant function \( c \) in \( C \) such that \( f(c) = 0 \); that is, an equilibrium point of (2). If \( D(\varphi) = \varphi(0) \), then the same conclusions are true for point dissipative.
References


