Technical Memorandum 33-601

Analysis of the Computed Torque Drive Method and Comparison With Conventional Position Servo for a Computer-Controlled Manipulator

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PREFACE

The work described in this report was performed by the Guidance and Control Division of the Jet Propulsion Laboratory.
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ABSTRACT

A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an artificial intelligence project. This development includes an analytical study of the control system software. This report presents a comparison of the computed torque method and the conventional position servo. No conclusion is made as to the preference of one system over the other, as it is dependent upon the application and the results of a sampled data analysis.
I. INTRODUCTION

A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an Artificial Intelligence project. This development includes an analytical study of the control system software. The purpose of this report is to present the results of this study.

The Stanford manipulator control system models the manipulator and its dynamics and computes the motor drive torque for each joint of the manipulator. If the model were exact, the manipulator could be driven open-loop with no error. But because the model is not exact, rate and position feedback is used in addition to the computed drive torque.

The manipulator operation is described from a control analysis viewpoint for the "Computed Torque" method in Section II and for a conventional position servo in Section III. The gains and output response for both systems are derived and discussed in detail. A comparison is made of the two control methods in Section IV. The Stanford gain values are used here for the "Computed Torque" method. But the final values may be different because the JPL application does not require as high turning rates, and the design is slightly different.

II. COMPUTED TORQUE SYSTEM

In the "Computed Torque" method the trajectory that the manipulator is to follow as a function of time is computed in the planning program. Each link angle (the angle of the manipulator joint) as a function of time, necessary to follow the desired trajectory, is also computed in the planning program. The drive motor torque required to achieve the link angle/time function is then computed based on a model of the manipulator hardware. The motor voltage/torque characteristic is also modeled and the computed torque is converted to applied motor voltage. This voltage is computed at a very high rate, applied to D/A converter, and finally to the motor input.

If the manipulator and motor models were exact, the response would be exact, but this is not possible because of modeling inaccuracies and parameter variations. Rate and position feedback is therefore used to compute correction torques which are summed with the main driving torque. The result is a sampled data control system with performance dependent on the feedback gain values.
Fig. 1. Physical Configuration of Computed Torque Method Control Loop
Fig. 2. Transfer Function Control Loop Configuration
1. **Transfer Function Derivation**

The interconnection of the physical components and the computer subroutines is shown in Fig. 1, and the detailed flow diagram for control analysis is shown in Fig. 2. The latter does not contain the data samplers because this analysis is for a linear continuous system. A subsequent sampled data analysis is required to determine the effect of the sampling frequency. The motor and gear train nonlinearities are modeled in the voltage torque characteristic which allows the linear analysis approach.

The control loop configuration can be reduced to the configuration of Fig. 3, with the branch transfer functions defined as follows:

\[
G_1(s) = \frac{\frac{1}{57.3}}{n} J' \frac{R'_M}{K_T} s^2
\]

\[
G_2(s) = \frac{57.3}{s \left( K_B + \frac{R_M}{K_T} J s \right)}
\]

\[
G_3(s) = \frac{1}{57.3} J' K_V \frac{R'_M}{K_T}
\]

\[
G_4(s) = \frac{1}{57.3n} J' \left( K_e + \frac{K_I}{s} \right) \frac{R'_M}{K_T}
\]

\[
G_5(s) = s
\]

\[
G_6(s) = \frac{1}{n} s
\]
\[ G_7(s) = 1 \]

\[ G_8(s) = -1 \]

\[ G_9(s) = \frac{K_B}{57.3} \]

\[ G_{10}(s) = -1 \]

\[ G_{11}(s) = n \]

From signal flow graph theory

\[ \frac{\theta_0}{\theta_1}(s) = \frac{G_1G_2G_{11} + G_6G_3G_2G_{11} + G_7G_4G_2G_{11}}{1 - G_5G_10G_3G_2 - G_8G_4G_2G_{11} - G_9G_2G_1} \]

\[ = \frac{1}{G_2} - \frac{(G_1 + G_6G_3 + G_7G_4)G_{11}}{G_2} \]

\[
\begin{align*}
J' \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} s + J' \left( K_e + \frac{K_I}{s} \right) \frac{R_M}{K_T} \\
&= \frac{J' \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} s + J' \frac{R_M}{K_T} \left( K_e + \frac{K_I}{s} \right) \frac{R_M}{K_T}}{K_B s + J' \frac{R_M}{K_T} s + J' \frac{R_M}{K_T} \left( K_e + \frac{K_I}{s} \right) \frac{R_M}{K_T}} - K_B s
\end{align*}
\]

\[
\begin{align*}
J' \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} s + J' \frac{R_M}{K_T} \left( K_e + \frac{K_I}{s} \right) \\
&= \frac{J' \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} s + J' \frac{R_M}{K_T} \left( K_e + \frac{K_I}{s} \right)}{J' \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} s + J' \frac{R_M}{K_T} \left( K_e + \frac{K_I}{s} \right)}
\end{align*}
\]

If \( J' = J \), \( \frac{R_M}{K_T} = \frac{R_M}{K_T} \).
Each of the individual branch transfer functions \((G_1 \rightarrow G_{11})\) will now be described in detail.

**\(G_2\): Drive motor and load.**

The voltage applied to the motor is

\[
V_a = E + i_M R_M
\]  

(4)

where

\[
E = \text{back e.m.f. voltage} \quad \text{(volts)}
\]
\[
i_M = \text{motor current} \quad \text{(DC amps)}
\]
\[
R_M = \text{motor resistance} \quad \text{(ohms)}
\]

The motor torque (referred to motor shaft) is a function of the current and the torque constant, \(K_T\) for a particular motor.

\[
T_M = K_T i_M
\]  

(5)

where

\[
K_T = \text{motor torque constant} \quad \text{(oz-in./amp)}
\]

The back e.m.f. is a function of another constant for the particular motor.

\[
E = K_B \dot{\theta}_M
\]  

(6)
where

\[ K_B = \text{motor back e.m.f. constant} \quad \left( \frac{\text{volts}}{\text{rad/sec}} \right) \]

\[ \dot{\theta}_M = \text{motor shaft rotation rate} \quad \left( \text{rad/sec} \right) \]

Substituting for \( E \) and \( i_M \) in Eq. (4)

\[ V_a = K_B \dot{\theta}_M + \frac{R_M}{K_T} \tau_M \quad (7) \]

The motor torque will produce an angular acceleration of the motor shaft.

\[ \tau_M = J \ddot{\theta}_M \quad (8) \]

where

\[ J = J_M + n^2 J_L \quad \text{(oz-in./rad/sec}^2 \equiv \text{oz-in.-sec}^2) \quad (9) \]

This total inertia is referred to the output by dividing by \( n^2 \)

\[ J_o = \frac{1}{n^2} J \quad (10) \]

\[ J_M = \text{motor rotor inertia} \quad \text{(oz-in.-sec}^2) \]

\[ J_L = \text{load inertia} \quad \text{(oz-in.-sec}^2) \]

\[ n = \frac{n_1}{n_2} = \text{ratio of motor gear teeth to load gear teeth} \]

\[ \ddot{\theta}_M = \text{motor shaft angular acceleration} \quad \left( \text{rad/sec}^2 \right) \]
As indicated above, the load inertia is referred to the motor shaft by multiplying by the gear ratio squared. This is simply derived by using the conservation of energy principle.

\[ \frac{1}{2} J_{LM} \dot{\theta}_M^2 = \frac{1}{2} J_L \dot{\theta}_o^2 \]  

(11)

where

\[ J_{LM} = \text{load inertia referred to motor shaft} \]
\[ \dot{\theta}_o = \text{load or output rotation rate} \]

Substituting \( \dot{\theta}_o = n \dot{\theta}_M \) results in

\[ J_{LM} = n^2 J_L \]  

(12)

Now, to derive the motor transfer function, substitute \( \tau_M \) from (8) into (7).

\[ V_a = K_B \dot{\theta}_M + \frac{R_M}{K_T} J \ddot{\theta}_M \]  

(13)

Taking the Laplace transform with I.C. = 0

\[ G_2(s) = \frac{\theta_M(s)}{V_a(s)} = \frac{57.3}{s \left( K_B + \frac{R_M}{K_T} J s \right)} \]  

(14)

The motor transfer function is in terms of radians output, but the program as presently defined uses degrees. Therefore, the 57.3 factor is introduced in the numerator.

\( G_1 \): Computed torque

The driving torque (not including error torques) is computed in \( G_1 \) from known motor and load parameters and then changed to applied motor voltage. For this analysis the gravity torque shown in fig. 2 is not
included in $G_1$. This is done because it can be handled as a known disturbance input which is then cancelled by the computed gravity torque, $\tau^g$. As shown in Fig. 2 the gravity torque $\tau^g$ is transferred from the output to the input for analysis purposes. It then has to be reduced by the gear ratio, $n$. The cancellation can be quite accurate except for the effect of an unknown weight being lifted by the manipulator, although the object weight can be determined from the error signal and then included in the computed gravity torque. This will be discussed under disturbance torque.

With the gravity loop eliminated, the remaining branch of $G_1$ computes the required motor torque. From known manipulator parameters and $\theta^i(t)$, the changing inertia $J^i(t)$ can be computed. Referring to Eq. (8), it is seen that the torque is proportional to the angular acceleration and the total inertia referred to the motor shaft. $J^o$ (referred to output) is computed in the program, therefore the $n^2$ factor is needed. Angular acceleration can be computed because $\theta^i(t)$ is analytic (third degree polynomial) and is represented by $s^2$. The $1/n$ factor is needed because $\theta^i(t)$ refers to the manipulator joint angle, but we are now referencing to the motor input. Therefore, $\theta^i$ (used to compute the motor torque) must be divided by the gear ratio. The final operation here is to change this required torque to applied motor voltage. This could be done as shown by using a constant $R_M/K_T$ (volts/oz-in.) which is the torque to voltage conversion constant for a particular motor neglecting motor friction and reduction gear losses. With only coulomb friction torque present (given in motor specs), the constant slope voltage torque curve is displaced up for plus $\dot{\theta}_M$ by the voltage to overcome friction and down for minus $\dot{\theta}_M$. But with the reduction gear, the characteristics are not so simple. An approximation to the curves are shown in Fig. 4. They will be determined from lab tests by measuring the output torque, keeping $\dot{\theta}_M$ constant and varying the applied voltage (the ordinate, $V_M$ does not include the back e.m.f. voltage). The solid lines are theoretical for no gear reduction and the dotted lines connect data points from lab data. Note that there are three different slopes or torque constants ($K_{M1}, K_{M2}, K_{M3}$). $K_{M1}$ differs from the theoretical value ($R_M/K_T$) by the constant $\mu$ which is the additional voltage required due to loss in the gear transmission. For $+\dot{\theta}_M$ and $+\tau_o$, $\mu$ is plus because the loss is opposing the desired output $\tau_o$, but for $-\tau_o$ the loss is aiding. The area covered by $K_{M3}$ is somewhat nonlinear and the value $K_{M3}$
(referring to $+\dot{\theta}_M$) is an approximation joining the ordinate cross point from $K_{M1}$ and a point with an ordinate $-V_f$ on $K_{M2}$. The point does not seem to have any theoretical relationship but was selected arbitrarily as coinciding with lab data. The final result of all this is that only three torque constants are needed ($K_M$, $\mu$, $K_{M3}$), along with the sign of $\dot{\theta}_M$ and the ordinate breakpoints for $\tau_o$, to convert this torque to voltage. Lab tests will be performed to obtain this characteristic.

From Eq. 4 under $G_2$ the applied motor voltage is

$$V_a = E + i_M R_M = E + V_M$$

(15)

With $V_M$ now computed the back e.m.f. ($E$) has to be computed to get $V_a$. This is done in $G_9$ using the motor rate $\dot{\theta}_M$ from the tachometer.
$G_6, G_5, G_{10}$: Rate error

The motor output rate is obtained from the tachometer ($G_5$) mounted on the motor shaft. The desired link angular rate ($\dot{\theta}_1$) is computed in the planning program and is multiplied by $1/n$ to reference it to the motor. The two are compared to indicate the rate error. The unity gain, $G_{10}$ is only for transfer function derivation purposes to separate the rate error from motor rate which is used in $G_9$. The tachometer output is in volts, and an A/D converter is required so the computer can interrogate the motor rate. The interrogation of $\dot{\theta}_M$ is at some finite rate, thus, we have a sampled data system.

$G_3$: Rate error feedback gain

The rate error is converted to radians/sec and multiplied by $J^1$ to remove this changing parameter from the system response characteristic. If it remained, the damping ratio and natural frequency (reducing to a second order system) would change with $J^1$. The system damping ratio is now a function of the constant, $K_v$ and of $K_e$.

$G_7, G_8$: Position error

The manipulator link rotation angle ($\theta_o$) is obtained from a potentiometer mounted on the link, and the desired link angle ($\theta_1$) is available from the planning program. The two are compared to indicate the position error which is later multiplied by $1/n$ to reference it to the motor. $\theta_o$ is in volts, thus, an A/D converter is required. As with $\dot{\theta}_M$ above, the interrogation is at some finite rate, which makes the system a sampled data system which can be analyzed as such using Z transform analysis methods. Any small delay time for reading in $\theta_o$ and $\dot{\theta}_M$ can be incorporated in the analysis also.

$G_4$: Position error and integral feedback gains

The position error is converted to radians and multiplied by $J^1$ as with $G_3$, but here it is to keep the system undamped natural frequency constant. The integral gain ($K_1$) which will be discussed in Section II-4 is also now a
constant. It should be noted that the integration \((K_I/s)\) is done numerically in the computer as:

(1) Initialize \(\tau_{II-1} = 0\)

\[
\tau_{II} = \tau_{II-1} + \frac{J}{57.3n} K_I \Delta t \theta_e
\]  

(16)

\[
\tau_{II-1} = \tau_{II}
\]

where \(\tau_{II}\) is the computed torque for the present computation cycle

\(\Delta t\) is the computation cycle time

\(\theta_e\) is the position error in degrees

\(G_2\): Motor back e.m.f. voltage

In Fig. 2 \(G_1, G_3\) and \(G_4\) do not include the voltage-torque conversion

\[
V_M = \left( \frac{R_M}{K_T} \right) \tau
\]

because in the computer program, the three outputs are summed and only one conversion is necessary. In the analysis the factor is included in each only for simplicity, but the total output voltage is \(V_M(s)\).

From the \(G_2\) derivation, the applied motor voltage is

\[
V_a = E + i_M R_M = E + V_M
\]

Thus, the back e.m.f. voltage, \(E\) has to be computed. This is done in \(G_9\) in which the known motor constant \(K_B\) is used with the tachometer motor rate.

\[
E = K_B \theta_o
\]
2. **Response to Input Forcing Functions**

As shown in Eq. (3) the output is identical to the input if the inertia, \( J' \) is computed correctly and if the voltage torque curve is exact, i.e.,

\[
\frac{R_M'}{K_T} = \frac{R_M}{K_T}
\]

Therefore, the output or link movement \( \theta_o(t) \) exactly follows the input, \( \theta_1(t) \) which was computed in the planning program. But practically, errors due to approximations in modeling and unknown system nonlinearities prevent this ideal response, consequently the error feedback loops were included.

It is obvious that if the motor torque is computed correctly, the output should respond correctly in an open loop manner and there would always be zero error. And the previous derivation of the system transfer function would be unnecessary. However, analyzing the response to disturbance torques or unknown gravity torques does require a transfer function. Also, it is possible to insert an inertia error, \( \Delta J' \) and determine the response error. Additional transfer functions are derived in the following sections.

3. **Transfer Function for Feedback Position and Rate Errors**

The two feedback transducers involved in each link control loop are a potentiometer and a tachometer which could have bias or null errors. The transfer functions for these errors can be derived easily from the system transfer function derived in Section II-1.
Position Feedback Null Error:

The flow diagram is indicated in the following figure:

Fig. 5. Signal Flow Diagram for Position Feedback Null Error

\[
\frac{\theta_o(s)}{\Delta \theta_1(s)} = \frac{G_4 G_{11}}{\frac{1}{G_2} - (G_5 G_{10} G_3 + G_8 G_4 G_{11} + G_9 G_5)} \tag{17}
\]

\[
\frac{J\left(K_e + \frac{K_I}{s}\right)}{s} R_M \frac{R_M}{K_T} s = \frac{J' \left(K_e + \frac{K_I}{s}\right) R_M}{J' \frac{R_M}{K_T} s^3 + J' K_v \frac{R_M}{K_T} s^2 + J' \frac{R_M}{K_T} K_e s + J' \frac{R_M}{K_T} K_I} \tag{18}
\]

for \(R_M'/K_T = R_M/K_T\), \(J' = J\)

\[
\frac{\theta_o(s)}{\Delta \theta_1(s)} = \frac{(K_e s + K_I)}{s^3 + K_v s^2 + K_e s + K_I} \tag{19}
\]
For $\Delta \theta_i(s) = \Delta \theta_i/s$

$$
\theta_o(s) = \frac{K \Delta \theta_i (s + K/K_e)}{s(s^2 + K_v s^2 + K_e s + K)}
$$

(20)

The steady state error is

$$
\theta_o(t) = s \theta_o(s) = \Delta \theta_i
$$

(21)

$$
t \to \infty \quad s \to 0
$$

with $K_I = 0$

$$
\theta_o(s) = \frac{K \Delta \theta_i}{s(s^2 + K_v s + K_e)}
$$

(22)

$$
\theta_o(t) = \Delta \theta_i
$$

(23)

Thus, the steady state error is the same with or without the integral gain. The second order transfer function can be in the form

$$
\frac{\theta_o(s)}{\Delta \theta_i(s)} = \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}
$$

(24)

where $\zeta$ is the damping ratio and $\omega_n$ the undamped natural frequency. The step function response for $\zeta = 1$ is shown in Fig. 6.

The response with $K_I \neq 0$ is altered very little if the resultant additional pole and zero are close together. Thus, the ratio of $K_I/K_e$ is the determining factor. It should be noted that the above is the system response if the link position is at some different position than $\theta_i$ which comes from the planning program.
Rate feedback error:

The signal flow diagram for a rate null error is

\[
\frac{\theta_o(s)}{\Delta \theta_M(s)} = \frac{(G_9 - G_3) G_{11}}{1 - \left[ G_5 (G_9 - G_3) + G_8 G_4 G_{11} \right]}
\]

(25)

\[
= \frac{n \left( - J^T \frac{R_M}{K_T} K_v + K_B \right) s}{J \frac{R_M}{K_T} s^3 + J^T \frac{R_M}{K_T} K_v s^2 + J^T \frac{R_M}{K_T} K_e s + J^T \frac{R_M}{K_T} K_I}
\]

(26)
For $R_M'/K_T = R_M/K_T$, $J' = J$

$$\frac{\theta_o(s)}{\Delta \theta_M(s)} = \frac{n \left( -K_v + \frac{K_T}{R_M} K_B \right)s}{s^3 + K_v s^2 + K_e s + K_I}$$

(27)

For a step input error, $\Delta \dot{\theta}_M(s) = \Delta \dot{\theta}_M/s$

$$\theta_o(s) = \frac{n \left( -K_v + \frac{K_T}{R_M} K_B \right) \Delta \dot{\theta}_M}{s^3 + K_v s^2 + K_e s + K_I}$$

(28)

for which

$$\lim_{t \to \infty} \theta_o(t) = 0$$

If $K_I = 0$

$$\theta_o(s) = \frac{n \left( -K_v + \frac{K_T}{R_M} K_B \right) \Delta \dot{\theta}_M}{s(s^2 + K_v s + K_e)}$$

(29)

for which

$$\lim_{t \to \infty} \theta_o(t) = \frac{n \left( -K_v + \frac{K_T}{R_M} K_B \right)}{K_e}$$

(30)
Thus, there is a steady state error if the integral feedback gain is zero. The time required to bring this error to zero depends on the magnitude of $K_I$.  

4. **Transfer Function for a Disturbance Torque**

A disturbance torque on the output can be expressed as a motor input for analysis purposes. It is then treated as a normal input but must be reduced by the gear ratio, $n$ ($n < 1$). An uncompensated gravity torque such as a load picked up by the manipulator is such a disturbance torque. The signal flow diagram for a disturbance torque is shown in Fig. 8.

Fig. 8. Signal Flow Diagram for a Disturbance Torque

\[ T_{dM}(s) \]

\[ G_{12} \]

\[ G_{11} \]

\[ G_3 \]

\[ G_5 \]

\[ G_9 \]

\[ G_{10} \]

\[ G_{11} \]

\[ G_4 \]

\[ G_6 \]

\[ \theta_0 \]

\[ \tau_{dM} \]

where $G_{12} = R_M / K_T$

\[
\frac{\theta_o(s)}{\tau_{dM}(s)} = \frac{G_{12}G_{11}}{G_2 - (G_5G_{10}G_3 + G_8G_4G_{11} + G_9G_5)}
\]

(31)

\[
57.3 \frac{R_M}{K_T} \text{ ns}
\]

(32)
Since $\tau_d \cdot M = n \cdot \tau_d$

$$\frac{\theta_o(s)}{\tau_{do}(s)} = \frac{57.3 \cdot n}{57.3 \cdot n} \cdot \frac{R_M}{K_T} \cdot s$$

(34)

For a step input, $\tau_{do}(s) = \tau_{do}/s$

$$\theta_o(t) = 0$$

$t \to \infty$

with $K_I = 0$

$$\theta_o(t) = \frac{57.3 \cdot n}{57.3 \cdot n} \cdot \frac{R_M}{K_T} \cdot \tau_{do}$$

(35)

The torque conversion factor was left in because in this case the $R_M/K_T$ does not cancel out if the computed value, $R_M/K_T$ (from the curves) is not equal to the real value, $R_M/K_T$. If they are equal.

$$\theta_o(t) = \frac{57.3 \cdot n}{57.3 \cdot n} \cdot \frac{\tau_{do}}{J'K_e}$$

(36)

where $\tau_{do}$ is the disturbance torque in oz-in., and $\theta_o$ is in degrees.

As might be expected, the steady state error is not a function of the changing inertia ($J$), but with the $J'$ multiplier the error is reduced by the resultant gain, $J'K_e$. 

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5. Feedback Gain Values and Resultant Response

The purpose of this section is to insert gain values in the various transfer functions and steady state error expressions to indicate realistic output error values. The gains used are the Stanford values but will be converted to eliminate the $\Delta t$ (jiffy)* element. This is necessary to use them in the transfer functions.

First, we will look at the characteristic equation.

$$s^3 + K_v s^2 + K_e s + K_I = 0$$  \hspace{1cm} (37)

If $K_I = 0$, we have

$$s^2 + K_v s + K_e = 0$$  \hspace{1cm} (38)

which is a second order system and can be expressed as

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = 0$$  \hspace{1cm} (39)

in which

$\zeta =$ the system damping ratio

$\omega_n =$ the system undamped natural frequency (rad/sec)

Common practice is to select $\zeta = 0.7$ and $\omega_n$ as high as practical to increase the response or tightness of the servo loop. Stanford has selected $\zeta = 1$ which is reasonable since the overshoot for a step input is then reduced from 7% ($\zeta = 0.7$) to zero. Doing this we have

$$2\omega_n = K_v$$

$$\omega_n^2 = K_e$$

*The term "jiffy" refers to the 1/60 second sampling interval used at Stanford.
or
\[ K_e = \left( \frac{K_v}{2} \right)^2 \]  

(40)

Now, selection of \( K_e \) or \( K_v \) determines \( \omega_n \). Remember from Fig. 6 that the response has reached its final value (practically speaking) at \( t = 7/\omega_n = 7/\sqrt{K_e} \).

This is all very nice, but as shown above the characteristic with \( K_1 \neq 0 \) is third order. Fortunately, (depending on the value of \( K_1 \)) the system can still be considered second order to get a measure of the system transient response. This will be elaborated upon further.

The gains used by Stanford for joint 1 are

\[ K_e = 0.038 \]
\[ K_v = 0.39 \]
\[ K_I = 0.005 \]  

(41)

These have the time element of 1/60 sec in them, therefore to eliminate this,

\[ K_v = 0.39 \times 60 = 24.4 \]
\[ K_e = 0.038 \times 60^2 = 132 \]  

(42)
\[ K_I = 0.005 \times 60^3 = 1080 \]

Substituting these values in the transfer function for a step displacement from Eq. 19

\[ \frac{\theta_o(s)}{\Delta \theta_i(s)} = \frac{132(s + 8)}{s^3 + 24s^2 + 132s + 1080} \]  

(43)
factoring (for Stanford gains)

\[
\frac{\theta_o(s)}{\Delta \theta_i(s)} = \frac{132(s + 8)}{(s + 20.2)(s^2 + 3.9s + 52)} \quad \begin{cases} 
K_I = 1080 \\
\zeta = 0.27 \\
\omega_n = 7.2 
\end{cases}
\] (44)

If \(K_I\) is doubled

\[
\frac{\theta_o(s)}{\Delta \theta_i(s)} = \frac{132(s + 16.3)}{(s + 22.4)(s^2 + 1.6s + 96)} \quad \begin{cases} 
K_I = 2160 \\
\zeta = 0.08 \\
\omega_n = 9.8 
\end{cases}
\] (45)

for \(K_I = 0\)

\[
\frac{\theta_o(s)}{\Delta \theta_i(s)} = \frac{K_e}{s^2 + 24.4s + 132} \quad \begin{cases} 
K_I = 0 \\
\zeta = 1.0 \\
\omega_n = 11.5 
\end{cases}
\] (46)

As \(K_I\) increases from 0 to 1080 the transient response is altered very little (although the steady state error will now go to zero over a longer time than the initial transient). At \(K_I = 1080\) the zero and pole added are not too close together, the \(\zeta\) is quite low, but the pole has a stabilizing effect to cancel the destabilizing effect of the zero and the decreased \(\zeta\). The result is a response similar to when \(K_I = 0\). But as \(K_I\) approaches 2160, the pole and zero approach each other and can be cancelled leaving just a second order system with very low damping. As \(K_I\) increases further the system becomes marginally stable, i.e., \(\zeta = 0\). Thus \(K_I\) cannot be increased much beyond 1000.

For stability purposes \(K_I\) has to be controlled as above, but to eliminate rate and position null errors and to reduce the output from a torque disturbance
to zero as rapidly as possible, $K_I$ should be large as feasible. The restriction for at least marginal stability is that

$$K_I \geq K_v K_e$$  \hspace{1cm} (47)

This is derived from the characteristic equation in Eq. (19). Let $K_I = K_v K_e$ in

$$s^3 + K_v s^2 + K_e s + K_I = 0$$

which then factors into

$$(s + K_v) (s^2 + K_e) = 0$$

This indicates that the two complex roots are on the imaginary axis of the complex $s$ plane, therefore the system is marginally stable.

If $K_v$ and $K_e$ are selected on the basis of considering only the second order response for $\zeta = 1$ (Eq. 40), then

$$K_I \leq \frac{K^3}{4}$$  \hspace{1cm} (48)

The above restriction on $K_I$ applies as well to a torque disturbance since the characteristic equation (Eq. 34) is the same. It is of some interest to consider the steady state error when $K_I = 0$.

$$\theta_o(t) = \frac{57.3 n^2 \tau_{do}}{\tau^T J K_e}$$  \hspace{1cm} (49)

In all the previous discussion the value of $J'$ did not matter since it cancels out if all gains and operations are as shown in Fig. 2. Here we have assumed that $J'$ not $J_o'$ is used as the gain multiplier, otherwise the $n$ factors would be different. Consequently, the above error is a function of $J'$, the computed inertia referred to the motor. This is of course $n^2 J_o$.

A value of $J_o'$ for the first link from the Stanford arm is derived as follows (the prime denoting computed is dropped).

Total inertia: $J_o = 698,000$ oz-in. jiffy

Motor inertia: $J_{Mo} = 285,000$ oz-in. jiffy

Load inertia: $J_{Lo} = 413,000$ oz-in. jiffy

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To eliminate the jiffy time element requires dividing by 60

\[ J_{Lo} = 115 \text{ oz-in.-sec}^2 \]

\[ J_{Mo} = 79 \text{ oz-in.-sec}^2 \]

The total inertia referred to the motor is

\[ J = n^2 (J_{Mo} + J_{Lo}) \]  \hspace{1cm} (50)

\[ = \frac{1}{(120)^2} (79 + 115) \]

\[ = 0.005 + 0.008 = 0.013 \text{ oz-in. sec}^2 \]

The value of n for the JPL arm (link 1) is 100 and the inertia values will be larger.

Substituting values in (49)

\[ \theta_o(t) = \frac{57.3 \tau_{do}}{(120)^2 (0.013)(132)} = 0.002 \tau_{do} \]  \hspace{1cm} (51)

If a force of 16 oz is applied with a lever arm of 30 inches, the resulting output angular error is 0.96 degrees. Thus if this magnitude of error cannot be tolerated, \( K_e \) would have to be increased, keeping in mind that \( J^i \) is a variable. The other alternative is to introduce the integral gain, \( K_I \) which would reduce the error to zero.

III. CONVENTIONAL POSITION SERVO

The positional servo described here uses the same sensors or transducers as the computed torque system but feeds this information directly to a summing amplifier which drives the motor, rather than to the computer.
Position, rate and integral feedback are used. A linear analysis is used which ignores the nonlinearity of the motor and gear train. A further analysis using nonlinear techniques is required to determine if and what type of compensation is needed to get the required performance despite these nonlinearities.

1. **Transfer Function Derivation**

   The components in a control loop for one manipulator link are shown in Fig. 9 and the form for analysis in Fig. 10.

   The forward gain function is

   \[ G(s) = \frac{nK_a(s + K_i)}{s^2 \left( K_B + \frac{R_M}{K_T}s \right)} \]  

   (52)

   The feedback gain function is

   \[ H(s) = \frac{K_v}{n} s + K_e \]  

   (53)

   ![Fig. 9. Physical Configuration of a Control Loop](image-url)
Hence, the transfer function is

\[
\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_c G(s)}{1 + G(s) H(s)} = \frac{nK_a K_e (s + K_I)}{R_M J s^3 + (K_B + K_a K_v) s^2 + K_a (nK_c + K_v K_I) s + nK_a K_c K_I}
\]  

(54)

2. Stability Criterion

If the gains are selected as in the computed torque system, then \( K_i \) is first assumed to be zero, reducing the system to second order.
\[
\begin{align*}
\theta_o(s) & = \frac{nK_aKe}{J} \frac{R_M}{K_T} \\
\theta_i(s) & = \frac{nK_aKe}{J} \frac{R_M}{K_T} \left( K_B + K_aK_v \right) s + \frac{nK_aKe}{J} \frac{R_M}{K_T} 
\end{align*}
\] (55)

where

\[
\omega_n = \left( nK_aKe \right) \frac{R_M}{K_T} \right)^{1/2} 
\] (56)

\[
\zeta = \frac{K_B + K_aK_v}{2 \left( nK_aKe \right) \frac{R_M}{K_T} \right)^{1/2}} 
\] (57)

selecting \( \zeta = 1 \) and solving for \( K_a \) and \( K_v \)

\[
K_a = \frac{-K_BK_v + 2nK_e \frac{R_M}{K_T} \pm 2 \sqrt{nK_e \frac{R_M}{K_T} \left( nK_e \frac{R_M}{K_T} - K_BK_v \right)}}{K_v^2} 
\] (58)

\[
K_v = \frac{-K_B + 2 \sqrt{nK_aKe \frac{R_M}{K_T}}}{K_a} 
\] (59)

In the above equations, all parameters except \( K_e \) are known, and a maximum value for this gain will be developed in Section 5.

From Eqs. 27, 47 and 54 the stability criterion with \( K_I \neq 0 \) is

\[
\frac{nK_aKeK_I}{R_M \frac{R_M}{K_T}} \leq \frac{K_a \left( K_B + K_aK_v \right)\left( nK_e + K_vK_I \right)}{\left( nK_e \frac{R_M}{K_T} \right)^2} 
\] (60)
or

\[ K_I \leq \frac{n K_e (K_B + K_a K_v)}{n J \frac{R_M}{K_T} K_e - K_v (K_B + K_a K_v)} \]  

(61)

Generally the second denominator term is very small and

\[ K_I \leq \frac{K_B + K_a K_v}{J \frac{R_M}{K_T}} \]  

(62)

This criterion only specifies a maximum \( K_I \) for conditional stability. Of course \( K_a \) and \( K_v \) did not have to be computed from Eqs. 58 and 59 for \( \zeta = 1 \) but that is a preferable way. The point to consider is the proper value of \( K_I \ll K_{I_{\text{max}}} \) to produce a transient response similar to that expected from the second order equation.

3. **Response to Input Forcing Functions**

The response to an input from the planning program, unlike the computed torque case, depends entirely on the feedback gains. The input is \( \theta_i(t) \) and the ability of \( \theta_o(t) \) to follow depends on the amplification of the resultant error signal which then drives the motor. A typical input function is shown in Fig. 11. As indicated the JPL requirement is for a slower response, at least for the present.

A measure of the error in \( \theta_o(t) \) can be obtained by approximating the input function with a sinusoid as shown, and then finding the frequency response to this input frequency, \( \omega \).

Substituting \( j\omega \) for \( s \) in Eq. (54).

\[ \frac{\theta_o(j\omega)}{\theta_i(j\omega)} = \frac{n K_a K_e (j\omega + K_I)}{n K_a K e K_I - (K_B + K_a K_v) \omega^2 + j} \]  

(63)

\[ \left[ K_a (n K_e + K_v K_I) \omega - \frac{R_M}{K_T} \omega^3 \right] \]

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The magnitude of this function is

$$\left| \frac{\theta_j(j\omega)}{\theta_i(j\omega)} \right|^2 = \frac{nK_a K e^{(\omega^2 + K_i^2)}}{\left[ nK_a K e_1 - (K_B + K_a K_v)\omega^2 \right]^2 + \left[ K_a (nK_e + K_v K_i)\omega - \frac{R}{K_T} J\omega^3 \right]^2}$$

(64)

For \( f \leq 1 \) cps and for anticipated values of gains, this function is so close to unity that it can be considered as such, and only the phase angle, \( \phi \) need be considered.

$$\phi = \tan^{-1}(\omega/K_i) - \tan^{-1}\left( \frac{K_a (nK_e + K_v K_i)\omega - \frac{R}{K_T} J\omega^3}{nK_a K e_1 - (K_e + K_a K_v)\omega^2} \right)$$

(65)
For \( f = 0.2 \) \((\omega = 1.356)\), \( K_v = 0.001\), \( K_a = 200\), \( K_B = 0.05\), \( J = 0.013\), \( R_M/K_T = 0.123\), which are nominal values, this reduces to

\[
\phi = \tan^{-1}(\omega/K_v) - \tan^{-1}\left[ \frac{\omega}{K_v} \left( \frac{K_v K_I}{nK_e} \right) \right]
\]

for \( K^2_I \gg \omega^2 \) \((K_I \approx 10)\)

\[
\phi = \tan^{-1}\left( \frac{-\omega K_v K_I}{nK_e (K_I^2 + \omega^2) + \omega^2 K_v K_I} \right)
\]

For \( n = 0.01\), \( K_v = 7\), \( K_I = 10\), \( K_v = 0.001\), \( \omega = 1.356\), which are nominal values,

\[
\phi = -\tan^{-1}\left( \frac{\omega K_v}{nK_e} \right) \quad \text{(rad)} \tag{66}
\]

This is the phase lag in \( \theta_o(t) \) for a sinusoidal input, \( \theta_i(t) = \theta_p \sin \omega t \). The error in \( \theta_o(t) \) can be expressed as

\[
\Delta \theta_o(t) = -\theta_p \sin \omega t + \theta_p \sin(\omega t + \phi), \quad \phi \text{ is neg.}
\]

\[
= \theta_p \left[ \sin \omega t (\cos \phi - 1) + \cos \omega t \sin \phi \right] \tag{67}
\]

For small \( \phi \)

\[
\Delta \theta_o(t) = \theta_p \phi \cos \omega t \tag{68}
\]
where

\( \theta_p \) is the peak angular sinusoidal rotation

\( \phi \) is the phase lag in radians from Eq. (66).

Another way of determining the response error is to assume a ramp input for \( \theta_1(t) \).

\[
\dot{\theta}_1(t) = kt
\]

\[
\theta_1(s) = \frac{k}{s} \tag{69}
\]

From Eq. (55) for \( \zeta = 1 \), we have

\[
\frac{\dot{\theta}_o(s)}{\theta_1(s)} = \frac{1}{\frac{s^2 + \frac{2}{\omega_n} s + 1}{s}} = \frac{\dot{\theta}_o(s)}{s \theta_1(s)} \tag{70}
\]

Therefore

\[
\dot{\theta}_o(s) = \frac{k}{s \left( \frac{s}{\omega_n} + 1 \right)^2} \tag{71}
\]

\[
\dot{\theta}_o(t) = k \left[ 1 - (1 + \omega_n t) e^{-\omega_n t} \right] \tag{72}
\]
The output error is

\[ \Delta \theta_o(t) = \int_0^t \left[ \dot{\theta}_i(t) - \dot{\theta}_o(t) \right] dt \]

\[
= k \left[ \frac{2}{\omega_n} \left( 1 - e^{-\omega_n t} \right) - t e^{-\omega_n t} \right] \tag{73}
\]

where \( \omega_n \) can be computed from the system parameters, and \( k \) is the ramp slope in degrees or radians per second.

4. **Transfer Function for a Disturbance Torque**

An output disturbance torque, \( \tau_{do} \) can be transferred to the motor input by multiplying it by the gear ratio.

\[ \tau_{dM} = n \tau_{do} \tag{74} \]

Therefore, applying \( \tau_{dM} \) at the motor input

\[ G(s) = \frac{n}{s \left( K_B + \frac{R_M}{K_T} s \right)} \tag{75} \]

\[ H(s) = K_a \left( K_e + \frac{K_v}{n} s + \frac{K_I}{s} \right) \tag{76} \]

\[ \frac{\theta_o(s)}{\tau_{do}(s)} = \frac{\frac{R_M}{K_T} n G(s)}{1 + G(s)H(s)} \]

\[ = \frac{\frac{R_M}{K_T} n^2 s}{\frac{R_M}{K_T} Js^3 + (K_B + K_a K_v)s^2 + nK_a K_e s + nK_a K_I} \tag{77} \]
For a step disturbance torque, the output error is

\[
\theta_o(s) = \frac{\frac{R_M}{K_T} \tau_{do} n^2}{\frac{R_M}{K_T} js^3 + (K_B + K_a v) s^2 + nK_a e s + nK_a I} \tag{78}
\]

The steady state output error is

\[
\theta_o(t) = s \theta_o(s) = 0 \quad \text{as } t \to \infty, \quad s \to 0
\]

5. **Gain Values and Resultant Response**

The determination of gains for this system is more complicated than for the computed torque system since more variables are involved and physical limitations come into play. Selecting gains based on a second order response as mentioned before means arbitrarily letting the damping ratio, \( \zeta = 1 \), and selecting an arbitrary undamped natural frequency, \( \omega_n \). The latter can be done by considering the response desired as shown in Fig. 6.

The equations for the gains (\( \zeta = 1 \)) are

\[
K_v = \frac{nK_e}{\omega_n} \left( 2 - \frac{K_B}{\frac{R_M}{K_T} \omega_n} \right) \tag{79}
\]

\[
K_a = J \frac{R_M}{K_T} \frac{\omega_n^2}{nK_e} \tag{80}
\]

Nominal values of the above parameters are

\[
J = 0.013 \quad \text{(oz-in. sec)}
\]

\[
\frac{R_M}{K_T} = 0.123 \quad \text{(v/oz-in.)}
\]

\[
K_B = 0.05 \quad \text{(v/rad/sec)}
\]

\[
n = 0.01 \quad \left( \frac{N_1}{N_2} \right) \text{ gear ratio}
\]

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The value of \( K_e \) is dependent on the angular range which is ±180° and the supply voltage = 24. Thus, the pot sensitivity is ±12/\( \pi \) (volts/rad). Adding some amplification with the same D.C. voltage supply can increase this to ±24/\( \pi \). A value of \( K_e = 7 \) will be used here. Inserting all these values in (79) and (80).

\[
K_v = \frac{0.07}{\omega_n} \left( 2 - \frac{31}{\omega_n} \right) \tag{82}
\]

\[
K_a = 0.023 \omega_n^2
\]

Selecting \( \omega_n = 100 \) for a very fast step response (Fig. 6).

\[
K_v = 0.0012 \tag{83}
\]

\[
K_a = 230
\]

Now \( K_I \) has to be selected

\[
K_I < \frac{K_B + K_a K_v}{R_M} = 204
\]

\[
J \frac{R_M}{K_T}
\]

Obviously, to get a response approaching \( \xi = 1 \), \( K_I \) has to be considerably less than this maximum value. Selecting \( K_I = 10 \), the system transfer function is:

\[
\frac{\theta_o(s)}{\theta_i(s)} = \frac{10,062(s + 10)}{s^3 + 203.7s^2 + 11,787s + 100,625}
\]

Factoring

\[
\frac{\theta_o(s)}{\theta_i(s)} = \frac{10,062(s + 10)}{(s + 10.2)(s^2 + 193.5s + 9813)} \tag{85}
\]
Obviously, the pole and zero essentially cancel and the response is second order with

\[ \omega_n = 99 \]

\[ \zeta = 0.98 \]  \hspace{1cm} (86)

The high values of \( K_v \) and \( K_a \) and the relatively low \( K_I \) has left the response the same as without \( K_I \). This is fine except that high amplifier gains create a noise problem. The transient response is much better than needed, but to control the error for a very rapidly changing input forcing function such as shown in Fig. 11, requires a high gain. From the sinusoidal input error function (Eq. 68) and Eq. 66,

\[
\Delta \theta_o(t) = \theta_p \phi \cos \omega t
\]

\[ = \theta_p \tan^{-1} \left( \frac{\omega K_v}{nK_e} \right) \cos \omega t \]  \hspace{1cm} (87)

Substituting the previously given parameter values

\[
\Delta \theta_o(t) = \theta_p \tan^{-1} (0.017 \omega) \cos \omega t
\]  \hspace{1cm} (88)

For \( \theta_p = 50 \) deg, \( \omega = 1.356 \)

\[ \Delta \theta_{op} = 50(0.023) = 1.15 \text{ degrees} \]  \hspace{1cm} (89)

For \( \omega = 6.28 \) which is a nominal input for the computed torque system at Stanford, this error increases to 5 degrees. Thus, it is obvious that, if the system is to compete with the Stanford computed torque system, the gains would have to be increased which is feasible.
A check on the above response error is to use error Eq. (73) for a ramp input. For the sinusoid used, the maximum $\dot{\theta}_0$ is about 60 deg/sec. The natural frequency from Eq. (86) is 100 rad/sec.

$$\Delta\theta_0(t) = k \left[ \frac{2}{\omega_n} \left( 1 - e^{-\omega_n t} \right) - t \cdot e^{-\omega_n t} \right]$$

$$= 60 \left[ \frac{2}{100} \left( 1 - e^{-100t} \right) - t \cdot e^{-100t} \right]$$

(90)

Thus, for $t > 0.1$ seconds, the error equation reduces to

$$\Delta\theta_0 = \frac{2k}{\omega_n} = 1.2 \text{ degrees}$$

(91)

which compares well with the previous value of 1.15 degrees from (89).

6. **Effect of Changing System Inertia**

The inertia changes as the manipulator moves links which extend or retract the arm. This is of no consequence for the computed torque system since it is computed and the computed value is used to eliminate this parameter from the system response. This is essentially computing a variable feedback gain. For the conventional system the natural frequency and damping are

$$\omega_n = \left( \frac{nK_a K_e}{J} \frac{R_M}{K_T} \right)^{1/2}$$

(92)

$$\zeta = \frac{K_B + K_a K_v}{2 \left( \frac{R_M}{K_T} \right)^{1/2}}$$
As discussed, this is the case if the gains are as shown earlier which essentially reduces the system to second order.

The inertias of the JPL arm are not yet known. They will be larger than the Stanford values which have been used in this report, but using Stanford values is most likely sufficient for this discussion. Supplied data indicates that the inertia value can about double for link No. 1. This means $\omega_n$ and $\xi$ are decreased by $\sqrt{2}$ which would not change the transient performance or decrease the stability any significant amount. This is even less significant for other links, except when a large inertia object is being manipulated.

IV. COMPARISON OF THE TWO DRIVE METHODS

The following paragraphs are a summary of the more important characteristics of the computed torque (CT) and conventional position servo (CS) systems. However, no final conclusion is made as to which system should be used because this depends on the particular application and on the results of a sampled data analysis.

1) Input Forcing Functions

CT: There is theoretically no output position error if the torque is computed correctly to produce the response computed in the planning program. This assumes the inertias are computed correctly. The advantage here is that very fast motions can be accomplished without depending on a feedback error.

CS: There is always a transient error which subsides to zero as the motion rate decreases to zero. This error is minimized by selecting the largest possible forward and feedback gains.

2) Disturbance Torque

CT: An unknown external force on the manipulator is a disturbance torque. Gravity torques are computed and compensated along with reaction torques from rotations in other non-perpendicular links. Lifting an object produces an unknown torque which results in an angular position error. The integral gain term then zeros this position error due to any unknown torque. The resulting output of the integral gain term indicates the torque value and thus the weight of the object.
CS: The feedback error signal from the integrating amplifier must compensate for all disturbance torques including gravity and reactions from other links. This is not too serious a problem with high enough gains. The reaction torques are only about 5% of the applied torque, and the gravity torques would not pose any problem since the output error is driven to zero by the integral term. Very fast motions with a large changing gravity and reaction torque would produce an error dependent on the gain values.

3) Gains

CT: The feedback gains (link 1) are such to produce a damping ratio \( \xi \) of 1 and an undamped natural frequency of about 10. This is not a very tight loop, but the only reason for possibly increasing \( \omega_n \) is for better response to disturbance torques, although the integral term eventually reduces the error to zero. There appears to be no reason for not making the gains for all the links the same or certainly not much different. These gains and their use take place in the computer, therefore can be easily changed with no effect on the hardware.

CS: The gains are considerably higher to produce an \( \omega_n = 100 \) rather than 10. This does not appear to be a problem as far as the hardware is concerned. The gains here are constant but the inertia changes which then changes \( \omega_n \) and \( \xi \). This maximum change factor is only about \( 1/\sqrt{2} \) (for the manipulator inertia alone). Changing gains means a physical adjustment on the amplifiers. A set of gains would have to be derived for each link since the inertias and motor torque constants are different.

4) Nonlinearities

CT: The nonlinearities in the drive motor, mainly the coulomb friction torque, and the gear train nonlinearity effects are all handled easily in the computed torque program. This is a distinct advantage of this method, assuming the nonlinearities can be modeled accurately. This does, of course, involve precise lab testing.
CS: The linear analysis presented in this memo does not consider these nonlinearities. To be sure of proper performance a nonlinear analysis is required and compensation introduced. This can probably be done with some limitation on performance.

5) Sampling and Delay

CT: The feedback position and rate are sampled at some rate depending on the time span of the fastest input function from the planning program. The input to the drive motor amplifier is also sampled at the same rate. The drive torque is computed which takes a finite time. Thus, the system is sampled data with a delay. If a longer time span is used for the JPL system (5—10 seconds for a large angular movement) the sampled time is much greater than the computation delay, and the latter can be ignored. But a sampled data analysis with one period delay in the feedback data is probably needed.

CS: Since the feedback takes place external to the computer, the system is continuous except for sampling of the input driving function from the planning program. Some consideration to this sampling rate is all that is required beyond the linear analysis, complemented by the nonlinear analysis.

6) Complexity

CT: Obviously, the large program for computing the drive torque is the negative aspect of this method. But if the effort of developing the program, debugging it, and the computation time and computer capacity are acceptable, considering the increased performance over a conventional servo, the system appears very attractive. Two A/D converters per link are required to get the position and rate information to the computer. A D/A converter provides the motor drive signal. Another factor is determination of very accurate inertia data and motor torque-voltage characteristics for essentially modeling the manipulator in the computed torque program.
CS: No torque program is needed, therefore, only approximate information on inertias is needed. Nonlinearities need to be known approximately so that the compensation and performance can be determined. The two A/D converters for feedback are not required. One or two additional amplifiers are needed per link.

7) Data Transfer

CT: Three data transfers per link are required. The rate of this transfer depends on how fast the arm is required to move and the desired accuracy. For a 5 second full angular movement of 180 degrees, a sampling rate of 20 to 30 per second is probably sufficient, but a detailed study is desirable using the results of this analysis and some sampled data analysis. At each sampling 3 words per link are transferred.

CS: Only the input function is transferred every sampling interval. This can probably be at the rate mentioned above, although here it is not so critical because the servo loop does all the control rather than the input.

8) Additional Analysis Required

CT: A sampled data analysis with one sampling interval delay plus a computation delay, although the computation delay is probably negligible.

CS: A Nonlinear analysis considering the motor friction torque and gear train effects.

REFERENCE

APPENDIX A

USE OF A CURRENT SOURCE FOR THE MOTOR DRIVE

The analysis in the body of this report assumes a voltage gain amplifier prior to the drive motor for both the computed torque system and the conventional position servo. A disadvantage of the voltage source is that a change in motor input resistance will affect system performance, whereas with a current source, the input current and thus torque is not affected. The effect of this resistance change on system performance is determined by examining the relevant equations in this report.

The first concern is with stability. This is determined by investigating the characteristic equation which is derived from (2) by assuming $J = J^1$ and that the torque constant ($K_T^1$) remains constant as the motor resistance ($R_M^1$) varies from the nominal value ($R_M$).

\[
\frac{R_M}{R_M^1} s^3 + K_v s^2 + K_e s + K_I = 0 \tag{A-1}
\]

A ratio change in $R_M/R_M^1$ from 1 to 1.2 which is the maximum anticipated change in $R_M$ results in

\[
1.2 s^3 + 24 s^2 + 132 s + 1080 = 0 \tag{A-2}
\]

or

\[
s^3 + 20 s^2 + 110 s + 900 = 0
\]

$K_I \leq 2200$.

Whereas with $R_M/R_M^1 = 1.0$

$K_I \leq 3220$. 
Factoring (A-1)

\[(s + 16.6)(s^2 + 3.4 s + 54.6) = 0\]  

(A-3)

The two complex roots have \(\omega_n = 7.3\) and \(\zeta = 0.23\) which compares to \(\omega_n = 7.2\) and \(\zeta = 0.27\) for no change in \(R_M\) (i.e., \(R_M/R_M^1 = 1\)). Thus, the change in the system stability due to a 20\% change in motor resistance, (and using a voltage drive) is very small. This effect can be reduced further by using a value of \(R_M\) midway between the 20\% variation. This actually means altering the voltage-torque characteristic slightly, assuming it is taken at the low end of the resistance change.

The effect of a change in \(R_M\) is much the same for a conventional system whose characteristic equation is

\[
\frac{R_M}{K_T} s^3 + (K_B + K_a K_v) s^2 + K_a (nK_e + K_v K_I) s + nK_a K_v K_I = 0
\]  

(A-4)

For the gains used previously and introducing \(R^1_M\) as the nominal motor resistance this becomes

\[
\frac{R_M}{R^1_M} s^3 + 203.7 s^2 + 11,787 s + 100,625 = 0
\]  

(A-5)

As indicated in (85) and (86) the gains here allow the system to be considered second order (to get the complex roots).

\[
\frac{R_M}{R^1_M} s^2 + 203.7 s + 11,787 = 0
\]  

(A-6)

Introducing \(\zeta^1\) and \(\omega^1_n\) as the nominal damping and natural frequency (\(R_M/R^1_M = 1\))

\[
\frac{R_M}{R^1_M} s^2 + 2\zeta^1 \omega^1_n s + \omega^1_n^2 = 0
\]  

(A-7)
Thus, the altered values, $\zeta$ and $\omega_n$ are

$$\omega_n = \left(\frac{R_M^1}{R_M^1}\right)^{1/2} \omega_n$$

$$\zeta = \left(\frac{R_M^1}{R_M^1}\right)^{1/2} \zeta^1$$

(A-8)

The slight increase in $\omega_n$ for the computed torque system from (A-3), rather than a decrease, is due to the lower gains which result in a system which cannot be considered second order as above. But if $R_M$ were increased by a larger factor, $\omega_n$ would decrease even for that case.

With stability evidently not a serious problem the next area to investigate is transient response and steady state errors.

**Driving Input Response**

The transfer function is

$$\frac{\theta_o(s)}{\theta_1(s)} = \frac{s^3 + K_v s^2 + K_e s + K_I}{R_M^1 s^3 + K_v s^2 + K_e s + K_I}$$

(A-9)

No attempt is made here to investigate transient response errors, since the inversion of Eq. (A-9) is very difficult for any analytic input function. Furthermore, the ratio $R_M / R_M^1$ is $< 1.2$ which is very likely negligible.

**Disturbance Torque**

The transfer function is

$$\frac{\theta_e(s)}{r_{do}(s)} = \frac{57.3 n^2 s}{J s^3 + J^1 R_M^1 K_v s^2 + J^1 R_M^1 K_e s + J^1 R_M^1 K_I}$$

(A-10)
\[ \theta_0(t) = 0 \]
\[ t \to \infty \]

But for \( K_I = 0 \)

\[ \theta_0(t) = 57.3 \frac{n^2 t_{do}}{J K_e} \frac{R_M}{R_M'} \quad (A-11) \]

With increased \( R_M' \), the steady state error increases proportionately.

**Position Feedback Null Error**

The transient response is altered slightly but the steady state error is unaltered from Eqs. (21) and (23) regardless of the value of \( K_I \) and change in \( R_M' \).

**Rate Feedback Error**

The transient response is altered slightly but the steady state error is unaltered from Eqs. (28) and (30) regardless of change in \( R_M' \).

If a current drive is used a slight change in the system occurs. The loop transfer function, \( G(\theta) \) in figure 3 is eliminated because the input is current, not voltage. The voltage-torque characteristic represented by \( R_M'/K_T \) now becomes \( 1/K_T \) which is oz-in/amp. This means the characteristic, obtained from lab tests, relates current rather than voltage to torque. Consequently, the motor speed does not have to be monitored during the test because the back e.m.f. does not have to be computed and subtracted from the applied voltage to get the motor voltage \( (V_M) \).
APPENDIX B

EFFECT OF MANIPULATING A LARGE INERTIA OBJECT

It has been shown that a change in system inertia has no effect on the system response (for the CT system), because this change in inertia is continuously computed and is used in the feedback loops to multiply the gains. This is a very desirable situation, but Stanford experience has shown that, if this gain becomes very high, the noise amplification becomes a problem and the system tends to become unstable. Joints (links) 4 and 5 are mostly affected because they experience a very large inertia change when a large inertia object is being manipulated. This assumes that the inertia of the object is known or is determined by the system prior to manipulation.

We are considering two problems here: (1) the noise problem with high inertia and therefore high gain, (2) the change in system response with an incorrect inertia due to an unknown object inertia.

The system characteristic equation is

\[ s^2 + \frac{J'}{J} K_v s^2 + \frac{J'}{J} K_e s + \frac{J'}{J} K_I = 0 \] (B-2)

Therefore, multiplying the gains \( K_v, K_e \) and \( K_I \) by the computed inertia, \( J' \) assures a system with uniform response. As the inertia changes the stability margin should theoretically remain the same. But if it seems desirable to reduce the response (and keep the damping constant at 1), a number of ways are available. One method is to not multiply \( K_e \) as before by \( J' \) and to divide \( K_I \) by \( J' \) rather than multiplying. Assuming that \( J' = J \) this results in

\[ s^3 + \frac{J' K_v}{J} s^2 + \frac{K_e}{J} s + \frac{K_I}{JJ'} = 0 \] (B-3)
Assuming a second order system and $\xi = 1$,

$$\omega_n = \left(\frac{K_e}{J}\right)^{1/2} \quad \text{(B-4)}$$

Thus, as $J$ increases, $\omega_n$ will decrease. But first, a desired or nominal frequency $\omega_n^\prime$ has to be selected, which corresponds to some nominal inertia, $J^\prime$. The gains can then be computed (for $\xi = 1$)

$$K_v^\prime = 2\omega_n^\prime \quad \text{(B-5)}$$

and

$$K_e^\prime = \frac{J K_v^\prime}{4} \quad \text{(B-6)}$$

Now, as $J$ changes, $K_e$ is constant but $K_v$ has to be recomputed each cycle to keep $\xi = 1$.

$$K_v = 2\left(\frac{K_e^\prime}{J}\right)^{1/2} \quad \text{(B-7)}$$

The system natural frequency is

$$\omega_n = \left(\frac{J}{J^\prime}\right)^{1/2} \omega_n^\prime \quad \text{(B-8)}$$

The result is a decrease in $K_v$ and the ratio of the rate error loop gain to nominal is

$$\frac{J' K_v}{J'' K_v} = \left(\frac{J'}{J''}\right)^{1/2} \quad \text{(B-9)}$$
The result of this gain procedure is that the above gain increases by the square root of the inertia change ratio rather than directly as before. The response is slower but the noise is not amplified as much. It is quite possible that this method which involves recomputing the gain each cycle may not be necessary with tachometers on links 4 and 5 because the noise is much less than if rate data is computed from position feedback data.

The above makes no mention of $K_T$, but its value is dependent on the other gains. From (47) the maximum value for $K_I$ is

$$K_I \leq K_v K_e$$

In this case it is

$$\frac{K_I}{J^2} \leq \frac{K_v K_e}{J} \quad \text{or} \quad K_I \leq J K_v K_e$$

Therefore if $J$ increases from the nominal $J''$ its value can theoretically increase. Its nominal value at $J''$ is some fraction of (B-10).

The second problem being discussed here is the possibility of not knowing the inertia of an object to be manipulated, and its effect on system performance.

For the gain method used in the body of this report ($\omega_n$ and $\zeta$ independent of a change in $J$), the transfer function, for an input function from the planning program, is

$$\frac{\theta_0(s)}{\theta_2(s)} = \frac{s^3 + K_v s^2 + K_e s + K_I}{JR_M s^3 + K_v s^2 + K_e s + K_I}$$

This is to show that an unknown change in motor resistance ($R_M$) can be lumped with a change in inertia ($J$). But the much greater change in $J$ when an object is manipulated makes the small 20% change in $R_M$ negligible. Therefore only $J$ will be used in this discussion.
The natural frequency and damping, assuming the gains are such that the system can be considered second order, are

\[ \omega_n = \left( \frac{J}{J_e} K_e \right)^{1/2} = \left( \frac{J'}{J} \right)^{1/2} \omega_n' \]  
\[ \zeta = \frac{K_v}{2} \left( \frac{J}{J_e} \right)^{1/2} = \left( \frac{J}{J} \right)^{1/2} \zeta' \]  

where the primed values are the computed and assumed values, whereas the unprimed are the actual values of the parameters.

For the gain computation method discussed previously in this appendix the parameters are

\[ \omega_n = (K_e/J)^{1/2} = \omega_n' \]  
\[ \zeta = \frac{K_v}{2} \frac{J}{(JK_e)^{1/2}} = \frac{J'}{J} \zeta' \]  

As shown in (B-13) the damping ratio decreases directly with increase in \( J \). Thus, this method of computing gains may accomplish its purpose if \( J \) is always known, but if not, the first method as shown in (B-12) does not result in as large a decrease in \( \zeta \). Obviously, other methods are easily incorporated, but they will not be discussed here.
APPENDIX C  
TEST FOR MAXIMUM MOTOR VOLTAGE

The JPL procedure at present is to use only third degree polynomials to fit the desired trajectory in the planning program. The polynomials are:

\[ \theta_i(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \]
\[ \dot{\theta}_i(t) = a_1 + 2a_2 t + 3a_3 t^2 \]  
\[ \ddot{\theta}_i(t) = 2a_2 + 6a_3 t \]  

(C-1)

There is some concern that the applied motor voltage necessary to generate the above functions may exceed the allowable maximum. This appendix is a very brief discussion of one method of checking this possibility.

The applied voltage is

\[ V_a = E + i M R_M \]
\[ = K_B \dot{\theta}_M + \tau_M K_T \]  

(C-2)

Since

\[ \dot{\theta}_M = \frac{1}{n} \dot{\theta}_i, \text{ and } \tau_M = J \ddot{\theta}_M \]

\[ V_a = \frac{K_B}{n} \dot{\theta}_i + J \frac{R_M}{nK_T} \ddot{\theta}_i \]  

(C-3)

Substituting from (C-1)

\[ V_a = \frac{1}{n} \left( K_B \left( a_1 + 2a_2 t + 3a_3 t^2 \right) \right) + J \frac{R_M}{K_T} \left( 2a_2 + 6a_3 t \right) \]  

(C-4)
Taking the derivative and setting it equal to zero, and solving for \( t \) results in

\[
   t = -\frac{J}{K_B} \left( \frac{R_M}{K_T} \right) - \frac{a_2}{3a_3}
\]  

(C-5)

Now, compute \( V_a \) from (C-4) using \( t \) from (C-5). If its value is larger than a given maximum, the trajectory is undesirable because the motor torque will not be as large as required to follow the trajectory in (C-1). Of course, the error signals will eventually bring the error to zero, but the resultant deviation before this may be undesirable. In any respect, depending on the error signals for large errors defeats the purpose of the computed torque system, and the operation is then much like a conventional position servo.
LIST OF SYMBOLS

\( J_L' \) Computed inertia of load (not including motor) referred to output

\( J_L \) Actual inertia of load (not including motor) referred to output

\( J' \) Total computed inertia referred to motor shaft

\( J \) Total actual inertia referred to motor shaft

\( J_o' \) Total computed inertia referred to output

\( J_M \) Motor inertia referred to motor shaft

\( K_v \) Rate error feedback gain

\( K_e \) Position error feedback gain

\( K_I \) Integral of position error feedback gain

\( K_B \) Motor back e.m.f. constant

\( n \) Gear speed reduction ratio (motor to output)

\( R_M \) Resistance of motor input winding

\( K_T \) Motor current - torque constant

\( R_M/K_T \) Motor voltage - torque constant (characteristic)

\( G \) Program for computing the gravitational torques

\( \tau_I \) Computed torque from the integral gain feedback

\( \tau_g \) Actual gravitational torque

\( \tau_I' \) Computed input driving function torque

\( V_M \) Motor voltage for driving torque and error torque

\( V_a \) Total applied motor voltage including back e.m.f.

\( \theta_M \) Motor shaft output angle

\( \theta_i \) Input angle from planning program

\( \theta_o \) Output angle

\( s \) Laplace operator

\( E \) Motor back e.m.f.

\( i_M \) Motor input current
\( \tau_M \) Motor torque  
\( \Delta \theta_i \) Bias error in position feedback signal  
\( \Delta \dot{\theta}_M \) Bias error in rate feedback signal  
\( \tau_{do} \) Disturbance torque referred to output  
\( \tau_{dM} \) Disturbance torque referred to motor  
\( \Delta t \) Computation cycle time  
\( \zeta \) Damping ratio  
\( \omega_n \) Undamped natural frequency  
\( K_a \) Amplifier gain of conventional position servo  
\( j\omega \) Complex frequency  
\( \theta_p \) Peak angular excursion of a sinusoidal input  
\( \Delta \theta_o \) Output angular error for sinusoidal input  
\( \Delta \theta_{op} \) Peak output error for sinusoidal input
A manipulator and its control system (modeled after a Stanford design) is being developed at JPL as part of an artificial intelligence project. This development includes an analytical study of the control system software. This report presents a comparison of the computed torque method and the conventional position servo. No conclusion is made as to the preference of one system over the other, as it is dependent upon the application and the results of a sampled data analysis.
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