FURTHER COMMENTS ON THE APPLICATION OF THE METHOD
OF AVERAGING TO THE STUDY OF THE ROTATIONAL MOTIONS
OF A TRIAXIAL RIGID BODY, PART 2

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1. Introduction

In [A.R.,1971], we described some of the results which we have obtained in applying the averaging technique described in [F.R.,1971] to the variational equations which arise in treating the perturbations of the free rotational motions of a triaxial rigid body. In [A.R.,1971] we carried out the first step of the averaging procedure and derived the averaged differential equations for a set of canonical variables ($\alpha_k, \beta_k$) for the problem of a triaxial body in a precessing, elliptic orbit about an attracting center. The development was carried out to the point that the averaged differential equations [A.R., 1971,(5.31)] are in a form which can readily be integrated if it is so desired. The second step of the averaging procedure was not carried out for these canonical variables ($\alpha_k, \beta_k$) because we were planning to use the averaging technique to develop first-order secular solutions for an alternative set of noncanonical variables. In [F.R.,1972] we began our discussion of the development of these secular solutions by carrying out the first step of the averaging procedure for a convenient set of noncanonical variables. In the present report, we complete the second and final step in the development of these first-order secular solutions.

2. First-Order. Secular Solution

Equations (4.24)$^+$ may be integrated numerically with a much longer integration step time than may be used with equations (4.3). In this way, we can obtain the first-order secular solutions for the rotational motion under the influence of the gravity-gradient torque. We can also, however, integrate the averaged system (4.24) analytically with the aid of the integral (4.30). We address ourselves to this problem in the remainder of this section.

To begin with, we attempt to integrate (4.24(b)), noting that $-s_\Theta^2$ is constant. We first express $c_\psi_H$ (and hence $s_\psi_H$) in terms of $\Theta_H$ by substituting (b) and (c) of (4.25) into (4.30). We find that, to first order,

$$c_\psi_H = - \frac{1}{s_{\Theta_H}} \left( \frac{c_{||} \Theta_H^2}{2b^{||}} - \frac{a_{||} \Theta_H}{b^{||}} \Theta_H + c_h \right), \quad (2.1)$$

It follows from (2.1) that

$$s_\psi_H = \pm \frac{(A^x)^{1/2}}{s_{\Theta_H}} \left( -c_{\Theta_H}^4 + A_{3}^x \Theta_H^3 + A_{2}^x \Theta_H^2 + A_{1}^x \Theta_H + A_{0}^x \right)^{1/2}, \quad (2.2)$$

where

$$A^x = \frac{c_{||} \Theta_H^2}{4b^{||}},$$

$^+$All equations designated (4.1j), i,j, nonnegative integers, refer to equations given in [F.R., 1972].
If (2.2) is substituted into (4.24(b)), we can write

\[ \frac{d\phi_H}{\left( -c_{\theta_H}^4 + A_{\phi_H}^3 c_{\theta_H}^3 + A_{\phi_H}^2 c_{\theta_H}^2 + A_{\phi_H} c_{\theta_H} + A_{\phi_H}^0 \right)^{1/2}} = -(A^*)^{1/2} \int_t^t \dot{\theta}_H dt. \] (2.3)

In order to integrate (2.3), we first note that the bi-quadratic equation

\[ - c_{\theta_H}^4 + A_{\phi_H}^3 c_{\theta_H}^3 + A_{\phi_H}^2 c_{\theta_H}^2 + A_{\phi_H} c_{\theta_H} + A_{\phi_H}^0 = 0 \] (2.4)

has the following allowable roots:

(i) four distinct real roots,

(ii) four real roots with two identical roots,
(iii) two distinct real roots with a pair of complex roots,
(iv) two identical real roots and a pair of complex roots.

In what follows, we consider each of the four cases.

Case (i): Four distinct real roots

In this case, we can write equation (2.3) in the Jacobian normal form for real roots [see (BF250.04) and (BF250.06)]

\[ g^* \text{du}^* = \frac{d\theta_H}{\sqrt{[(a_1-c_H^*)(a_2-c_H^*)(a_3-c_H^*)(c_H^*-a_4^*)]}} = \frac{(\Lambda^*)^{1/2}}{\theta_H} \text{dn} \theta_H \frac{\text{dn} \theta_H}{\theta_H} \cdot \theta_H, \]

where

\[ g^* = \frac{2}{\sqrt{[(a_1-a_3)(a_2-a_4)]}} \]  \hspace{1cm} (2.5)

if \( a_1, a_2, a_3 \) and \( a_4 \) are the real roots of the biquadratic equation (4.4), and it is assumed that \( a_1 > a_2 > a_3 > a_4 \). The variable \( c_H^* \) is related to \( u^* \) through the equation

\[ c_H^* = \frac{A_1 + A_2 \text{sn}^2 u^*}{A_3 + A_4 \text{sn}^2 u^*}, \quad 0 \leq u^* \leq K^*. \]  \hspace{1cm} (2.6)

Reference to equations in H and Book of Elliptic Integrals for Engineers and Physicists, Byrd, P. F. and Friedman, M. D., Springer, Berlin, 1954, are prefixed by the notation BF.
Here $K^*$ is defined by the integral

$$K^* = \int_0^{\pi/2} \frac{d\xi}{(1-k^* \sin^2 \xi)^{1/2}},$$

where $k^*$ is the modulus of Jacobian elliptic functions and integrals. Explicit values of $k^*$ and $A_i$, $i=1,2,3,4$ will be given in the study of the special cases which follows.

Since the time rate of change of $c_{\theta H}$ is real, the value of $c_{\theta H}$ must either lie between $a_1$ and $a_2$ inclusive or between $a_3$ and $a_4$ inclusive. We will analyze (2.5) in the subcases which follow.

(1) $a_1 > c_{\theta H} > a_2$ : The $A_i$, $i=1,2,3,4$, have the values

$$A_1 = a_1(a_2-a_4),$$

$$A_2 = a_4(a_1-a_2),$$

$$A_3 = a_2-a_4,$$

$$A_4 = a_1-a_2,$$

and $k^{*2}$ has the value

$$k^{*2} = \frac{(a_1-a_2)(a_3-a_4)}{(a_1-a_3)(a_2-a_4)}$$

(2.8)
for this and the remaining three possibilities under Case (i).

If we integrate (2.5) with respect to \( u^* \) from 0 to \( u^* \) (i.e., integrating with respect to time \( t \) from \( t_l \) to \( t \)), we find [see (BF257.00)] that

\[
\frac{u_1^*}{u_1^*} = \pm \frac{1}{\rho^*} (A^*)^{1/2} \int_{t_l}^t \Omega (t-t_1),
\]

where \( t_1 \) is the value of \( t \) at which \( c_{\theta_H} = a_1 \).

Substituting (2.9) into equation (2.6), we obtain the first-order secular solution for \( c_{\theta_H} \) and thus the first-order secular solution for \( \theta_H \).

(2) \( a_1 > c_{\theta_H} > a_2 \) : The \( A_i \), \( i=1,2,3,4 \), have the values

\[
\begin{align*}
A_1 &= a_2(a_3 - a_1), \quad (a) \\
A_2 &= a_3(a_1 - a_2), \quad (b) \\
A_3 &= a_3 - a_1, \quad (c) \\
A_4 &= a_1 - a_2. \quad (d)
\end{align*}
\]

If we integrate (2.5) with respect to time from \( t_2 \) to \( t \), we find [see(BF256.00)] that

\[
\frac{u_2^*}{u_2^*} = \pm \frac{1}{\rho^*} (A^*)^{1/2} \int_{t_2}^t \Omega (t-t_2),
\]

where \( t_2 \) is the value of \( t \) at which \( c_{\theta_H} = a_2 \).
Substituting (2.11) into equation (2.6), we obtain the first-order secular solution for $\theta_H$.

(3) $a_3 < c_{\theta_H} < a_4$: The $A_i$, $i=1,2,3,4$, have the values

$$A_1 = a_3(a_4 - a_2), \quad (a)$$
$$A_2 = a_2(a_3 - a_4), \quad (b)$$
$$A_3 = a_4 - a_2, \quad (c)$$
$$A_4 = a_3 - a_4, \quad (d)$$

If we integrate (2.5) with respect to time $t$ from $t_3$ to $t$, we have [see (BF253.00)] that

$$u_3^* = \pm \frac{1}{g^*} (A^*)^{1/2} s_0^* \Omega(t-t_3), \quad (2.13)$$

where $t_3$ is the value of $t$ at which $c_{\theta_H} = a_3$.

Substituting (2.15) into equation (2.6), we obtain the first-order secular solution for $\theta_H$.

(4) $a_3 \leq c_{\theta_H} < a_4$: The $A_i$, $i=1,2,3,4$, have the values

$$A_1 = a_4(a_1 - a_3), \quad (a)$$
$$A_2 = a_1(a_3 - a_4), \quad (b)$$
$$A_3 = a_1 - a_3, \quad (c)$$
$$A_4 = a_3 - a_4, \quad (d)$$
If we integrate (2.5) with respect to time $t$ from $t_4$ to $t$, we find [see(BF252,00)] that

$$u_4^* = \frac{1}{\sqrt{2\pi}} (A^*)^{1/2} \sum_{\theta} \frac{n^*}{t-t_4}, \quad (2.15)$$

where $t_4$ is the value of $t$ at which $\theta = a_4$.

Substituting (2.15) into equation (2.6), we obtain the first-order secular solution for $\theta_H$.

The associated first-order secular solution for the variable $\psi_H$ for each of the subcases described above, is readily obtained from (2.1). This eliminates the need to integrate (4.24(a)) directly. Four more integrals of (4.24) remain to be determined. We consider next the variable $\phi_H$.

In order to integrate equation (4.24(c)), we first rewrite it in the form

$$x_3 = \left( s_{\theta} \cdot \Omega \right) \frac{c_{\psi_H}}{s_{\theta_{H}}} - (x_{1''} - 3x_{3''}) c_{\theta H}^2 - x_{3''}. \quad (2.16)$$

We note that if we can integrate $c_{\psi_H}/s_{\theta_H}$ and $c_{\theta H}^2$ with respect to time, to first-order, we can then obtain the solution for $\phi_H$.

From relations (4.25(b)) and (4.30), it is found, to first-order, that

$$\frac{c_{\psi H}}{s_{\theta H}} - \frac{h_{\psi}}{h_{\theta H}} = - \frac{h_{\psi}}{h - h_{\theta H}} = - \frac{h (c_{1''} - 3x_{3''} - h_{z_0} - h_{x_0} - C_0)}{h^2 - h_{x_0}^2}. \quad (2.17)$$
If we introduce the identities

\[
\frac{1}{h^2-h_{z_o}^2} = \frac{1}{2h} \left( \frac{1}{h+h_{z_o}} + \frac{1}{h-h_{z_o}} \right), \quad (a)
\]

\[
\frac{h_{z_o}}{h^2-h_{z_o}^2} = \frac{1}{2} \left( \frac{1}{h-h_{z_o}} - \frac{1}{h+h_{z_o}} \right), \quad (b) \quad (2.18)
\]

\[
\frac{h^2}{h^2-h_{z_o}^2} = \frac{h}{2} \left( \frac{1}{h+h_{z_o}} + \frac{1}{h-h_{z_o}} \right) - 1, \quad (c)
\]

equation (2.17) takes the separated form

\[
\frac{c \psi_H}{s_{\theta_H}} = \frac{D_1}{h} \frac{1}{h-c_{\theta_H}} + \frac{D_2}{h} \frac{1}{h+c_{\theta_H}} + D_3, \quad (2.19)
\]

where

\[
D_1 = - \frac{1}{2} \left( a''''h^2 - b'''h + c'' \right),
\]

\[
D_2 = - \frac{1}{2} \left( a''''h^2 + b'''h + c'' \right),
\]

\[
D_3 = a''''h.
\]

We can express the right hand side of (2.19) in terms of \( u^* \) if we replace \( c_{\theta_H} \) through the use of (2.6). We use (2.5) to replace \( t \) by \( u^* \) as the variable of integration. We then obtain the differential form
\[ \frac{c \gamma_H}{s_{\theta_H}} \frac{d t}{d t} = \frac{g^*}{(A^*)^{1/2}} \frac{1}{s_{\theta_0}^*} \left[ \frac{D_1}{h(A_3-A_1)} + \frac{A_3+A_4}{1-\gamma^2_1} \right] + \frac{D_2}{h(A_3+A_1)} \frac{A_3+A_4}{1-\gamma^2_2} \left[ A_3 \left( V_1(u_1^{*}, \gamma^2_1) - V_1(0, \gamma^2_1) \right) - A_4 \left( W_1(u_1^{*}, \gamma^2_1) - W_1(0, \gamma^2_1) \right) \right] + \frac{D_2}{h(A_3+A_1)} \left[ A_3 \left( V_1(u_1^{*}, \gamma^2_2) - V_1(0, \gamma^2_2) \right) - A_4 \left( W_1(u_1^{*}, \gamma^2_2) - W_1(0, \gamma^2_2) \right) \right] - D_3 u_1^x \]
where $i=1, 2, 3, 4$, and
\[ V_1(u^*_1, \gamma^*_j) = II(u^*_1, \gamma^*_j) \quad (j=1, 2) \quad (2.22) \]
\[ W_1(u^*_1, \gamma^*_j) = \frac{1}{\gamma^*_1} \left[ II(u^*_1, \gamma^*_j) - F(u^*_1) \right] \quad (b) \]

Here $II(u^*_1, \gamma^*_j)$, $j$, a positive integer is Legendre's incomplete integral of the third kind and $\gamma^*_j \neq 1$ or $\gamma^*_j \neq k_1^2$. For the special cases where $\gamma^*_j = 1$ or $\gamma^*_j = k_1^2$, the reader may refer to (BF111.06) for appropriate formulas. It can be evaluated by using Formulas (BF430) through (BF440).

If, next, we square (2.6), we have that
\[ c^2_{\phi H} = \frac{A_1^2}{A_2^3} \left( 1 - \gamma^2 \frac{\text{sn}^2 u^*_1}{\gamma^*_1} \right)^2 \quad (2.23) \]

where
\[ -\gamma^2_3 = \frac{A_2}{A_1} , \quad -\gamma^2_4 = \frac{A_4}{A_3} \quad (2.24) \]

Repeating the procedure used to obtain $I_{11}$, we find that (use BF340.02)
\[ I_{2i} = \int_{t_1}^{t} c^2_{\phi H} \frac{dt}{(A)^{1/2} \text{sn}^2} \left[ \frac{A_1^2}{A_3} \frac{1}{\gamma^*_4} \left( \frac{1}{\gamma^*_3} + \right) \right] 
+ 2 \gamma^*_3 (\gamma^*_4 - \gamma^*_3) \left[ V_1(u^*_1, \gamma^*_4) - V_1(0, \gamma^*_4) \right] \quad (2.25) \]
\[ + \left( \gamma_4^2 - \gamma_3^2 \right)^2 \left[ \frac{\partial^2}{\partial u_1^* \partial u_1^*} \left( V_2(u_1^*, \gamma_4^2) - V_2(0, \gamma_4^2) \right) \right] \]  

where

\[ V_1(u_1^*, \gamma_4^2) = \text{II}(u_1^*, \gamma_4^2), \]  

(a)

\[ V_2(u_1^*, \gamma_4^2) = \frac{1}{2(y_4^2 - 1)(k^* - \gamma_4^2)} \left[ \gamma_4^2 E(u_1^*) + (k^* - \gamma_4^2) u_1^* \right. \]  

\[ + (2\gamma_4^2 k^* - 2\gamma_4^2 - \gamma_4^2 - 3k^2) \text{II}(u_1^*, \gamma_4^2) \]  

(2.26)

\[ - \frac{\gamma_4^2 \text{sn} u_1^* \text{cn} u_1^* \text{dn} u_1^*}{1 - \gamma_4^2 \text{sn}^2 u_1^*} \]  

(b)

If we take the unperturbed solution \((\phi_H)_o\) as our initial value of \(\phi_H\), the initial value of \(x_3\) will be zero and we can write

\[ x_3 = \phi_H - (\phi_H)_o = (s_n^o \text{sn}) I_{14} - (x_1'' - 3x_3'') I_{21} - x_3'' t. \]  

(2.27)

The remaining three integrals of (4.24) follow easily. They are explicitly

\[ x_4 = \theta' - (\theta')_o = 0 \]  

(2.28)

\[ x_5 = \phi' - (\phi')_o = -x_5'''(t - 3I_{21}) \]  

(2.29)
while the unperturbed solutions are taken as initial values of the relevant variables.

Summarizing, we have the secular, first-order solution for Case (i):

\[ \phi' = (\phi')_0 + (s_c^2 \theta'_c) t - (x_1'' + 3x_3'' + x_3') t, \]

\[ \theta' = (\theta'_c)_0, \]

\[ h = (h)_0, \]

where \( u^*_1 \) is given by whichever of (2.9), (2.11), (2.13) or (2.15) applies to the appropriate subcase.

Case (ii): Four real roots with identical roots

This case can be treated as a special case of (i) and can be further grouped into two subcases.
(1) $a_1 = a_2 \neq c_0$ or $a_3 = a_4 \neq c_0$. It is seen from (2.8) that $k^* = 0$. Thus all the elliptic functions reduce to trigonometric functions (i.e., $\text{sn} u^* = \sin u^*$, etc.). In either case, equation (2.6) becomes

$$c_{\theta H} = \frac{A_1 + A_2 \sin^2 u_1^*}{A_1 + A_4 \sin^2 u_1^*}, \quad 0 \leq u_1^* \leq \pi/2, \quad (2.32)$$

where the $A_i$, $i=1,2,3,4$, are given by either (2.12) or (2.14) if $a_1 = a_2$. They are given by either (2.7) or (2.10) if $a_3 = a_4$. Then $u_1^*$ is given by (2.13) or (2.15) if $a_1 = a_2$ and it is given by (2.9) or (2.11) if $a_3 = a_4$.  

We note that if we replace (2.31(b)) by (2.32) and if, for this case ($k^* = 0$), we can evaluate $I_{11}$ and $I_{21}$ which correspond to (2.21) and (2.25). Then equations (2.31) will give us the first-order secular solutions. It is also seen from equations (2.21), (2.22), (2.25) and (2.26) that if we evaluate both $II(u_i^*, \gamma_j^*)$, $j=1,2,3,4$ and $F(u_i^*)$ at $k^* = 0$ then these equations will determine both $I_{11}$ and $I_{21}$. Formulas for the elliptic integrals $II(u_i^*, \gamma_j^2)$ and $F(u_i^*)$ can be found in [1]. Explicitly, they are [see (BF11.01) and (BF21.01)].

$$F(u_i^*) = u_i^*,$$

$$II(u_i^*, \gamma_j^2) = u_i^*, \quad \text{if} \quad \gamma_j^2 = 0 \quad (2.33)$$
\[ \tan^{-1} \left[ \frac{(1 - \gamma^2)^{1/2} \tan u_1^*}{(1 - \gamma^2)^{1/2}} \right], \quad \text{if } \gamma^2 < 1, \]

\[ \tan^{-1} \left[ \frac{((\gamma_j^2 - 1)^{1/2} \tan u_1^*)}{(\gamma_j^2 - 1)^{1/2}} \right], \quad \text{if } \gamma_j^2 > 1. \]  

(2.33)

If \( \gamma_j^2 = 1 \), we can use (BF111.01), (BF121.01) and (BF111.06), and write

\[ I_2(u_1^*, 1) = \tan u_1^*. \]  

(2.34)

Thus, the integrals \( I_{11} \) and \( I_{21} \) are determined and equations (a), (b), (c), (d), (e) and (f) of (2.31), together with (2.32), give the first-order secular solutions for the six variables of interest.

(2) \( a_2 = a_3 \): It follows from (2.8) that \( k^* = 1 \). Thus all elliptic functions reduce to hyperbolic functions (i.e., \( \text{sn} u^* = \text{tanh} u^*, \text{cn} u^* = \text{sech} u^* \)). Equation (2.6) takes the form

\[ c_\theta_H = \frac{A_1 + A_2 \tan h^2 u_1^*}{A_3 + A_4 \tan h^2 u_1^*}, \quad 0 \leq u_1^* \leq \infty \]  

(2.35)

where the \( A_i, i=1,2,3,4 \), are given by either (2.7), (2.10), (2.12) or (2.14) and \( u_1^* \) is given by the associated relation (2.9), (2.11), (2.13) or (2.15).
The analysis proceeds as in the preceding subcase, and we have, from (BF111.04), that if \( k^x = 1 \)

\[
F(u_1^x) = \ln(\tan \phi_1 - \sec \phi_1), \quad \phi_1 = \text{am} u_1^x \quad (a)
\]

\[
\text{II}(u_i^x, \gamma_j^2) = \frac{1}{1 - \gamma_j^2} [\ln(\tan \phi_1 + \sec \phi_1)
- \gamma_j \ln \left( \frac{1 + \gamma_j \sec \phi_1}{1 - \gamma_j \sec \phi_1} \right)^{1/2} ] \quad (b) \quad (\gamma_j^2 \neq 1).
\]

[Here \( \gamma_j^2 \) cannot take on the value one since \( \text{II}(u_i^x, 1) = \infty \).]

Thus the integrals \( I_{11} \) and \( I_{21} \) can be determined by (2.21), (2.22) and (2.25), (2.26), respectively, and therefore equations (a), (b), (c), (d), (e), (f) coupled with (2.35), give the first-order secular solutions.

Case (iii): Two distinct real roots and a pair of complex roots

Let \( a_1, a_2 \) be the real roots and let \( a_3, a_3^* \) be the complex roots and assume that \( a_1 > a_2 \).

We can write equation (2.3) in the Jacobian normal form for complex roots [see (BF250.05) and (BF250.06)]

\[
g^x du^x = \frac{dc\theta_H}{\left[(a_1-c\theta_H)(c\theta_H-a_2)(c\theta_H-a_3)(c\theta_H-a_3^*)\right]^{1/2}} = \frac{(A^x)^{1/2}}{\gamma_0} \int_{\theta_0}^{\theta} dt.
\]

(2.37)
where \( g^* \) is given by the equation
\[
g^* = \frac{1}{\sqrt{(A' B')}}
\] (23.8)

and
\[
A' = [(a_1 - b^*)^2 + a^*]^1/2 \quad , \quad \quad (a)
\]
\[
B' = [(a_2 - b^*)^2 + a^*]^1/2 \quad , \quad \quad (b)
\]

\[
a^* = -\frac{1}{4} (a_3 - k^*)^2 \quad , \quad \quad (c)
\]
\[
b^* = \frac{1}{2} (a + a^*) \quad . \quad \quad (d)
\]

The variable \( c_{\theta_H} \) is now related to \( u^* \) through the equation
\[
c_{\theta_H} = \frac{A_1 + A_2 \cn u^*}{A_3 + A_4 \cn u^*} \quad , \quad 0 \leq u^* \leq 2K^* \quad . \quad (2.40)
\]

where \( K^* \) has the same definition as in Case (i) and \( k^* \) is the new modulus of Jacobian elliptic functions and integrals.

Since the time rate of change of \( c_{\theta_H} \) is real, the value of \( c_{\theta_H} \) must lie between \( a_1 \) and \( a_2 \). If \( a_1 > c_{\theta_H} > a_2 \), we have
\[
A_1 = a_1 B' + a_2 A' \quad , \quad (a)
\]
\[
A_2 = a_2 A' - a_1 B' \quad , \quad (b)
\]
\[
A_3 = A' + B' \quad , \quad (c)
\]
\[
A_4 = A' - B' \quad , \quad (d)
\] (2.41)
and $k^{*2}$ has the value

$$k^{*2} = \frac{(a_1 - a_2)^2 - (A'B')^2}{4A'B'} .$$

(2.42)

If we integrate (2.37) with respect to time $t$ from $t_2$ to $t$, we obtain [see (BF259.00)]

$$u = - \frac{1}{g H} (A^*)^{1/2} \int_{t_2}^{t} \frac{1}{s_{\theta_H}} (t-t_2) ,$$

(2.43)

where $t_2$ is the value of $t$ at which $c_{\theta_H} = a_2$. Substituting (2.43) into equation (2.40) we obtain the first-order secular solution for $c_{\theta_H}$. With the time dependence of $c_{\theta_H}$ known, equation (2.1) gives the first-order secular solution for $\psi_H$.

Proceeding as in Case (i), and using equations (2.40) and (2.43) in conjunction with equation (2.19), we find that, to first-order,

$$\frac{c \psi_H}{s_{\theta_H}} \frac{dt}{d\theta_H} = - \frac{g}{(A^*)^{1/2} s_{\theta_H}} \left[ \frac{D_1 A_3}{h(A_3 - A_1)} \frac{1 + \gamma_3 \text{cn} u^*}{1 + \gamma_1 \text{cn} u^*} \right]$$

$$+ \frac{D_2 A_3}{h(A_3 + A_1)} \frac{1 + \gamma_3 \text{cn} u^*}{1 + \gamma_2 \text{cn} u^*} du^* .$$

(2.44)
where

\[ \gamma_1 = \frac{A_4 - A_2}{A_3 - A_1}, \quad (a) \]

\[ \gamma_2 = \frac{A_4 + A_2}{A_3 + A_1}, \quad (b) \]

\[ \gamma_3 = \frac{A_4}{A_3}, \quad (c) \]

To integrate equation (2.44) with respect to time \( t \), we can use \((BF361.62)\) and rearrange it in the form

\[
\frac{c}{s} \psi_H \frac{dt}{\theta_H} = - \frac{g^*}{(A^*)^{1/2} s_{90}^O} \left[ \frac{D_1 A_3}{h(A_3-A_1)} \left( \frac{\gamma_3 + 1 - \gamma_3/\gamma_1}{1 - \gamma_1 \tan u^*} \right) \right.

\left. + \frac{D_2 A_3}{h(A_3+A_1)} \left( \frac{\gamma_3 + 1 - \gamma_3/\gamma_2}{1 + \gamma_2 \tan u^*} \right) + D_3 \right] d u^*. \quad (2.44) \]

If \( \gamma_1 \neq 1 \) and \( \gamma_2 \neq 1 \), we find, from \((BF361.54)\) [or from \((BF341.03)\)], that

\[ I_{11} = \int_{t_1}^{t} \frac{c}{s} \frac{\psi_H}{\theta_H} dt \]

\[ = - \frac{g^*}{(A^*)^{1/2} s_{90}^O} \left\{ \left( \frac{D_1 A_3 \gamma_2}{h(A_3-A_1) \gamma_1} + \frac{D_2 A_3}{h(A_3+A_1)} \frac{\gamma_3}{\gamma_2} \right) u_1^* + \right\} \]
\[
+ \frac{D_1 A_3}{h(A_3 - A_1)} \cdot \frac{\gamma_1 - \gamma_3}{\gamma_1} \left[ R_1(u^*, \gamma_1) - R_1(0, \gamma_1) \right]
\]
\[
+ \frac{D_2 A_3}{h(A_3 + A_1)} \cdot \frac{\gamma_2 - \gamma_3}{\gamma_2} \left[ R_1(u^*, \gamma_2) - R_1(0, \gamma_2) \right]
\] , (2.46)

where

\[
R_1(u^*, \gamma_1) = \frac{1}{1 - \gamma_1^2} \left[ II(u^*, \gamma_1^2) - \gamma_1 \cdot f_1(u^*, \gamma_1^2) \right] , \quad (a)
\]

\[
f_1(u^*, \gamma_1^2) = \left( \frac{1 - \gamma_1^2}{k^{*2} + k^{*12} \gamma_1^2} \right)^{1/2} \tan^{-1} \left[ \left( \frac{k^{*2} + k^{*12} \gamma_1^2}{1 - \gamma_1^2} \right)^{1/2} \right] ,
\]

if

\[
\frac{\gamma_1^2}{\gamma_1^2 - 1} < k^{*2}
\]

= \text{sd} \ u^* , \quad \text{if} \quad \frac{\gamma_1^2}{\gamma_1^2 - 1} = k^{*2} \quad ,(b) \quad (2.47)
\]

\[
= \left( \frac{\gamma_1^2 - 1}{k^{*2} + k^{*12} \gamma_1^2} \right)^{1/2} \ln \left[ \frac{(k^{*2} + k^{*12} \gamma_1^2)^{1/2} \text{dn} \ u^*}{(k^{*2} + k^{*12} \gamma_1^2)^{1/2} \text{dn} \ u} \right] \left( \frac{\gamma_1^2 - 1}{\gamma_1^2 - 1} \right)^{1/2} \sn \ u^* \]

, \quad \text{if} \quad \frac{\gamma_1^2}{\gamma_1^2 - 1} > k^{*2} .
If either $\gamma^2_1 = 1$ or $\gamma^2_2 = 1$, the integral in (2.46) is to be replaced with the integral given by either (BF361.51) or (BF341.53).

Equation (2.40) can be used to write

$$I_{21} = \int_{t_2}^{t} c^2 dt$$

$$= - \frac{g^*}{(A^*)^{1/2}} \frac{A_1^2}{A_3} \int_{0}^{u^*} \left[ \frac{\gamma^2_5}{\gamma^2_4} - \frac{2 \gamma_5 (\gamma^{1/2}_4 - \gamma_5)}{\gamma^2_4} \frac{1}{1 + \gamma^{1/2}_4 \text{cn} u^*} + \right.$$

$$\left. + \left( \frac{\gamma^{1/2}_4 - \gamma_5}{\gamma^2_4} \right)^2 \frac{1}{(1 - \gamma^2_4 \text{cn} u^*)^2} \right] du^* \ , \quad (2.48)$$

where

$$\gamma_4 = \frac{A_4}{A_2} \ , \quad \text{(a)}$$

$$\gamma_5 = \frac{A_2}{A_3} \ . \quad \text{(b)}$$

Integrating with respect to time, we find from (BF341.03) and (BF341.04) that

$$I_2 = - \frac{g^*}{(A^*)^{1/2}} \frac{A_1^2}{A_3} \left\{ \frac{\gamma^2_5}{\gamma^2_4} u^* + \frac{2 \gamma_5 (\gamma^{1/2}_4 - \gamma_5)}{\gamma^2_4} \frac{1}{\gamma^2_4 (1 - \gamma^2_4)} \right\}$$

$$R_1(u^*, \gamma^2_4) -$$
\[ -R_1(0, \gamma_4^2) + \left( \frac{\gamma_4 - \gamma_5}{\gamma_4} \right)^2 \frac{\gamma_4^2 (2k^2 - 1) - 2k^2}{(\gamma_4^2 - 1)(k^2 + \gamma_4^2 k^2)} \int R_1(u^*, \gamma_4^2) - \]

\[ -R_1(0, \gamma_4^2) + 2k^* \left[ R_1(u^*, \gamma_4^2) - R_1(0, \gamma_4^2) \right] + \frac{\gamma_4^2 \text{sn} u^* \text{dn} u^*}{1 - \gamma_4 \text{cn} u^*} \]

\[ -k^* \left[ R_2(u^*, \gamma_4^2) - R_2(0, \gamma_4^2) \right] \]  

(2.50)

where

\[ R_1(u^*, \gamma_4^2) = u^* + \frac{\gamma_4}{k^*} \cos^{-1}(\text{dn} u^*) \], \hspace{1cm} (a)

\[ R_2(u^*, \gamma_4^2) = \frac{1}{k^*} \left[ (k^* - \gamma_4^2 k^* k^2) u^* + \gamma_4^2 E(u^*) \right] \]

(2.51)

\[ + 2 \gamma_4 k^* \cos^{-1}(\text{dn} u^*) \],

and \( \gamma_4^2 \neq 1 \).

If \( \gamma_4^2 = 1 \), the integrals in (2.48) are to be replaced with the integrals given by (BF341.53) and (BF341.54). Thus equations (a), (b), (c), (d), (e), (f) of (2.31) together with (2.40), (2.46) and (2.50) give the associated first-order secular solutions for the six variables.
Case (iv): Two identical real roots and a pair of complex roots

This is a special case of Case (iii). It can be shown in a straightforward manner from (2.40) that if $a_1 = a_2$ then $c_\theta H = a_1$ and $\theta H$ is a constant of the motion. Consequently $\psi'_H, \theta_H, \theta'$ and $h$ are all constants of the motion and $\phi_H$ and $\phi'$ are linear functions of time.

BIBLIOGRAPHY