COINCIDENCE DEGREE AND PERIODIC SOLUTIONS
OF NEUTRAL EQUATIONS

by

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1. Introduction

This paper is devoted to the problem of existence of periodic solutions for some nonautonomous neutral functional differential equations. It is essentially an application of a basic theorem on the Fredholm alternative for periodic solutions of some linear neutral equations recently obtained by one of the authors [2] and of a generalized Leray-Schauder theory developed by the second one [3, 4]. Although their proofs are surprisingly simple, the obtained results are nontrivial extensions to the neutral case of a number of recent existence theorems for periodic solutions of functional differential equations. In particular, section 3 generalizes some existence criteria due to one of the authors [5] and a corresponding recent extension by J. Cronin [6], the example following Theorem 4.1 improves a condition for existence given by Lopes [14] for the equation of a transmission line problem, and Theorem 5.1 generalizes a result due to R. E. Fennell [7]. Lastly, criteria analogous to Theorem 5.2 for the retarded case can be found in [8]. For partly related results concerning periodic solutions of neutral functional differential equations, see [9].

2. Fredholm Alternative for Linear Equations

Let $C([a,b],\mathbb{R}^n)$ be the space of continuous functions from $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. For a fixed $r \geq 0$, let $C = C([-r,0],\mathbb{R}^n)$ with norm $|\varphi| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$. 
for \( \varphi \in C \). If \( x \in C([\sigma-r,\sigma+\delta],\mathbb{R}^n) \) for some \( \delta > 0 \), let \( x_t \in C \), \( t \in [\sigma,\sigma+\delta] \) be defined by \( x_t(\theta) = x(t+\theta), \theta \in [-r,0] \). Suppose \( \omega > 0 \) fixed, \( A: \mathbb{R} \times C \to \mathbb{R}^n \) is continuous, \( A(t+\omega)\varphi = A(t)\varphi \) for all \( (t,\varphi) \in \mathbb{R} \times C \), \( A(t)\varphi \) is linear in \( \varphi \) and there exists a continuous function \( \gamma: [0,\infty) \to \mathbb{R}, \gamma(0) = 0 \), such that

\[
|A(t)\varphi^s| \leq \gamma(s)|\varphi^s|, \quad 0 \leq s \leq r
\]

for all \( t \in \mathbb{R} \) and all functions \( \varphi^s \in C \) such that \( \varphi^s(\theta) = 0 \) for \( \theta \in [-r,-s] \). Let \( D: \mathbb{R} \times C \to \mathbb{R}^n \) be defined by \( D(t)\varphi = \varphi(0) - A(t)\varphi \).

The operator \( D \) is said to be \textit{stable} if the zero solution of the functional equation \( D(t)y_t = 0 \) is uniformly asymptotically stable; that is, there are constants \( K, \alpha > 0 \) such that if \( y(\varphi) \) is the solution of \( D(t)y_t = 0 \) with \( y_0 = \varphi \), then

\[
|y_t(\varphi)| \leq Ke^{-\alpha t}|\varphi|, \quad t \geq 0, \quad \varphi \in C. \tag{2.1}
\]

Let \( \mathcal{P}_\omega = \{ x \in C(\mathbb{R},\mathbb{R}^n): x(t+\omega) = x(t), t \in \mathbb{R} \} \), \( \mathcal{H}_\omega = \{ H \in C(\mathbb{R},\mathbb{R}^n): H(0) = 0 \) and \( H(t) = \alpha t + h(t) \) for some \( \alpha \in \mathbb{R}^n, h \in \mathcal{H}_\omega \} \). For any \( h \in \mathcal{P}_\omega \), let \( |h| = \sup_{0 \leq t \leq \omega} |h(t)| \) and for any \( H \in \mathcal{H}_\omega \), \( H(t) = \alpha t + h(t), \alpha \in \mathbb{R}, h \in \mathcal{P}_\omega \), let \( |H| = |\alpha| + |h| \).

**Proposition 2.1.** If \( D \) is stable, then, for any \( c \in \mathbb{R}^n \), there is a unique solution \( M_c \) of the equation \( D(t)x_t = c \) in \( \mathcal{P}_\omega \).

Furthermore, \( M \) is a continuous linear operator from \( \mathbb{R}^n \) to \( \mathcal{P}_\omega \).
Proof: Following the proof of Lemma 3.4 in [10], there are constants \( b > 0, a > 0 \) and an appropriate equivalent norm in \( C \) such that the solution \( x(\varphi, c) \) of \( D(t)x_t = c, x_0 = \varphi \), satisfies \( |x_t(\varphi, c)| \leq |c|b + |\varphi|\exp(-at), t \geq 0, \varphi \in C, c \in \mathbb{R}^n \). If \( T\varphi = x_0(\varphi, c) \), then \( T \) is a contraction mapping. Thus, if \( d > 0 \) is sufficiently large that \( |c|b + d \exp(-ao) < d \), then \( T \) has a unique fixed point such that \( |\varphi| < d \). Consequently, there is a solution of the equation in \( \mathcal{P}_\omega \). The fact that \( D \) is stable implies the uniqueness, linearity and continuous dependence on \( c \).

Let us rephrase Proposition 2.1 in a different way. Let \( L: \mathcal{P}_\omega \to \mathcal{P}_\omega \) be the continuous linear mapping defined by

\[
Lx(t) = D(t)x_t - D(0)x_0, \quad t \in \mathbb{R}.
\]

Proposition 2.1 implies that

\[
\ker L = \{x \in \mathcal{P}_\omega: \exists c \in \mathbb{R}^n \text{ with } x = Mc\}
\]

is an \( n \)-dimensional subspace of \( \mathcal{P}_\omega \). Let \( P: \mathcal{P}_\omega \to \mathcal{P}_\omega \) be a continuous projection onto \( \ker L \).

For the statement of the next proposition, let \( Q: \mathcal{H}_\omega \to \mathcal{H}_\omega \) be the continuous projection defined by

\[
QH(t) = \omega^{-1}H(\omega)t, \quad t \in \mathbb{R}.
\]
Proposition 2.2. If D is stable, then $\text{Im } L = \ker Q$ and there is a continuous linear operator $K: \text{Im } L \to \ker P$ such that $K$ is a right inverse of $L$. Thus, $L$ is a Fredholm operator with index 0.

Proof: The second proof given in [2] of the Fredholm alternative holds equally well for the equation $D(t)x_t = H(t)$. Thus, from [2], $\dim \ker L = \text{codim } \text{Im } L$. Proposition 2.1 implies $\dim \ker L = n$.

For the equation $Lx = H$ to have a solution, it is clearly necessary that $H \in \ker Q$. Since $\text{codim } \ker Q = n$, it follows that $\text{Im } L = \ker Q$. The existence of the bounded right inverse follows from [2] or one may apply the closed graph theorem to $L(I-P)^\omega$.

For the applications, it is necessary to be able to compute $\ker L$. In some simple cases, this is easily accomplished. For example, if $a(t) = a(t+\omega)$, $t \in \mathbb{R}$, is an $n \times n$ matrix with $|a(t)| < k < 1$ for $t \in \mathbb{R}$, then the unique solution $M_C$ in $\mathcal{D}_\omega$ of

$$x(t) - a(t)x(t-r) = c$$

is given by

$$(M_C)(t) = \left[ I + \sum_{k=0}^{\infty} \prod_{j=0}^{k} a(t-ju) \right] c. \quad (2.3)$$

Another case particularly interesting in the applications is when $D(t)\varphi$ is independent of $t$. Then $\ker L = \{ \text{constant functions in } \mathcal{D}_\omega \}$. 
3. Existence Theorems for Nonlinear Equations

With the above notations, let us consider the neutral functional differential equation

$$\frac{d}{dt} D(t)x_t = f(t, x_t)$$  \hspace{1cm} (3.1)

where $D$ is stable and $f: \mathbb{R} \times C \rightarrow \mathbb{R}^n$ is $\omega$-periodic with respect to $t$, continuous and takes bounded sets into bounded sets. If we define $N_\omega: P_\omega \rightarrow K_\omega$ by

$$N_x(t) = \int_0^t f(s, x_s) ds, \quad t \in \mathbb{R},$$

it is clear that finding $\omega$-periodic solutions of (3.1) is equivalent to solving the operator equation $Lx = Nx$ in $P_\omega$ with $L$ defined in (2.7). To apply coincidence degree theory to this problem still requires that $N$ should be compact, i.e. continuous and taking bounded sets of $P_\omega$ into relatively compact sets of $P_\omega$.

Proposition 3.1. Under the conditions listed above, $N$ is compact.

Proof: The continuity is obvious. If $S > 0$ and $x \in P_\omega$ is such that $|x| \leq S$, then $|x_t| \leq S$ for every $t \in S$ and thus $|f(s, x_s)| \leq T$ for some $T > 0$ and every $s \in \mathbb{R}$. It then follows easily that
\[ |N_x(t)| \leq T(1+2\omega), \quad t \in \mathbb{R} \]

and

\[ |N_x(t_1) - N_x(t_2)| \leq T|t_1 - t_2|, \quad t_1, t_2 \in \mathbb{R} \]

and Proposition 3.1 is a consequence of the Arzela-Ascoli theorem.

A direct application of Propositions 2.2, 3.1 above and of Theorem 5.1(i) of [3] gives the following

**Theorem 3.1.** If there exists an open bounded set \( \Omega \subset \mathcal{P} \) whose boundary \( \partial \Omega \) contains no \( \omega \)-periodic solution of (3.1) and if the \( \mathcal{I}_+ \)-coincidence degree \( d[(L,N),\Omega] \) is not zero, then equation (3.1) has at least one \( \omega \)-periodic solution in \( \Omega \).

This result is quite general but requires the solution of two difficult problems, namely, finding \( \Omega \) (it is an a priori bound problem) and estimating \( d[(L,N),\Omega] \). Theorem 7.2 of [3] reduces this last question to the study of Brouwer degree of some well-defined finite-dimensional mapping if the a priori estimate condition is slightly strengthened. Let \( g: \mathbb{R} \times C \times [0,1] \to \mathbb{R}^n, \quad (t,\varphi,\lambda) \to g(t,\varphi,\lambda) \) be \( \omega \)-periodic with respect to \( t \), continuous, taking bounded sets into bounded sets and such that

\[ g(t,\varphi,1) \equiv f(t,\varphi), \quad (t,\varphi) \in \mathbb{R} \times C. \]  (3.2)
Let $M: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the mapping defined in Proposition 2.1 and define $\mathcal{G}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$\mathcal{G}(a) = \omega^{-1} \int_0^\omega g(t, (Ma)_t, 0)dt.$$

If $D(t)\varphi$ is independent of $t$, one can put $M = I$, the identity in this definition. Theorem 7.2 of [3] implies the following

Theorem 3.2. Suppose there exists an open bounded set $\Omega \subset \mathbb{R}^n$ for which the following conditions are satisfied.

1. For each $\lambda \in (0,1)$, the equation

$$\frac{d}{dt} D(t)x_t = \lambda g(t, x_t, \lambda)$$

has no $\omega$-periodic solution on $\partial \Omega$.

2. $\mathcal{G}(a) \neq 0$ for every $a \in \mathbb{R}^n$ such that $Ma$ belongs to $\partial \Omega$.

3. The Brouwer degree $d_B[\mathcal{G}, \tilde{\Omega}, 0]$ is not zero, where

$$\tilde{\Omega} = \{a \in \mathbb{R}^n: Ma \text{ belongs to } \Omega\}.$$

Then equation (3.1) has at least one $\omega$-periodic solution in $\Omega$.

Another useful special case of Theorem 3.1 follows at once from Theorem 7.3 of [3]. Suppose that the mapping $g$ defined above verifies (3.2) and the supplementary condition
Theorem 3.3. Suppose there exists an open bounded set \( \Omega \subset \mathbb{R} \) symmetric with respect to the origin, containing it and such that \( \partial \Omega \) contains no \( \omega \)-periodic solution of each equation

\[
\frac{d}{dt} D(t)x_t = g(t,x_t,\lambda), \lambda \in [0,1]
\]

with \( g \) verifying (3.2) and (3.3). Then equation (3.1) has at least one \( \omega \)-periodic solution in \( \Omega \).

Let us note that (3.3) will always be satisfied if \( g(t,\varphi,0) \) is linear with respect to \( \varphi \). Also, Theorems 3.1, 3.2 and 3.3 are respective generalizations of Theorems 2, 3 and 4 of [5] which all correspond to the case of retarded functional differential equations, i.e. \( D\varphi = \varphi(0) \), and \( \Omega \) an open ball. Also, an extension to the neutral case of Theorem 1 of [6] is easily obtained by a suitable choice of \( \Omega \) and the properties of coincidence degree.

4. An Application

Let us consider the neutral equation

\[
\frac{d}{dt} \left[ x(t) - \sum_{k=1}^{N} A_k x(t-\tau_k) \right] = \text{grad} \bigvee [x(t)] + e(t) \quad (4.1)
\]
where, $V: \mathbb{R}^n \to \mathbb{R}$ is of class $C^1$, $e \in P$, $\tau_k \in [-r,0)$
($k = 1, 2, \ldots, N$) and the $n \times n$ constant matrices $A_k$ are such
that
\[ \sum_{k=1}^{N} |A_k| = 1 - \alpha, \quad \alpha > 0. \tag{4.2} \]

Let $\bar{e}$ be the mean value of $e$.

**Theorem 4.1.** If there exists $R > 0$ such that
\[ \bar{e} + \omega^{-1} \int_0^\omega \text{grad} V[x(t)] \, dt \neq 0 \quad \text{for every } x \in P \text{ satisfying } \inf |x(t)| > R \]
and if the Brouwer degree $d_B[\bar{e} + \text{grad} V, B(0,R),0]$ is not zero, then equation (4.1) has at least one $\omega$-periodic solution.

**Proof:** Let $| \cdot |$ and $\langle \cdot , \cdot \rangle$ respectively denote the Euclidean norm and the inner product in $\mathbb{R}^n$. It is well known [10] that condition (4.2) implies that the operator
\[ D: \varphi \to \varphi(0) - \sum_{k=1}^{N} A_k \varphi(-\tau_k) \]
is stable and the right hand side of (4.1) clearly takes bounded sets into bounded sets. Let us consider the family of equations

\[ \frac{dx(t)}{dt} - \sum_{k=1}^{N} A_k x(t-\tau_k) = \lambda \text{grad} V[x(t)] + \lambda e(t), \quad \lambda \in (0,1). \tag{4.3} \]

If $x$ is any $\omega$-periodic solution of (4.3) for some $\lambda \in (0,1)$ then
\( x(t) \) must have a continuous first derivative (see [14]) and

\[
\omega^{-1} \int_0^\omega \langle \dot{x}(t) - \sum_{k=1}^N a_k \dot{x}(t-k), \dot{x}(t) \rangle dt =
\]

\[
= \lambda \omega^{-1} \int_0^\omega \langle \nabla V[x(t)], \dot{x}(t) \rangle dt + \omega^{-1} \int_0^\omega \langle e(t), \dot{x}(t) \rangle dt,
\]

which implies, using Schwarz inequality and \((4.2)\),

\[
\left( \omega^{-1} \int_0^\omega |\dot{x}(t)|^2 dt \right)^{1/2} \leq \alpha^{-1} \eta
\]

with \( \eta^2 = \omega^{-1} \int_0^\omega |e(t)|^2 dt \). Then, for every \( t, t' \in [0,\omega] \), we have

\[
|x(t) - x(t')| \leq \alpha^{-1} \eta. \tag{4.4}
\]

On the other hand, every \( \omega \)-periodic solution of \((4.3)\) verifies the equation

\[
-\ddot{e} + \omega^{-1} \int_0^\omega \nabla V[x(t)] dt = 0
\]

and hence there must exist some \( \sigma \in [0,\omega] \) for which \( |x(\sigma)| < R \). Taking \( t' = \sigma \) in \((4.4)\) we obtain

\[
|x| < R + \alpha \alpha^{-1} \eta = S
\]

for every \( \omega \)-periodic solution of \((4.3)\). The result then follows
from Theorem 3.2 by taking for $\Omega$ the open ball of center 0 and radius $R_0$.

As an application of Theorem 4.1, let us consider the special case of a scalar equation with one delay,

\[
\frac{d}{dt} [x(t) + ax(t-r)] = p(x) + e(t)
\]

where $|a| < 1$, $e \in \mathcal{P}_\omega$ and $p(x)$ is a given function of $x$. This equation arises in a transmission line problem with a shunt across the line (see [13], [14]). Then, if $p$ is any continuous function such that $|p(x)| \to \infty$ if $|x| \to \infty$ and $p(x)p(-x) < 0$ for all $x$ with $|x|$ sufficiently large, there will exist one $\omega$-periodic solution. Using Liapunov functions, Lopes [14] has obtained the existence of an $\omega$-periodic solution of this special equation for $|a| < 1/2$ and $xp(x) \to +\infty$ as $|x| \to \infty$.


We shall consider in this section $\omega$-periodic equations of the form

\[
\frac{d}{dt} D(t)x_t = b(t, x_t) + f(t, x_t)
\]  

(5.1)

where $D$ satisfies the conditions in section 2, $b: \mathbb{R} \times C \to \mathbb{R}^n$, $(t, \varphi) \to b(t, \varphi)$ is linear with respect to $\varphi$ and continuous,
\( f: \mathbb{R} \times C \to \mathbb{R}^n \) is continuous, takes bounded sets into bounded sets and is such that

\[
\limsup_{|\varphi| \to \infty} \left( |\varphi|^{-1} |f(t,\varphi)| \right) = \inf_{0<\rho<\infty} \left( \sup_{|\varphi| \geq \rho} |f(t,\varphi)| \right) = 0 \quad (5.2)
\]

uniformly in \( t \in \mathbb{R} \).

Let us recall that a mapping \( f: X \to Y \) between normed spaces is quasi-bounded if the number \( \|f\| = \limsup \|x\|^{-1} |f(x)| \) is finite, in which case it is called the quasinorm of \( f \) \([12]\). We shall use in this section a mapping theorem of Granas for compact quasibounded perturbations of the identity \([12]\) and a special case of its generalization in the frame of coincidence degree theory \([4]\).

**Proposition 5.1.** If \( f \) satisfies the conditions above, then the mapping \( N: \mathcal{B}_c \to \mathcal{B}_c \) defined by \( N(t) = \int \int_0^t f(s, x_0) ds, \) \( t \in \mathbb{R} \), is compact, quasibounded and \( \|N\| = 0 \).

**Proof:** The compactness follows from Proposition 3.1. Now, if \( \varepsilon > 0 \), it follows from (5.1) and the fact that \( f \) takes bounded sets into bounded sets that there exist \( r(\varepsilon) > 0 \) such that, for every \((t, \varphi) \in \mathbb{R} \times C,\)

\[ |f(t,\varphi)| \leq \varepsilon |\varphi| + r. \]
Hence, for every $x \in \mathcal{P}_\omega$,

$$|Nx| = \left| \omega^{-1} \int_0^\omega f(s,x_s)ds \right| + \sup_{t \in [0,\omega]} |\int_0^t [f(t',x_{t'}) - \omega^{-1} \int_0^\omega f(s,x_s)ds]dt'| \leq (1+2\omega)[\varepsilon|x| + \gamma(\varepsilon)]$$

which clearly implies $\|Nx\| = 0$.

Now we can prove the following

**Theorem 5.1.** With $D$, $b$ and $f$ as above, suppose the linear equation

$$\frac{d}{dt} D(t)x_t = b(t,x_t) \tag{5.3}$$

has no nontrivial $\omega$-periodic solution. Then equation (5.1) has at least one $\omega$-periodic solution.

**Proof.** The result is equivalent to solving the equation $Lx - Bx = Nx$ in $\mathcal{P}_\omega$ with $B: \mathcal{P}_\omega \to \mathcal{L}_\omega$ defined by $Bx(t) = \int_0^t b(s,x_s)ds$, $t \in \mathbb{R}$, and $L,N$ as above. From Proposition 3.1 we know that $B$ is a compact mapping and $L$ being a continuous Fredholm mapping of index zero, the same is true for $L - B$ [11]. As $L - B$ is one-to-one by our assumption on (5.3) it will necessarily be onto and such that $(L-B)^{-1}: \mathcal{L}_\omega \to \mathcal{P}_\omega$ is continuous. The proof of Theorem 5.1 is then equivalent to the fixed point problem $x = (L-B)^{-1}Nx$ in $\mathcal{P}_\omega$ with
(L-B)^{-1} N clearly compact, quasibounded and of quasinorm zero. The
result then follows from Granas' theorem.

An interesting problem is now to try to drop the assumption
about the nonexistence of nontrivial \( \omega \)-periodic solutions for (5.3).

It is clear from the Fredholm alternative that conditions upon \( f \)
will then be needed. We consider here the simplest case, i.e.
\( b(t, \varphi) \equiv 0 \). Let us define \( \mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by
\( \mathcal{F}(a) = \omega^{-1} \int_0^\omega f(t, (Ma)_t) dt \).
If \( D(t) \varphi \) is independent of \( t \), take \( M = I \), the identity, in the
definition of \( \mathcal{F} \).

\textbf{Theorem 5.2.} Let \( D \) and \( f \) be as above and suppose there exists
\( \mu > 0 \) such that \( |(Mc)(t)| \geq \mu |c| \) for every \( t \in \mathbb{R} \) and every
\( c \in \mathbb{R}^n \). If there exists \( R_1 > 0 \) such that \( \int_0^\omega f(s, x_s) ds \neq 0 \) for
every \( x \in \mathcal{F} \) verifying \( \inf_{t \in \mathbb{R}} |x(t)| \geq R_1 \) and if \( d_B[\mathcal{F}, \Omega_R, 0] \) is
not zero, where \( \Omega_R = \{ a \in \mathbb{R}^n : Ma \in B(0, R) \} \) and \( R = \mu^{-1}|M| R_1 \), then
the equation

\[ \frac{d}{dt} D(t)x_t = f(t, x_t) \quad (5.4) \]

has at least one \( \omega \)-periodic solution.

\textbf{Proof.} We will use Propositions 3.1 and 5.1 above and Theorem 4.1
of [14]. The proof will be complete if we show the existence of
\( \alpha \geq 0 \) and \( R > 0 \) such that every \( \omega \)-periodic solution \( x \) of (5.4)
satisfies the inequality

$$|Px| < c|(I-P)x| + R.$$  \hspace{1cm} (5.5)  

If $x$ is any $\omega$-periodic solution of (5.4), then $\int_{0}^{\omega} f(s, x_{s}) \, ds = 0$ and hence there will exist $\sigma \in [0, \omega]$ such that $|x(\sigma)| < R_{1}$.

Therefore, if $c \in \mathbb{R}^{n}$ is such that $Px = Mc$, we have

$$\mu |c| \leq |(Mc)(\sigma)| < R_{1} + |(I-P)x(\sigma)| \leq R_{1} + |(I-P)x|,$$

which implies

$$|Px| \leq |M| |c| < \mu^{-1} |M| R_{1} + \mu^{-1} |M| |(I-P)x|$$

and (5.5) holds with $\alpha = \mu^{-1} |M|$ and $R = \mu^{-1} |M| R_{1}$.

Let us remark that if $D(t)\phi$ is independent of $t$, $\ker L$ is the subspace of $\mathcal{G}_{\omega}$ of constant functions and the positive number $\mu$ involved above always exists and can be taken equal to one. Hence, a simple example for Theorem 5.2 is given by the scalar equation

$$\frac{d}{dt} [x(t) - ax(t-r)] = g(x_{t}) + e(t) \hspace{1cm} (5.6)$$

where $a \in (-1, 1)$, $e \in \mathcal{G}_{\omega}$ has mean value zero, $g: \mathbb{C} \rightarrow \mathbb{R}^{n}$ is
continuous, quasibounded with quasinorm zero, takes bounded sets into bounded sets and is such that, for some \( R > 0 \), either
\[ g(\varphi)\varphi(\theta) > 0 \quad \text{or} \quad g(\varphi)\varphi(\theta) < 0, \]
for every \( \theta \in [-r,0] \) and every \( \varphi \in C \) such that
\[ \inf_{[-r,0]} |\varphi(\theta)| \geq R. \] It is the case, for example,

for the equation

\[
\frac{d}{dt} [x(t) + ax(t-r)] = b \frac{x(t-r)}{|x(t-r)|^{1/2}} + e(t)
\]

if \( |a| < 1, b \neq 0, e \in \mathcal{B} \) has mean value zero and \( y/|y|^{1/2} \) is extended by 0 at \( y = 0 \).

To apply Theorem 5.2 to a scalar equation of the form

\[
\frac{d}{dt} [x(t) - a(t)x(t-r)] = g(x_t) + e(t)
\]

with \( g \) and \( e \) as above and \( a \in \mathcal{B} \), the crucial point is to prove the existence of \( \mu > 0 \) such that \( |(Mc)(t)| \geq \mu |c| \) for every \( t \in \mathbb{R} \) and every \( c \in \mathbb{R} \). The following propositions give answers to this problem. For the sake of brevity, we shall say that the operator \( M \) associated with the scalar equation

\[
x(t) - a(t)x(t-r) = c \]

has property \( \mu \) if there exists \( \mu > 0 \) such that \( |(Mc)(t)| \geq \mu |c| \) for every \( t \in \mathbb{R} \) and every \( c \in \mathbb{R} \).

**Proposition 5.2.** If \( |a(t)| \leq k \) for all \( t \in \mathbb{R} \) and \( k \in [0,1/2) \), then \( M \) has property \( \mu \).
Proof. From the relation

\[(M_c)(t) - a(t)(M_c)(t-r) = c\]

one obtains easily \(|M_c| \leq (1-k)^{-1}|c|\) and hence

\[|(M_c)(t)| \geq |c| - k|(M_c)(t-r)| \geq |c|(1-2k)(1-k)^{-1}\]

for every \(t \in \mathbb{R}\) and every \(c \in \mathbb{R}\).

The following example will show that Proposition 5.2 is the best possible without supplementary assumptions on the oscillatory character of \(a(t)\). Let \(\omega = p, p\) a positive integer, \(r = 1\) and \(a(t)\) be a \(p\)-periodic continuous function such that \(|a(t)| \leq k < 1, t \in \mathbb{R}, a(0) = -k, a(m) = k (m = 1, 2, \ldots, p-1)\). Then, if \(x(t)\) is the solution of \(x(t) - a(t)x(t-1) = 1\), property \(\mu\) clearly will not hold if we exhibit one \(t \in [0, p]\) such that \(x(t) = 0\). Using formula (2.3) and the form of \(a(t)\) we have

\[x(0) = 1 - k(1+k+k^2+\ldots+k^{p-1}-k^p(1+k+\ldots+k^{p-1})\ldots\]

\[= 1 - k\frac{1-k^p}{1-k} - k^p\frac{1-k^p}{1-k} - \ldots\]

\[= 1 - k(1-k^p + k^2p - \ldots) = 1 - k(1-k^p)(1-k)^{-1}(1+k^p)^{-1}\]

\[= (1+k^p)^{-1}(1-k)^{-1}(1-2k+k^p) = \gamma(k)\].
It is easy to show that $r(k)$ is strictly positive in $[0,1/2)$ and is strictly negative in a neighborhood of 1. Thus, $x(0) = 0$ for some $k \in [1/2,1)$ and this zero is arbitrary close to $1/2$ if we take $p$ sufficiently large, as follows at once from the form of $r(k)$.

It is, however, possible to improve the condition upon $k$ when $a(t)$ has a constant sign as follows from

**Proposition 5.3.** If $|a(t)| \leq k < 1$ and $a(t)$ has constant sign, then $M$ has property $\mu$.

**Proof.** Let us first consider the case where $0 \leq a(t) \leq k$ for every $t \in \mathbb{R}$. Then $M$ has property $\mu$ because

$$|(Mc)(t)| = |1+a(t)+a(t)a(t-r)+\ldots| |c| \geq |c|.$$ 

Now suppose that $-k \leq a(t) \leq 0$ for every $t \in \mathbb{R}$. It is clear that the unique $\omega$-periodic solution $x$ of $x(t) - a(t)x(t-r) = c$ is the limit of the sequence $\{x^m(t)\}$ of $\omega$-periodic functions defined by

$$x^0(t) = c, \quad x^{m+1}(t) = c + a(t)x^m(t-r), \quad m = 0,1,2,\ldots.$$ 

If $c > 0$, then

$$x^1(t) = [1+a(t)]c \geq (1-k)c > 0, \quad x^2(t) = c + a(t)x^1(t-r) \geq [1-k(1-k)]c = (1-k+k^2)c > 0, \quad \text{and if} \quad x^m(t) \geq [1-k+k^2+\ldots+(-1)^m]c > 0,$$
then \( x^{m+1}(t) = c + a(t)x^m(t-r) \geq c\{1-k[1-k+\ldots+(-1)^{m+1}]\} = [1-k+k^2+\ldots+(-1)^{m+1}]c > 0 \). Hence, by induction and passing to the limit, we have \(|(Mc)(t)| \geq (1+k)^{-1}|c|\). Finally, suppose that \(c < 0\). Then,

\[
c \leq x^1(t) = c + a(t)c \leq (1-k)c < 0
\]

and hence

\[
c \leq x^2(t) = c + a(t)x^1(t-r) \leq (1-k)c < 0.
\]

If we suppose that \(c \leq x^{m-1}(t) \leq (1-k)c < 0\), then \(0 \leq a(t)x^{m-1}(t-r) \leq -ck\) and hence

\[
0 > (1-k)c \geq x^m(t) = c + a(t)x^{m-1}(t) \geq c.
\]

By induction and passing to the limit we have \(0 > (1-k)c \geq (Mc)(t) \geq c\) and hence \(|(Mc)(t)| \geq (1-k)|c|\), which achieves the proof.

**Corollary 5.1.** If \(a\) is a constant verifying \(0 < |a| < 1\) then, for every \(b \in \mathcal{P}_0\) such that \(|b(t)| < \min(|a|,|1-a|)\), \(t \in \mathbb{R}\), the mapping \(M\) associated with \(x(t) - [a+b(t)]x(t-r) = c\) has property \(\mu\).
REFERENCES


