BIFURCATION AND STABILITY FOR
A NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATION

by

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List of symbols

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Mathematical symbols

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This note is a brief report on some research conducted by the author and E.F. Infante in 1971. A complete report on this same research is scheduled to appear in a separate article [1].

Let $f$ be a given function continuously mapping the real line $\mathbb{R}$ into itself. Let $\lambda$ be a given non-negative real number. Let $\phi: [0, \pi] \rightarrow \mathbb{R}$ be any $C^1$-smooth function such that $\phi(0) = \phi(\pi) = 0$. We shall be discussing the following problem. Find a function $u$ continuously mapping the domain $\{(x,t): 0 \leq x \leq \pi, 0 \leq t < \infty\}$ into $\mathbb{R}$ such that (i) the partial derivatives $u_t$ and $u_{xx}$ are defined and continuous on $[0, \pi] \times (0, \infty)$; (ii) $u$ satisfies the equations

\begin{align*}
u_t(x,t) &= u_{xx}(x,t) + \lambda f(u(x,t)) \quad (0 < x < \pi, 0 < t < \infty) \quad (1a) \\
u(0,t) &= u(\pi,t) = 0 \quad (0 < t < \infty) \quad (1b) \\
u(x,0) &= \phi(x) \quad (0 < x < \pi). \quad (1c)
\end{align*}

By a solution of (1) we mean a function $u$ having the properties just specified.

Our primary goal in studying (1) is to determine the asymptotic behavior of solutions $u$ of (1) as $t \rightarrow \infty$. The investigation takes place under the following hypotheses concerning $f$.

\begin{enumerate}
\item[(H_1)] $f$ is a $C^2$-smooth function mapping $\mathbb{R}$ into itself.
\item[(H_2)] $f(0) = 0$ and $f'(0) > 0$.
\item[(H_3)] $\limsup_{|\xi| \rightarrow \infty} \xi^{-1} f(\xi) = 0$.
\item[(H_4)] $\text{sgn } f''(\xi) = -\text{sgn } \xi$ for all $\xi \in \mathbb{R}$.
\end{enumerate}

In that which follows we shall let $X$ denote the space of all $C^1$-smooth
functions \( \phi: [0, \pi] \to \mathbb{R} \) such that \( \phi(0) = \phi(\pi) = 0 \). On \( X \) we impose a norm \( || \cdot ||_1 \) by setting \( ||\phi||_1 = \sup\{|\phi'(x)| : 0 \leq x \leq \pi\} \) for all \( \phi \in X \). \( X \) is a Banach space under \( || \cdot ||_1 \).

It can be shown that, for any \( \phi \in X \) and \( \lambda \in [0, +\infty) \), Eqs. (1) have a unique solution \( u(\phi, \lambda) \) defined on \([0, \pi] \times [0, +\infty)\). A non-trivial aspect of this assertion is the statement that the domain of definition for \( u(\phi, \lambda) \) is all of \([0, \pi] \times [0, +\infty)\). We shall briefly return to this matter below.

For any \( \phi \in X \), \( \lambda \in [0, +\infty) \), \( x \in [0, \pi] \), and \( t \in [0, +\infty) \), we can let \( u(x, t; \phi, \lambda) \) denote the value of \( u(\phi, \lambda) \) at \((x, t)\). With this in mind, we can define, for any \( \lambda \in [0, +\infty) \), a nonlinear semigroup \{\( U_\lambda(t) \)\} on \( X \) by setting \( U_\lambda(t)\phi = u(\cdot, t; \phi, \lambda) \) for all \( \phi \in X \) and \( t \in [0, +\infty) \). It can be shown that \{\( U_\lambda(t) \)\} is strongly continuous.

Let \( \lambda \in [0, +\infty) \). By an equilibrium solution of (1) (corresponding to \( \lambda \)) we mean a function \( u_0 \in X \) such that \( U_\lambda(t)u_0 = u_0 \) for all \( t \in [0, +\infty) \). By virtue of (H\(_2\)), the origin \( \phi_0 = 0 \) in \( X \) is an equilibrium solution of (1) for every \( \lambda \in [0, +\infty) \).

To discuss the existence of other equilibrium solutions for (1), we introduce a sequence of real numbers \( \{\lambda_n\}^{\infty}_{n=1} \) by setting \( \lambda_n = n^2 / f'(0) \) for each integer \( n \geq 1 \). By virtue of (H\(_2\)), we have \( 0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ... \). We are now ready to state our first theorem.

**Theorem 1.** For any integer \( n \geq 1 \) and any number \( \lambda \in [\lambda_n, +\infty) \), Eqs. (1) have two equilibrium solutions \( u^\pm_n(\lambda) \) possessing the following three properties:

(i) \( u^\pm_n(\lambda) = 0 \) if and only if \( \lambda = \lambda_n \).

(ii) The mappings \( \lambda \mapsto u^\pm_n(\lambda) \) from \([\lambda_n, +\infty) \) into \( X \) are each continuous. In particular, \( u^\pm_n(\lambda) \to 0 \) as \( \lambda \to \lambda_n \). Also, \( ||u^\pm_n(\lambda)||_1 \to +\infty \) as \( \lambda \to +\infty \).
(iii) For any $\lambda \in \mathbb{R}_{+}^{\infty}$, $u_{n}^{\pm}(\lambda)$ has exactly $n+1$ zeros $x_{n}^{\pm}(\lambda)$, $x_{n}^{\pm}(\lambda)$ in $[0, \pi]$ with $0 = x_{0}^{\pm}(\lambda) < x_{1}^{\pm}(\lambda) < \ldots < x_{n}^{\pm}(\lambda) = \pi$.

Moreover, for each integer $q = 0, 1, \ldots, n-1$, we have
\[ (-1)^{q}u_{n}^{+}(x;\lambda) > 0 \text{ if } x_{n}^{+}(\lambda) < x < x_{n+1}^{+}(\lambda) \] and we have
\[ (-1)^{q}u_{n}^{-}(x;\lambda) < 0 \text{ if } x_{n}^{-}(\lambda) < x < x_{n+1}^{-}(\lambda). \]

In addition to the preceding assertions, we have that for any $\lambda \in [0, +\infty)$ Eqs. (1) have no equilibrium solutions other than the zero solution $u_{0} = 0$ and those elements $u_{n}^{\pm}(\lambda)$, $n \geq 1$, such that $\lambda_{n} < \lambda$.

On the basis of Assertion (ii) in Theorem 1, we may state that, for any integer $n \geq 1$, the two equilibrium solutions $u_{n}^{\pm}(\lambda)$ bifurcate from the zero solution as $\lambda$ increases from $\lambda_{n}$.

Now we come to our second theorem.

**Theorem 2.** For any $\phi \in X$ and any $\lambda \in [0, +\infty)$, there exists an equilibrium solution $u_{0}(\phi, \lambda)$ of (1) such that $U_{\lambda}(t)\phi \to u_{0}(\phi, \lambda)$ as $t \to +\infty$.

The question arises, given $\phi \in X$ and $\lambda \in [0, +\infty)$, to which of the equilibrium solutions described in Theorem 1 is $u_{0}(\phi, \lambda)$ equal? A partial answer to this query is given in the following theorem.

**Theorem 3.** For any $\lambda \in [0, \lambda_{1}]$, the zero solution $u_{0} = 0$ of (1) is globally asymptotically stable in the sense of Liapunov. In particular, for each $\phi \in X$ and $\lambda \in [0, \lambda_{1}]$, we have $||U_{\lambda}(t)\phi||_{1} \to 0$ as $t \to +\infty$. For any $\lambda \in (\lambda_{1}, +\infty)$, the zero solution $u_{0} = 0$ of (1) is unstable. For any $\lambda \in (\lambda_{1}, +\infty)$, the solutions $u_{n}^{\pm}(\lambda)$ are each asymptotically stable in the sense of Liapunov. Finally, for any integer $n \geq 2$ and any $\lambda \in [\lambda_{n}, +\infty)$, the solutions $u_{n}^{\pm}(\lambda)$ are each unstable.
Theorems 1-3 are proved in the article [1] already mentioned. We shall not repeat the proofs here but shall rather confine ourselves to making the following remarks.

Our approach to studying Eqs. (1) is to interpret (1) as a dynamical system on X and then to apply certain methods associated with the Liapunov theory of stability. The methods we have in mind are set forth in [2], [3] and [U] and are often referred to as the invariance principle in stability theory.

An essential tool in our use of the invariance principle is the following Liapunov functional:

\[ V_{\lambda}(\phi) = \int_0^\pi \left( \frac{1}{2} \phi'(x)^2 - \lambda \int_0^x f(\xi)d\xi \right)dx \quad (\phi \in X, \lambda \in [0, +\infty)). \quad (2) \]

For each \( \lambda \in [0, +\infty) \), Eq. (2) defines a functional \( V_{\lambda} \) mapping \( X \) into \( \mathbb{R} \). For any \( \phi \in X \) and \( \lambda \in [0, +\infty) \), it can be shown that

\[ \dot{V}_{\lambda}(U_{\lambda}(t)\phi) = -\int_0^\pi |u(t, x; \phi, \lambda)|^2 dx \quad (t > 0). \quad (3) \]

Consider any \( \phi \in X \) and \( \lambda \in [0, +\infty) \). Using \( V \lambda \) one can show that the solution \( u(\phi, \lambda) \) is defined everywhere on \([0, \pi] \times [0, +\infty)\). This is a matter which we have mentioned earlier in this note. Of more immediate interest is the fact that, using \( V \lambda \), one can show that \( u(\phi, \lambda) \) has a nonempty compact connected invariant \( \omega \)-limit set \( \omega(\phi, \lambda) \subseteq X \). Here, one also uses the invariance principle referred to two paragraphs above. That same principle together with Eq. (3) tells us that any element in \( \omega(\phi, \lambda) \) must be an equilibrium solution of (1).
Therefore, one now seeks the equilibrium solutions of Eqs. (1). This means that one studies the two-point boundary-value problem

\[ u''(x) + \lambda f(u(x)) = 0 \quad (0 < x < \pi, \lambda > 0). \]  
\[ u(0) = u(\pi) = 0 \]  

The results of our investigation are stated in Theorem 1. In particular, we see that, for any \( \lambda \in [0, +\infty) \), each equilibrium solution of (1) is isolated in \( X \). Hence, for any \( \lambda \in [0, +\infty) \), the set \( \omega(\phi, \lambda) \) consists of exactly one equilibrium solution of (1). From this there follows Theorem 2.

Theorem 3 is established using arguments from the classical theory of calculus of variations. We shall not attempt to describe these arguments here.

References


