OPTIMUM THREE IMPULSE TRAJECTORY GENERATOR

WITH PATCHED CONIC TRAJECTORY MODEL

DRA

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SUMMARY

This report documents a portion of the work accomplished in Contract NAS5-11385 to investigate the use of optimal multi-impulse trajectories as a nominal about which one may expand using asymptotic expansion techniques to obtain approximations to optimal low thrust trajectories. The work documented herein consists of the analysis and description of an optimal 3-impulse trajectory program. Unlike other optimum multi-impulse programs, this one incorporates a patched-conic trajectory model and is specifically designed for compatibility with the subsequent addition of the low thrust expansion approximation. A report presenting the equations for the expansion of an impulse into an optimal finite burn is being published concurrently with this report.

TABLE OF CONTENTS

I. INTRODUCTION

In this report we describe in detail a procedure for obtaining an optimal three-impulse, patched conic trajectory, from orbital motion about a "launch" planet (taken as the earth) to orbital motion about a target planet. A computer program implementing this procedure has been constructed, and some numerical results are presented and discussed in the last section.

Since we are using a patched conic formulation, the trajectory will consist of a sequence of Kepler arcs along each of which the vector Λ, adjoint to the velocity vector, and its time derivative $\dot{\Lambda}$ propagate as solutions of the Kepler variational equations. The end points of the successive Kepler arcs define a sequence of five instants in time at which various boundary and transversality conditions must be met:

1.   At $t = t_0$ insertion is made from orbital motion about the earth to an escape hyperbola. It is assumed that this insertion will be made by a high thrust, low specific impulse engine, with an option to jettison after use.

2. t_X is the time of exit from the earth's SOI (sphere of influence). Between t_0 and t_X the motion of the vehicle is along an earth focused Kepler hyperbola. At the SOI the attracting gravitational field switches from that of the earth to that of the sun, and at the same time transformation from geocentric to heliocentric coordinates is made. The vector Λ and all state variables are continuous across the SOI, but the transversality conditions require a discontinuity in $\dot{\Lambda}$ at this point.

3. t_I is the time at which a midcourse impulse is executed. Between t_X and t_I the vehicle moves along a sun focused ellipse. It is assumed that this midcourse impulse is made by a low thrust, high specific impulse engine, again

with an option to jettison after use. The presence of the impulse implies a discontinuity in velocity at t_I; the transversality conditions require continuity for both Λ and $\dot{\Lambda}$ and also that Λ and $\dot{\Lambda}$ be perpendicular.

4. t_N is the time of entry into the target's SOI. Between t_I and t_N the vehicle moves along a second sun focused Kepler ellipse. At t_N the attractive gravitational field switches from that of the sun to that of the target planet, and a transformation from heliocentric to planetocentric coordinates is made. All state variables and Λ are continuous, but, again, the transversality conditions require a discontinuity in $\dot{\Lambda}$.

5. t_f is the time for the retro-maneuver. The vehicle moves along a target-focused hyperbola between t_N and t_f, and is inserted into orbital motion about the target at t_f. It is assumed that this maneuver is executed by a high thrust, low specific impulse engine, with an option to jettison.

The three engines used for the impulses at t_o, t_I and t_f are characterized by their exhaust velocities c_1, c_2 and c_3 respectively. The dead weights (which may be jettisoned) are taken proportional to the initial mass, with proportionality factors k_1, k_2, k_3 respectively. Assuming that all three jettison options are exercised (and this is the case for the numerical results) the ratio of the final mass to the initial mass is easily shown to be

$$\prod_{i=1}^{3} \left[1 - (1 + k_i)(1 - e^{-\Delta v_i/c_i}) \right] \tag{1.1}$$

where the Δv_i, $i = 1, 2, 3$, are the magnitudes of the velocity discontinuities associated with the impulses at t_o, t_I and t_f respectively. The goal of the optimization procedure will be to maximize this mass ratio (or minimize its negative), subject to a set of initial and terminal constraints on the earth and target orbital motions, by proper selection of the three impulses.

Because of the constraints and discontinuities associated with the five special points, defined by t_o, t_X, t_I, t_N and t_f, and the fact that three coordinate systems are involved a rather elaborate system of notation has been developed and is given in Section II. Also included in Section II is the pertinent information on the coordinate transformations used, the equations of motion for the vehicle, the differential equations for the adjoint variables and the transversality conditions (without derivation) which are explicitly used in Section III. Derivations of the transversality conditions is made in Section VI, and a more complete discussion is given there.

The procedure for constructing the optimal three impulse orbit to orbit trajectory consists of a double iteration in which the time t_I and the position R_I of the midcourse impulse are the control parameters to be differentially corrected at the end of each pass through the double iteration. We outline this procedure, discuss the role of terminal constraints and motivate our choice of control parameters in Section III. An initial guess on the control parameters is, of course, required to start the double iteration. The method used for this initialization is presented in Section IV.

The first part of the double iteration is an iteration (also requiring initialization, given in Section IV) to find a sequence of Kepler arcs satisfying the terminal constraints and the current values of the control parameters. This iteration is described in Section V, where also complete details are given for the choice of terminal constraints implemented in our computer program. The second part of the double iteration involves the determination of inconsistencies in the propagation of the adjoint variables due to non-optimality of the current sequence of Kepler arcs. In our formulation these inconsistencies appear as failure of the adjoint variables to satisfy the transversality conditions at t_I. The calculations involved in the second part of the iteration are given in Section VI. The last part of the double iteration procedure is to differentially correct the control parameters, and

the basic idea involved is given in Section III. Since the optimization problem is sensitive, and the procedure for obtaining initial guesses is somewhat crude, a sophisticated differential correction procedure is required. The one selected for our program is described in Reference 1.

Finally, in Section VII we present and discuss the numerical results for an optimized three impulse trajectory from an orbit about the earth to an orbit about Mars.

II.   NOTATION AND BASIC EQUATIONS

We use a number of subscripts and superscripts, listed below, which may be appended either individually or in combination to the various symbols which follow:

Subscripts X, I, N refer to quantities evaluated at exit from the earth's sphere of influence, the midcourse impulse point and entry to the target's sphere of influence, respectively.

Subscripts o, f refer to quantities evaluated at the initial and final times respectively.

Subscripts E and T refer to quantities associated with earth and target respectively, including references to the earth centered and target centered coordinate systems. No subscript is used for solar reference.

Superscripts $+$ and $-$ refer to evaluation just before and just after a discontinuity.

R, r	are the position vector of the vehicle and its magnitude, except that R_I also refers to the position of the midcourse impulse.
$\dot{R}$, v	are the velocity vector of the vehicle and its magnitude.
t	is time
Λ, λ	the primer vector, adjoint to $\dot{R}$, and its magnitude. A subscript V is used to distinguish this variable in Section VI.
$\dot{\Lambda}$	time derivative of the primer vector and the negative of Λ_R, the vector adjoint to R.

$\bar{r}_E$, $\bar{r}_T$	radii of the earth and target spheres of influence, respectively.
μ_E, μ, μ_T	G times mass of earth, sun and target, respectively.
P, $\dot{P}$, $\ddot{P}$	planetary position, velocity and acceleration vectors.
$\bar{k}_E$, $\bar{k}_T$	unit polar vectors for earth and target.
c_1, c_2, c_3	exhaust velocities of the engines used to produce impulses at t_o, t_I and t_f, respectively.
k_1, k_2, k_3	masses of these engines and associated propellant tankage divided by m_o.
ΔV_1, ΔV_2, ΔV_3	magnitudes of velocity differences produced by these impulses.
Π	cost function = - final mass/initial mass.
a, e, i	semimajor axis, eccentricity, inclination.
H, h	angular momentum vector and its magnitude.
Ω	longitude of ascending node of planetocentric hyperbola on planet's equator.
ω	angle from ascending node to spacecraft on planetocentric hyperbolic trajectory, measures in the direction of motion.
γ	planetocentric flight path angle.

Subscripts E and T refer to <u>orbital</u> motion about earth and target, respectively. An additional subscript H refers to the hyperbolic trajectories.

Other symbols, used locally, are introduced as needed.

The planetary coordinates and velocities will be needed in the coordinate transformations and are available in various ephemerides. In our program we

used an approximate analytic ephemeris from J.P.L.[2]. The coordinate systems used in our program for the earth focused, sun focused and target focused segments of the trajectory are, respectively, geocentric, heliocentric and target centered, all with their axes parallel to a geocentric equatorial system. To transform a vector V and its time derivative $\dot{V}$ among these coordinate systems at a time $\bar{t}$, we use the formulas

$$V(\bar{t}) = V_E(\bar{t}) + P_E(\bar{t}) = V_T(\bar{t}) + P_T(\bar{t}) \tag{2.1}$$

$$\dot{V}(\bar{t}) = \dot{V}_E(\bar{t}) + \dot{P}_E(\bar{t}) = \dot{V}_T(\bar{t}) + \dot{P}_T(\bar{t})$$

The equations of motion of the vehicle are

$$\ddot{R}_E = -\mu_E R_E / r_E^3 \qquad t_o \le t \le t_X$$

$$\ddot{R} = -\mu R / r^3 \qquad t_X \le t \le t_N \tag{2.2}$$

$$\ddot{R}_T = -\mu_T R_T / r_T^3 \qquad t_N \le t \le t_f$$

with, in accordance with Eqs. (2.1),

$$R_X = R_{EX} + P_{EX}, \qquad \dot{R}_X = \dot{R}_{EX} + \dot{P}_{EX}$$

$$R_N = R_{TN} + P_{TN}, \qquad \dot{R}_N = \dot{R}_{TN} + \dot{P}_{TN} \tag{2.3}$$

implying continuity of the state variables, R, $\dot{R}$ over the spheres of influence. The accelerations are, however, not continuous. Using the approximate Keplerian nature of the motion of the earth and target about the sun, we have

$$\ddot{P}_{EX} = -(\mu + \mu_E) \frac{P_{EX}}{p_{EX}^3}$$

$$\ddot{P}_{TN} = -(\mu + \mu_T) \frac{P_{TN}}{p_{TN}^3} \tag{2.4}$$

where the p's are the magnitudes of the P's, and hence

$$\Delta\ddot{R}_X = \ddot{R}_X^+ - \ddot{R}_X^- = -\frac{\mu R_X}{r_X^3} + \frac{\mu_E R_{EX}}{r_{EX}^3} + (\mu + \mu_E) \frac{P_{EX}}{p_{EX}^3}$$

$$\tag{2.5}$$

$$\Delta\ddot{R}_N = \ddot{R}_N^+ - \ddot{R}_N^- = +\mu \frac{R_N}{r_N^3} - \mu_T \frac{R_{TN}}{r_{TN}^3} - (\mu + \mu_T) \frac{P_{TN}}{p_{TN}^3}$$

The primer vector Λ satisfies the Kepler variational equations:

$$\ddot{\Lambda} = -\mu_E \Lambda/r_E^3 + 3\mu_E (\Lambda \cdot R_E) R_E/r_E^5 \qquad t_o \leq t \leq t_X$$

$$\ddot{\Lambda} = -\mu \Lambda/r^3 + 3\mu (\Lambda \cdot R) R/r^5 \qquad t_X \leq t \leq t_N \tag{2.6}$$

$$\ddot{\Lambda} = -\mu_T \Lambda/r_T^3 + 3\mu_T (\Lambda \cdot R_T) R_T/r_T^5 \qquad t_N \leq t \leq t_f$$

Transversality conditions at the sphere of influence require discontinuities in $\dot{\Lambda}$ at these points, given by

$$\dot{\Lambda}_X^+ = \dot{\Lambda}_X^- + R_{EX} (\Lambda_X \cdot \Delta\ddot{R}_X) / (R_{EX} \cdot \dot{R}_{EX})$$

$$\tag{2.7}$$

$$\dot{\Lambda}_N^+ = \dot{\Lambda}_N^- + R_{TN} (\Lambda_N \cdot \Delta\ddot{R}_N) / (R_{TN} \cdot \dot{R}_{TN})$$

The primer vector Λ is continuous at the spheres of influence. Transversality conditions at the midcourse impulse point require

$$\Lambda = \Lambda_I = \lambda_I \frac{\dot{R}_I^+ - \dot{R}_I^-}{\Delta v_2} \quad \text{continuous}$$

$$\dot{\Lambda} = \dot{\Lambda}_I \qquad\qquad\qquad \text{continuous} \qquad\qquad\qquad (2.8)$$

$$\dot{\lambda}_I = \Lambda_I \cdot \dot{\Lambda}_I / \lambda_I = 0$$

The initial and final constraints provide the boundary conditions for the state variables, while the initial and final transversality conditions provide boundary values for the primer vector and its time derivative. Discussion of the state variable constraints appears in Sections IV and V and the initial and final transversality conditions are given in Section VI.

III.   THE DOUBLE ITERATION PROCEDURE

The optimization of the patched conic trajectory of a space vehicle, from orbit about the earth to orbit about a target planet, is a two point boundary value problem involving the six state variables, R and $\dot{R}$, and their six adjoint variables, Λ and $\dot{\Lambda}$ as dependent variables, with time as the independent variable. There are, thus, thirteen parameters associated with each terminus of the trajectory: the terminal time and the terminal values of the twelve dependent variables for a total of twenty six terminal values. In this problem t_o is a significant parameter since the time origin determines the configuration of the sun and planets, in whose gravitational fields the vehicle moves. Since there are six state variables a maximum of seven constraints (six state and one time) may be specified at each of the terminal points. For each unspecified, or "free", function of state variables and time at either terminus a transversality condition on the adjoint variables holds. Hence, in all cases, there will be seven initial and seven final conditions on the aggregate of state variables, adjoint variables and time.

The first basic idea involved in the double iteration procedure is to decouple the state and adjoint variables in such a way that we can find a sequence of Kepler arcs satisfying the terminal state constraints and then generate a time history of the adjoint variables along these Kepler arcs. This idea led us to characterize the midcourse impulse by the time t_I and the position R_I at which it occurs. These parameters t_I and R_I are regarded as "free", that is as unconstrained, although we must have current estimates for their values at each entry into this calculation. The second basic idea is now to decouple the trajectory problem (finding the sequence of Kepler arcs) into a "forward" and a "backward" part:

(1)   Find Kepler arcs from initial state and time values to t_I, R_I.

(2)   Find Kepler arcs from t_I, R_I to final state and time values.

The solution to each of these problems requires a total of four terminal values consistent with the terminal constraints imposed on the problem, and augmented, if necessary, by current estimates of "free" terminal parameters. In our implementation of this procedure the initial and final times are specified, together with three initial state constraints and three final state constraints. There is, therefore, no need to obtain current estimates on "free" terminal parameters. The modifications necessary for other distributions of terminal constraints are discussed briefly at the end of this section.

In the next section we give the procedure for obtaining an initial estimate for t_I and R_I, as well as the initialization procedure required for the trajectory iteration described in Section V. An iteration procedure is used for the trajectory calculation because the changes in coordinate system and the equations of motion across the earth and target spheres of influence render the problem awkward for analytic treatment.

The details for the generation of time histories for Λ and $\dot{\Lambda}$ are given in Section VI. Since the sequence of Kepler arcs obtained from the trajectory calculation cannot be assumed optimal, one would not expect to be able to propagate the adjoint variables by the Kepler variational equations along these arcs and also match all the transversality conditions. In fact, one would expect that exactly four transversality conditions would be violated because the eight time and terminal state constraints assumed would (in general) yield a unique sequence of Kepler arcs for each choice of the four free parameters t_I, R_I. Further, since these are the four free parameters for the trajectory calculation we choose the transversality conditions at t_I

-11-

$\dot{\Lambda}_I$ continuous

$$\dot{\lambda}_I = \frac{d}{dt} |\Lambda_I| = \frac{\Lambda_I \cdot \dot{\Lambda}_I}{\lambda_I} = 0 \qquad (3.1)$$

to measure the "mismatch". After completing a pass through both the trajectory and Λ, $\dot{\Lambda}$ calculations, differential corrections for t_I and R_I are computed to give the current estimates for the next pass. The principle involved is simple: defining the 4-vectors X and U as

$$X = \begin{pmatrix} \dot{\Lambda}_I^+ - \dot{\Lambda}_I^- \\[2mm] \tfrac{1}{2} \Lambda_I \cdot (\dot{\Lambda}_I^+ + \dot{\Lambda}_I^-)/\lambda_I \end{pmatrix} \qquad (3.2)$$

$$U = \begin{pmatrix} t_I \\[2mm] R_I \end{pmatrix}$$

one must find U such that X vanishes. Since X is a very complicated non-linear function of U, an analytic solution is not feasible. Linearized corrections to U may be obtained from

$$\Delta U = \begin{pmatrix} \Delta t_I \\[2mm] \Delta R_I \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial X}{\partial U} \end{pmatrix}^{-1} X \qquad (3.3)$$

and we update t_I and R_I by

$$\text{New } t_I = \text{old } t_I + \Delta t_I$$

$$(3.4)$$

$$\text{New } R_I = \text{old } R_I + \Delta R_I$$

The success of this method depends on having current estimates to t_I and R_I that are close to the values for which X vanishes. If this condition is not satisfied a more sophisticated correction procedure is required. The details of the correction method used in our program are given in Reference 1.

Sufficient detail has now been given to enable us to summarize the double iteration procedure succinctly, as follows:

I.   Generate, by iteration, a sequence of Kepler arcs satisfying initial and final time constraints and three constraints each on the initial and final states, and passing through the current estimate of R_I at the current estimate of t_I.

II.   Generate a time history of the adjoint variables, Λ and $\dot{\Lambda}$, along the Kepler arcs generated in I, in such a way as to satisfy the initial and final transversality conditions and the transversality conditions at the earth and target spheres of influence.

III.   Compute the X vector, which measures the failure of the adjoint variables to satisfy the transversality conditions at t_I, R_I and then differentially correct t_I and R_I to drive X to zero. Return to step I and iterate until convergence is obtained.

As noted above this procedure refers to the case for which the initial and final times and three state constraints at each of the terminal points are satisfied. To generalize the procedure is not difficult. For example, to leave one or more of these eight constraints unspecified requires that we impose the corresponding transversality conditions on the Λ, $\dot{\Lambda}$ calculation, and augment the "free" control variables with the unspecified parameters. That is, we augment the X vector, using the additional transversality conditions and the U vector by the additional free variables. An initial estimate for the additional free variables must be made, with subsequent current estimates coming from Step III. Just

how the initial estimate is made would depend on the selection of the free variables. To impose more than four constraints at either terminus, one selects four for use in the trajectory calculation and uses the mismatch in the remaining ones to augment the X vector. Note that if more than a total of eight terminal constraints are imposed the four free parameters t_I and R_I will, in general, be insufficient to eliminate all mismatches.

IV. INITIALIZATION PROCEDURES

Because apse and nodal crossings are known to represent optimum impulse points for certain types of maneuvers, procedures for obtaining initial estimates of t_I and R_I, given t_o and t_f, for each of these points were incorporated into the program. For impulses at either of the apses a heliocentric two-body trajectory between the ecliptic projections of the launch and target planets is computed by solving Lambert's problem. The indicated apse (i.e., either periapse or apoapse) time and position are then employed as the guesses for t_I and R_I if that apsis is passed between the specified times t_o and t_f. The nodal crossing points are obtained by forming the cross product of the angular momenta of the two planetary orbits. The intersection of the indicated node with the heliocentric two-body trajectory resulting from the solution of Lambert's problem above then yields the estimates of time and position t_I and R_I. If the indicated apse or node is not contained within the ecliptic two-body trajectory, a search for one of the other possible choices is then made and, if available within the trajectiry, is used to commence the iteration.

Another initialization procedure must be carried out at each entry to Step I of the double iteration procedure. As will be seen in the next section, the trajectory iteration requires current estimates of $\dot{R}_X$ and $\dot{R}_N$, the velocities of the vehicle at exit from the earth's SOI and at entry to the target's SOI, respectively. In obtaining our initial estimate for these velocities, we assume that the initial estimates for t_I and R_I are already available. Using again the ephemeris values for P_{Eo} and P_{Tf} (positions of earth and target at t_o and t_f respectively), we solve two heliocentric Lambert problems:

$$P_{Eo} \text{ to } R_I \text{ in transit time } (t_I - t_o)$$

(4.1)

$$R_I \text{ to } P_{Tf} \text{ in transit time } (t_f - t_I)$$

to obtain initial and final velocity vectors $\dot{\bar{P}}_{Eo}$, $\dot{\bar{P}}_{Tf}$ (which, of course, differ from the planetary velocities at these times, and hence are denoted by bars). These Lambert problems give a gross approximation to transfer from the earth to the target planet via R_I at time t_I, and have two characteristics that are of use to us:

(1)   the composite trajectory pierces both spheres of influence; manipulation of velocities at t_I, R_I could easily miss the earth and target altogether.

(2)   it turns out that the $\dot{R}_X$ and $\dot{R}_N$ associated with these Lambert problems are excellent approximations to their final values after convergence of the iteration.

Thus our first estimates on t_X, R_X, t_N, R_N are obtained simply by propagating one heliocentric Kepler ellipse forward from P_{Eo}, $\dot{P}_{Eo}$ to a time t_X such that

$$|R_X - P_{Eo}| = \bar{r}_E \tag{4.2}$$

and a second Kepler ellipse backward from P_{TF}, $\dot{P}_{TF}$ to a time t_N such that

$$|R_N - P_{TF}| = \bar{r}_T \tag{4.3}$$

and evaluating $\dot{R}_X$ as the velocity on the first ellipse at t_X and $\dot{R}_N$ as the velocity on the second ellipse at t_N.

The information so far generated in this section, t_I, R_I, t_X, $\dot{R}_X$, t_N, $\dot{R}_N$ provides the input for the first pass through Step I of the double iteration procedure. Updated values of t_X, t_N, $\dot{R}_X$ and $\dot{R}_N$, for subsequent passes, are calculated just before leaving Step I, and updated values of T_I and R_I are generated in Step III.

One further initialization is required, and should, perhaps, have been mentioned first. The coordinate systems used in the analysis depend on t_o and t_f and are defined in terms of the relations among the earth-equatorial, ecliptic and target-equatorial systems at these times. Since many terminal constraints are given most conveniently in terms of elliptic orbital parameters (e.g. inclination) relative to the planets involved, all such coordinate systems must be initialized. Note that if the generalization is made to "free" t_o and/or t_f, this initialization must precede every entry into Step I where the terminal constraints are used in the trajectory calculation.

V. THE TRAJECTORY ITERATION

The trajectory calculation presented in this section is a modified version of that given by S. Pines[3]. This report deals with a direct transfer from earth orbit to Mars orbit with no midcourse impulse, and uses terminal constraints which are different from ours. The basic idea of using positions and velocities at the spheres of influence to control the iteration is retained.

As noted at the end of the previous section, the input quantities for the trajectory iteration are t_o, t_X, t_I, t_N, t_f, $\dot{R}_X$, $\dot{R}_N$ and R_I, of which t_o and t_f are assumed given and fixed. The remaining parameters are estimated by the initialization procedure for the first pass through the trajectory iteration. For succeeding passes t_X, t_N, $\dot{R}_X$ and $\dot{R}_N$ are updated at the end of the trajectory calculation, while t_I and R_I are updated in Step III of the double iteration procedure.

Before presenting the iteration, we specify the constraints to be imposed at the terminal times t_o and t_f. The constraints selected for our program are the inclination and periapse distances for each of the orbital motions. The "free" terminal parameters are thus argument and time of periapse and longitude of the ascending node. The transversality conditions associated with the free parameters dictate that the initial and final impulses take place at the pericenters of the respective hyperbolic trajectories, that the pericenters and inclinations of the hyperbolic trajectories be also the pericenters and inclinations of the corresponding elliptic orbits. These conditions imply pericenter passage times of t_o and t_f for both ellipse and hyperbola at the earth and target, respectively.

The trajectory iteration consists of two steps which are alternated until convergence is obtained:

1. (a) Transform $\dot{R}_X$ and $\dot{R}_N$ to planetocentric coordinates using the current values of t_X and t_N to obtain from the ephemeris the necessary planetary velocities:

$$\dot{R}_{EX} = \dot{R}_X - \dot{P}_{EX}$$

$$\dot{R}_{TN} = \dot{R}_N - \dot{P}_{TN}$$

<div align="right">(5.1)</div>

1. (b)   Using the $\dot{R}_{EX}$ and $\dot{R}_{TN}$ found in 1. (a) and the terminal constraints find R_{EX} and R_{TN} and updated values for t_X and t_N such that

$$|R_{EX}| = \bar{r}_X \qquad |R_{TN}| = \bar{r}_T$$

and satisfying the conditions that R_{EX}, $\dot{R}_{EX}$, t_X define an earth focused Kepler hyperbola with the specified inclination, perigee distance, and perigee passage time t_o and that R_{TN}, $\dot{R}_{TN}$, t_N define a target focused Kepler hyperbola with the specified inclination, pericenter distance, and pericenter passage time t_f. The formulas for these calculations are derived below.

2. (a)   Using the updated t_X, t_N and the R_{EX}, R_{TN} found in 1. (b) transform R_{EX} and R_{TN} to heliocentric coordinates (again using the ephemeris)

$$R_X = R_{EX} + P_{EX}$$

<div align="right">(5.2)</div>

$$R_N = R_{TN} + P_{TN}$$

2. (b)   Solve the Lambert problems

$$R_X \rightarrow R_I \text{ with transit time } t_I - t_X$$

$$R_I \rightarrow R_N \text{ with transit time } t_N - t_I$$

to obtain updated values for $\dot{R}_X$ and $\dot{R}_N$.

3.   Return to step 1 and iterate until convergence on $\dot{R}_X$, $\dot{R}_N$, t_X and t_N are obtained. (12 iterations was typical for 16 digit convergence for our test cases). After convergence is obtained, extend the Kepler calculations of 1.(b) to obtain the initial conditions R_{Eo}, $\dot{R}^+_{Eo}$ for the earth hyperbola and final conditions R_{Tf}, $\dot{R}^-_{Tf}$ for the target hyperbola. Then the constraints on the orbital motions enable us to write

$$\dot{R}^-_{Eo} = \left[\frac{\mu_E \ (1+e_E)}{a_E \ (1-e_E)} \right]^{\frac{1}{2}} \frac{\dot{R}^+_{Eo}}{|\dot{R}^-_{Tf}|}$$

$$\dot{R}^+_{Tf} = \left[\frac{\mu_T \ (1+e_T)}{a_T \ (1-e_T)} \right]^{\frac{1}{2}} \frac{\dot{R}^-_{Tf}}{|\dot{R}^-_{Tf}|}$$

(5.3)

The semimajor axes and eccentricities, a_E, a_T, e_E, e_T of the elliptic orbits about earth and target are easily calculated from the periapse constraints imposed on these orbits, and the magnitudes of the pericenter velocities (given in Eq. (5.3)) are then readily verified.

      The solution of the Lambert problem is routine and we do not give any details here. The two point boundary value problem used in step 1.(b) (and also in step 3) is less well known. Following Pines[3], we wish to fit Kepler hyperbolas to given sphere of influence velocities and pericenter constraints. That is, for the earth we are given

$$\dot{R}_{EX}, \quad |R_{EX}| = \bar{r}_E \quad \text{at } t_X$$

$$|R_{Eo}| = r_o, \quad R_{Eo} \cdot \dot{R}^+_{Eo} = 0 \text{ at } t = 0$$

$$\text{inclination} = i_o$$

while for the target we are given

$$\dot{R}_{TN}, \quad |R_{TN}| = \bar{r}_T \text{ at } t = t_N$$

$$|R_{Tf}| = r_f, \quad R_{Tf} \cdot \dot{\bar{R}}_{Tf} = 0 \text{ at } t = t_f$$

letting

$$v^2_{EX} = \dot{R}_{EX} \cdot \dot{R}_{EX}, \quad v^2_{TN} = \dot{R}_{TN} \cdot \dot{R}_{TN} \qquad (5.4)$$

the semimajor axes of these hyperbolas are

$$\frac{1}{a_{EH}} = \frac{2}{\bar{r}_E} - \frac{v^2_{EX}}{\mu_E}$$

$$(5.5)$$

$$\frac{1}{a_{TH}} = \frac{2}{\bar{r}_T} - \frac{v^2_{TN}}{\mu_T}$$

where we use the subscript H to distinguish hyperbolic elements from elliptic elements. Also since r_o and r_f, the perigee and target pericenter distances are given we can obtain e_{EH} and e_{TH} from

$$r_o = a_{EH} (1-e_{EH}) \quad e_{EH} = \frac{a_{EH} - r_o}{a_{EH}}$$

$$(5.6)$$

$$r_f = a_{TH} (1-e_{TH}) \quad e_{TH} = \frac{a_{TH} - r_f}{a_{TH}}$$

Hence the magnitudes of the angular momentum vectors for these hyperbolas are given by

$$h_{EH} = \sqrt{\mu_E a_{EH} (1-e_{EH}^2)} = \sqrt{\mu_E r_o (1+e_{EH})} = \sqrt{\mu_E r_o \left(2 - \frac{r_o}{a_{EH}}\right)}$$

$$h_{TH} = \sqrt{\mu_T r_T \left(2 - \frac{r_f}{a_{TH}}\right)}$$

(5.7)

Also, from the definition of angular momentum

$$h_{EH}^2 = (R_{EX} \times \dot{R}_{EX})^2 = \bar{r}_E^2 v_{EX}^2 - (R_{EX} \cdot \dot{R}_{EX})^2$$

$$h_{TH}^2 = (R_{TN} \times \dot{R}_{TN})^2 = \bar{r}_T^2 v_{TN}^2 - (R_{TN} \cdot \dot{R}_{TN})^2$$

(5.8)

which yield for $R_{EX} \cdot \dot{R}_{EX}$ and $R_{TN} \cdot \dot{R}_{TN}$

$$R_{EX} \cdot \dot{R}_{EX} = \sqrt{\bar{r}_E^2 v_{EX}^2 - h_{EH}^2}$$

$$R_{TN} \cdot \dot{R}_{TN} = -\sqrt{\bar{r}_T^2 v_{TN}^2 - h_{TH}^2}$$

(5.9)

when use is made of Eq. (5.7) for h_{EH} and h_{TH}. The difference in the signs prefixing the radicals in Eq. (5.8) arises because at t_X the vehicle exits from the earth's SOI, while at t_N it is entering the target's SOI.

The angular momentum vectors for the hyperbolic orbits must be perpendicular to the velocity vectors $\dot{R}_{EX}$ and $\dot{R}_{TN}$. This means that, using the unit polar vectors $\bar{k}_E$ and $\bar{k}_T$, for earth and target, respectively, we may write

$$H_{EH} = \alpha_E (\bar{k}_E \times \dot{R}_{EX}) + \beta_E \dot{R}_{EX} \times (\bar{k}_E \times \dot{R}_{EX})$$

$$H_{TH} = \alpha_T (\bar{k}_T \times \dot{R}_{TN}) + \beta_T \dot{R}_{TN} \times (\bar{k}_T \times \dot{R}_{TN})$$

(5.10)

where the α's and β's are to be so selected that, using the magnitudes of the angular momenta, h_{EH} and h_{TH} of Eqs. (5.7) and the specified inclinations i_o and i_f

$$\bar{k}_E \cdot H_{EH} = h_{EH} \cos i_o \qquad \bar{k}_T \cdot H_{TH} = h_{TH} \cos i_f$$

$$H_{EH} \cdot H_{EH} = h_{EH}^2 \qquad H_{TH} \cdot H_{TH} = h_{TH}^2$$

(5.11)

Dropping subscripts temporarily, we thus seek α and β such that

$$H = \alpha (\bar{k} \times \dot{R}) + \beta \dot{R} \times (\bar{k} \times \dot{R})$$

(5.12)

with $\quad \bar{k} \cdot H = h \cos i$ given and $H \cdot H = h^2$ given.

From the first of these conditions

$$\bar{k} \cdot H = \beta (\bar{k} \times \dot{R})^2 = \beta v^2 \sin^2 j = h \cos i$$

(5.13)

where

$$\cos j = \frac{\bar{k} \cdot \dot{R}}{v}$$

(5.14)

and hence

$$\beta = \frac{h \cos i}{v^2 \sin^2 j}$$

(5.15)

From the second condition

$$\alpha^2 v^2 \sin^2 j + \beta^2 v^4 \sin^2 j = h^2$$

(5.16)

and, using the expression (5.15) for β

$$\alpha^2 = \frac{1}{v^2 \sin^2 j} \left[h^2 - \frac{h^2 \cos^2 i}{v^4 \sin^4 j} \cdot v^4 \sin^2 j \right]$$

<div align="right">(5.17)</div>

$$= \frac{h^2}{v^2 \sin^2 j} \left[1 - \frac{\cos^2 i}{\sin^2 j} \right]$$

This equation for α possesses

no solution for $\cos^2 i > \sin^2 j$

one solution, $\alpha = 0$, for $\cos^2 i = \sin^2 j$

<div align="right">(5.18)</div>

two solutions, $\alpha = \frac{\pm h}{v |\sin j|} \sqrt{1 - \frac{\cos^2 i}{\sin^2 j}}$ for $\cos^2 i < \sin^2 j$

Clearly, the ranges of i for which no solutions exist correspond to inclinations which are unattainable from a given velocity asymptote without an additional velocity impulse. Although such cases are very real possibilities, we exclude them from further consideration here because the special treatment required for this inclusion detracts from the primary purpose of the report.

The angular momentum vector, by definition, is

$$H = R \times \dot{R}$$

<div align="right">(5.19)</div>

and hence

$$\dot{R} \times H = R v^2 - \dot{R} (R \cdot \dot{R})$$

<div align="right">(5.20)</div>

or

$$R = \frac{\dot{R} \times H}{v^2} + \dot{R} \frac{R \cdot \dot{R}}{v^2}$$

<div align="right">(5.21)</div>

But, since H may now be assumed known in terms of α and β

$$\dot{R} \times H = \alpha \dot{R} \times (\bar{k} \times \dot{R}) + \beta \dot{R} \times (\bar{R} \times (\bar{k} \times \dot{R}))$$

$$= \alpha v^2 \bar{k} - \alpha (\bar{k} \cdot \dot{R}) \dot{R} - \beta v^2 (\bar{k} \times \dot{R})$$

(5.22)

so that

$$R = \alpha \bar{k} - \beta (\bar{k} \times \dot{R}) + \frac{\dot{R}}{v^2} \left[(R \cdot \dot{R}) - \alpha (\bar{k} \cdot \dot{R}) \right]$$

(5.23)

Restoring subscripts, we have

$$\cos j_E = \frac{\bar{k}_E \cdot \dot{R}_{EX}}{v_{EX}} \qquad \cos j_T = \frac{\bar{k}_T \cdot \dot{R}_{TN}}{v_{TN}}$$

$$\beta_E = \frac{h_{EH} \cos i_o}{v_{EX}^2 \sin^2 j_E} \qquad \beta_T = \frac{h_{TH} \cos i_f}{v_{TN}^2 \sin^2 j_T}$$

$$\alpha_E^2 = \frac{h_{EH}^2}{v_{EX}^2 \sin^2 j_E} \left(1 - \frac{\cos^2 i_o}{\sin^2 j_E} \right)$$

$$\alpha_T^2 = \frac{h_{TH}^2}{v_{TH}^2 \sin^2 j_T} \left(1 - \frac{\cos^2 i_f}{\sin^2 j_T} \right)$$

(5.24)

to obtain the angular momenta from Eq. (5.10). The positions R_{EX} and R_{TN} are given by Eq. (5.23) as

$$R_{EX} = \alpha_E \bar{k}_E - \beta_E (\bar{k}_E \times \dot{R}_{EX}) + \frac{\dot{R}_{EX}}{v_{EX}^2} \left[R_{EX} \cdot \dot{R}_{EX} - \alpha_E \cos j_E \right]$$

$$R_{TN} = \alpha_T \bar{k}_T - \beta_T (\bar{k}_T \times \dot{R}_{TN}) + \frac{\dot{R}_{TN}}{v_{TN}^2} \left[R_{TN} \cdot \dot{R}_{TN} - \alpha_T \cos j_T \right]$$

(5.25)

where the dot products may be evaluated from Eqs. (5.9).

To update the times t_X and t_N, we make use of the known semimajor axes, eccentricities and perigee passages times (t_o and t_f) of the hyperbolas to determine the eccentric anomalies E_X and E_T of the SOI points:

$$\bar{r}_E = a_{EH}(1-e_{EH}\cosh E_X) \qquad \bar{r}_T = a_{TH}(1-e_{TH}\cosh E_T) \tag{5.26}$$

and then use Kepler's equation to get the times:

$$n_E(t_X - t_o) = e_{EH}\sinh E_X - E_X$$

$$\tag{5.27}$$

$$n_T(t_N - t_f) = e_{TH}\sinh E_T - E_T$$

where

$$n_E = \sqrt{\frac{\mu_E}{|a_{EH}|^3}} \qquad n_T = \sqrt{\frac{\mu_T}{|a_{TH}|^3}} \tag{5.28}$$

Finally, after the iteration is complete, we carry the Kepler hyperbola calculation one step further to obtain

R_{Eo}, $\dot{R}_{Eo}^{+}$ by backwards propagation of the converged values for R_{EX}, $\dot{R}_{EX}$, t_X, to t_o

R_{Tf}, $\dot{R}_{Tf}^{-}$ by forwards propagation of the converged values for R_{TN}, $\dot{R}_{TN}$, t_N to t_f.

This completes the description of the calculations necessary to implement the trajectory iteration, Step I of the double iteration procedure, the updating of t_X, t_N, $\dot{R}_X$, $\dot{R}_N$ for the next entry into Step I, and the calculation of all parameters necessary for Step II of the double iteration procedure.

VI.   PRIMER VECTOR CALCULATIONS

The ultimate purpose of the optimum three impulse solution is to provide
a reference trajectory about which to expand a truncated series solution which
will closely approximate a low thrust trajectory. This approximate trajectory
should provide an adequate first guess for initiating the iterative numerical so-
lution of a low thrust trajectory optimization problem using an indirect optimi-
zation technique. Therefore, even though the three impulse optimization problem
may be formulated and solved within the context of ordinary calculus, it is help-
ful to pose the problem in terms of the variational calculus to facilitate imple-
mentation of the results.

Past experience with optimization problems involving patched conic tra-
jectories has indicated that the primer vector and its time derivative are extremely
sensitive in the proximity of a planet. This presents tremendous difficulties
in the solution of the boundary value problem because the transversality conditions
containing these variables evaluated at points near the planets become highly non-
linear functions of the independent parameters. As a consequence the boundary
value problem is unstable and virtually impossible to solve using differential
correction techniques. A method has been found [4], however, which greatly
alleviates the sensitivity of the boundary value problem for patched conic tra-
jectories. This method involves the use of the standard conic equations to re-
write constraints in the state at a highly sensitive point in terms of the state at
a less sensitive point. Specifically, it has been found that expressing constraints
at closest approach to a planet (e.g., specified passage distance) in terms of con-
ditions at the sphere of influence is sufficient to permit the solution of the boundary
value problem. It is for this reason that the optimization problem that follows is
formulated commencing at exit of Earth's sphere of influence and terminating at
entrance of the target's sphere of influence.

Proceeding formally as in Reference 4, we define a complete set of state variables for the two heliocentric trajectory segments which are joined at the midcourse impulse point. In the analysis to follow, the pre-subscripts 1 and 2 refer to the segments before and after the impulse, respectively. Thus, the state equations may be written

$$_i\dot{V} = - \frac{\mu}{_ir^3} \; _iR$$

$$_i\dot{R} = \; _iV$$

(6.1)

for $i = 1$ and 2. Then, defining $_1\tau$ as the time from exit of the Earth's SOI to the impulse point, $_2\tau$ as the time from the impulse point to entrance of the target planet's SOI, and s as the independent variable of integration, where $0 \leq s \leq 1$, such that

$$_2t = \; _2t^{(0)} + \; _2\tau s$$

$$_1t = \; _1t^{(0)} + \; _1\tau s$$

(6.2)

one may rewrite the state equations with s as the independent variable of integration as follows:

$$_iV' = - \; _i\tau \; \frac{\mu}{_ir^3} \; _iR$$

$$_iR' = \; _i\tau \; _iV$$

(6.3)

$$_it' = \; _i\tau$$

$$_i\tau' = 0$$

where the prime denotes differentiation with respect to s. This transformed problem fits well within the framework of the indirect method.

The variational Hamiltonian for the transformed problem is

$$h_V = \sum_{i=1}^{2} \left[{}_i\Lambda_V \cdot {}_iV' + {}_i\Lambda_R \cdot {}_iR' + {}_i\lambda_t \, {}_it' \right]$$

$$= \sum_{i=1}^{2} {}_i\tau \left[-\frac{\mu}{{}_ir^3} ({}_i\Lambda_V \cdot {}_iR) + {}_i\Lambda_R \cdot {}_iV + {}_i\lambda_t \right] \qquad (6.4)$$

Since h_V is known to be a constant of the motion, it is clear that the bracketed term for each leg is also a constant. In fact, the constant

$${}_ih = -\frac{\mu}{{}_ir^3} ({}_i\Lambda_V \cdot {}_iR) + {}_i\Lambda_R \cdot {}_iV + {}_i\lambda_t \qquad (6.5)$$

is the more familiar Hamiltonian for the problem in which time is the independent variable of integration. The adjoint equations may be written down using the general formula

$$\Lambda'_X = - \partial h_V / \partial X. \qquad (6.6)$$

That is,

$${}_i\Lambda'_V = - {}_i\tau \, \Lambda_R$$

$${}_i\Lambda'_R = {}_i\tau \, \frac{\mu}{{}_ir^3} \left[{}_i\Lambda_V - \frac{3}{{}_ir^2} ({}_i\Lambda_V \cdot {}_iR) \, {}_iR \right]$$

$$\qquad (6.7)$$

$${}_i\lambda'_t = 0$$

$${}_i\lambda'_\tau = - {}_ih$$

The equations (6.7) are known to possess analytic solutions. The variables $_i\lambda_t$ are obviously constants, the values of which we will subsequently determine to be arbitrary. Also, the variables $_i\lambda_\tau$ are seen to have the solutions

$$_i\lambda_\tau (s) - \,_i\lambda_\tau (0) = - \,_i h \, s. \tag{6.8}$$

Upon transforming the first two of equations (6.7) to derivatives with respect to time, it becomes clear that

$$_i\Lambda_R = - \,_i\dot{\Lambda}_V \tag{6.9}$$

which leads to the second order equation

$$_i\ddot{\Lambda}_V = \frac{3\mu}{_i r^5} (_i\Lambda_V \cdot \,_i R) \,_i R - \frac{\mu}{_i r^3} \,_i\Lambda_V. \tag{6.10}$$

and since this equation is identical in form to that of the state variational equations, its solution will also be of the same form as that of the variational equations. In particular, if one partitions the state transition matrix into the four 3 x 3 matrices

$$_i A = \frac{\partial_i R(t_2)}{\partial_i R(t_1)} \; ; \quad _i B = \frac{\partial_i R(t_2)}{\partial_i \dot{R}(t_1)} \; ; \quad _i C = \frac{\partial_i \dot{R}(t_2)}{\partial_i R(t_1)} \; ; \quad _i D = \frac{\partial_i \dot{R}(t_2)}{\partial_i \dot{R}(t_1)} \tag{6.11}$$

and if one is given the values of $_i\Lambda_V(t_1)$ and $_i\dot{\Lambda}_V(t_1)$, then the solution to (6.10) is

$$_i\Lambda_V(t_2) = \,_i A \,_i\Lambda_V(t_1) + \,_i B \,_i\dot{\Lambda}_V(t_1)$$

$$_i\dot{\Lambda}_V(t_2) = \,_i C \,_i\Lambda_V(t_1) + \,_i D \,_i\dot{\Lambda}_V(t_1) \tag{6.12}$$

This solution is applicable for t_2 less than or greater than t_1.

The transversality conditions that are sought are derived from the general condition

$$d\Pi + \sum_{i=1}^{2} \left[{}_i\Lambda_V \cdot d_i\dot{V} - {}_i\dot{\Lambda}_V \cdot d_iR + {}_i\lambda_t \, d_i t + {}_i\lambda_\tau \, d_i\tau \right]_0^1 = 0 \qquad (6.13)$$

in conjunction with the specified constraints on the boundary conditions, where Π denotes the performance index that is to be minimized. For purposes of illustration, we choose Π to be the negative of the ratio of final to initial mass. That is

$$\Pi = -\left[1-(1+k_1)(1-e^{-\Delta v_1/c_1}) \right]\left[1-(1+k_2)(1-e^{-\Delta v_2/c_2}) \right]\left[1-(1+k_3)(1-e^{-\Delta v_3/c_3}) \right] \qquad (6.14)$$

with notation as defined in Section II. For simplicity, we shall assume that the launch and capture orbit injection maneuvers are each coplanar with their respective parking orbits and that the periapses of the hyperbolic and elliptic orbits at each terminal are coincident. Under these assumptions the first and third impulses are written

$$\Delta v_1 = \sqrt{v_{\infty E}^2 + 2\mu_E/a_E(1-e_E)} - \sqrt{\mu_E(1+e_E)/a_E(1-e_E)}$$

$$\Delta v_3 = \sqrt{v_{\infty T}^2 + 2\mu_T/a_T(1-e_T)} - \sqrt{\mu_T(1+e_T)/a_T(1-e_T)} \qquad (6.15)$$

where $v_{\infty E}$ and $v_{\infty T}$ are the hyperbolic excess speeds of launch and target hyperbolic trajectories, respectively. The second impulse is equal to the difference in the heliocentric velocities on each side of the midcourse impulse, i. e.,

$$\Delta v_2 = \left| {}_2V(0) - {}_1V(1) \right| \qquad (6.16)$$

Thus, if the launch and target parking orbits are specified in terms of semi-major axis and eccentricity, it is seen that Π may be written functionally as

$$\Pi = \Pi(v_{\infty E}, \; _1V(1), \; _2V(0), \; v_{\infty T}) \tag{6.17}$$

such that

$$d\Pi = \Pi_{v_{\infty E}} dv_{\infty E} + \Pi_{v_{\infty T}} dv_{\infty T} + \Pi_{_1V(1)} \cdot d_1 V(1) + \Pi_{_2V(0)} \cdot d_2 V(0) \tag{6.18}$$

with

$$\Pi_{v_{\infty E}} = \frac{\partial \Pi}{\partial \Delta v_1} \; \frac{v_{\infty E}}{\sqrt{v_{\infty E}^2 + 2\mu_E / a_E (1 - e_E)}}$$

$$\Pi_{v_{\infty T}} = \frac{\partial \Pi}{\partial \Delta v_3} \; \frac{v_{\infty T}}{\sqrt{v_{\infty T}^2 + 2\mu_T / a_T (1 - e_T)}} \tag{6.19}$$

$$\Pi_{_2V(0)} = - \Pi_{_1V(1)} = \frac{_2V(0) - _1V(1)}{\Delta v_2} \; \frac{\partial \Pi}{\partial \Delta v_2}$$

where

$$\frac{\partial \Pi}{\partial \Delta v_i} = \left[1 - (1 + k_j)(1 - e^{-\Delta v_j / c_j}) \right] \left[1 - (1 + k_m)(1 - e^{-\Delta v_m / c_m}) \right] \frac{(1 + k_i)}{c_i} e^{-\Delta v_i / c_i} \tag{6.20}$$

$$i, \; j, \; m = 1, \; 2, \; 3; \quad i \neq j \neq m$$

The specified boundary conditions of the problem affect the transversality conditions through the differentials of the state appearing in equation (6.13). Employing the notation of Section II, one will recognize that

$$_1R(0) = R_X = R_{EX} + P_{EX} \quad ; \quad _1V(0) = \dot{R}_X = \dot{R}_{EX} + \dot{P}_{EX}$$

(6.21)

$$_2R(1) = R_N = R_{TN} + P_{TN} \quad ; \quad _2V(1) = \dot{R}_N = \dot{R}_{TN} + \dot{P}_{TN}$$

and since the planetary positions and velocities may be considered functions only of time, we may write

$$d\,_1R(0) = dR_{EX} + \dot{P}_{EX}d\,_1t(0) \quad ; \quad d\,_1V(0) = d\dot{R}_{EX} + \ddot{P}_{EX}d\,_1t(0)$$

(6.22)

$$d\,_2R(1) = dR_{TN} + \dot{P}_{TN}d\,_2t(1) \quad ; \quad d\,_2V(1) = d\dot{R}_{TN} + \ddot{P}_{TN}d\,_2t(1)$$

A standard formula for decomposing the differential of a position or velocity vector into the differentials of polar coordinates is given in Reference 4, and leads to

$$dR_{EX} = (\bar{k}_E \times R_{EX})\, d\Omega_E + (\bar{h}_E \times R_{EX})\, d\omega_{EX} + \left(\frac{\bar{k}_E \times \bar{h}_E}{\sin i_E} \times R_{EX}\right)d\,i_E$$

$$d\dot{R}_{EX} = \frac{\dot{R}_{EX}}{v_{EX}}\, dv_{EX} + (\bar{k}_E \times \dot{R}_{EX})\, d\Omega_E + (\bar{h}_E \times \dot{R}_{EX})(d\omega_{EX} - d\gamma_{EX}) + \left(\frac{\bar{k}_E \times \bar{h}_E}{\sin i_E} \times \dot{R}_{EX}\right)d\,i_E$$

$$dR_{TN} = (\bar{k}_T \times R_{TN})\, d\Omega_T + (\bar{h}_T \times R_{TN})\, d\omega_{TN} + \left(\frac{\bar{k}_T \times \bar{h}_T}{\sin i_T} \times R_{TN}\right)d\,i_T$$

(6.23)

$$d\dot{R}_{TN} = \frac{\dot{R}_{TN}}{v_{TN}}\, dv_{TN} + (\bar{k}_T \times \dot{R}_{TN})\, d\Omega_T + (\bar{h}_T \times \dot{R}_{TN})(d\omega_{TN} - d\gamma_{TN})$$

$$+ \left(\frac{\bar{k}_T \times \bar{h}_T}{\sin i_T} \times \dot{R}_{TN}\right)d\,i_T$$

where Ω, ω, i, and γ are the longitude of ascending node on the equator, the argument of position at the sphere of influence, the inclination relative to the equator, and the flight path angle at the sphere of influence, respectively, of the planetocentric hyperbola, and $\bar{h}$ denotes a unit vector in the direction of the planetocentric angular momentum. Boundary conditions which link the two trajectory segments are

$$_2R(0) = {}_1R(1); \qquad _2t(0) = {}_1t(1) \tag{6.24}$$

which imply

$$d_2R(0) = d_1R(1); \qquad d_2t(0) = d_1t(1). \tag{6.25}$$

Because the heliocentric flight times $_i\tau$ are parameters

$$d_i\tau(1) = d_i\tau(0) = d_i\tau \tag{6.26}$$

and these differential may be replaced in (6.13) using the relations

$$d_i\tau = d_it(1) - d_it(0) \tag{6.27}$$

The times at which the spheres of influence are crossed are generally of little immediate importance to the trajectory analyst. Of greater interest are the times at which the launch and target injection maneuvers are performed, and it is useful to replace the differentials of $_1t(0)$ and $_2t(1)$ with the differentials of the launch time t_o and the target injection time t_f. Denoting $t_{\infty E}$ and $t_{\infty T}$ as the times within the spheres of influence of Earth and the target planet, respectively, then

$$_1t(0) = t_o + t_{\infty E} \quad ; \quad _2t(1) = t_f - t_{\infty T}$$

$$d_1t(0) = dt_o + dt_{\infty E}; \quad d_2t(1) = dt_f - dt_{\infty T} \tag{6.28}$$

where

$$t_{\infty E} = (\mu_E/v_{\infty E}^3)(e_{EH} \sinh f_E - f_E)$$

$$e_{EH} = 1 + a_E (1-e_E) v_{\infty E}^2 /\mu_E$$

$$f_E = \cosh^{-1}\left[(1+\bar{r}_E v_{\infty E}^2/\mu_E)/e_{EH}\right]$$

$$t_{\infty T} = (\mu_T/v_{\infty T}^3)(e_{TH} \sinh f_T - f_T)$$ (6.29)

$$e_{TH} = 1+a_T (1-e_T) v_{\infty T}^2 /\mu_T$$

$$f_T = \cosh^{-1}\left[(1+\bar{r}_T v_{\infty T}^2 /\mu_T)/e_{TH}\right]$$

Thus the times within the spheres of influence are seen to be functions of the hyperbolic excess speeds plus other known functions and we may write

$$dt_{\infty E} = \frac{\partial t_{\infty E}}{\partial v_{\infty E}} \, dv_{\infty E}$$

(6.30)

$$dt_{\infty T} = \frac{\partial t_{\infty T}}{\partial v_{\infty T}} \, dv_{\infty T}$$

where

$$\frac{dt_{\infty E}}{dv_{\infty E}} = -\frac{3 t_{\infty E}}{v_{\infty E}} + \frac{2}{v_{\infty E}^2 e_{EH} \sinh f_E} \left[(e_{EH} \cosh f_E -1)\bar{r}_E - (e_{EH}-\cosh f_E)(1-e_E)a_E\right]$$

(6.31)

$$\frac{dt_{\infty T}}{dv_{\infty T}} = -\frac{3 t_{\infty T}}{v_{\infty T}} + \frac{2}{v_{\infty T}^2 e_{TH} \sinh f_T} \left[(e_{TH} \cosh f_T -1)\bar{r}_T - (e_{TH}-\cosh f_T)(1-e_T)a_T\right]$$

Finally, it is not difficult to note that the speed and the flight path angle at the sphere of influence are functions only of the hyperbolic excess speed and other known parameters. Consequently, one may replace the differentials of these two functions, which appear in (6.23), with the differential of the excess speed. Employing the relations

$$r_o = a_E (1-e_E); \qquad v_{oH}^2 = v_{\infty E}^2 + 2\mu_E/r_o$$

$$r_f = a_T (1-e_T); \qquad v_{fH}^2 = v_{\infty T}^2 + 2\mu_T/r_f \qquad (6.32)$$

$$\bar{r}_E v_{EX} \cos \gamma_{EX} = r_o v_{oH}; \qquad \bar{r}_T v_{TN} \cos \gamma_{TN} = r_f v_{fH}$$

one obtains

$$\frac{\dot{R}_{EX}}{v_{EX}} d v_{EX} - (\bar{h}_E \times \dot{R}_{EX}) d \gamma_{EX} = \frac{v_{\infty E}}{v_{oH}^2} \left[\dot{R}_{EX} + \frac{v_{oH}^2 - v_{EX}^2}{(R_{EX} \cdot \dot{R}_{EX})} R_{EX} \right] d v_{\infty E}$$

$$(6.33)$$

$$\frac{\dot{R}_{TN}}{v_{TN}} d v_{TN} - (\bar{h}_T \times \dot{R}_{TN}) d \gamma_{TN} = \frac{v_{\infty T}}{v_{fH}^2} \left[\dot{R}_{TN} + \frac{v_{fH}^2 - v_{TN}^2}{(R_{TN} \cdot \dot{R}_{TN})} R_{TN} \right] d v_{\infty T}$$

Upon substituting the results of equations (6.18) – (6.33) into the general equation (6.13) and setting the terms associated with the remaining independent differentials to zero individually yields the appropriate transversality conditions for our problem.

(a) $\left({}_2\dot{\Lambda}_V(0) - {}_1\dot{\Lambda}_V(1)\right) \cdot d\,{}_2R(0) = 0$

(b) $\left(\Pi_{{}_1V(1)} + {}_1\Lambda_V(1)\right) \cdot d\,{}_1V(1) = 0$

(c) $\left(\Pi_{{}_2V(0)} - {}_2\Lambda_V(0)\right) \cdot d\,{}_2V(0) = 0$

(d) $\left({}_2\Lambda_V(0) \cdot {}_2\dot{V}(0) - {}_1\Lambda_V(1) \cdot {}_1\dot{V}(1) - {}_2\dot{\Lambda}_V(0) \cdot {}_2V(0) + {}_1\dot{\Lambda}_V(1) \cdot {}_1V(1)\right) d\,{}_2t(0) = 0$

(e) $\left({}_1\Lambda_V(0) \cdot \ddot{R}_{EX} - {}_1\dot{\Lambda}_V(0) \cdot \dot{R}_{EX}\right) dt_o + \left({}_2\dot{\Lambda}_V(1) \cdot \dot{R}_{TN} - {}_2\Lambda_V(1) \cdot \ddot{R}_{TN}\right) dt_f = 0$

(f) $\bar{k}_E \cdot \left(R_{EX} \times {}_1\dot{\Lambda}_V(0) - \dot{R}_{EX} \times {}_1\Lambda_V(0)\right) d\,\Omega_E = 0$

(g) $\bar{h}_E \cdot \left(R_{EX} \times {}_1\dot{\Lambda}_V(0) - \dot{R}_{EX} \times {}_1\Lambda_V(0)\right) d\,\omega_{EX} = 0$

(h) $\dfrac{\bar{k}_E \times \bar{h}_E}{\sin i_E} \cdot \left(R_{EX} \times {}_1\dot{\Lambda}_V(0) - \dot{R}_{EX} \times {}_1\Lambda_V(0)\right) d\,i_E = 0$

(6.34)

(i) $\left\{\Pi_{v_{\infty E}} - \dfrac{v_{\infty E}}{v_{oH}^2} \left[\dot{R}_{EX} \cdot {}_1\Lambda_V(0) + \dfrac{v_{oH}^2 - v_{EX}^2}{(R_{EX} \cdot \dot{R}_{EX})} \left(R_{EX} \cdot {}_1\Lambda_V(0)\right)\right]\right.$

$\left. - \dfrac{dt_{\infty E}}{dv_{\infty E}} \left({}_1\dot{\Lambda}_V(0) \cdot \dot{R}_{EX} - {}_1\Lambda_V(0) \cdot \ddot{R}_{EX}\right)\right\} d\,v_{\infty E} = 0$

(j) $\bar{k}_T \cdot \left(\dot{R}_{TN} \times {}_2\Lambda_V(1) - R_{TN} \times {}_2\dot{\Lambda}_V(1)\right) d\,\Omega_T = 0$

(k) $\bar{h}_T \cdot \left(\dot{R}_{TN} \times {}_2\Lambda_V(1) - R_{TN} \times {}_2\dot{\Lambda}_V(1)\right) d\,\omega_{TN} = 0$

(l) $\dfrac{\bar{k}_T \times \bar{h}_T}{\sin i_T} \cdot \left(\dot{R}_{TN} \times {}_2\Lambda_V(1) - R_{TN} \times {}_2\dot{\Lambda}_V(1)\right) d\,i_T = 0$

(m) $\left\{ \Pi_{v_{\infty T}} + \dfrac{v_{\infty T}}{v_{fH}^2} \left[\dot{R}_{TN} \cdot {}_2\Lambda_V{}^{(1)} + \dfrac{v_{fH}^2 - v_{TN}^2}{R_{TN} \cdot \dot{R}_{TN}} \left(R_{TN} \cdot {}_2\Lambda_V{}^{(1)} \right) \right] \right.$

$\left. - \dfrac{dt_{\infty T}}{dv_{\infty T}} \left({}_2\dot{\Lambda}_V{}^{(1)} \cdot \dot{R}_{TN} - {}_2\Lambda_V{}^{(1)} \cdot \ddot{R}_{TN} \right) \right\} dv_{\infty T} = 0$

(6.34)
cont

Applying the standard interpretation to these equations, we regard a boundary condition to imply that the differential of the state variable which is specified must be zero, thereby satisfying identically the associated equation in (6.34). If, however, no constraint is imposed, then the differential is arbitrary and its coefficient must vanish, and it is this vanishing of the coefficient that represents the transversality condition. For the problem under consideration, we assume that no constraints are placed on the time and location of the midcourse impulse nor on the velocities on each side of the impulse. Consequently, the coefficients of the differentials in Equations (a) - (d) in (6.34) must each vanish individually. From (a) we determine that the time derivative of the primer is continuous over the second impulse. The equations (b) and (c) define the primer vector explicitly at the midcourse impulse. They indicate that, for the performance index of our problem, the primer is continuous over the impulse and directed along the impulse. The use of the results of (a) - (c) in equation (d) leads to the conclusion that, on the optimal trajectory, the derivative of the primer at the second impulse is perpendicular to the impulse and therefore to the primer itself. The equations (e) - (m) yield transversality conditions associated with parameters at the launch and target planets, and are used as described below. Note that λ_t has disappeared entirely from the equations; hence the earlier statement that λ_t is arbitrary.

The evaluation of the adjoint variables along the trajectory is accomplished with the use of equations (6.12) which require that both the primer vector and its time derivative be known at one instant in time. Parts (b) and (c) of (6.34) give the primer vector at one point in time for each trajectory segment. To define the time derivative of the primer vector on each segment at the same time requires a total of six independent equations (i.e., three for each segment). For this purpose

we choose parts (f), (g) and (i) for use on the first trajectory segment and parts (j), (k) and (m) for the second segment. The equations (h) and (l) are not used for this purpose because the trajectory iteration procedure described in Section V assumes that inclination at both ends are specified. In defining the partitioned transition matrices for the first segment, let the t_1 of equations (6.11) represent the time at the impulse and let t_2 represent the time at exit of the Earth's sphere of influence. Then using (6.12) to substitute for ${}_1\Lambda_V(0)$ and ${}_1\dot{\Lambda}_V(0)$ in (f), (g) and (i), and using the vector operation identity

$$\mathbf{X} \cdot (\mathbf{Y} \times \mathbf{Z}) = (\mathbf{X} \times \mathbf{Y}) \cdot \mathbf{Z}$$

yields the equations

$$(\bar{k}_E \times R_{EX}) \cdot ({}_1C_1\Lambda_V(1) + {}_1D_1\dot{\Lambda}_V(1)) - (\bar{k}_E \times \dot{R}_{EX}) \cdot ({}_1A_1\Lambda_V(1) + {}_1B_1\dot{\Lambda}_V(1)) = 0$$

$$(\bar{h}_E \times R_{EX}) \cdot ({}_1C_1\Lambda_V(1) + {}_1D_1\dot{\Lambda}_V(1)) - (\bar{h}_E \times \dot{R}_{EX}) \cdot ({}_1A_1\Lambda_V(1) + {}_1B_1\dot{\Lambda}_V(1)) = 0 \qquad (6.35)$$

$$\Pi_{v_{\infty E}} - \left[\frac{v_{\infty E}}{v_{oH}^2}\left(\dot{R}_{EX} + \frac{v_{oH}^2 - v_{EX}^2}{R_{EX} \cdot \dot{R}_{EX}} R_{EX}\right) - \frac{dt_{\infty E}}{dv_{\infty E}} \ddot{R}_{EX}\right] \cdot \left({}_1A_1\Lambda_V(1) + {}_1B_1\dot{\Lambda}_V(1)\right)$$

$$- \frac{dt_{\infty E}}{dv_{\infty E}} \dot{R}_{EX} \cdot \left({}_1C_1\Lambda_V(1) + {}_1D_1\dot{\Lambda}_V(1)\right) = 0$$

Upon noting the identity

$$\mathbf{Y} \cdot (\mathbf{Q}\mathbf{X}) = (\mathbf{Q}^T \mathbf{Y}) \cdot \mathbf{X} \qquad (6.36)$$

where Q is a matrix and X and Y are vectors, and reorganizing equations (6.35), one obtains

$$\left[{}_1D^T(\bar{k}_E \times R_{EX}) - {}_1B^T(\bar{k}_E \times \dot{R}_{EX})\right] \cdot {}_1\dot{\Lambda}_V^{(1)} = \left[{}_1A^T(\bar{k}_E \times \dot{R}_{EX}) - {}_1C^T(\bar{k}_E \times R_{EX})\right] \cdot {}_1\Lambda_V^{(1)}$$

$$\left[{}_1D^T(\bar{h}_E \times R_{EX}) - {}_1B^T(\bar{h}_E \times \dot{R}_{EX})\right] \cdot {}_1\dot{\Lambda}_V^{(1)} = \left[{}_1A^T(\bar{h}_E \times \dot{R}_{EX}) - {}_1C^T(\bar{h}_E \times R_{EX})\right] \cdot {}_1\Lambda_V^{(1)}$$

$$\left\{\frac{dt_{\infty E}}{dv_{\infty E}} {}_1D^T\dot{R}_{EX} + {}_1B^T\left[\frac{v_{\infty E}}{v_{oH}^2}\left(\dot{R}_{EX} + \frac{v_{oH}^2 - v_{EX}^2}{R_{EX} \cdot \dot{R}_{EX}} R_{EX}\right) - \frac{dt_{\infty E}}{dv_{\infty E}} \ddot{R}_{EX}\right]\right\} \cdot {}_1\dot{\Lambda}_V^{(1)} = \Pi_{v_{\infty E}}$$

(6.37)

$$- \left\{\frac{dt_{\infty E}}{dv_{\infty E}} \cdot {}_1C^T\dot{R}_{EX} + {}_1A^T\left[\frac{v_{\infty E}}{v_{oH}^2}\left(\dot{R}_{EX} + \frac{v_{oH}^2 - v_{EX}^2}{R_{EX} \cdot \dot{R}_{EX}} R_{EX}\right) - \frac{dt_{\infty E}}{dv_{\infty E}} \ddot{R}_{EX}\right]\right\} \cdot {}_1\Lambda_V^{(1)}$$

But these three equations, when written in matrix form, are

$$Q {}_1\dot{\Lambda}_V^{(1)} = Y \tag{6.38}$$

which has the solution

$$\tag{6.39} {}_1\dot{\Lambda}_V^{(1)} = Q^{-1}Y$$

Proceeding similarly for the second leg with t_1 again representing the time at the impulse but t_2 representing the time at entry of the target sphere of influence, the equations for the second segment corresponding to (6.37) are

$$\left[{}_2D^T(\bar{k}_T \times R_{TN}) - {}_2B^T(\bar{k}_T \times \dot{R}_{TN})\right] \cdot {}_2\dot{\Lambda}_V^{(0)} = \left[{}_2A^T(\bar{k}_T \times \dot{R}_{TN}) - {}_2C^T(\bar{k}_T \times R_{TN})\right] \cdot {}_2\Lambda_V^{(0)}$$

$$\left[{}_2D^T(\bar{h}_T \times R_{TN}) - {}_2B^T(\bar{h}_T \times \dot{R}_{TN})\right] \cdot {}_2\dot{\Lambda}_V^{(0)} = \left[{}_2A^T(\bar{h}_T \times \dot{R}_{TN}) - {}_2C^T(\bar{h}_T \times R_{TN})\right] \cdot {}_2\Lambda_V^{(0)}$$

$$\left\{\frac{dt_{\infty T}}{dv_{\infty T}} {}_2D^T\dot{R}_{TN} - {}_2B^T\left[\frac{v_{\infty T}}{v_{fH}^2}\left(\dot{R}_{TN} + \frac{v_{fH}^2 - v_{TN}^2}{R_{TN} \cdot \dot{R}_{TN}} R_{TN}\right) + \frac{dt_{\infty T}}{dv_{\infty T}} \ddot{R}_{TN}\right]\right\} \cdot {}_2\dot{\Lambda}_V^{(0)} = \Pi_{v_{\infty T}}$$

(6.40)

$$- \left\{\frac{dt_{\infty T}}{dv_{\infty T}} {}_2C^T\dot{R}_{TN} - {}_2A^T\left[\frac{v_{\infty T}}{v_{fH}^2}\left(\dot{R}_{TN} + \frac{v_{fH}^2 - v_{TN}^2}{R_{TN} \cdot \dot{R}_{TN}} R_{TN}\right) + \frac{dt_{\infty T}}{dv_{\infty T}} \ddot{R}_{TN}\right]\right\} \cdot {}_2\Lambda_V^{(0)}$$

The procedure for evaluating the primer vector and its time derivative along a trajectory resulting from the iteration of Section V is now clear. From equations (6.34) parts (b) and (c), the primer vector at the impulse point is evaluated. This, in conjunction with information pertaining to the trajectory (including the transition matrices for the two segments), is used in equations (6.37) and (6.40) to obtain the $\dot{\Lambda}_V$'s on both sides of the impulse. These will, in general, not satisfy either (6.34), part (a) or part (d). A differential correction scheme is then employed to adjust the time and position of the second impulse until the four equations represented by parts (a) and (d) are satisfied.

The preceding is sufficient to cover the problem with specified launch and target arrival dates and specified inclination at the two terminals. One may, however, optimize any combination of these parameters using the remaining equations in (6.34) by simply expanding the order of the boundary value problem implied in the preceding paragraph. For example, the planetocentric inclinations at launch and/or target may be optimized by driving to zero the coefficients of (h) and (l), respectively. One simply adds these coefficients to the other list of boundary conditions, parts (a) and (d), and treats i_E and i_T as independent parameters along with the position and time of the impulse. Similarly, the coefficients of dt_o and dt_f in part (e) must vanish independently if t_o and t_f are to be optimized. If both t_o and t_f are left open but the flight time $t_f - t_o$ is constrained, then $dt_o = dt_f$, and the sum of the two coefficients in (e) must vanish, thereby providing a single boundary condition with either t_o or t_f serving as the independent parameter.

Once the boundary value problem is solved, one may elect to obtain the primer vector within the planetocentric segments of the trajectory. This may be accomplished by proceeding in both directions from the second impulse along the heliocentric arcs using equations (6.12). At the spheres of influence the discontinuities in the $\dot{\Lambda}_V$'s are applied as defined by equations (2.7). Then the equations (6.12) once again apply for propagating the adjoint variables along the planetocentric conic trajectory.

VII. NUMERICAL RESULTS

To test the approach described in the preceding sections, an Earth-Mars transfer was chosen for illustrative purposes. A trajectory leaving Earth orbit at noon, January 21, 1977 and arriving in Mars orbit 270 days later on October 18, 1977 was selected. For computational purposes, the altitudes of the initial and final circular orbits were both taken to be zero. The assumed characteristics of the first stage included a jet exhaust speed c_1 of 4.4 km/sec and a structural factor k_1 of 0.1. The corresponding parameters of the second and third stages were assumed to be as follows:

$$c_2 = 30 \text{ km/sec}, \quad k_2 = 0.03, \quad c_3 = 3 \text{ km/sec}, \quad k_3 = 0.1.$$

The heliocentric ecliptic transfer angle for this mission is approximately 315 degrees, commencing at a longitude of 121 degrees and reaching Mars at a longitude of 76 degrees. The longitude of ascending node of Mars orbit on the ecliptic plane is about 49 degrees while the longitude of aphelion of the heliocentric ecliptic two-body trajectory connecting Earth and Mars occurs at a longitude of 65 degrees. Thus, it is clear that both apsides and both nodes are contained within the heliocentric ecliptic transfer and therefore each of the four programmed starting guesses for t_I and R_I may be invoked for this case. But, because the longitudes of the line of nodes and the line of apsides differ by only 17 degrees, the starting guesses obtained by using the ascending node and the apoapse point are very close as are the guesses associated with the descending node and the periapse point.

The solution to the problem posed may be characterized as follows. About 99 days following departure from Earth the intermediate impulse of 3.8 km/sec is applied at a longitude of 259 degrees and a latitude of -1 degree. The heliocentric distance at the impulse is about 0.64 AU. The impulses at launch and arrival are 4.36 km/sec and 4.16 km/sec, respectively, and the final to initial mass ratio is 0.047.

The programmed starting guess closest to this solution is the periapse point which lies 14 degrees in longitude from the solution. The periapse distance of the heliocentric ecliptic trajectory is about 0.425 AU. Although this would seem to be a reasonably close first guess, it was found that the program would not converge to the solution from the ecliptic periapse point, nor from any of the other programmed first guesses for that matter. A principal reason for this is that the mathematical definition of the performance index, as stated in equation (6.14), is not always descriptive of the actual situation. For example, it is clear that if a velocity impulse is sufficiently large, the propellant mass requirements can exceed the difference in the initial stage mass and the engine and tankage mass such that the corresponding term within the square brackets of (6.14) becomes negative. Although this represents a physically unrealizable situation, the circumstance should not hamper convergence because the partial derivatives of the performance index with respect to the velocity impulse will still be negative. However, if two velocity impulses are sufficiently large that the corresponding bracketed terms in (6.14) are both negative, then the partial derivatives of Π with respect to each of those impulses will be positive and hence indicate that the desired solution is in the direction of increasing velocity impulses. The result is that subsequent iterations are unable to move the trial impulse point and time to a region where physically meaningful solutions are available, and the technique simply fails to converge. The importance of this problem is placed in perspective when one realizes that each of the four programmed starting guesses for the problem posed results in precisely this situation.

This problem may be circumvented by first solving the problem with the structural factors, k_i, set to zero. This prevents the individual stage payload fractions from becoming negative and thereby avoids the problem. The resulting solution should then represent a reasonable first guess to the problem with non-zero structural factors. This approach was attempted with each of the

programmed starting guesses. As might be expected for this case, starting from either the ascending node or the aphelion point, the program failed to converge because the starting guesses were far from the known solution. Starting from either the descending node or the periapse point, successive iterations did proceed toward the solution. However, as the transfer angle on the second trajectory segment (i.e., following the intermediate impulse) approached 180 degrees (note that either first guess results in an initial angle of greater than 180 degrees) the improvement per iteration slowed significantly and finally halted altogether. This was due to the fact that the near 180 degree three-dimensional transfer between specified points represents a highly sensitive region in which the behavior of the state and adjoint variables are transient. This results in irregularities in the multi-dimensional surface representing the end conditions, and these irregularities can present an insurmountable obstacle using techniques which depend on local slopes of that surface, such as the one presented here.

Various approaches were attempted to alleviate this latter problem. Direct minimization of the performance index (a feature of the iterator employed [1]) starting from the periapse point was unsuccessful because progress halted at the 180 degree transfer point. Employing a first guess on the heliocentric ecliptic trajectory that was in the appropriate region in terms of angular position was also unsuccessful because the curvature of the surface is such that the iterator over corrects in longitude causing the next trial trajectory to cross the offensive barrier once again.

It is believed that the specific problem investigated displays the major difficulties that one would expect to encounter with the approach described herein. The results of this case imply that, to use the approach effectively, a better method for obtaining initial guesses for the time and location of the midcourse impulse must be developed, because the radius of convergence for some problems will obviously be very small. Alternatively, improved techniques for controlling the

magnitude and direction of the steps taken by the iterator could be developed. It is suggested that more extensive use of the program be made for a variety of missions as the need and opportunity arises, and that results of these studies be employed to assist in the development of the necessary improved starting techniques.

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