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MODAL ANALYSIS FOR LIAPUNOV STABILITY OF ROTATING ELASTIC BODIES

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UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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MODAL ANALYSIS FOR LIAPUNOV STABILITY OF ROTATING ELASTIC BODIES

André D. Colin

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Prepared for: George C. Marshall Space Flight Center
Marshall Space Flight Center, Alabama 35812

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ABSTRACT

The study defined under Contract NAS8-28358 consisted of four parallel efforts: (1) Modal analyses of elastic continua for Liapunov stability analysis of flexible spacecraft; (2) Development of general-purpose simulation equations for arbitrary spacecraft; (3) Evaluation of alternative mathematical models for elastic components of spacecraft; and (4) Examination of the influence of vehicle flexibility on spacecraft attitude control system performance.

This report includes a complete record of achievements under tasks (1) and (3) above, in the form of technical appendices, and a summary description of progress to date under tasks two and four.

Task (1) has provided the basis for the Ph. D. dissertation of André Colin (see Appendix 3, in Volume 2 of this report). This task in itself required two phases of investigation: modal analysis and stability analysis. The modal analysis is accomplished for a range of continuum models (strings, beams and thin plates with various boundary conditions on spinning spacecraft) by means of singular perturbation methods, and the stability analysis is accomplished by using Liapunov theorems with the momentum-constrained Hamiltonian as the testing function.

Task (2) is the basis for the Ph. D. dissertation of Arthur S. Hopkins, which is still in progress.

Task (3) is the subject of two technical papers by the Principal Investigator, included here as Appendices 1 and 2. In these papers the range of applicability of various discrete and continuous models of nonrigid spacecraft is examined. It is concluded that there is a domain of engineering applicability for each of the models considered, but that finite elements models are generally the most valuable for flexible spacecraft simulations.

Task (4) is currently receiving primary attention by the Principal Investigator and a postdoctoral scholar, Dr. Yoshiaki Ohkami. Results will be described in forthcoming documents.

UNIVERSITY OF CALIFORNIA

Los Angeles

MODAL ANALYSIS FOR LIAPUNOV STABILITY OF ROTATING ELASTIC BODIES

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering

by

André Daniel Colin

1973

ACKNOWLEDGMENT

First, I would like to thank Dr. Willems of the University of Louvain who introduced me to space vehicle problems.

I also wish to express my deepest gratitude to my advisor, Dr. Likins, for his guidance throughout the development of this dissertation, for his collaboration and his availability. I am also grateful to Drs. Lange, Leondes, Cole and Eisenberg for serving on my doctoral committee.

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NOMENCLATURE

w	transverse displacement
$P(\ )$	external axial load
P_α	harmonic function of the vibration
$(\cdot)$	derivative with respect to time
$(')$	spatial derivative with respect to dimensionless variables
y,s,x,r	dimensionless variables
$h_i(\ )$	terms of the expansion of the modes away from the boundaries
$g_i(\ )$	terms of the boundary layer expansions
$f_i(\ )$	terms of the boundary layer expansions
$\tilde{y},y^*$	boundary layer coordinates
$\tilde{x},x^*$	boundary layer coordinates
$\tilde{r},r^*$	boundary layer coordinates
y_η,x_η,r_η	fixed quantity introduced in the matching process
u	radial displacement
h	thickness of the disk
m	coefficient of the argument in the circular dependence of the modes
$\mathscr{L}_1, \mathscr{L}_2$	Legendre functions of the first and second kind, respectively.
Ai, Bi	Airy integrals
T	built-in tension
T_0	minimum built-in tension
k	defined by $T = k^2 T_0$
TST	transcendentally small terms
EI	flexural stiffness of the beam

NOMENCLATURE (Cont'd)

E	modulus of elasticity in the modal analysis and identity matrix in the stability analysis
ε	perturbation parameter
$\nu_i(\varepsilon), \kappa_i(\varepsilon)$	asymptotic sequences
$\mu_i(\varepsilon), \delta_i(\varepsilon)$	asymptotic sequences
μ	linear density of the beam or mass per unit volume for the membrane
ξ, μ, ζ	reference axis for the description of the transverse displacement
ρ, θ	polar coordinates of an element dm of the membrane
ν	Poisson modulus
ϕ_α	mode shapes for the beam
$\phi_{\alpha,m}$	radial dependence of the membrane mode shapes
ω_α	eigenfrequency of the beam vibrations
$\omega_{\alpha,m}$	eigenfrequency of the membrane vibrations
Ω	nominal spin rate
$\lambda_\alpha(\varepsilon)$	eigenfrequency of the beam in the transformed equation
λ_i	terms of the expansion of the eigenfrequence
μ_0^2	defined by $\lambda_0^2 = k^2\mu_0^2$
$\lambda_{\alpha,m}(\varepsilon)$	eigenfrequency of the membrane in the transformed equation
$\varepsilon_\theta\ \varepsilon_\rho$	tangential and radial strains, respectively
$\sigma_\theta\ \sigma_\rho$	tangential and radial stresses, respectively
∇_ρ^2	Laplacian operator $\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$
∇_m^2	Linear differential operator $\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}$

NOMENCLATURE (Cont'd)

γ	defined by $\gamma = 1 + m$
k_1^2	defined by $k_1^2 = \dfrac{3+\nu}{1+2k^2+\nu}$
k_2^2	defined by $k_2^2 = \dfrac{1+3\nu}{1+2k^2+\nu}$
k_3^2	defined by $k_2^2 = k_1^2\, k_3^2$
r_1	$k_1 r$
$F(\alpha,\beta,\gamma;x)$	hypergeometric function
CM	center of mass
N	location of system CM when steadily spinning
Body B	portion of system identified as rigid (core)
Body A	portion of system identified as flexible (appendages)
T	kinetic energy of complete system
H	Hamiltonian
H_c	constrained Hamiltonian
V	potential energy
U_d	potential energy of deformation
K	general spinning stiffness matrix
$\bar{K}$	spinning stiffness matrix for the planar model
$\mathscr{M}$	mass of complete system
$\underline{p}$	inertial generic position vector
$\{\underline{b}\}$	vector basis fixed in Body B identified as $(\hat{\underline{b}}_1,\hat{\underline{b}}_2,\hat{\underline{b}}_3)^T$
$\{\underline{n}\}$	inertial vector basis
$\underline{c}$	vector from CM to N
$\underline{u}$, u_i	deformation vector and its representation in $\{\underline{b}\}$

NOMENCLATURE (Cont'd)

$\underline{h}, {}^{b}h, h$	angular momentum vector, its representation in $\{\underline{b}\}$ and its magnitude
$\underline{\rho}$	body generic position vector
Θ	orthogonal transformation relating $\{\underline{b}\}$ to $\{\underline{n}\}$
$\theta_1, \theta_2, \theta_3$	Euler (attitude) angles
$\underline{\omega}, \omega_i$	inertial angular velocity vector and its representation in vector basis $\{\underline{b}\}$
$\underline{\Gamma}, \Gamma_i$	position vector of element dm subsequent to spin and its representation in $\{\underline{b}\}$
I_0^N	inertia matrix about point N of the complete system at steady state
$\underline{I}^N, I^N$	inertia dyadic of complete system about point N and its inertia matrix representation is $\{\underline{b}\}$
$\underline{I}_A^N, I_A^N$	inertia dyadic of appendage about point N and its inertia matrix representation in $\{\underline{b}\}$
$\underline{I}_B^N, I_B^N$	inertia dyadic of core about point N and its inertia matrix representation in $\{\underline{b}\}$
A,B,C	diagonal elements of I_0^N
Δ	inertia matrix about point N consisting of first order appendage terms
U	inertia matrix about point N consisting of second order appendage terms
ω_{SS}	spin rate at steady state
Ω^2	diagonal matrix made of the squares of the modal frequencies
β_ν	generalized coordinates of deformation
ϕ_ν	mode shapes representation in the stability analysis
β	column matrix of the modal coordinates
ϕ	column matrix of the mode shapes

NOMENCLATURE (Cont'd)

M	symmetric matrix defined by $\left[\int_A \phi\phi^T dm\right]$
m	symmetrix matrix defined by $\left[\int_A \phi\ dm\right]\left[\int_A \phi^T dm\right]$ in the stability analysis
Λ_1	a column matrix defined by $\int_A \Gamma_1\phi\ dm$
Λ_2	a column matrix defined by $\int_A \Gamma_2\phi\ dm$
Π_1	symmetrix matrix defined by $\Lambda_1\Lambda_1^T$
Π_2	symmetrix matrix defined by $\Lambda_2\Lambda_2^T$

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ABSTRACT OF THE DISSERTATION

Modal Analysis for Liapunov Stability

of Rotating Elastic Bodies

by

André Daniel Colin

Doctor of Philosophy in Engineering

University of California, Los Angeles, 1973

Professor Peter W. Likins, Chairman

The determination of the attitude stability of spinning elastic bodies is considered here. A complete study of this problem requires in general two parallel investigations. In the first part, a modal analysis is accomplished for a class of special problems and finally, a derivation for literal attitude stability criteria for idealized spinning flexible spacecraft is performed.

The modal analysis is accomplished for a class of structures idealized by beams or disks of small flexural rigidity. In the description of the flexible appendage, a continuous elastic model is used, and the vibrations in the direction of the spin axis are analyzed for a fully constrained base rotation. The study is specialized to the cases where the high spin rate is combined to the small flexural rigidity to produce a small parameter ε. The method of matched asymptotic expansions represents the general framework used in obtaining the solution for the several structures studied. The determination of

the eigenfrequencies and free vibration mode shapes is obtained by the concurrent use of the orthogonality of the mode shapes and the conditions derived from the matching process.

Also, the stability of spinning flexible satellites in a force-free environment is analyzed. The satellite is modeled as a rigid core having attached to it a flexible appendage described by a continuous elastic model, such as those mentioned earlier. A Liapunov stability procedure is used where the Hamiltonian of the system, constrained through the angular momentum integral so as to admit complete damping, is used as a testing function. Closed form stability criteria are generated for the first mode of a restricted appendage modal lying in a plane containing the system center of mass and orthogonal to the spin axis.

CHAPTER 1

STATEMENT OF THE PROBLEM

The influence of flexibility on spinning structure has been the subject of numerous technical investigations since the unexpected behavior of Explorer I. It was, indeed, the deformation of flexible antennas that caused the spacecraft to deviate from its prescribed motion. The variety of tasks fulfilled by modern satellites implies the presence of large appendages whose weights, for obvious reasons, are kept as low as possible. As the complexity and the number of spacecraft orbiting in space increase, the previous concern becomes more and more important.

In the development of dynamic models, describing the motion of flexible spacecraft, many authors employ a system of hybrid coordinates wherein discrete coordinates (for the translation and rotation of rigid bodies or reference frames) are used together with distributed or modal coordinates (for the deformations of elastic bodies). In the modeling of the vehicle appendages, various types of idealizations have been adopted: the elastic continuum model, the distributed-mass finite element model, and the elastically interconnected nodal body model. Common to all types of modeling, the modal analysis requires the derivation of the linearized equations of small oscillations from the constant state of deformation, induced by spin, and the transformation of these equations into a system of uncoupled equations of motion in terms of "normal mode coordinates." They represent an infinite

system for a continuous model, and a finite set for a discretized model, but in either case consideration of only a reduced number of them is a practical necessity.

It is mainly towards the first type of modeling — the elastic continuum model — that this dissertation is oriented, and towards the derivation of a stability analysis when such a model is used. The use of the elastic continuum model presents with respect to the other types of modeling a noticeable advantage when applied to some specific structures. It is, on the other hand, more difficult to implement for a general structure. Our concern in this work is to deal with some of those applications where the continuum model prevails.

The use of elastic continuum model for a rotating structure has generally been developed in the context of radial or axial beams. The investigation of continuous models for rotating planar structures such as membranes is more recent, and has generally been affected by mathematical difficulties arising in the development of the modes.

In the development of modal analysis of a rotating radial beam, we should mention the work done by R.T. Yntema.[1] The author used in his analysis of the modes of radial beams a Galerkin method, where he expands the modes in terms of the nonrotating beam modes. Interested in the same problem, J.E. Rakowski and M.L. Renard,[2] and P.C. Hughes and J.C. Fung,[3] used similar procedures to obtain the eigenfrequencies of a rotating radial beam. They used the linear property of the equations of the deformations in order to generate a family of solutions. The satisfaction of the boundary conditions of the problem gave them a procedure to converge to the exact value of the eigenfrequencies.

The generation of modes for circular membranes using a continuous model has mainly been developed for membranes of zero stiffness. Most probably, the earliest work was done by H. Lamb and R.V. Southwell,[4] in 1921, where the authors derived the modes for a spinning disk of no flexural rigidity. They, also, used an approximate method to bound the lowest frequency in the case where both bending and membrane effects are important. Among the other authors, whose contributions should be mentioned are J.G. Simmonds,[5] and W. Eversman.[6] Both of them spent a great deal of effort to solve the problem of a spinning membrane when it is clamped at its center. FInally, W. Eversman and R.O. Dodson[7] studied in 1969 the free vibrations of a centrally clamped spinning circular disk where they introduce flexural rigidity into their analysis. Here, again, the linear property of the problem is used in deriving the general solution. All of the above references dealing with beams or membranes of nonzero stiffness employ procedures requiring extensive numerical computations.

In contrast to those past achievements, the problem of the elastic continuum model has been approached here along directions guided by future applications. It is clear that flexible spinning spacecraft will continue to be designed and flown in the future. It is expected that many of these satellites will exhibit large flexible appendages such as antenna arrays or solar panels. Many of these applications are conceived on the verge of instability, and it becomes more and more important to develop proper stability criteria to cope with these future applications. An important class includes flexible spacecraft with high spin and low flexural rigidity. The partial differential

equations describing the motion of flexible appendages for the previous class of problems are characterized by a small parameter appearing in the coefficient of the highest derivative. This common feature will constitute the basis for our analytical approach to modal analysis. The resulting modal coordinates will be used in the determination of attitude stability criteria by means of Liapunov's direct method, employing as testing function the Hamiltonian constrained by the angular momentum integral.

The method of matched asymptotic expansions[8] represents the general framework and the main mathematical tool used in obtaining modal coordinates. Briefly, the method consists of separating the region of interest into "boundary-layer" regions near the boundaries where flexural rigidity effects are important and a central region in which the terms expressing the centripetal acceleration dominate. The solution is then developed in asymptotic expansions in terms of the small parameter observed in the highest derivative term. The approximation to the solution is then obtained by truncating the expansion to a finite number of terms, the error being small for sufficiently small values of the parameter. The use of asymptotic expansions for the description of the modes of rotating structures is not new, and we should mention two references using the previous techniques to study the transverse vibrations of rotating structures.

The first reference to survey is the work done by W.E. Boyce and G.H. Handelman in 1961,[9] where they approached the problem of the rotating beam with tip mass. This paper is mainly conceived as an application of a previous paper published by J. Moser,[10] wherein

uniformly valid asymptotic expansions are derived for an equation of the type treated there. In this last paper Moser derives a set of linearly independent solutions for the problem and, then, uses the linear property of the problem to come up with a general solution. The linearly independent solutions are taken in the form of $B(x,\eta)\exp[\eta^{-1}h(x)]$ where η is of the order of magnitude of the small perturbation parameter of the problem and $B(x)$ and $h(x)$ represent functions to be determined. Applying this procedure to the rotating beam with tip mass, Boyce and Handelman were mainly interested in what is called the zeroth order solution — the first term of the expansion. The solution to this last problem is then obtained through an energy method after the observation that one of the boundary conditions in the problem represents a natural boundary condition. What is meant by natural boundary condition is that the minimization problem arising in the Rayleigh-Ritz method gives us as transversality condition for the tip mass, the boundary condition of the zeroth order problem. In Reference 9, the authors do not proceed to the next term of their asymptotic expansion.

Another relevant reference using the method of asymptotic expansion for rotating structures is given by J.H. Abel and W.C. Kerr.[11] In their paper, they applied the technique of asymptotic expansion to a rotating cable-counterweighted space station in orbit. The problem without any flexural rigidity was already solved by V. Chobotov[12] in 1963, and represents in fact the zeroth order solution for their problem. In contrast to the method used in Reference 9, it is by matching the central and boundary layer solutions that they were able

to come up with the sufficient number of conditions for the determination of all unknown constants introduced by the integration of the several equations.

It is by using the conditions coming from the matching of the solution valid in the boundary-layer and the solution valid in the central region, and also by considering the orthogonality relationship between the eigenfunctions, that we plan to obtain explicit expressions for the eigenvalues and free vibration mode shapes for several elementary structures. For all the structures studied, we concentrate on the "transverse vibrations," involving oscillatory motions which are parallel to the nominal spin axis and transverse to a plane established by the structure, since these vibrations are most critical for the stability of a spinning structure. We also make throughout the study the assumption that the motion of the rigid rotating base to which the elastic appendage is attached is not affected by the transverse vibrations of the flexible appendage. For a free spinning spacecraft, the center of mass of the entire system remains at rest in inertial space, but it does move within the core for some modes. Similarly, the core body rotates for some of the system normal modes, but our modal analysis is based on a fully constrained base rotation. This represents in fact a convenient assumption — which is necessary if our answers are to have value for hybrid coordinate analysis of appendages on arbitrary spinning bodies — and permits the obtention of uncoupled modes — conditions under which we can transform the partial differential equations into a set of ordinary differential equations.

In Chapter 2, the derivation of the eigenmodes and eigenfrequencies for two unidimensional types of flexible appendages is presented. At first, we looked at the problem of the radial uniform classical beam. The classical assumptions consisting of neglecting the shear deformations and the rotatory inertia were made. We also limited our study to a radial beam clamped at its root on the spin axis. After the completion of this first case — where the spinning motion acts as an element stiffening the structure — we looked at the case of a uniform cable clamped at both extremities, and spinning about a central axis. The spinning motion introduces in this structure a softening effect and requires the introduction of a built-in tension. This configuration seems to be of little direct application, but it has been examined in order to familiarize ourselves with the next problem of wider application, discussed in the following chapter.

In Chapter 3, the modal analysis of a rotating membrane is considered. Two cases are investigated. In the first one, the flexible appendage consists of a circular membrane clamped along its edge, and spinning about a central axis normal to its plane. Here, again, the rotation introduces a decrease in the eigenfrequency of the nonrotating structure and justifies the introduction of a built-in tension. This last effect is generally referred as the effect of preload and is known to be highly configuration-dependent. Here, again, the assumption is made that the motion of the spinning rigid rim is uninfluenced by membrane vibrations. Finally, the freely spinning membrane is analyzed and it is observed that for this last structure, the eigenfrequencies of the rotating membrane are less affected by the flexural

rigidity than in the case with outer rim constrained by a spinning rigid ring.

Finally, in Chapter 4, a general derivation is given for the stability analysis of rotating structures when the deformations of the flexible parts are expressed in terms of the modes of the rotating structure when determined from a continuous elastic model. Also, some of the restrictions introduced in our derivation are justified. It is shown, for instance, that for the class of problems considered in Chapters 2 and 3, the wobbling (nutational) motion separates from the spinning motion in the linearized equations. For any flexible appendage lying in a plane perpendicular to the spin axis, and passing through the system center of mass, the linearized equations of motion separate into two groups. The wobbling motion consists of the motion described by the nutation angles and the transverse vibrations, and the spinning motion consists of the rotation along the spin axis and what is often referred as the in-plane deformations. It is, in fact, this last point that justifies the consideration of only the transverse vibration in the previous study. For most of the aerospace applications, the attitude stability of the spacecraft is of main interest and is only affected by the out-of-plane deformations. It is also observed that the attitude stability is affected by the anti-symmetric modes of symmetric spacecraft only, and through this, the assumption made earlier to consider the translational motion of the central rigid core of the system as being not affected by the deformations, is made acceptable. The anti-symmetric modes are, indeed, those which keep

the center of mass of the total system at rest with respect to the rigid core.

The development of stability criteria for free spinning bodies has been the basis of numerous technical papers. Rigid body analysis prior to the flight of Explorer I predicts a stable free rotation in inertial space if the angular velocity vector is directed parallel to a principal axis of either maximum or minimum moment of inertia. The analysis following Explorer I led to the general conclusion that for a flexible spinning satellite to exhibit stable free motion its axis of spin must be restricted to that of the principal axis of maximum inertia; this proposition is sometimes referred to as "the greatest moment of inertia" rule. However, one would expect that the last criterion is not sufficient to assure stability and it is not surprising at all to find that spacecraft with very large flexible appendages are less stable than quasi-rigid ones. It is in the examination and development of stability criteria involving the modes of vibration and the natural frequencies of the structures that most of the recent papers are oriented.

It would be too lengthy to cover the numerous publications dealing with this last task; Reference 13 provides a current bibliography on this subject. We should, however, mention here that the problem is generally approached through two different procedures. Some authors examine the stability problem by using a Routh-Hurwitz analysis, and some prefer to use a Liapunov analysis. In our study, we decided to employ the latter approach, proceeding in parallel with the work done by F.J. Barbera and P.W. Likins.[13] In their work, the authors develop

general stability criteria for a flexible appendage described by a collection of particles. They used Liapunov's second method as the basic analytical tool and by specializing in the more restrictive case wherein the appendage lies in a plane containing the center of mass and orthogonal to the spin axis, they are able to come up with some analytical criteria.

The stability analysis developed in Chapter 4 is based on a similar approach. Liapunov stability theorems are employed with the Hamiltonian of the system, constrained through the angular momentum integral, used as a testing function — a method proposed by R. Pringle,[14] in order to circumvent problems related to the negative definiteness of the Hamiltonian time derivative. In our formulation, the flexible appendage is described as a continuous elastic body, as opposed to a collection of elastically interconnected particles. We kept the derivation as general as possible. However, in order to come up with specific criteria, our study has been restricted to more particular cases. Finally, the analysis was brought down to a level amenable to literal stability criteria by truncating the number of normal coordinates to a single mode. The stability criteria emerging from this study represents, due to the special form adopted for the Hamiltonian, conditions for stability that are sufficient and (except for a few recognizable singular cases) necessary as well.

CHAPTER 2

DYNAMICS OF ROTATING ELASTIC BEAMS AND CABLES

2.1 Introduction

In the following chapter, we face the problem of a dynamical study of two elementary structures characterized by a dominant effect of the forces induced by spin over the flexural rigidity. In the first part, a modal analysis is done for a rotating beam clamped normal to the spin axis. The study is oriented towards the application in rotation-stabilized space vehicles having a nondeformable frame to which flexible rods are attached. In the second part, the modal analysis of a taut cable clamped at its extremities is done.

In both studies, we will consider vibrations only in the meridional direction. We will also ignore deformations which are present only because of the Poisson effect. We will assume, at the outset, that the motion of the base to which the elastic structure is attached is uninfluenced by the elastic vibrations of the appendage.

2.2 Dynamics of a Rotating Elastic Beam

We now specialize in the study of a deformable element which consists of a long flexible beam undergoing transverse vibrations. When an elastic beam is normal to the spin axis of its inertially rotating base, it does sustain deformations in the steady state configuration in which it remains straight and aligned with a radial line emanating from its base. Therefore, as emphasized by P.W. Likins, F.J. Barbera and V. Baddeley,[15] one must consider nonlinear strain-displacement equations if deformation variables are to be measured

from the undeformed state. The requirement for the retention of second degree terms comes from the fact that steady state deformations induced by constant spin are not arbitrarily small, and cannot be included with the arbitrarily small deviations from the steady state deformations in the linearization process. This great difficulty of nonlinear elasticity explains generally why applications in the literature are restricted to beams. For those particular cases, the equations of motion are typically derived by means of procedures which rely, from the outset, upon the availability of solutions for the steady-state load distribution and deformation of the elastic continuum.

In our analysis, we neglect rotation of the transverse cross section of the thin circular rod or beam and consider the linear density and the flexural stiffness EI to be constant. The rods are also considered as being much longer than the frame dimension. We therefore assume the point at which the rods are attached coincides with the axis of rotation of the frame, as in Figure 1.

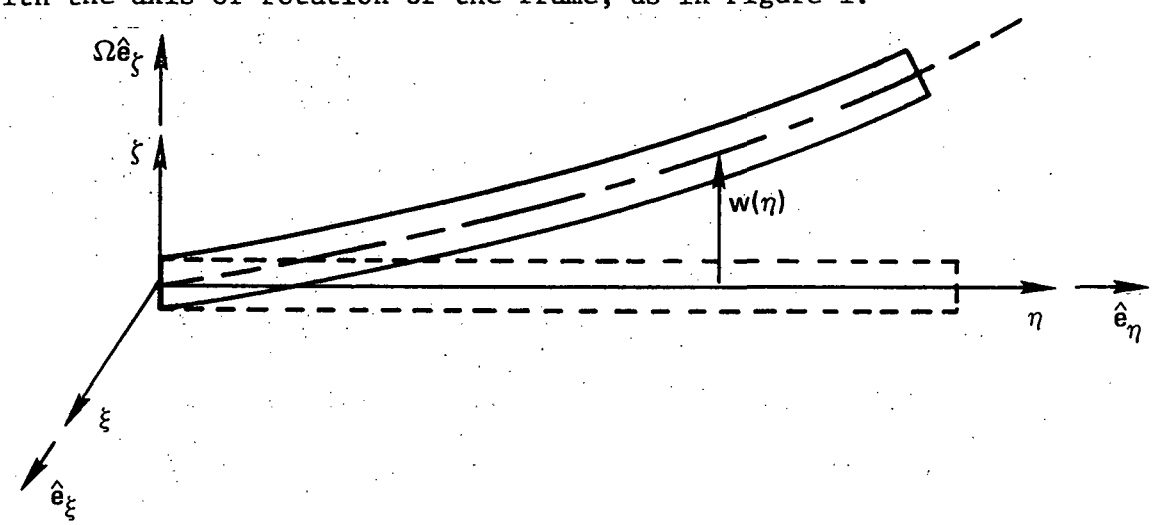

Figure 1. Rotating Uniform Elastic Beam.

Under those assumptions, the transverse vibrations of the classical (Euler-Bernoulli) beam, subject to an external axial load $P(\eta)$ is given in general form by (see Reference 15)

$$EI \frac{\partial^4 w}{\partial \eta^4} - \frac{\partial}{\partial \eta} \left[P(\eta) \frac{\partial w}{\partial \eta}\right] + \mu \frac{\partial^2 w}{\partial t^2} = 0$$

where EI is the flexural stiffness of the rod

μ is the linear density

$P(\eta)$ is the external axial load .

The boundary conditions are given for a rotating rod under the assumptions of one cantilevered end and one free end by

$$w(0) = 0 = \frac{\partial w}{\partial \eta}(0) = \frac{\partial^2 w}{\partial \eta^2}(L) = \frac{\partial^3 w}{\partial \eta^3}(L) \quad .$$

For the rotating uniform radial beam, the influence of steady-state centripetal accelerations can be represented by an "effective force" or "centrifugal force" given by

$$P(\eta) = \int_{\eta}^{L} \mu \, \Omega^2 \, \eta^1 d\eta^1 = \frac{1}{2} \mu \Omega^2 (L^2 - \eta^2)$$

so that the previous equation takes the form

$$EI \frac{\partial^4 w}{\partial \eta^4} - \frac{1}{2} \mu \, \Omega^2 \left[(L^2 - \eta^2) \frac{\partial^2 w}{\partial \eta^2} - 2\eta \frac{\partial w}{\partial \eta}\right] + \mu \frac{\partial^2 w}{\partial t^2} = 0.$$

The artifice of the "effective axial load" permits this derivation of the vibration equation to be accomplished without the reliance on nonlinear strain-displacement equations noted previously to be required for general derivations of small-vibration equations for rotating elastic continua.

Two remarks should be made about this last equation. First of all, this relation expresses the transverse motion of the beam free of any external forces or deviations of base motion from simple spin, coming, for instance, from the coupling between the transverse vibrations and any nutational motion of the rigid core. Also, it should be emphasized that in the previous relation, the deformations along the η axis represent the result of two separate deformations: the steady-state extensional deformation from the undeformed state and the deviation from that steady-state. This last point could be overlooked in the previous relation due to the artifice of the "effective axial force," but is an unavoidable fact of the general continuous model. In the derivations done in Chapter 3, this last remark will become more apparent in the sense that the steady state deformation has to be computed first and the equations of the motion are then represented by the deviation from this steady-state.

We use, now, the method of separation of variables:

$$w(\eta,t) = \phi_\alpha(\eta)\, P_\alpha(t) \quad .$$

Substituting into the previous equation, we have:

$$\ddot{P}_\alpha + \omega_\alpha^2 P_\alpha = 0$$

$$\frac{EI}{\mu} \frac{d^4\phi_\alpha}{d\eta^4} - \frac{1}{2}\Omega^2 \left[(L^2 - \eta^2) \frac{d^2\phi_\alpha}{d\eta^2} - 2\eta \frac{d\phi_\alpha}{d\eta}\right] - \omega_\alpha^2 \phi_\alpha = 0$$

where $(\dot{\ })$ stands for derivative with respect to time

ω_α represents the eigenfrequency of the vibration

and the boundary conditions become

$$\phi_\alpha(0) = 0 = \frac{d\phi_\alpha}{d\eta}(0) = \frac{d^2\phi_\alpha}{d\eta^2}(L) = \frac{d^3\phi_\alpha}{d\eta^3}(L) \quad .$$

With the introduction of the dimensionless variable, $y = \frac{\eta}{L}$, and division by Ω^2, the previous relation becomes:

$$\varepsilon\, \phi_\alpha^{IV} - \frac{1}{2}\,[(1-y^2)\phi_\alpha']' - \lambda_\alpha^2\, \phi_\alpha = 0 \tag{2.1}$$

where (') stands for derivative with respect to the dimensionless variable y

$$\varepsilon = \frac{EI}{\mu\Omega^2 L^4}$$

$$\lambda_a^2 = \frac{\omega_\alpha^2}{\Omega^2}$$

with the boundary conditions

$$\phi_\alpha(0) = 0 = \phi_\alpha'(0) = \phi_\alpha''(1) = \phi_\alpha'''(1)$$

In the parameter ε, the high spin rate and the low flexural rigidity combine their effects to produce a small quantity. A typical value for the parameter ε is given by the range of values $10^{-3}\ldots10^{-4}$ (see Reference 2). As a result, the previous equation is suitable for the use of singular perturbation theory by the presence of a small parameter in the highest derivative term.

Let us consider for the central portion of the beam, a solution for which analytical dependence on the small parameter ε is given by

$$\phi_\alpha(y,\varepsilon) = h_0(y) + \nu_1(\varepsilon)\, h_1(y) + \nu_2(\varepsilon)\, h_2(y) + \ldots$$

where $\nu_1(\varepsilon)$ is a small $o(\varepsilon)$, and the $\nu_i(\varepsilon)$ constitutes an asymptotic sequence of functions.

By definition, $\phi(\mu) = o\ (\Psi(\mu))$ as $\mu \to \mu_o$ if, given any $\delta > 0$, there exists a neighborhood N_δ of μ_o such that $|\phi| \leq \delta|\Psi|$ for μ in the neighborhood; thus $\phi(\mu) = o(\Psi(\mu))$ if $\phi/\Psi \to 0$ as $\mu \to \mu_o$. Also a sequence $\phi_n(\mu)$, n=1,2,... is called an asymptotic sequence if

$$\phi_{n+1}(\mu) = o\Big(\phi_n(\mu)\Big) \text{ as } \mu \to \mu_o \ .$$

By definition, two functions ϕ,Ψ of μ belong to the same equivalence class in a neighborhood of μ_o if

$$0 < \lim_{\mu \to \mu_o} \frac{\phi(\mu)}{\Psi(\mu)} < \infty \ .$$

If this double inequality is satisfied, we adopt the notation

$$\text{ord } \phi(\mu) = \text{ord } \Psi(\mu) \quad .$$

A partial ordering of equivalence classes is given by

$$\text{ord } \phi(\mu) < \text{ord } \Psi(\mu) \text{ if } \lim_{\mu \to \mu_o} \frac{\phi}{\Psi} \to 0$$

so that if $\phi(\mu) = o\ \Big(\Psi(\mu)\Big)$ then ord $\phi(\mu) <$ ord $\Psi(\mu)$.

To shorten the writing, the subscript α, characteristic number of the mode will be omitted momentarily and we will use this subscript explicitly only when needed, in order to differentiate between different modes.

Similarly the eigenvalue λ_α^2 of the problem will be written simply as λ^2 and it will be expanded in terms of the small parameter ε in the following way:

$$\lambda^2(\varepsilon) = \lambda_o^2 + \kappa_1(\varepsilon)\lambda_1^2 + \kappa_2(\varepsilon)\lambda_2^2 + \ldots$$

where $\kappa_i(\varepsilon)$ constitutes an asymptotic sequence.

The corresponding differential equations defining $h_0(y)$ and $h_1(y)$ are given by

$$\frac{1}{2}\,[(1-y^2)h_0']' + \lambda_0^2\, h_0^2 = 0$$

and

$$\frac{1}{2}\,[(1-y^2)h_1']' + \lambda_0^2\, h_1 = \begin{cases} -\lambda_1^2\, h_0 & \text{if } \dfrac{\kappa_1(\varepsilon)}{\nu_1(\varepsilon)} = 1 \\ 0 & \text{if } \lim\limits_{\varepsilon \to 0} \dfrac{\kappa_1(\varepsilon)}{\nu_1(\varepsilon)} = 0 \end{cases}$$

where the different possibilities were limited to the case where ord $\nu_1(\varepsilon)$ > ord ε, which will be verified later. We also neglect the possibility that $\nu_1(\varepsilon) = o\big(\kappa_1(\varepsilon)\big)$, because this would imply that $\lambda_1^2 = 0$.

At this stage, one remark can already be made on the solution for h_0. There are no apparent boundary conditions to the equation defining h_0. But this equation is known as the Legendre equation, the solution to which is well known in terms of the Legendre polynomials (see Reference 17). Even though it might seem that there would be a complete freedom in the values of λ_0^2, we can see from here, already, that the values of λ_0^2 are limited to a discrete spectrum based on the observation that the point y=1 constitutes a singularity for the Legendre polynomials, and only a discrete set of values of λ_0^2 will give the value of the Legendre polynomials to be bounded at y=1.

In order to make those observations more specific, we have to concentrate for a while on the boundary layers existing at both extremities, whose solutions will give us the constants of integration we need for the previous integration.

Boundary Layer Near y = 0

In order to study the boundary layer near the origin, we have to introduce a stretched coordinate

$$\tilde{y} = \frac{y}{\sigma(\varepsilon)} \qquad \text{assuming } \sigma(\varepsilon) \to 0 \quad \text{when } \varepsilon \to 0 \quad .$$

The corresponding asymptotic expansion valid near y = 0 is given by

$$\phi_\alpha(y,\varepsilon) = \mu_0(\varepsilon)g_0(\tilde{y}) + \mu_1(\varepsilon)g_1(\tilde{y}) + \mu_2(\varepsilon)g_2(\tilde{y}) + \ldots$$

where the $\mu_i(\varepsilon)$ constitute an asymptotic sequence. Replacing the previous asymptotic expansion into the differential equation of the mode shape $\phi(y)$ and considering that $y = \sigma(\varepsilon)\,\tilde{y}$ and $dy = \sigma(\varepsilon)\,d\tilde{y}$, we obtain:

$$\varepsilon\left[\frac{\mu_0(\varepsilon)}{\sigma^4(\varepsilon)}\,\frac{d^4g_0(\tilde{y})}{d\tilde{y}^4} + \frac{\mu_1(\varepsilon)}{\sigma^4(\varepsilon)}\,\frac{d^4g_1(\tilde{y})}{d\tilde{y}^4} + \ldots\right]$$

$$- \frac{1-\sigma^2(\varepsilon)\tilde{y}^2}{2}\left[\frac{\mu_0(\varepsilon)}{\sigma^2(\varepsilon)}\,\frac{d^2g_0(\tilde{y})}{d\tilde{y}^2} + \frac{\mu_1(\varepsilon)}{\sigma^2(\varepsilon)}\,\frac{d^2g_1(\tilde{y})}{d\tilde{y}^2} + \ldots\right]$$

$$+ \sigma(\varepsilon)\tilde{y}\left[\frac{\mu_0(\varepsilon)}{\sigma(\varepsilon)}\,\frac{dg_0(\tilde{y})}{d\tilde{y}} + \frac{\mu_1(\varepsilon)}{\sigma(\varepsilon)}\,\frac{dg_1(\tilde{y})}{d\tilde{y}} + \ldots\right]$$

$$- \lambda_0^2\,[\mu_0(\varepsilon)g_0(\tilde{y}) + \mu_1(\varepsilon)g_1(\tilde{y}) + \ldots]$$

$$- \kappa_1(\varepsilon)\lambda_1^2\,[\mu_0(\varepsilon)g_0(\tilde{y}) + \mu_1(\varepsilon)g_1(\tilde{y}) + \ldots] = 0 \quad .$$

A suitable boundary layer coordinate is chosen by the requirements that the higher derivatives are of the same order of magnitude near the origin or by considering the previous equation, in writing

$$\frac{\varepsilon\mu_0(\varepsilon)}{\sigma^4(\varepsilon)} = \frac{\mu_0(\varepsilon)}{\sigma^2(\varepsilon)}$$

or $\qquad \sigma^2(\varepsilon) = \varepsilon \qquad$ and $\qquad \sigma(\varepsilon) = \sqrt{\varepsilon} \quad .$

The boundary layer coordinate is thus given by:

$$\boxed{\tilde{y} = \frac{y}{\sqrt{\varepsilon}}}$$

The dominant boundary-layer equation is given by:

$$\frac{d^4 g_0(\tilde{y})}{d\tilde{y}^4} - \frac{1}{2}\frac{d^2 g_0(\tilde{y})}{d\tilde{y}^2} = 0$$

Both boundary conditions at $\tilde{y} = 0$ have to be satisfied by g_0:

$$\frac{dg_0(\tilde{y})}{d\tilde{y}} = g_0 = 0 \qquad \text{at } \tilde{y} = 0$$

Defining $\dfrac{d^2 g_0(\tilde{y})}{d\tilde{y}^2} = g_0{}^*(\tilde{y})$, we have

$$\frac{d^2 g_0{}^*(\tilde{y})}{d\tilde{y}^2} - \frac{1}{2} g_0{}^*(\tilde{y}) = 0 \quad .$$

Using the fact that the exponential growth ($e^{\tilde{y}/\sqrt{2}}$) cannot match as $\tilde{y} \to \infty$ and taking into account the boundary condition, we obtain successively

$$\frac{d^2 g_0(\tilde{y})}{d\tilde{y}^2} = C_o\, e^{-\frac{1}{\sqrt{2}}\tilde{y}}$$

and

$$g_0(\tilde{y}) = 2\, C_o \left[\frac{\tilde{y}}{\sqrt{2}} - 1 + e^{-\frac{\tilde{y}}{\sqrt{2}}}\right]$$

which represents a solution with one arbitrary constant but doesn't depend on the eigenvalue.

We are now in a position where we can try to match this boundary layer expansion and the solution we found previously for the central portion of the beam.

An intermediate limit suitable for matching near $\tilde{y} = 0$ is given by y_η fixed:

$$y_\eta = \frac{y}{\eta(\varepsilon)} \qquad \eta(\varepsilon) \to 0 \qquad \text{and} \qquad \frac{\eta(\varepsilon)}{\sqrt{\varepsilon}} \to \infty \quad \text{as} \quad \varepsilon \to 0$$

so that

$$y = \eta\, y_\eta \to 0 \qquad \text{and} \qquad \tilde{y} = \frac{\eta}{\sqrt{\varepsilon}}\, y_\eta \to \infty \,.$$

Matching near $y = 0$ takes the form:

$$\lim_{\substack{\varepsilon \to 0 \\ y_\eta \text{ fixed}}} \left\{ \begin{aligned} & h_0(\eta y_\eta) + \nu_1(\varepsilon)\, h_1(\eta y_\eta) + \ldots \\ & \qquad - \mu_0(\varepsilon) g_0\left(\frac{\eta}{\sqrt{\varepsilon}}\, y_\eta\right) - \mu_1(\varepsilon) g_1\left(\frac{\eta}{\sqrt{\varepsilon}}\, y_\eta\right) + \ldots \end{aligned} \right\} = 0.$$

Expanding h_0 and h_1 as Taylor series near the origin and using the solution previously found for $g_0(\tilde{y})$, we have:

$$\lim_{\substack{\varepsilon \to 0 \\ y_\eta \text{ fixed}}} \left\{ \begin{aligned} & h_0(0) + \eta y_\eta\, h_0'(0) + \ldots + \nu_1(\varepsilon) h_1(0) + \nu_1(\varepsilon) h_1'(0) \eta y_\eta + \ldots \\ & \qquad -2\mu_0(\varepsilon) C_o \left[\frac{\eta}{\sqrt{2}\,\sqrt{\varepsilon}}\, y_\eta - 1 + e^{-\frac{\eta}{\sqrt{2}\sqrt{\varepsilon}}\, y_\eta} \right] + \ldots \end{aligned} \right\} = 0 \,.$$

From here, we can see that the matching is possible only if the following equalities are satisfied:

$$h_0(0) = 0$$

$$h_0'(0) = \sqrt{2}\; C_o$$

and

$$\mu_0(\varepsilon) = \sqrt{\varepsilon} \quad .$$

Another point can be noticed from the previous equation. The term $2\,\mu_0(\varepsilon)\,C_o$ appearing in the expression of g_0 can not be matched except by a suitable value for $h_1(0)$ and this requires that

$$\nu_1(\varepsilon) = \mu_0(\varepsilon) = \sqrt{\varepsilon} \quad \text{and} \quad h_1(0) = -2\,C_o$$

which satisfies our requirement that $\nu_1(\varepsilon) \to 0$ when $\varepsilon \to 0$, and implies ord $\nu_1(\varepsilon) >$ ord ε an inequality that we still had to prove.

We are now in a position where we can go back to the solution valid in the central region of the beam.

The equation defining h_0

$$\frac{1}{2}\,[(1-y^2)h_0']' + \lambda_0^2\,h_0 = 0$$

is a Legendre equation.

Defining the Legendre functions of the first and second kind by:

$$\begin{cases} \mathscr{L}_1(2\lambda_0^2,\ y) = 1 - (\frac{1}{2!})\ (2\lambda_0^2)y^2 + (\frac{1}{4!})\ (2\lambda_0^2)(2\lambda_0^2 - 6)y^4 - \ldots \\ \mathscr{L}_2(2\lambda_0^2,\ y) = y - (\frac{1}{3!})\ (2\lambda_0^2 - 2)y^3 + \ldots \end{cases}$$

we see directly that only the second expression $\mathscr{L}_2(2\lambda_0^2,\ y)$ satisfies the requirement $h_0(0) = 0$. We thus have:

$$h_0(y) = k_0\ \mathscr{L}_2(2\lambda_0^2,\ y)\ .$$

The set of discrete values λ_0^2 which represents the zeroth order of the asymptotic expansion of the eigenvalue are in fact those values which truncate the Legendre polynomials. For all other value of λ_0^2, the serie becomes infinite and unbounded for $y = 1$, which could not make physical sense for our problem. This last point will be shown later.

As a result the set of discrete values feasible for λ_0^2 is given by

$$\lambda_0^2 = 1,6,15...$$

or in general

$$\lambda_0^2 = \frac{n(n+1)}{2} \quad \text{for n odd.}$$

Now that an expression has been found for $h_0(y)$, let us look at the expression of $h_1(y)$.

Before any computation for h_1, we will need a property of the mode of the previous problem, i.e. their orthogonality. In order to prove the orthogonality of the modes, we have to consider the following equations and boundary conditions:

$$\varepsilon\, \phi_\alpha^{IV} - \frac{1}{2}\, [(1-y^2)\phi_\alpha']' - \lambda_\alpha^2\, \phi_\alpha = 0$$

$$\varepsilon\, \phi_\beta^{IV} - \frac{1}{2}\, [(1-y^2)\phi_\beta']' - \lambda_\beta^2\, \phi_\beta = 0$$

$$\phi_\alpha(0) = \phi_\alpha'(0) = \phi_\alpha''(1) = \phi_\alpha'''(1) = 0$$

$$\phi_\beta(0) = \phi_\beta'(0) = \phi_\beta''(1) = \phi_\beta'''(1) = 0 \quad .$$

Multiplying the first equation by ϕ_β and integrating by parts from 0 to 1, we obtain:

$$\varepsilon\, \phi_\alpha'''\phi_\beta \Big|_0^1 - \varepsilon \int_0^1 \phi_\alpha'''\phi_\beta'\, dy - \frac{1}{2}\, [(1-y^2)\phi_\alpha']\phi_\beta \Big|_0^1$$

$$+ \frac{1}{2} \int_0^1 (1-y^2)\phi_\alpha'\phi_\beta'\, dy - \lambda_\alpha^2 \int_0^1 \phi_\alpha\phi_\beta\, dy = 0 \quad .$$

The boundary conditions and a new integration by parts gives us

$$-\varepsilon\phi_\alpha''\phi_\beta' \Big|_0^1 + \varepsilon \int_0^1 \phi_\alpha''\phi_\beta''\, dy + \frac{1}{2} \int_0^1 (1-y^2)\phi_\alpha'\phi_\beta'\, dy - \lambda_\alpha^2 \int_0^1 \phi_\alpha\phi_\beta\, dy = 0.$$

A similar expression can be obtained from the second equation and for $\lambda_\alpha^2 \neq \lambda_\beta^2$ we have from the difference of the two expressions

$$\int_0^1 \phi_\alpha \phi_\beta \, dy = 0 \quad .$$

The previous result has to be completed by the relation defining the norm of the modes and for orthonormal modes we also have

$$\int_0^1 \phi_\alpha^2 \, dy = 1 \quad .$$

Expanding the modes through the asymptotic expansion we defined before, and considering that the previous expression has to be true for every small ε, the last expression becomes, for instance:

$$\int_0^1 h_0^2(y) \, dy = 1$$

$$\int_0^1 h_0(y) h_1(y) \, dy = 0$$

and similar expression for $h_2 \ldots$

We have to be careful in the use of the previous relation because we have to include under the integral sign the contribution coming from the boundary-layer solution. For our purpose, we already showed that the boundary layer near the origin is of thickness $\sqrt{\varepsilon}$ and, as a result does not contribute in the second integral. We will prove later that the boundary layer near the free end does not contribute to the previous integral too, and the two integrals we just wrote will, from there, be useful. From

$$\int_0^1 h_0^2 \, dy = 1, \text{ we conclude}$$

$$\int_0^1 k_0^2 \, \mathscr{L}_2^2 \, (2\lambda_0^2, y) dy = 1$$

and

$$\boxed{k_0 = \pm \sqrt{\frac{1}{\int_0^1 \mathscr{L}_2^2 (2\lambda_0^2, y) dy}}} \quad .$$

Considering now the differential equation defining $h_1(y)$ we have

$$\frac{1}{2} \left[(1-y^2)h_1'\right]' + \lambda_0^2 \, h_1 = \begin{cases} -\lambda_1^2 \, h_0 & \text{if } \dfrac{\kappa_1(\varepsilon)}{\sqrt{\varepsilon}} = 1 \\ 0 & \text{if } \dfrac{\kappa_1(\varepsilon)}{\sqrt{\varepsilon}} \to 0 \text{ as } \varepsilon \to 0 \end{cases} .$$

The second possibility is now to be cancelled if we want to have any chance to satisfy the integral

$$\int_0^1 h_0(y) \, h_1(y) \, dy = 0 \quad .$$

As a result, the function $\kappa_1(\varepsilon)$ is well defined and we have

$$\kappa_1(\varepsilon) = \sqrt{\varepsilon}$$

and

$$\frac{1}{2} \left[(1-y^2)h_1'\right]' + \lambda_0^2 \, h_1 = -\lambda_1^2 \, h_0 \quad . \tag{2.2}$$

We also found before in the matching process near the cantilevered end:

$$h_0'(0) = \sqrt{2} \, Co = - \frac{h_1(0)}{\sqrt{2}} \quad \text{or}$$

$$k_0 \, \mathscr{L}_2' \, (2\lambda_0^2, 0) = \sqrt{2} \, Co = - h_1(0) / \sqrt{2}$$

$$h_1(0) = -k_0 \, \mathscr{L}_2' \, (2\lambda_0^2, 0)\sqrt{2} \quad .$$

We will now determine the value of λ_1^2 through the use of the differential equation (2) and its boundary conditions and also by using the orthonormality condition on h_0.

Multiplying Equation (2) by h_0 and integrating by parts from 0 to 1, we have

$$\frac{1}{2}\,[(1-y^2)h_1']h_0\Big|_0^1 - \frac{1}{2}\int_0^1 [(1-y^2)h_1']h_0'\,dy + \lambda_0^2\int_0^1 h_1h_0dy = -\lambda_1^2\int_0^1 h_0^2\,dy$$

Assuming $h_1'(1)$ to be bounded, the use of the boundary condition of h_0 and its normality gives us

$$\frac{1}{2}\int_0^1 (1-y^2)\,h_1'\,h_0'\,dy - \lambda_0^2\int_0^1 h_0h_1\,dy = \lambda_1^2 \quad .$$

Integrating by parts once again, we have

$$\frac{1}{2}\,(1-y^2)h_0'h_1\Big|_0^1 - \frac{1}{2}\int_0^1 [(1-y^2)h_0']'\,h_1\,dy - \lambda_0^2\int_0^1 h_0h_1\,dy = \lambda_1^2 \quad .$$

The differential equation defining h_0

$$-\frac{1}{2}\,[(1-y^2)\,h_0']' = +\,\lambda_0^2\,h_0$$

brings us the final expression:

$$-\frac{1}{2}\,h_0'(0)\;h_1(0) = \lambda_1^2$$

or

$$\lambda_1^2 = \frac{1}{2\sqrt{2}}\,h_1^2(0)$$

or

$$\lambda_1^2 = \frac{1}{\sqrt{2}}\;\frac{[\mathscr{L}_2'(2\lambda_0^2,\,0)]^2}{\int_0^1 \mathscr{L}_2^2\,(2\lambda_0^2,y)\,dy} \quad .$$

Considering the definition of $\mathscr{L}_2(2\lambda_0^2, y)$, we can see that $\mathscr{L}_2'(2\lambda_0^2, 0) = 1$ and

$$\lambda_1^2 = \frac{1}{\sqrt{2}\int_0^1 \mathscr{L}_2^2(2\lambda_0^2, y)\,dy} \quad .$$

For the first mode, corresponding to $\lambda_0^2 = 1$, the previous relation becomes

$$\lambda_1^2 = \frac{1}{\sqrt{2}\int_0^1 y^2 dy} = \frac{3}{\sqrt{2}\, y^3\Big|_0^1} = \frac{3}{\sqrt{2}} \approx 2.12 \quad .$$

For the second mode or for $\lambda_0^2 = 6$, we have:

$$\lambda_1^2 = \frac{1}{\sqrt{2}\int_0^1 (y - \frac{5}{3}y^3)^2 dy} = \frac{63}{4\sqrt{2}} \approx 11.18 \quad .$$

We should emphasize at this stage that the previous expression found for λ_1^2 is simply based on the properties of the differential equation and did not use any other relation. It is easy to prove that the previous values found for λ_0^2 and λ_1^2 are also those which satisfy the orthogonality relationship between different modes. For the zero-order terms of the expansion, the relation is simply given by the orthogonality property of the Legendre polynomials in the interval from o to 1. For the first order terms, the derivation is lengthy, but the values of λ_1^2 are coming out from those relations. Now that an expression has been found for λ_1^2, we can go back to Equation (2.2)

$$\frac{1}{2}\left[(1-y^2)h_1'\right]' + \lambda_0^2 h_1 = -\lambda_1^2 h_0 \quad . \qquad (2.2)$$

The general solution of the previous equation is given by a particular solution $\bar{h}_1$ which satisfies the boundary condition

$$h_1(0) = -k_0 \ \mathscr{L}_2' \ (2\lambda_0^2, 0) \ \sqrt{2}$$

and the general solution of the homogeneous part of (2) multiplied by a constant k_1. This last solution is nothing else than h_0, the solution of the zero-order equation. So, in general,

$$h_1(y) = \bar{h}_1(y) + k_1 h_0(y) \ .$$

Out of the family of solutions, we have to choose the solution which satisfies the orthogonality relationship between h_0 and h_1 for one mode or

$$0 = \int_0^1 h_0 h_1 \, dy = \int_0^1 h_0 \, \bar{h}_1 \, dy + k_1 \int_0^1 h_0^2 \, dy$$

or, finally

$$k_1 = - \int_0^1 h_0 \, \bar{h}_1 \, dy \qquad .$$

The knowledge of k_1 specifies entirely the function h_1 and as we will see later, this last step will be needed when we will match the central part expansion to the boundary-layer expansion near the free end.

Boundary Layer Near the Free End

We, now, have to look at the boundary layer needed at the free end in order to satisfy the boundary condition and confirm what has been said before about the value of h_0 at this extremity. In order to proceed, we have to introduce another suitable boundary layer coordinate near y=1. Let, then $y^* = \frac{1-y}{\phi(\varepsilon)}$ where $\phi(\varepsilon) \to 0$ when $\varepsilon \to 0$. We

thus have $\phi y^* = 1-y$ and $dy = -\phi dy^*$. The corresponding asymptotic expansion valid near $y = 1$ is taken as

$$\phi_\alpha(y,\varepsilon) = \delta_0(\varepsilon)\ f_0(y^*) + \delta_1(\varepsilon) f_1(y^*) + \ldots$$

where the $\delta_i(\varepsilon)$ constitute an asymptotic sequence.

Replacing the previous asymptotic expansion into the differential equation of the mode, we obtain for the basic equation

$$\begin{aligned}
&\varepsilon\left[\frac{\delta_0(\varepsilon)}{\phi^4(\varepsilon)}\ \frac{d^4 f_0(y^*)}{dy^{*4}} + \frac{\delta_1(\varepsilon)}{\phi^4(\varepsilon)}\ \frac{d^4 f_1(y^*)}{dy^{*4}} + \ldots\right] \\
&- \frac{1-(1-\phi y^*)^2}{2}\left[\frac{\delta_0(\varepsilon)}{\phi^2(\varepsilon)}\ \frac{d^2 f_0(y^*)}{dy^{*2}} + \frac{\delta_1(\varepsilon)}{\phi^2(\varepsilon)}\ \frac{d^2 f_1(y^*)}{dy^{*2}} + \ldots\right] \\
&- (1-\phi y^*)\left[\frac{\delta_0(\varepsilon)}{\phi(\varepsilon)}\ \frac{d\ f_0(y^*)}{dy^*} + \frac{\delta_1(\varepsilon)}{\phi(\varepsilon)}\ \frac{d\ f_1(y^*)}{dy^*} + \ldots\right] \\
&- \lambda_0^2[\delta_0(\varepsilon) f_0(y^*) + \delta_1(\varepsilon) f_1(y^*) + \ldots] \\
&-\kappa_1(\varepsilon)\lambda_1^2\ [\delta_0(\varepsilon) f_0(y^*) + \delta_1(\varepsilon) f_1(y^*) + \ldots] = 0
\end{aligned} \tag{2.3}$$

Rewriting $1-(1-\phi y^*)^2$ as $2\phi y^* - \phi^2 y^{*2}$ in the previous equation, we choose a suitable boundary layer coordinate by the requirements that the higher derivative terms are of the same order of magnitude near the free end. Thus by considering the previous equation, we have

$$\frac{\varepsilon\delta_0(\varepsilon)}{\phi^4(\varepsilon)} = \frac{\delta_0(\varepsilon)}{\phi(\varepsilon)}$$

or

$\phi^3(\varepsilon) = \varepsilon$ and

$\phi\ (\varepsilon) = \varepsilon^{1/3}$ so the boundary layer coordinate is given by

$$y^* = 1-y/\varepsilon^{1/3}\ .$$

The dominant boundary layer equation is given by

$$\frac{d^4 f_0(y^*)}{dy^{*4}} - y^* \frac{d^2 f_0(y^*)}{dy^{*2}} - \frac{df_0(y^*)}{dy^*} = 0 \quad .$$

Both boundary conditions at $y = 1$ have to be satisfied, by $f_0(y^*)$ at $y^* = 0$

$$\frac{d^2 f_0(y^*)}{dy^{*2}} = \frac{d^3 f_0(y^*)}{dy^{*3}} = 0 \qquad \text{at } y^* = 0 \quad .$$

The previous equation can be written

$$\frac{d^4 f_0(y^*)}{dy^{*4}} = \frac{d}{dy^*}\left[y^* \frac{df_0}{dy^*}\right] \quad .$$

A first integration gives us as a solution

$$\frac{d^3 f_0(y^*)}{dy^{*3}} = y^* \frac{df_0}{dy^*}$$

where we already used one of the boundary conditions in removing the constant of integration. Defining

$$\frac{df_0}{dy^*} = f_0^*(y^*),$$

we have

$$\frac{d^2 f_0}{dy^{*2}} = y^* f_0^*$$

with the boundary condition $df_0^*/dy^* = 0$ for $y^* = 0$.

This last equation is known as the Airy equation, the solution of which is given in terms of the Airy integrals. We, thus, have

$$f_0^* = A_0 \, Ai(y^*) + B_0 Bi(y^*) \quad .$$

The asymptotic expansions of the Airy integrals can be found in the literature (see Reference 18) and are given by

$$Ai(z) \sim \frac{1}{2} \Pi^{-1/2} z^{-1/4} e^{-\xi} {}_2F_0(\frac{1}{6}, \frac{5}{6}; -\frac{1}{2\xi})$$

and

$$Bi(z) \sim \Pi^{-1/2} z^{-1/4} e^{\xi} {}_2F_0(\frac{1}{6}, \frac{5}{6}; \frac{1}{2\xi})$$

where

$$\xi = \frac{2}{3} z^{3/2}$$

and ${}_2F_0$ stands for the generalized Hypergeometric series.

A direct observation of the expression for Bi(z) shows that it is inappropriate for matching due to its exponential growth, and only

$$f_0^* = A_0 \, Ai(y^*)$$

has the exponential decay property.

We also have to satisfy the boundary condition

$$\frac{df_0^*}{dy^*} = 0 \quad \text{for} \quad y^* = 0$$

and the power series expansion of Ai'(z) near z = 0 is given by the following expression:

$$Ai'(z) = -b \, {}_0F_1(; \frac{1}{3}; \xi^2/4) + \frac{1}{2} a z^2 \, {}_0F_1(; \frac{5}{3}; \xi^2/4)$$

where ξ has the same meaning as previously and the constants a and b take respectively the values 0.355 and 0.258.

As a result, the satisfaction of the last boundary condition requires that $A_0 = 0$ and we are left with

$$f_0^*(y^*) = 0$$

or

$$f_0 = D_0.$$

If we spent some time in the development of the solution for f_0, it is due to the fact that the previous development will be found helpful in the search for the solution of the other functions in the boundary-layer.

To investigate a little further the boundary layer solution, we have to try to match the solution valid in the central region of the beam to the expansion valid near the free end. In order to accomplish this last objective, we have to expand the solution found in the central region near the free end. We have:

$$\phi_\alpha(y,\varepsilon) = h_0(y) + \sqrt{\varepsilon}\ h_1(y)+\ldots$$

Defining the variable $y' = 1-y$, we have:

$$\phi_\alpha(y',\varepsilon) = h_0(1-y') + \sqrt{\varepsilon}\ h_1(1-y')+\ldots$$

or

$$\phi_\alpha(y',\varepsilon) = h_0(1) - h_0'(1)y' + h_0''(1)\ \frac{y'^2}{2!} +\ldots$$

$$\pm \sqrt{\varepsilon}\ h_1(1) - \sqrt{\varepsilon}\ h_1'(1)y'+\ldots$$

For matching, an intermediate limit suitable for matching near $y = 1$ is given by y_η fixed:

$$y_\eta = \frac{y'}{\eta(\varepsilon)} \qquad \text{where } y' \text{ is always positive.}$$

$$\eta(\varepsilon) \to 0 \qquad \text{and} \qquad \frac{\eta}{\varepsilon^{1/3}} \to \infty$$

so that $y' = \eta\ y_\eta \to 0$ and $y^* = \dfrac{\eta\ y_\eta}{\varepsilon^{1/3}} \to \infty$.

Matching near y'=0 takes the form:

$$\lim_{\substack{\varepsilon \to 0 \\ y_\eta \text{ fixed}}} \left\{ \begin{array}{l} h_0(1) - h_0'(1)\eta\, y_\eta + h_0''(1)\, \dfrac{\eta^2 y_\eta^2}{2!} + \ldots + \sqrt{\varepsilon}\, h_1(1) \\ - \sqrt{\varepsilon}\;\; h_1'(1)\eta\, y_\eta + \ldots \\ \qquad - \delta_0(\varepsilon) D_0 - \delta_1(\varepsilon)\; f_1\left(\dfrac{\eta\, y_\eta}{\varepsilon^{1/3}}\right) - \delta_2(\varepsilon) f_2\left(\dfrac{\eta\, y_\eta}{\varepsilon^{1/3}}\right) - \ldots \end{array} \right\} = 0 \quad .$$

From here, we can see that the matching suggests the use of the following equalities:

$$\begin{array}{lll} \delta_0(\varepsilon) = \varepsilon^0 & \text{and} & h_0(1) = D_0 \\ \delta_1(\varepsilon) = \varepsilon^{1/3} & & \delta_3(\varepsilon) = \varepsilon^{2/3} \\ \delta_2(\varepsilon) = \varepsilon^{1/2} & & \delta_4(\varepsilon) = \varepsilon^{5/6} \ldots \end{array}$$

As a result, we can see that the determination of D_0 seems to be free and the conjecture made earlier about the value of $h_0(1)$ is perfectly valid.

The boundedness of the value of $h_0(1)$ is the real boundary condition that should be used at the free end, and the value of D_0 is thus known!

Now that the matching process has suggested an order of magnitude of $\varepsilon^{1/3}$ for $\delta_1(\varepsilon)$ in order to match the linear term in $\eta\, y_\eta$ appearing in the previous relation, let us compute the expression for $f_1(y^*)$.

The differential equation that $f_1(y^*)$ must satisfy is obtained by picking the term of order of magnitude

$$\frac{\varepsilon\, \delta_1(\varepsilon)}{\phi^4(\varepsilon)} = \varepsilon^0$$

in the general expression (3)

$$\frac{d^4 f_1(y^*)}{dy^{*4}} - y^* \frac{d^2 f_1(y^*)}{dy^{*2}} + \frac{y^{*2}}{2} \frac{d^2 f_0(y^*)}{dy^{*2}} + y^* \frac{df_0(y^*)}{dy^*}$$

$$- \frac{df_1(y^*)}{dy^*} - \lambda_0^2 f_0(y^*) = 0$$

with the boundary conditions

$$\left.\begin{aligned} \frac{d^2 f_1(y^*)}{dy^{*2}} &= 0 \\ \frac{d^3 f_1(y^*)}{dy^{*3}} &= 0 \end{aligned}\right\} \quad \text{for } y^* = 0$$

Using the constant D_0 for $f_0(y^*)$, we are left with

$$\frac{d^4 f_1(y^*)}{dy^{*4}} - y^* \frac{d^2 f_1}{dy^{*2}} - \frac{df_1}{dy^*} = \lambda_0^2 D_0 \quad .$$

In this last linear differential equation, the homogeneous part is exactly the same as the one defining f_0.

A particular solution is given by taking a linear relation for $f_{1h}(y^*)$ such as

$$f_{1h}(y^*) = -\lambda_0^2 D_0 y^* \quad .$$

The boundary conditions on $f_1(y^*)$ cancel once more the two Airy Integrals, solutions of the homogeneous part and as a result

$$f_1(y^*) = -\lambda_0^2 h_0(1) y^* \quad .$$

Going back to the expression computed for the matching near $y'=0$ we have to show that

$$-h_0'(1) + \lambda_0^2 h_0(1) = 0.$$

The differential equation defining h_0 is given by

$$\frac{1}{2}(1-y^2)h_0'' - y\,h_0' + \lambda_0^2\,h_0 = 0,$$

evaluated at y=1, we have:

$$-h_0'(1) + \lambda_0^2\,h_0(1) = 0.$$

In the matching, valid near the free end, we observe that the limit process suggests to introduce a term of order of magnitude $\delta_2(\varepsilon) = \sqrt{\varepsilon}$ in order to take care of the presence of $\sqrt{2}\,h_1(1)$. We will thus introduce here into the asymptotic expansion valid near the free end the expression

$$f_2(y^*) = D_1$$

where D_1 is equal to $h_1(1)$. There is, as expected, no new information added to the problem, the value of h_1 being already completely specified.

Proceeding to the next term, the matching process of the quadratic term in $\eta^2 y_\eta^2$ suggests for $\delta_3(\varepsilon)$ an order of magnitude of $\varepsilon^{2/3}$. The equation defining $f_3(y^*)$ is given by considering the terms of order of magnitude

$$\frac{\varepsilon\,\delta_3(\varepsilon)}{\phi^4(\varepsilon)} = \frac{\varepsilon\varepsilon^{2/3}}{\varepsilon^{4/3}} = \varepsilon^{1/3}$$

in the expansion near y = 1 or

$$\frac{d^4 f_3(y^*)}{dy^{*4}} - y^*\,\frac{d^2 f_3}{dy^{*2}} + \frac{y^{*2}}{2}\,\frac{d^2 f_1}{dy^{*2}} - \frac{df_3}{dy^*} + y^*\,\frac{df_1}{dy^*} - \lambda_0^2\,f_1(y^*) = 0$$

with the boundary conditions

$$\left.\begin{aligned} \frac{d^2 f_3}{dy^{*2}} &= 0 \\ \frac{d^3 f_3}{dy^{*3}} &= 0 \end{aligned}\right\} \text{ for } y^* = 0$$

We already found that $f_1(y^*) = -\lambda_0^2\, h_0(1) y^*$. So the previous expression becomes:

$$\frac{d^4 f_3(y^*)}{dy^{*4}} - y^* \frac{d^2 f_3}{dy^{*2}} - \frac{df_3}{dy^*} = -\lambda_0^4\, h_0(1) y^* + \lambda_0^2\, h_0(1) y^* \quad .$$

The solution of this equation is given by a particular solution and the general solution of the homogeneous part. The particular solution is taken as

$$f_{3P}(y^*) = D_3 y^{*2}$$

where

$$D_3 = \frac{\lambda_0^2}{4} (\lambda_0^2 - 1)\, h_0(1).$$

The general solution of the homogeneous part has to satisfy the requirement on the third derivative due to the choice of the particular solution and as a result, the general solution is given by

$$f_{3h}(y^*) = A_1 \int_0^{y^*} Ai(\xi) d\xi + D_4$$

as can be seen from previous development, where the other Airy Integral has been cancelled for its exponential growth.

The final expression for f_3 is then

$$f_3(y^*) = A_1 \int_0^{y^*} Ai(\xi)d\xi + \frac{\lambda_0^2}{4} (\lambda_0^2-1)\ h_0(1)y^{*2} + D_4 \quad .$$

The determination of A_1 is given by the use of the second boundary condition

$$\frac{d^2f_3}{dy^{*2}} = 0 \qquad \text{for } y^* = 0$$

or

$$A_1\ Ai'(y^*) + \frac{\lambda_0^2}{2} (\lambda_0^2 - 1)\ h_0(1) = 0 \qquad \text{for } y^* = 0$$

or

$$A_1 = - \frac{\lambda_0^2(\lambda_0^2 - 1)\ h_0(1)}{2\ Ai'(o)}$$

where the value of Ai'(o) can be found in tables. (Reference 18).

Going back to the matching procedure, we note that the term

$$A_1 \int_0^{\eta y_\eta/\varepsilon^{1/3}} Ai(\xi)d\xi + D_4$$

presents the exponential decay which allows us to neglect its contribution in the process, and we are left with

$$h_0''(1)\ \frac{\eta^2 y_\eta^2}{2!} - \frac{\lambda_0^2}{4} (\lambda_0^2 - 1)\ h_0(1)\eta^2 y_\eta^2 = 0.$$

This last relation represents an identity as we now show.

Considering the expression defining h_0 and expressing this relation in terms of the variable y', we obtain:

$$\frac{2y' - y'^2}{2} h_0'' - (1-y')h_0' + \lambda_0^2\ h_0 = 0.$$

Expanding h_0, h_0' and h_0'' in a Taylor series expansion valid near $y'=0$, we have

$$h_0(1-y') = h_0(1) - h_0'(1)y' + \frac{h_0''(1)}{2!} y'^2 + \ldots$$

$$h_0'(1-y') = h_0'(1) - h_0''(1)y' + \frac{h_0'''(1)}{2!} y'^2 + \ldots$$

$$h_0''(1-y') = h_0''(1) - h_0'''(1)y' + \ldots$$

Introducing the last expansions into our previous equations, we obtain the following identities:

$$-h_0'(1) + \lambda_0^2 h_0(1) = 0$$

$$2h_0''(1) + h_0'(1) - \lambda_0^2 h_0'(1) = 0 \quad .$$

This last expression can also be written:

$$2h_0''(1) = (\lambda_0^2 - 1)\ \lambda_0^2 h_0(1) \ ,$$

which represents precisely the equality coming from the matching process.

Pursuing the matching even further, the next term to consider is $f_4(y^*)$ corresponding to $\delta_4(\varepsilon) = \varepsilon^{5/6}$. The equation that $f_4(y^*)$ must satisfy is represented by the terms of order of magnitude:

$$\frac{\varepsilon\delta_4(\varepsilon)}{\phi^4(\varepsilon)} = \frac{\varepsilon\varepsilon^{5/6}}{\varepsilon^{4/3}} = \varepsilon^{1/2}$$

in the development (3), or

$$\frac{d^4 f_4(y^*)}{dy^{*4}} - y^* \frac{d^2 f_4(y^*)}{dy^{*2}} + y^{*2} \frac{d^2 f_2(y^*)}{dy^{*2}} - \frac{df_4(y^*)}{dy^*}$$

$$+ y^* \frac{df_2(y^*)}{dy^*} - \lambda_0^2 f_2(y^*) - \lambda_1^2 f_0(y^*) = 0.$$

Taking into account the fact that

$$f_2(y^*) = h_1(1) \quad \text{and}$$
$$f_0(y^*) = h_0(1) \ ,$$

we have

$$\frac{d^4 f_4(y^*)}{dy^{*4}} - y^* \frac{d^2 f_4}{dy^{*2}} - \frac{df_4(y^*)}{dy^*} = \lambda_0^2 \, h_1(1) + \lambda_1^2 \, h_0(1) \ .$$

The use of the boundary condition for f_4 limits the solution for $f_4(y^*)$ to

$$f_4(y^*) = \lambda_0^2 \, h_1(1) \, y^* - \lambda_1^2 \, h_0(1) \, y^* \ .$$

In the matching process, we are left with

$$-h_1'(1) + \lambda_0^2 \, h_1(1) + \lambda_1^2 \, h_0(1)$$

which is identically equal to zero if we consider the equation defining h_1 evaluated at the point y=1 or

$$-h_1'(1) + \lambda_0^2 \, h_1(1) = -\lambda_1^2 \, h_0(1) \quad .$$

To complete the study of the boundary-layer solution near the free end, let us discuss briefly the term f_5 of the development of the boundary-layer solution. For this purpose, we have that $\delta_5(\varepsilon) = \varepsilon$, and the equation defining $f_5(y^*)$ is given by

$$\frac{d^4 f_5(y^*)}{dy^{*4}} - y^* \frac{d^2 f_5(y^*)}{dy^{*2}} + \frac{y^{*2}}{2} \frac{d^2 f_3(y^*)}{dy^{*2}} - \frac{df_5(y^*)}{dy^*}$$
$$+ y^* \frac{df_3(y^*)}{dy^*} - \lambda_0^2 \, f_3(y^*) = 0$$

with the Boundary Conditions:

$$\left.\begin{aligned} \frac{d^2 f_5(y^*)}{dy^{*2}} &= 0 \\ \frac{d^3 f_5(y^*)}{dy^{*3}} &= 0 \end{aligned}\right\} \quad \text{for} \quad y^* = 0$$

Only the asymptotic behavior as $y^* \to \infty$ is essential for matching and this is easily found.

Writing for the expression already found for $f_3(y^*)$ the following expression

$$f_3(y^*) = \frac{\lambda_0^2}{4} (\lambda_0^2 - 1)\, h_0(1)\, y^{*2} + \text{TST}$$

where TST stands for transcendentally small terms, always neglected for matching, the equation defining f_5 is given by

$$\frac{d^4 f_5(y^*)}{dy^{*4}} - y^* \frac{d^2 f_5(y^*)}{dy^{*2}} - \frac{df_5(y^*)}{dy^*} = \frac{\lambda_0^4}{4} (\lambda_0^2 - 1) h_0(1) y^{*2} - \frac{3\lambda_0^2}{4} (\lambda_0^2 - 1) h_0(1) y^{*2} + \text{TST}.$$

The equation for $f_5(y^*)$ can be integrated once and gives us:

$$\frac{d^3 f_5(y^*)}{dy^{*3}} - y^* \frac{df_5}{dy^*} = \left[\frac{\lambda_0^4}{4} (\lambda_0^2 - 1)\, h_0(1) - \frac{3\lambda_0^2}{4} (\lambda_0^2 - 1) h_0(1) \right] \frac{y^{*3}}{3} + D_5 + \text{TST}.$$

The calculation for f_5 is carried out simply from the differential euqation, noting that

$$\frac{d^3 f_5(y^*)}{dy^{*3}} \ll y^* \frac{df_5}{dy^*} \quad \text{as } y^* \to \infty .$$

The rationale beyond this last argument is exactly the reason why the matching works, so that we have

$$\frac{df_5}{dy^*} \cong -\frac{(\lambda_0^2-1)}{4}(\lambda_0^4-3\lambda_0^2)h_0(1)\frac{y^{*2}}{3} - \frac{D_5}{y^*} + \frac{1}{y^*}\frac{d^3f_5(y^*)}{dy^{*3}} + \text{TST} .$$

It follows that

$$\frac{d^2f_5}{dy^{*2}} \cong -\frac{(\lambda_0^2-1)}{2}(\lambda_0^4-3\lambda_0^2)h_0(1)\frac{y^*}{3} + \frac{D_5}{y^{*2}} - \frac{1}{y^{*2}}\frac{d^3f_5}{dy^{*3}} + \frac{1}{y^*}\frac{d^4f_5}{dy^{*4}} + \text{TST},$$

and

$$\frac{d^3f_5}{dy^{*3}} \cong -\frac{(\lambda_0^2-1)}{6}(\lambda_0^4-3\lambda_0^2)h_0(1) + O(\frac{1}{y^{*3}})+\ldots$$

Finally, we have:

$$\frac{df_5}{dy^*} \cong -\frac{(\lambda_0^2-1)}{12}(\lambda_0^4-3\lambda_0^2)\,h_0(1)y^{*2} - \left[D_5 + \frac{(\lambda_0^2-1)}{6}(\lambda_0^4-3\lambda_0^2)h_0(1)\right]\frac{1}{y^*} + O(\frac{1}{y^{*4}}) + \text{TST} ,$$

$$f_5 \cong -\frac{(\lambda_0^2-1)}{36}(\lambda_0^4-3\lambda_0^2)h_0(1)y^{*3} - \left[D_5 + \frac{(\lambda_0^2-1)}{6}(\lambda_0^4-3\lambda_0^2)h_0(1)\right]\ln y^* + D_6 + O(\frac{1}{y^{*3}}) + \text{TST} .$$

The constants of integration D_5 and D_6 are then found from the matching. The other constants of integration for each solution are in the transcendentally small terms and are found from the boundary conditions at the origin.

In the matching conditions, expanding the various terms and neglecting transcendentally small terms, we have:

$$D_5 = -\frac{(\lambda_0^2 - 1)}{6} (\lambda_0^4 - 3\lambda_0^2)\ h_0(1)$$

$$D_6 = h_2(1).$$

We also have to show the identity:

$$\frac{h_0'''(1)}{3!} + \frac{1}{36} (\lambda_0^2 - 1)(\lambda_0^4 - 3\lambda_0^2)\ h_0(1) = 0 \quad .$$

All the other terms omitted vanish more rapidly than those matched. By going back to the development we already used for the identity found for f_3, we also have:

$$3\ h_0'''(1) = (\lambda_0^2 - 3)\ h_0''(1)$$

or

$$3\ h_0'''(1) = \frac{(\lambda_0^2 - 1)}{2} (\lambda_0^4 - 3\lambda_0^2)\ h_0(1) \quad .$$

The last identity is thus proven.

Having now completed the modal analysis of the uniform classical beam, we summarize the results by recording the final expression for the eigenfrequencies, or

$$\omega_\alpha^2 = \lambda_\alpha^2\ \Omega^2 = \Omega^2(\lambda_0^2 + \sqrt{\varepsilon}\ \ \lambda_1^2 + \ldots)$$

where $\lambda_0^2 = \frac{n(n+1)}{2}$ for n odd,

and $$\lambda_1^2 = \frac{1}{\sqrt{2} \int_0^1 \mathscr{L}_2^2\ (2\lambda_0^2, y)\,dy}$$

This section concludes with a comparison of previous work and that of the author using the results presented by F.R. Vigneron.[19] In

Reference 19, the author presents the relation for the lowest loaded natural frequency of a radial boom as

$$\omega_1^2 = \omega_{NR}^2 + 1.193\ \Omega^2,$$

where ω_{NR}^2 stands for the unloaded lowest frequency for a uniform beam. The quantity ω_{NR}^2 is recognized as $\omega_{NR}^2 = (3.515)^2\ EI/\mu L^4$, so the expression of ω_1^2 becomes

$$\omega_1^2 = (3.515)^2 \frac{EI}{\mu L^4} + 1.193\ \Omega^2 \quad .$$

The definition of ε gives us

$$\frac{EI}{\mu L^4} = \varepsilon\Omega^2 ,$$

so this expression can be written

$$\omega_1^2 = \Omega^2(1.193 + (3.515)^2\ \varepsilon) \quad .$$

Comparing the above expression with our development of ω_1^2 or

$$\omega_1^2 = \Omega^2(1 + 2.12\ \sqrt{\varepsilon}\ ),$$

we obtained the curves of Figure 2.

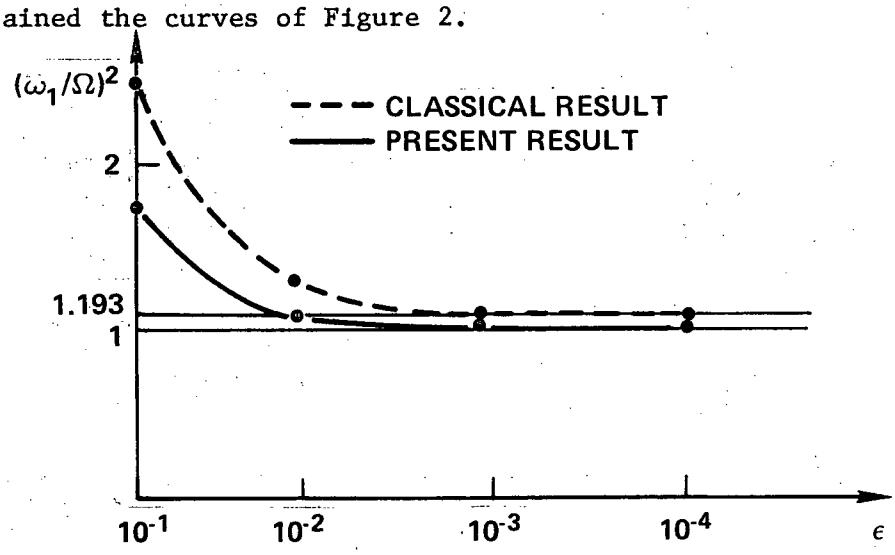

Figure 2. Correspondence With Classical Results

In the interpretation of these results, we have to keep in mind that the result presented by Vigneron represents a better approximation for increasing values of ε while our approximation becomes more precise as ε decreases. The limiting behavior of a beam when ε goes to zero is given by the cable and the true value of the ratio $(\omega_1/\Omega)^2$ is given, for that case, by 1 (as opposed to 1.193).

After having considered the case of the classical rotating beam, we will now consider the case of a rotating cable of small flexural rigidity when clamped at both ends.

2.3 Dynamics of a Taut, Rotating, Elastic Cable

In this derivation, the same assumptions as those made earlier, i.e. to consider the motion of the spinning rigid ring to which the cable is attached as not being affected by the transverse vibration of the cable, and also to ignore the rotation of the transverse cross-section, will be made. The total system is also considered as being in rotation with an angular velocity Ω around the axis of symmetry of the ring to which the cable is clamped (see Figure 3).

For the special type of configuration under investigation, it must be emphasized that the spin has essentially a destabilizing effect. It is known that the influence of spin on flexible spacecraft is characterized by the presence of three effects: preload, Coriolis coupling and centripetal acceleration. The effect of preload is of main concern for this structure, because of its dependence upon the configuration (orientation of the flexible appendage with respect to the rigid core). For this particular structure, the preload modifies seriously the stiffness properties of the structure and justifies the

introduction of a built-in tension that we will assume to be big enough to ensure a tension everywhere within the cable.

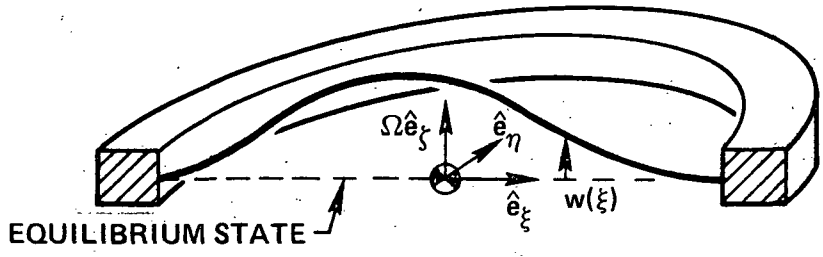

Figure 3. Rotating Uniform Taut Cable.

In what follows we do intend to invest the cable with some small flexural stiffness, so that the governing equations of motion are the fourth order partial differential equations of the beam, rather than the classical second order partial differential equations of the taut string. Under the previous assumptions, the transverse vibrations of the classical (Euler-Bernoulli) beam subjected to an external axial load $P(\xi)$ is given in general form by

$$EI \frac{\partial^4 w}{\partial \xi^4} - \frac{\partial}{\partial \xi}\left[P(\xi) \frac{\partial w}{\partial \xi}\right] + \mu \frac{\partial^2 w}{\partial t^2} = 0$$

where EI is the flexural stiffness of the beam or cable

μ is the linear density

$P(\xi)$ is the external axial load.

We should notice here that the equilibrium position of the taut cable is given when the total system is rotating at an angular velocity Ω, with the cable straight along the ξ axis. No displacement can occur either in the η direction or the ζ direction. The position vector ζ appearing in the last relation is in fact the position vector at equilibrium. If we stress this last point, it is due to the fact that

the equilibrium state is not a zero-stress state but there exists a steady state stretching into the cable, steady state stretching which is already included into our position vector ξ. We assume that this last fact doesn't alterate the constancy of the linear density or the flexural rigidity.

The boundary conditions are given for this problem by:

$$w(\pm L) = 0 = \frac{\partial w}{\partial \xi}(\pm L)$$

where L stands for half of the total length of the cable.

For the rotating uniform taut cable, the steady state axial force is given by the constant built-in tension T applied to the cable and by a compression force R resulting from the integral of the "centrifugal forces" applied to the elementary masses. If we consider a cross-section of the cable, located at a position vector ξ, the system of stress applied to the whole section is given by

$$P(\xi) = T - \int_0^{\xi} \mu\Omega^2 \, \xi' d\xi'$$

or

$$P(\xi) = T - \mu\Omega^2 \frac{\xi^2}{2} \quad .$$

One remark should be mentioned here. One of our assumptions, when we introduce the built-in tension T, was to make sure that every elementary element along the cable was in a state of tension. With the definition of the axial load, it is the same constraint as assuming that

$$P(\xi) \geq 0 \qquad \text{for every } \xi \quad \text{or}$$

$$T \geq \int_0^L \mu\Omega^2 \xi' d\xi' = \mu\Omega^2 \frac{L^2}{2} = T_o \quad .$$

Introducing the value just found for $P(\xi)$ into the previous relation, we have:

$$EI \frac{\partial^4 w}{\partial \xi^4} - \frac{\partial}{\partial \xi} \left[\left(T - \mu\Omega^2 \frac{\xi^2}{2} \right) \frac{\partial w}{\partial \xi} \right] + \mu \frac{\partial^2 w}{\partial t^2} = 0.$$

The introduction of the dimensionless variable $s = \frac{\xi}{L}$ transforms the previous equation into

$$\frac{EI}{L^4} \frac{\partial^4 w}{\partial s^4} - \frac{\mu\Omega^2}{2} \frac{\partial}{\partial s} \left[(k^2 - s^2) \frac{\partial w}{\partial s} \right] + \mu \frac{\partial^2 w}{\partial t^2} = 0,$$

where we introduce the definition $T = k^2 T_o$. The constant k^2 is a known quantity which has to be greater than one.

The last relation can also be written:

$$\varepsilon \frac{\partial^4 w}{\partial s^4} - \frac{\partial}{\partial s} \left[\left(1 - \frac{s^2}{k^2} \right) \frac{\partial w}{\partial s} \right] + \frac{2}{\Omega^2 k^2} \frac{\partial^2 w}{\partial t^2} = 0,$$

where ε stands for $2EI/\mu\Omega^2 L^4 k^2$.

Once again, we observe that the low flexural stiffness is combined with the high spin rate to produce the small parameter ε.

Written in a slightly different way, the parameter ε is also

$$\varepsilon = \frac{EI}{L^2 T}$$

and is a measure of the ratio of the flexural stiffness to the built-in tension.

The boundary conditions for the problem are also

$$w(\pm 1) = \frac{\partial w}{\partial s} (\pm 1) = 0,$$

conditions which express the physical clamping of the cable at its extremities.

Now that the dynamic equation of the rotating cable has been obtained through the use of the well-known relation of a classical beam subjected to an external load, we could in a completely similar way have derived the previous relation by solving first the steady stretching of the cable, and then determine the transverse vibration as being a perturbation with respect to that steady-state of deformations. The last derivation would give in general a partial differential equation with nonlinear coefficients, and the first approximation of the coefficients would give exactly the same relation as the one we derived before. This last feature was mentioned before as being a recognizable fact of continuous modeling, and already included into the general derivation of the transverse vibrations where the substitution

$$\frac{P(\xi)}{A} = E \frac{\partial u}{\partial \xi}$$

has been done into the derivation, where

u expresses the deformation along the cable axis

A expresses the cross-section area of the cable

E expresses the modulus of elasticity.

The procedure we just outlined was the approach used by Abel and Kerr,[11] when they derived the dynamic equation for the transverse vibration.

We now solve the transverse vibration equation, by using the separation of variables

$$w(s,t) = \phi_\alpha(s)\, P_\alpha(t) \ .$$

Substituting into the previous equation, we have

$$\ddot{P}_\alpha + \omega_\alpha^2 P_\alpha = 0,$$

$$\varepsilon \frac{d^4\phi_\alpha}{ds^4} - \frac{d}{ds}\left[\left(1 - \frac{s^2}{k^2}\right)\frac{d\phi_\alpha}{ds}\right] - \frac{2\omega_\alpha^2}{\Omega^2 k^2}\phi_\alpha = 0,$$

where $(\cdot)$ stands for time derivative, ω_α represents the eigenfrequency of the vibration. The boundary conditions become at the same time:

$$\phi_\alpha(\pm 1) = \frac{d\phi_\alpha}{ds}(\pm 1) = 0.$$

As announced in Chapter 1, the method of matched asymptotic expansion will be used in this problem, but a suitable change of variables is recommended first, so we define

$$s = kx.$$

We then have

$$\frac{\varepsilon}{k^4}\phi_\alpha^{IV} - \frac{1}{k^2}[(1-x^2)\phi_\alpha']' - \frac{2\omega_\alpha^2}{\Omega^2 k^2}\phi_\alpha = 0$$

where $(')$ stands for spatial derivative with respect to x, or

$$\frac{\varepsilon}{k^2}\phi_\alpha^{IV} - [(1-x^2)\phi_\alpha']' - \lambda_\alpha^2\phi_\alpha = 0 \tag{2.4}$$

where λ_α^2 replaces $2\,\omega_\alpha^2/\Omega^2$.

The boundary conditions become: $\phi_\alpha(\pm k^{-1}) = \phi_\alpha'(\pm k^{-1}) = 0$. The parameter k is always bigger than one and consequently k^{-1} is ≤ 1.

Let us consider for the central part of the beam, a solution for which analytic dependence on the small parameter ε is given by:

$$\phi_\alpha(x,\varepsilon) = h_0(x) + \nu_1(\varepsilon)h_1(x) + \nu_2(\varepsilon)h_2(x) + \ldots$$

Once again, the subscript α, characteristic number of the mode, will be omitted when not needed. Similarly the eigenvalue λ_α^2 of the problem can also be expanded in terms of the small parameter ε in the following way:

$$\lambda_\alpha^2(\varepsilon) = \lambda_0^2 + \kappa_1(\varepsilon)\lambda_1^2 + \kappa_2(\varepsilon)\lambda_2^2 + \ldots$$

where $\kappa_i(\varepsilon)$ constitute an asymptotic sequence.

The differential equation defining $h_0(x)$ is then given by

$$[(1-x^2)h_0']' + \lambda_0^2\, h_0 = 0 \tag{2.5}$$

and is a Legendre equation.

Defining the Legendre function of the first and second kind by:

$$\mathscr{L}_1(\lambda_0^2, x) = 1 - (\tfrac{1}{2!})\, \lambda_0^2\, x^2 + (\tfrac{1}{4!})\, \lambda_0^2(\lambda_0^2 - 6)x^4 + \ldots$$

$$\mathscr{L}_2(\lambda_0^2, x) = x - (\tfrac{1}{3!})\, (\lambda_0^2 - 2)x^3 + \ldots,$$

we see directly that $\mathscr{L}_1$ corresponds to the even modes and $\mathscr{L}_2$ corresponds to the odd modes.

We will now discuss briefly some of the characteristics that we are able to deduce from the zeroth order expansion of the modes based on the fact that we are expecting boundary conditions for the Legendre polynomials of the form $h_0(\pm k^{-1}) = 0$. This last point can be seen physically in the sense that we are not expecting a jump for the modes near the extremities and will be shown later by the process of matching the asymptotic expansions.

Based on the boundary conditions $h_0(\pm k^{-1}) = 0$, we see that the eigenfrequencies are the solution of the transcendental equation:

$$\mathscr{L}_1(\lambda_0^2, k^{-1}) = 0 \quad \text{and} \quad \mathscr{L}_2(\lambda_0^2, k^{-1}) = 0 \quad .$$

One remark should be made here. Because k^{-1} can take on only positive values less than one, we observe that the singularity of the Legendre polynomials at x=1 is avoided.

We first study the even modes. Defining the ratio $\lambda_0^2/k^2 = \mu_0^2$, we will try to plot the variation of μ_0^2 with respect to k^2. The relation $\mathscr{L}_1(\lambda_0^2, k^{-1}) = 0$ becomes

$$1 - \frac{1}{2!}\mu_0^2 + \frac{1}{4!}\mu_0^2(\mu_0^2 - \frac{6}{k^2}) - \frac{1}{6!}\mu_0^2(\mu_0^2 - \frac{6}{k^2})(\mu_0^2 - \frac{20}{k^2})\ldots = 0$$

Some discrete values for the plotting are easily found like:

$$\mu_0^2 = \frac{6}{k^2} \quad \text{gives us} \quad 1 - \frac{\mu_0^2}{2} = 0 \quad \text{or} \quad \mu_0^2 = 2 \quad \text{and} \quad k^2 = 3$$

$$\mu_0^2 = \frac{20}{k^2} \quad \text{gives us similarly} \begin{cases} \mu_0^2 = 14.8 & \text{and} \quad k^2 = 1.35 \\ \mu_0^2 = 2.32 & \text{and} \quad k^2 = 8.6 \end{cases} .$$

We remark easily here that the value $k^2 = 8.6$ corresponds to the third mode and the value $k^2 = 1.35$ corresponds to the first mode. By first mode, we mean that the corresponding solution has no point of zero displacement except the extremities and by third mode, we mean that along the span of the cable, there is two points with a zero transverse displacement.

Some interest lies also in the limiting case given when the parameter k^2 goes to infinity or when it goes down to one. In the limiting case, when $k^2 \to \infty$, the expression $\mathscr{L}_1(\lambda_0^2, k^{-1}) = 0$ becomes the expansion of $\cos\mu_0$, and as a result the corresponding value of μ_0^2, valid for the first mode, is given by

$$\mu_0^2 = \frac{\Pi^2}{4} \quad .$$

This last result is reasonable, because the case $k^2 \to \infty$, corresponds to the ordinary vibrating string. The effect of the distributed forces induced by rotation becomes indeed negligible.

The other limiting case $k^2 \to 1$ is more difficult to handle due to the singularity at $x = 1$. In order to study this last limiting case, it is indicated to look for the intersection of the curve we are looking for with the hyperbola defined by $\mu_0^2 = \varepsilon_0/k^2$ where ε_0 is a small parameter.

The intersection of $\mathscr{L}_1(\lambda_0^2, k^{-1}) = 0$ with $\mu_0^2 = \frac{\varepsilon_0}{k^2}$ is given by

$$1 - \frac{\varepsilon_0}{2!k^2} + \frac{\varepsilon_0}{4!k^2}\left(\frac{\varepsilon_0}{k^2} - \frac{6}{k^2}\right) - \frac{\varepsilon_0}{6!k^2}\left(\frac{\varepsilon_0}{k^2} - \frac{6}{k^2}\right)\left(\frac{\varepsilon_0}{k^2} - \frac{20}{k^2}\right) \ldots = 0 \quad .$$

Neglecting the higher order terms in ε_0, we have

$$1 - \frac{\varepsilon_0}{k^2}\left[\frac{1}{2} + \frac{1}{4k^2} + \frac{1}{6k^4} + \frac{1}{8k^6} + \ldots\right] = 0$$

which can be written as

$$1 + \frac{\varepsilon_0}{2}\log\left(1 - \frac{1}{k^2}\right) = 0 \quad \text{or} \quad 1 - \frac{1}{k^2} = e^{-2/\varepsilon_0}$$

so $$k^2 = \frac{1}{1-e^{-2/\varepsilon_0}} = 1 + e^{-2/\varepsilon_0} \approx 1 \quad .$$

As a result, we thus see that when $\mu_0^2 \to 0$, we have that k^2 differs from one by a transcendentally small quantity.

We finally have for the first mode the following graph (Figure 4). The previous graph shows also the asymptote when $k^2 \to \infty$, which also represents the value of the eigenfrequency of the nonrotating taut

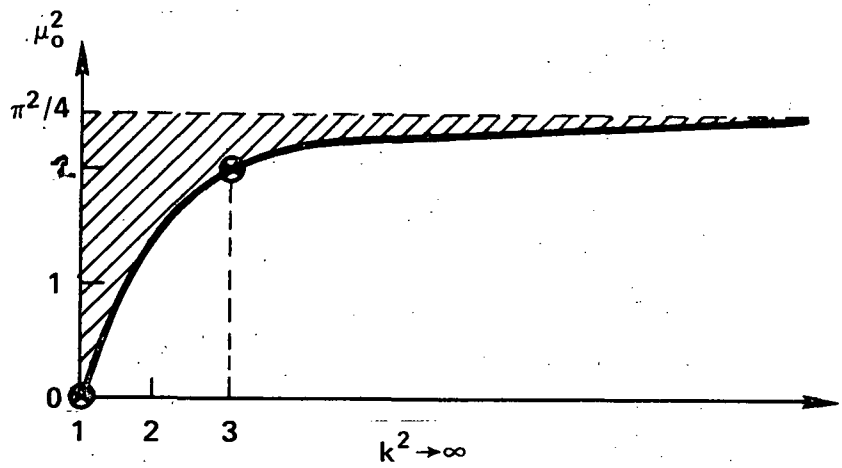

Figure 4. Eigen Frequencies Corresponding to First Mode.

string. The hashed part of the graph represents then the decrease in the eigenfrequency due to the rotation, decrease in the eigenfrequency which is often referred to as the destabilizing effect of rotation. As expected by the previous remark, the destabilizing effect decreases when k^2 increases, which corresponds to the decrease of the effect of the forces induced by spin. We could also show how the first even mode varies as a function of the parameter k^2 (Figure 5).

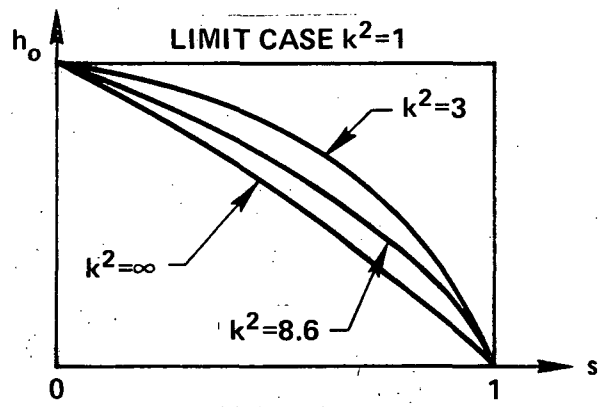

Figure 5. Mode Shapes Corresponding to First Mode.

A similar development could be done for the odd modes.

The relation $\mathscr{L}_2(\lambda_0^2, k^{-1}) = 0$ becomes, after defining $\lambda_0^2/k^2 = \mu_0^2$

$$1 - \frac{1}{3!}\left(\mu_0^2 - \frac{2}{k^2}\right) + \frac{1}{5!}\left(\mu_0^2 - \frac{2}{k^2}\right)\left(\mu_0^2 - \frac{12}{k^2}\right) = 0.$$

We could once again derive some discrete value of μ_0^2 and k^2. The limiting case, when $k^2 \to \infty$, reproduces here again, the result we might expect from the taut vibrating string. In the case, where $k^2 \to 1$, the corresponding value of μ_0^2 is once again obtained by looking for the intersection of the graph with the hyperbola $\mu_0^2 = 2+\varepsilon_0/k^2$ and as we might expect, the resulting value of μ_0^2 is 2. All the results are summarized in the following graph (Figure 6), which represents the variation of μ_0^2 with respect to k^2 for the second mode.

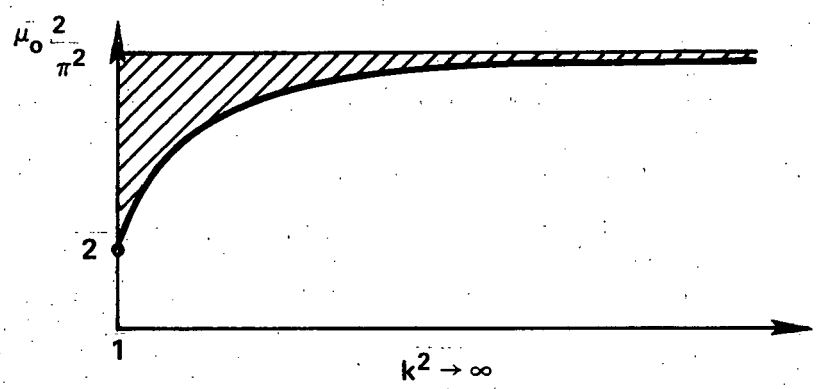

Figure 6. Eigen Frequencies Corresponding to the Second Mode.

The destabilizing effect introduced by the rotation is again apparent on the previous graph. By second mode, we mean, that there exists along the span of the cable one point which doesn't experience any transverse vibration. We could also show how the second mode varies, as a function of the parameter k^2 (Figure 7). After this discussion on the zero-order solution of the eigenvalue problem, we will now turn our attention towards the next order of magnitude of the solution. However, before any attempt could be done in this sense, we will have to look at the solution valid near the extremities of the cable where we still have to prove that $h_0(\pm k^{-1}) = 0$.

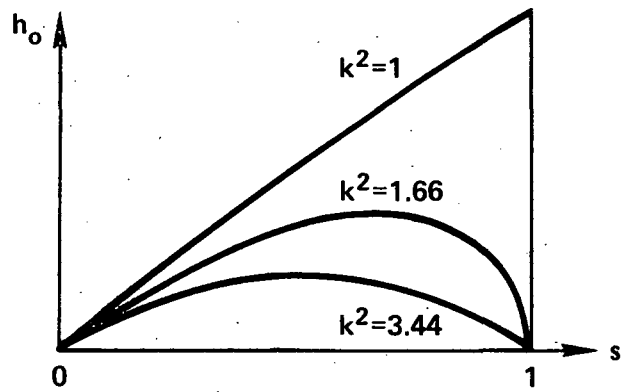

Figure 7. Mode Shapes Corresponding to the Second Mode.

Boundary Layer Near $x = k^{-1}$

We now plan to look at the boundary layer solution near $x=k^{-1}$. It is obvious that the boundary layer near $x=-k^{-1}$ could be obtained in a completely similar way and will thus be omitted.

In order to construct the boundary layer expansion, a suitable boundary-layer coordinate has to be chosen, such that the higher order derivative terms dominate the equation.

Let

$$\tilde{x} = \frac{k^{-1} - x}{\phi(\varepsilon)} \qquad \text{where} \quad \phi(\varepsilon) \to 0 \quad \text{when} \quad \varepsilon \to 0$$

so $$\phi\tilde{x} = k^{-1} - x \qquad \text{and} \qquad dx = -\phi d\tilde{x} \quad .$$

The corresponding asymptotic expansion near $x = k^{-1}$ is chosen such that

$$\phi_\alpha(\tilde{x},\varepsilon) = \delta_0(\varepsilon)\, f_0(\tilde{x}) + \delta_1(\varepsilon) f_1(\tilde{x}) + \ldots$$

where the $\delta_i(\varepsilon)$ form an asymptotic sequence. The basic equation becomes with the previous substitutions:

$$\frac{\varepsilon}{k^2}\left[\frac{\delta_0(\varepsilon)}{\phi^4}\frac{d^4f_0}{d\tilde{x}^4} + \frac{\delta_1(\varepsilon)}{\phi^4}\frac{d^4f_1}{d\tilde{x}^4} + \ldots\right]$$

$$- [1 - (k^{-1}-\phi\tilde{x})^2]\left[\frac{\delta_0(\varepsilon)}{\phi^2}\frac{d^2f_0}{d\tilde{x}^2} + \frac{\delta_1(\varepsilon)}{\phi^2}\frac{d^2f_1}{d\tilde{x}^2} + \ldots\right]$$

$$- 2(k^{-1}-\phi\tilde{x})\left[\frac{\delta_0(\varepsilon)}{\phi}\frac{df_0}{d\tilde{x}} + \frac{\delta_1(\varepsilon)}{\phi}\frac{df_1}{d\tilde{x}} + \ldots\right]$$

$$- \lambda_0^2[\delta_0(\varepsilon)f_0(\tilde{x}) + \delta_1(\varepsilon)f_1(\tilde{x}) + \ldots]$$

$$- \kappa_1(\varepsilon)\lambda_1^2\,[\delta_0(\varepsilon)f_0(\tilde{x}) + \delta_1(\varepsilon)f_1(\tilde{x}) + \ldots] = 0 \qquad (2.6)$$

Expanding the coefficient of the second derivative, we have

$$1-(k^{-1}-\phi\tilde{x})^2 = 1 - k^{-2} + 2\phi k^{-1}\tilde{x} - \phi^2\tilde{x}^2 \ .$$

We now have to specify some information about the order of magnitude of $1-k^{-2}$. We will assume throughout the rest of the study that

$$1 - \frac{1}{k^2} = O(1) \ .$$

By assuming this, we consider the built-in tension to be greater than T_0, by a "sensible" factor. The thickness of the boundary layer is then obtained by considering the highest derivative terms or

$$\frac{\varepsilon\delta_0(\varepsilon)}{\phi^4} \approx \frac{\delta_0(\varepsilon)}{\phi^2}$$

so $\qquad \phi = \sqrt{\varepsilon} \qquad$ and $\quad \tilde{x} = \dfrac{k^{-1}-x}{\sqrt{\varepsilon}} \ .$

The dominant boundary layer equation is given by:

$$\frac{1}{k^2}\frac{d^4f_0}{d\tilde{x}^4} - \left(1 - \frac{1}{k^2}\right)\frac{d^2f_0}{d\tilde{x}^2} = 0 \ .$$

The boundary conditions are:

$$\left\{ \begin{array}{l} f_0(\tilde{x}) = 0 \\ \dfrac{df_0}{d\tilde{x}}(\tilde{x}) = 0 \end{array} \right\} \qquad \text{for } \tilde{x} = 0$$

Defining $d^2f_0/d\tilde{x}^2 = f_0^*(\tilde{x})$ and $(k^2-1) = k^{*2}$, we have

$$\frac{d^2f_0^*}{d\tilde{x}^2} - k^2 f_0^*(\tilde{x}) = 0 \quad .$$

Cancelling, among the two independent solutions of the last equation, the exponential growth which is unfitted for the matching process, we are left with

$$\frac{d^2f_0}{d\tilde{x}^2} = f_0^* = C_o\, e^{-k^*\tilde{x}} \quad .$$

A first integration and one boundary condition gives us:

$$\frac{df_0}{d\tilde{x}} = \frac{C_o}{k^*}(1 - e^{-k^*\tilde{x}}) \quad .$$

A second integration and the second boundary conditions gives us:

$$f_0(\tilde{x}) = \frac{C_o}{k^{*2}}(k^*\,\tilde{x} - 1 + e^{-k^*\tilde{x}})$$

The matching condition with the solution previously found for the central portion of the cable enables us to define the constant C_o. We now expand the outer solution, near $x = k^{-1}$. We define the variable $x' = k^{-1} - x$. We have for the even modes

$$\mathscr{L}_1(\lambda_0^2, k^{-1} - x') = \mathscr{L}_1(\lambda_0^2, k^{-1}) - \frac{d}{dx}\mathscr{L}_1(\lambda_0^2, k^{-1})x' + \ldots$$

and for the odd modes:

$$\mathscr{L}_2(\lambda_0^2,\ k^{-1} - x') = \mathscr{L}_2(\lambda_0^2,\ k^{-1}) - \frac{d}{dx}\ \mathscr{L}_2(\lambda_0^2,\ k^{-1})x' + \ldots$$

An intermediate limit suitable for the matching near $x = k^{-1}$ is defined by

$$x_\eta = \frac{x'}{\eta(\varepsilon)} \qquad \text{with } x_\eta \text{ fixed.}$$

Taking into account $x' = \phi(\varepsilon)\ \tilde{x} = \sqrt{\varepsilon}\ \ x'$, we also have

$$x_\eta = \frac{\sqrt{\varepsilon}\ \tilde{\tilde{x}}}{\eta(\varepsilon)} \qquad \text{where} \qquad \frac{\eta}{\sqrt{\varepsilon}} \to \infty \qquad \text{when } \varepsilon \to 0 \ .$$

The behavior near $x = k^{-1}$ takes the form:

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} \left\{ \begin{array}{l} h_0(k^{-1}-x') + \nu_1(\varepsilon)h_1(k^{-1}-x') + \ldots \\ \\ \quad - \delta_0(\varepsilon)f_0\left(\dfrac{\eta x_\eta}{\sqrt{\varepsilon}}\right) - \delta_1(\varepsilon)f_1\left(\dfrac{\eta x_\eta}{\sqrt{\varepsilon}}\right) \ldots \end{array} \right\} = 0$$

Replacing h_0 and f_0 by the solution previously found, we have

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} \left\{ \begin{array}{l} D_0 \mathscr{L}_{\frac{1}{2}}\ (\lambda_0^2,\ k^{-1}) - D_0 \dfrac{d}{dx}\ \mathscr{L}_{\frac{1}{2}}\ (\lambda_0^2,\ k^{-1})\eta x_\eta + \ldots \\ \\ \quad + \nu_1(\varepsilon)h_1(k^{-1}) - \nu_1(\varepsilon)\ \dfrac{d}{dx}\ h_1(k^{-1})\eta x_\eta + \ldots \\ \\ \qquad - \delta_0(\varepsilon)\ \dfrac{C_0}{k^{*2}}\left(k^*\ \dfrac{\eta x_\eta}{\sqrt{\varepsilon}} - 1 + e^{-k^*\ \frac{\eta x_\eta}{\sqrt{\varepsilon}}}\right) \\ \qquad\qquad - \delta_1(\varepsilon)f_1\left(\dfrac{\eta x_\eta}{\sqrt{\varepsilon}}\right) \ldots \end{array} \right\} = 0 \ .$$

The matching is then accomplished by taking

$$\mathscr{L}_{\frac{1}{2}}\ (\lambda_0^2,\ k^{-1}) = 0$$

which is the relation we used before but was still to be proven and also

$$\delta_0(\varepsilon) = \sqrt{\varepsilon}$$

$$C_0 = -k^* D_0 \frac{d}{dx} \mathscr{L}_{\frac{1}{2}} (\lambda_0^2, k^{-1})$$

where $\mathscr{L}_{\frac{1}{2}}$ stands for both Legendre polynomials.

Now that the linear terms have been matched, we can see that $\nu_1(\varepsilon)$ has to be of the same order of magnitude as $\delta_0(\varepsilon)$ so

$$\nu_1(\varepsilon) = \sqrt{\varepsilon}$$

and also

$$h_1(k^{-1}) = -\frac{C_0}{k^{*2}} .$$

With the information obtained through matching, we are now able to look at the next order of magnitude for the expansion in the central area. Applying a procedure similar to the one used for the rotating cantilevered beam, we plan to consider the orthogonality property of the modes.

Let us consider two different modes ϕ_α and ϕ_β defined by the following relations.

$$\begin{cases} \frac{\varepsilon}{k^2} \phi_\alpha^{IV} - [(1-x^2)\phi_\alpha']' - \lambda_\alpha^2 \phi_\alpha = 0 \\ \frac{\varepsilon}{k^2} \phi_\beta^{IV} - [(1-x^2)\phi_\beta']' - \lambda_\beta^2 \phi_\beta = 0 \end{cases}$$

and the boundary conditions

$$\phi_\alpha(\pm k^{-1}) = \phi_\alpha'(\pm k^{-1}) = \phi_\beta(\pm k^{-1}) = \phi_\beta'(\pm k^{-1}) = 0.$$

Multiplying the relation defining ϕ_α by ϕ_β and integrating over the whole span, we have

$$\frac{\varepsilon}{k^2} \int_{-k^{-1}}^{k^{-1}} \phi_\alpha^{IV}\phi_\beta \, dx - \int_{-k^{-1}}^{k^{-1}} [(1-x^2)\phi_\alpha']'\phi_\beta \, dx - \lambda_\alpha^2 \int_{-k^{-1}}^{k^{-1}} \phi_\alpha\phi_\beta = 0$$

The second term can be written:

$$\int_{-k^{-1}}^{k^{-1}} [(1-x^2)\phi_\alpha']'\phi_\beta \, dx = [(1-x^2)\phi_\alpha']\phi_\beta \Big|_{-k^{-1}}^{k^{-1}} - \int_{-k^{-1}}^{k^{-1}} [(1-x^2)\phi_\alpha'\phi_\beta' \, dx.$$

The use of the boundary conditions limits the previous expression to the last term. Integration by parts and the use of the boundary conditions transforms the first terms into the following sequences of expression

$$\int_{-k^{-1}}^{k^{-1}} \phi_\alpha^{IV}\phi_\beta \, dx = \phi_\alpha''' \, \phi_\beta \Big|_{-k^{-1}}^{k^{-1}} - \int_{-k^{-1}}^{k^{-1}} \phi_\alpha''' \, \phi_\beta' \, dx =$$

$$- \phi_\alpha''\phi_\beta' \Big|_{-k^{-1}}^{k^{-1}} + \int_{-k^{-1}}^{k^{-1}} \phi_\alpha''\phi_\beta'' \, dx \quad .$$

We thus have finally

$$\frac{\varepsilon}{k^2} \int_{-k^{-1}}^{k^{-1}} \phi_\alpha''\phi_\beta'' \, dx + \int_{-k^{-1}}^{k^{-1}} (1-x^2)\phi_\alpha'\phi_\beta' \, dx = \lambda_\alpha^2 \int_{-k^{-1}}^{k^{-1}} \phi_\alpha\phi_\beta dx \quad .$$

The symmetry of the left hand side suggest that it is also equal to

$$\lambda_\beta^2 \int_{-k^{-1}}^{k^{-1}} \phi_\alpha\phi_\beta dx$$

as may be confirmed by a parallel development. Combining those two

expressions, we obtain the expected result i.e.

$$\int_{-k^{-1}}^{k^{-1}} \phi_\alpha \phi_\beta \, dx = 0 \qquad \text{for} \quad \alpha \neq \beta \quad .$$

The orthogonality relationship between the modes has to be completed by the relation normalizing the modes. We are thus free to choose the modes such that

$$\int_{-k^{-1}}^{k^{-1}} \phi_\alpha^2 \, dx = 1 \quad .$$

Considering now the asymptotic expansions used for the modes

$$\phi_\alpha(x,\varepsilon) = h_0(x) + \sqrt{\varepsilon}\, h_1(x) + \ldots,$$

the previous relation becomes, for instance:

$$\int_{-k^{-1}}^{k^{-1}} h_0^2 \, dx = 1$$

$$\int_{-k^{-1}}^{k^{-1}} h_0 h_1 \, dx = 0 \quad .$$

Now that the orthogonality has been proved between the different modes and also between the different terms of the asymptotic expansion, we can come back to the main problem.

Once more, it should be emphasized that in the expansion used for the modes in the orthogonality relationship, a uniformly valid expansion should be used, a uniformly valid expansion which takes into account the contribution of the boundary layer into the mode. But for the problem at hand, the contributions of the boundary layer do

not affect the relations found before, because we limit ourselves to the expansion up to $\sqrt{\varepsilon}$.

The differential equation defining $h_1(x)$ is obtained by replacing the asymptotic expansions valid for $\lambda_\alpha^2(\varepsilon)$ and $\phi_\alpha(x,\varepsilon)$ into the differential equation of the modes. We thus have:

$$[(1-x^2)h_1']' + \lambda_0^2\, h_1 = \begin{cases} -\lambda_1^2\, h_0 & \text{if } \kappa_1(\varepsilon) = \sqrt{\varepsilon} \\ 0 & \text{if } \dfrac{\kappa_1(\varepsilon)}{\sqrt{\varepsilon}} \to 0 \end{cases} .$$

The relation of orthogonality between h_0 and h_1 found before, limits the possibility to the case where $\kappa_1(\varepsilon) = \sqrt{\varepsilon}$. The equation we have to look at is now given by

$$[(1-x^2)h_1']' + \lambda_0^2\, h_1 = -\lambda_1^2\, h_0 . \tag{2.7}$$

From this equation, we notice that an even correction h_1 corresponds to an even function h_0 and reciprocally an odd correction to an odd function. With the remark noted, we can now use the relation obtained with the orthogonality property. The relation $\int_{-k}^{k^{-1}} h_0^2\, dx=1$ gives us

$$\int_{-k^{-1}}^{k^{-1}} D_0^2\, \mathscr{L}_{\frac{1}{2}}^2\, (\lambda_0^2,\, x)\, dx = 1$$

or

$$D_0^2 = \frac{1}{\int_{-k^{-1}}^{k^{-1}} \mathscr{L}_{\frac{1}{2}}^2\, (\lambda_0^2, x)\, dx}$$

Now, considering the equation defining h_1, Equation (7), and multiplying this relation by h_0 and integrating over the whole span we have

$$\int_{-k^{-1}}^{k^{-1}} [(1-x^2)h_1']' h_0 dx + \lambda_0^2 \int_{-k^{-1}}^{k^{-1}} h_1 h_0 dx = -\lambda_1^2 \int_{-k^{-1}}^{k^{-1}} h_0^2 dx = -\lambda_1^2$$

where the normalization relation has been used.

An integration by parts and the use of the relation $\int_{-k^{-1}}^{k^{-1}} h_1 h_0 \, dx = 0$ gives us

$$[(1-x^2)h_1']h_0 \Big|_{-k^{-1}}^{k^{-1}} - \int_{-k^{-1}}^{k^{-1}} (1-x^2)h_1' h_0' = -\lambda_1^2 \quad .$$

The use of the boundary condition and a new integration by parts gives us

$$-(1-x^2)h_0' h_1 \Big|_{-k^{-1}}^{k^{-1}} + \int_{-k^{-1}}^{k^{-1}} h_1 [(1-x^2)h_0']' dx = -\lambda_1^2 \quad .$$

Considering now the differential equation defining h_0, Equation (5), and using the orthogonality between h_0 and h_1, we obtain

$$\lambda_1^2 = (1-x^2)h_0' h_1 \Big|_{-k^{-1}}^{k^{-1}} \quad .$$

We mentioned before that to even modes corresponds even corrections and vice-versa. As a result the expression $h_0' h_1$ takes on different signs when evaluated at the end points k^{-1} and $-k^{-1}$ and we are thus left with

$$\lambda_1^2 = 2(1-k^{-2})h_0'(k^{-1})h_1(k^{-1}) \quad .$$

Considering now the information we found in the matching near $x = k^{-1}$ we have that

$$h_0'(k^{-1}) = D_0 \frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1})$$

$$h_1(k^{-1}) = -\frac{C_0}{k^{*2}} = \frac{D_0}{k^*} \frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) \quad .$$

We thus have

$$\lambda_1^2 = 2\left(\frac{k^2-1}{k^2}\right) \frac{D_0}{\sqrt{k^2-1}} \left[\frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1})\right]^2 \quad .$$

The final expression for λ_1^2 is then

$$\lambda_1^2 = 2 \frac{\sqrt{k^2-1}}{k^2} \frac{\left[\frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1})\right]^2}{\int_{-k^{-1}}^{k^{-1}} \mathscr{L}_{\frac{1}{2}}^2(\lambda_0^2, x)\,dx} \quad .$$

The values found for λ_0^2 and λ_1^2 are also those values which guarantee the orthogonality between different modes. This derivation is less straightforward and is presented in Appendix I, so to lighten the presentation. Now that the value of λ_1^2 has been found, the determination of $h_1(x)$ is completely determined with the help of the boundary conditions we obtained in the matching process.

To complete the study of the present problem, we will now look at the next term of the asymptotic expansion used in the boundary layer. The matching near $x = k^{-1}$ suggests as the order of magnitude for $\delta_1(\varepsilon)$ the value ε. The differential equation defining f_1 is given by considering the terms of order of magnitude

$$\frac{\varepsilon\delta_1(\varepsilon)}{\phi^4} \approx \frac{\varepsilon^2}{\varepsilon^2} = \varepsilon^o$$

in the boundary layer expansion, Equation (6), or

$$\frac{1}{k^2}\frac{d^4f_1}{d\tilde{x}^4} - (1-k^{-2})\frac{d^2f_1}{d\tilde{x}^2} - 2\,k^{-1}\tilde{x}\frac{d^2f_0}{d\tilde{x}^2} - 2\,k^{-1}\frac{df_0}{d\tilde{x}} = 0 \quad .$$

The expression previously found for f_0 is

$$f_0(\tilde{x}) = \frac{C_0}{k^{*2}}\,[k^*\,\tilde{x} - 1 + e^{-k^*\tilde{x}}]$$

where C_0 is a known constant. We thus have:

$$\frac{d^4f_1}{d\tilde{x}^4} - (k^2-1)\frac{d^2f_1}{d\tilde{x}^2} = 2k\,C_0\left[\tilde{x}\,e^{-k^*\tilde{x}} + \frac{1}{k^*} - \frac{e^{-k^*\tilde{x}}}{k^*}\right] \quad .$$

It would be instructive to compute the complete solution valid for f_1. To reach this objective let us define

$$\frac{d^2f_1}{d\tilde{x}^2} = f_1^* \; .$$

We have, taking into account the definition of $k^{*2} = k^2-1$,

$$\frac{d^2f_1^*}{d\tilde{x}^2} - k^{*2}\,f_1^* = 2k\,C_0\left[\tilde{x}\,e^{-k^*\tilde{x}} + \frac{1}{k^*} - \frac{e^{-k^*\tilde{x}}}{k^*}\right] \quad .$$

The particular solution f_{1P}^* is taken under the form

$$f_{1P}^* = 2k\,C_0[a_0 + a_1\,\tilde{x}\,e^{-k^*\tilde{x}} + a_2\,\tilde{x}^2\,e^{-k^*\tilde{x}}] \quad .$$

Introducing this development into the equation defining f_1^*, we obtain the following three relations:

$$a_0 = -\frac{1}{k^{*3}}\;;\quad a_2 = -\frac{1}{4k^*}\quad \text{and}\quad a_1 = \frac{1}{4k^{*2}} \quad .$$

The total solution is now written as:

$$f_1^* = \frac{d^2 f_1}{d\tilde{x}^2} = C_1\, e^{-k^*\tilde{x}} + 2k\, C_0 \left[-\frac{1}{k^{*3}} + \frac{1}{4k^{*2}}\, \tilde{x}\, e^{-k^*\tilde{x}} - \frac{1}{4k^2}\, \tilde{x}^2 e^{-k^*\tilde{x}} \right].$$

The boundary condition for f_1, are given by:

$$\frac{df_1}{d\tilde{x}}(\tilde{x}) = f_1(\tilde{x}) = 0 \quad \text{for} \quad \tilde{x} = 0.$$

Integration of the previous equations gives us:

$$\frac{df_1}{d\tilde{x}} = -\frac{C_1}{k^*}\, e^{-k^*\tilde{x}} - \frac{2k\, C_0}{k^{*3}}\, \tilde{x} + \frac{k\, C_0}{2k^{*2}}\, e^{-k^*\tilde{x}} \left[\frac{\tilde{x}}{-k^*} - \frac{1}{k^{*2}} \right]$$

$$- \frac{k\, C_0}{2k^*}\, e^{-k^*\tilde{x}} \left[\frac{\tilde{x}^2}{-k^*} - \frac{2\tilde{x}}{k^{*2}} - \frac{2}{k^{*3}} \right] + C_2 \quad .$$

The use of one boundary condition transforms this expression into

$$\frac{df_1}{d\tilde{x}} = -C_2\, e^{-k^*\tilde{x}} - \frac{2k\, C_0}{k^{*3}}\, \tilde{x} + \frac{k\, C_0}{2k^{*3}}\, \tilde{x}\, e^{-k^*\tilde{x}} + \frac{k\, C_0}{2k^{*2}}\, \tilde{x}^2\, e^{-k^*\tilde{x}} + C_2$$

Integrating once more we have:

$$f_1 = \frac{C_2}{k^*}\, e^{-k^*\tilde{x}} - \frac{k\, C_0}{k^{*3}}\, \tilde{x}^2 + \frac{k\, C_0}{2k^{*3}}\, e^{-k^*\tilde{x}} \left[\frac{\tilde{x}}{-k^*} - \frac{1}{k^{*2}} \right]$$

$$+ \frac{k\, C_0}{2k^{*2}}\, e^{-k^*\tilde{x}} \left[\frac{\tilde{x}^2}{-k^*} - \frac{2\tilde{x}}{k^{*2}} - \frac{2}{k^{*3}} \right] + C_2\tilde{x} + C_3 \; .$$

The boundary condition gives us

$$\frac{C_2}{k^*} - \frac{k\, C_0}{2k^{*5}} - \frac{k\, C_0}{k^{*5}} + C_3 = 0 \quad \text{or} \quad C_2 = 3\, \frac{k\, C_0}{k^{*4}} - k^* C_3 \quad .$$

So we finally have

$$f_1 = -\frac{k\,C_0}{2k^{*3}}\,\tilde{x}^2\,e^{-k^*\tilde{x}} - \frac{3k\,C_0}{2k^{*4}}\,\tilde{x}\,e^{-k^*\tilde{x}} - C_3 e^{-k^*\tilde{x}} - \frac{k\,C_0}{k^{*3}}\,\tilde{x}^2$$

$$+\left(\frac{3k\,C_0}{k^{*4}} - k^* C_3\right)\tilde{x} + C_3 \quad .$$

We can see that in the previous relation, only one constant is left unknown, C_3, and this last constant will now be determined through the matching process!!

The matching near $x = k^{-1}$ is given by the following expression

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} \left\{ \begin{array}{l} h_0(k^{-1} - x') + \nu_1(\varepsilon) h_1(k^{-1} - x') + \ldots \\ \qquad - \delta_0(\varepsilon) f_0\left(\frac{\eta x_\eta}{\sqrt{\varepsilon}}\right) - \delta_1(\varepsilon) f_1\left(\frac{\eta x_\eta}{\sqrt{\varepsilon}}\right) + \ldots \end{array} \right\} = 0 \quad .$$

Some of the terms appearing in the previous expressions have already been matched, so let us consider the remaining ones; we then have

$$\lim_{\substack{\varepsilon \to 0 \\ x_\eta \text{ fixed}}} \left\{ \begin{array}{l} \ldots D_0 \frac{d^2}{dx^2} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) \frac{\eta^2 x_\eta^2}{2} + \ldots + \varepsilon^{1/2}\left[h_1(k^{-1}) - \frac{d}{dx} h_1(k^{-1}) \eta x_\eta \right. \\ \qquad \left. + \ldots \right] - \sqrt{\varepsilon}\, \frac{C_0}{k^{*2}} \left[-1 + e^{-k^* \frac{\eta x_\eta}{\sqrt{\varepsilon}}} \right] \\ -\varepsilon \left[-\frac{k\,C_0}{k^{*3}} \frac{\eta^2 x_\eta^2}{\varepsilon} + \left(\frac{3k\,C_0}{k^{*4}} - k^* C_0\right) \frac{\eta x_\eta}{\sqrt{\varepsilon}} + C_3 \right. \\ \qquad - \frac{k\,C_0}{2k^{*3}} \frac{x_\eta \eta^2}{\varepsilon} e^{-k^* \frac{\eta x_\eta}{\sqrt{\varepsilon}}} - \frac{3k\,C_0}{2k^{*4}} \frac{x_\eta \eta}{\sqrt{\varepsilon}} e^{-k^* \frac{\eta x_\eta}{\sqrt{\varepsilon}}} \\ \qquad \left. - C_3 e^{-k^* \frac{\eta x_\eta}{\sqrt{\varepsilon}}} \right] + \ldots \end{array} \right\} = 0 \quad .$$

The matching is obtained by the satisfaction of the following equalities

$$\begin{cases} \frac{1}{2} D_0 \frac{d^2}{dx^2} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) + \frac{k\, C_0}{k^{*3}} = 0 \\ h_1(k^{-1}) + \frac{C_0}{k^{*2}} = 0 \\ -\frac{d}{dx} h_1(k^{-1}) - \left(\frac{3k\, C_0}{k^{*4}} - k^* C_3 \right) = 0 \end{cases} .$$

The last equality defines the value of C_3 and the last constant to be determined is thus given by

$$C_3 = \frac{1}{k^*} \left[\frac{d}{dx} h_1(k^{-1}) + \frac{3k\, C_0}{k^{*4}} \right] .$$

The second equality was already found before and used. We are thus left with the last equality or replacing C_0 by its expression

$$\frac{1}{2} D_0 \frac{d^2}{dx^2} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) - \frac{k}{k^{*3}} k^* D_0 \frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) = 0 .$$

This last relation can also be written

$$k^{*2} \frac{d^2}{dx^2} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) - 2k \frac{d}{dx} \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) = 0 ,$$

and this last relation represents an identity as we can see from the differential equation defining h_0 or

$$(1-x^2) h_0'' - 2x\, h_0' + \lambda_0^2\, h_0 = 0 .$$

Evaluated at $x=k^{-1}$, the previous relation becomes

$$(1-k^{-2})\, h_0''(k^{-1}) - 2\, k^{-1} h_0'(k^{-1}) + \lambda_0^2\, h_0(k^{-1}) = 0.$$

We already know that $h_0(k^{-1})$ is zero so we have

$$(k^2-1)\, \frac{d^2}{dx^2}\, \mathscr{L}_{\frac{1}{2}}(\lambda_0^2, k^{-1}) - 2k\, \frac{d}{dx}\, \mathscr{L}_{\frac{1}{2}}\, (\lambda_0^2, k^{-1}) = 0$$

which is the last identity.

With this, we have completed the study of the modal analysis for the transverse vibration of a rotating cable of small flexural rigidity clamped at both ends. This last problem has been introduced mainly to become more familiar with those types of problems and gain more insight into the physics of the problem. The experience is mainly to be used in the next step of our study which will be dealing with a two dimensional approach for similar problems.

We conclude this chapter by summarizing some of the results of this last section. The final expression for the eigenfrequencies is reported as

$$\omega_\alpha^2 = \frac{1}{2}\, \Omega^2\, \lambda_\alpha^2 = \frac{1}{2}\, \Omega^2 (\lambda_0^2 + \sqrt{\varepsilon}\, \lambda_1^2 + \ldots)$$

where λ_0^2 is the solution of the transcendental equations

$$\mathscr{L}_1(\lambda_0^2, k^{-1}) = 0 \quad \text{and} \quad \mathscr{L}_2(\lambda_0^2, k^{-1}) = 0$$

and

$$\lambda_1^2 = 2\, \frac{\sqrt{k^2-1}}{k^2}\, \frac{\left[\frac{d}{dx}\, \mathscr{L}_{\frac{1}{2}}\, (\lambda_0^2, k^{-1})\right]^2}{\int_{-k^{-1}}^{k^{-1}} \mathscr{L}_{\frac{1}{2}}^2\, (\lambda_0^2, x)\, dx}\, .$$

The determination of the constant k^2 is obtained by recalling the definition $T = k^2 T_0$, where $T_0 = \mu\, \Omega^2 L^2/2$.

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CHAPTER 3

DYNAMICS OF ROTATING THIN PLATES AND MEMBRANES

3.1 Introduction

In this chapter, we concentrate our attention on the problem of a dynamical study of two elementary planar structures characterized by a dominant effect of the forces induced by spin over the flexural rigidity.

We focus our attention mainly on two problems. In the first one, a modal analysis is done for a rotating very thin circular plate or membrane with some flexural rigidity, clamped along its edges. This type of study is oriented towards the possible use of large spinning membrane-like disks as optical and radar reflectors for space vehicles. This last use of spinning membranes has led to a renewed interest in the problem of calculating the transverse vibrations of an elastic disk rotating at a constant speed — a problem which traditionally has been studied in connection with gas and steam turbines. In the second part of this chapter, the modal analysis of a free spinning membrane is accomplished. In both of those studies, only vibrations usually referred as "out-of-plane" are considered.

3.2 Dynamics of a Taut Rotating Membrane in a Circular Ring

The deformable element consists of a large flexible circular membrane undergoing transverse vibrations, while rotating with an angular velocity Ω with respect to the axis of symmetry, defined when the structure is at equilibrium.

The following equations of the motion are given in a reference frame located at the center of mass of the total system and rotating with the system at the constant angular velocity Ω. The direction of the ξ axis will be chosen in such a way that there will be no need for a phase angle in the equation of the motion. We are free to choose such an axis due to the symmetry of the particular structure under investigation. The physical characteristics of the membrane will be assumed constant throughout the whole membrane.

It must be emphasized here that the effect of the spin on the membrane is a destabilizing effect: the rotation not only introduces a decrease in the stiffness but also instability can occur due to the creation of a state of compression into the membrane. To take care of this last problem, we introduce a built-in tension — a similar procedure was used for the rotating cable — such that the membrane, when rotating, is everywhere under tension.

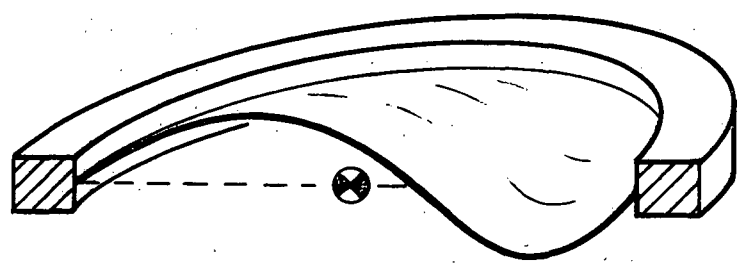

Figure 8. Rotating Taut Membrane.

In the study, we will disregard the stresses, which occur on account of the mutual pressure of horizontal layers of the disk and, consequently, the strains in the direction of its thickness; the

problem is thus simplified and the generalized Hooke's laws give us the following relations:

$$\sigma_\rho = \frac{E}{1 - \nu^2} [\varepsilon_\rho + \nu\varepsilon_\theta]$$

$$\sigma_\theta = \frac{E}{1 - \nu^2} [\varepsilon_\theta + \nu\varepsilon_\rho]$$

where E stands for the modulus of elasticity,

ν stands for the Poisson modulus

ε_θ and ε_ρ stand for the tangential and radial strains respectively

σ_θ and σ_ρ stand for the tangential and radial stresses respectively.

Also, because the thickness of the disk is small in comparison with its radius, the variation of radial and tangential stresses over the thickness can be neglected. Also, a hypothesis analogous to the hypothesis of plane section in a rod will be used. A linear element within the disk perpendicular to the middle plane remains straight and normal to this plane after its deflection during bending. All those previous assumptions are part of the approximate theory of thin plates.

In this study, the problem of the stretching of the thin disk is solved first and the dynamical equations for the transverse vibrations are then derived as deviations with respect to the steady state. This last remark constitutes a characteristic of continuous modeling and will be materialized in the following pages. We also assume that the stretching of the disk does not alter the constancy of its physical characteristics.

The stress distribution in the membrane must satisfy the following expression, as an equilibrium condition for the forces acting along the radius.

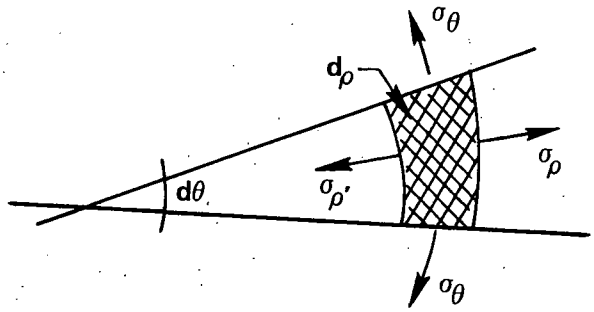

Figure 9. Distribution of Stresses.

$$\frac{d}{d\rho}(\rho\sigma_\rho) - \sigma_\theta + \mu\,\Omega^2\rho^2 = 0 \tag{3.1}$$

where μ is the mass per unit volume of the material of the disk. Because of the symmetry, there is no dependence in the angular variable. This relationship and the following results from classical thin plate theory can be found in Reference 20.

For a circular disk, the strain components, in the case of symmetry, are well known and given by

$$\varepsilon_\rho = \frac{du}{d\rho} \quad \text{and} \quad \varepsilon_\theta = \frac{u}{\rho}$$

where u represents the radial displacement. The generalized Hooke's laws and the expressions for the strain components give us:

$$\sigma_\rho = \frac{E}{1-\nu^2}\left(\frac{du}{d\rho} + \nu\,\frac{u}{\rho}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{\rho} + \nu\,\frac{du}{d\rho}\right) .$$

Combining those expressions with the equilibrium condition (3.1), we obtain the equation for the radial displacement of the disk u:

$$\rho^2 \frac{d^2u}{d\rho^2} + \rho \frac{du}{d\rho} - u = - \frac{1-\nu^2}{E} \mu \Omega^2 \rho^3 .$$

The general solution of this equation is

$$u = \frac{1}{E} \left[(1-\nu) C\rho - (1+\nu) \frac{C_1}{\rho} - \frac{1-\nu^2}{8} \mu \Omega^2 \rho^3 \right]$$

where C and C_1 are arbitrary constants. The corresponding stress components are now found from:

$$\sigma_\rho = C + \frac{C_1}{\rho^2} - \frac{3+\nu}{8} \mu \Omega^2 \rho^2$$

$$\sigma_\theta = C - \frac{C_1}{\rho^2} - \frac{1+3\nu}{8} \mu \Omega^2 \rho^2 .$$

Because we are dealing with a complete disk, we must take $C_1 = 0$ to have u=0 at the center.

In the determination of the constant C, we have to be very cautious. The reason is coming from the interpretation of the boundary condition for the problem

$$u = 0 \quad \text{when } \rho = a.$$

This last boundary condition is, indeed, only valid after the membrane has been stretched statically by the built-in tension T. As a result, the constant C contains two parts, one coming from the built-in tension and the other from the satisfaction of the previous boundary condition for the radial displacement originated by the radial forces induced by spin, only. The last reasoning is perfectly valid if we consider that the generalized Hooke's laws represent a linear dependence between stress and strain. We know that in a circular membrane under a

constant tension T, the stress components are the same and equal to the applied tension. We, thus, have

$$C = T + C' ,$$

where T is the constant built-in tension. The determination of the constant C' is now determined by using the previous boundary condition, but by considering in the constant C appearing in the expression of the radial displacement only the contribution of C'. The total displacement is then obtained by the summation of the displacement produced by the built-in tension and the displacement we just derived. So

$$(1-\nu)C'a = \frac{1-\nu^2}{8} \mu \Omega^2 a^3$$

and

$$C' = \frac{1+\nu}{8} \mu \Omega^2 a^2 \quad .$$

The final expressions for the radial and tangential stresses are given by

$$\sigma_\rho = T + \frac{1+\nu}{8} \mu \Omega^2 a^2 - \frac{3+\nu}{8} \mu \Omega^2 \rho^2 \tag{3.2}$$

$$\sigma_\theta = T + \frac{1+\nu}{8} \mu \Omega^2 a^2 - \frac{1+3\nu}{8} \mu \Omega^2 \rho^2 \quad . \tag{3.3}$$

One comment should be added here. Namely, we assumed before that the built-in tension must be such that a positive tension is present everywhere in the disk or

$$\sigma_\rho = T + \frac{1+\nu}{8} \mu \Omega^2 a^2 - \frac{3+\nu}{8} \mu \Omega^2 \rho^2 \geq 0$$

for every ρ such that $0 \leq \rho \leq a$, or

$$T + \frac{1+\nu}{8}\mu\,\Omega^2 a^2 - \frac{3+\nu}{8}\mu\,\Omega^2 a^2 \geq 0,$$

which gives us

$$T \geq \frac{\mu\,\Omega^2 a^2}{4} \quad .$$

For simplicity, we will write the tension T under the form

$$T = \frac{k^2 \mu\,\Omega^2 a^2}{4}$$

where the positive constant k^2 is greater than one. With this last remark, we have

$$\sigma_\rho = \frac{1 + 2k^2 + \nu}{8}\mu\,\Omega^2 a^2 - \frac{3+\nu}{8}\mu\,\Omega^2 \rho^2$$

$$\sigma_\theta = \frac{1 + 2k^2 + \nu}{8}\mu\,\Omega^2 a^2 - \frac{1+3\nu}{8}\mu\,\Omega^2 \rho^2 \quad .$$

We thus have illustrated here the fact mentioned several times earlier that the equation of the transverse vibration for a continuous model represents in fact the perturbation of the system with respect to its steady-state deformation.

Now, that the distribution of the radial and tangential stresses is known, the dynamic equation for the transverse vibration of a circular plate subject to radial and tangential stresses is given in general form by: (see Reference 20)

$$-\frac{E\,h^2\,\nabla_\rho^4\,w}{12(1-\nu^2)} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\sigma_\rho\,\rho\,\frac{\partial w}{\partial\rho}\right) + \frac{\sigma_\theta}{\rho^2}\frac{\partial^2 w}{\partial\theta^2} = \mu\,\frac{\partial^2 w}{\partial t^2}$$

where h represents the thickness of the disk and

∇_ρ^4 is the Laplacian operator which expressed in polar coordinates represents

$$\nabla_\rho^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} \quad .$$

The boundary conditions for this problem are given by expressing the clamping condition along the edges or

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial w}{\partial\rho} &= 0 \end{aligned}\right\} \quad \text{when } \rho = a \quad .$$

Let us now introduce the dimensionless quantity r, defined by the relation:

$$\rho = ar.$$

The previous expression becomes then:

$$-\frac{Eh^2}{12a^4(1-\nu^2)}\nabla^4 w + \frac{1}{a^2 r}\frac{\partial}{\partial r}\left[\sigma_r r \frac{\partial w}{\partial r}\right] + \frac{\sigma_\theta}{a^2 r^2}\frac{\partial^2 w}{\partial\theta^2} = \mu\frac{\partial^2 w}{\partial t^2} \qquad (3.4)$$

where

$$\sigma_r = \frac{\mu\,\Omega^2 a^2}{8}\,[(1 + 2k^2 + \nu) - (3+\nu)r^2]$$

and

$$\sigma_\theta = \frac{\mu\,\Omega^2 a^2}{8}\,[(1 + 2k^2 + \nu) - (1+3\nu)r^2] \ .$$

Dividing through by $\mu\,\Omega^2$ we obtain:

$$-\frac{Eh^2}{12\mu\Omega^2 a^4(1-\nu^2)}\nabla^4 w + \frac{1}{8r}\frac{\partial}{\partial r}\left\{[(1 + 2k^2 + \nu)r - (3+\nu)r^3]\frac{\partial w}{\partial r}\right\}$$
$$+ \frac{1}{8r^2}[(1+2k^2+\nu) - (1+3\nu)r^2]\frac{\partial^2 w}{\partial\theta^2} = \frac{1}{\Omega^2}\frac{\partial^2 w}{\partial t^2} \quad .$$

This last equation is appropriate for the use of separation of variables, providing the initial conditions are also separable. Let us then use for the solution an expression of the form:

$$w = \phi_{\alpha,m}(r) e^{i(\omega_{\alpha,m} t + m\theta)} .$$

This last expression is completely general and there is no need for a phase angle for the reason explained earlier.

The equation of main interest for our purpose is given by the differential equation defining $\phi_{\alpha,m}(r)$:

$$- \frac{Eh^2}{12(1-\nu^2)a^4\mu\Omega^2} \nabla_m^4 \phi_{\alpha,m} + \frac{(1+2k^2+\nu)}{8r} \frac{d}{dr}\left[r\left(1 - \frac{3+\nu}{1+2k^2+\nu} r^2\right)\frac{d\phi_{\alpha,m}}{dr}\right]$$

$$- \frac{(1+2k^2+\nu)}{8} \frac{m^2}{r^2}\left[1 - \frac{1+3\nu}{1+2k^2+\nu} r^2\right] \phi_{\alpha,m} = - \frac{\omega_{\alpha,m}^2}{\Omega^2} \phi_{\alpha,m}$$

where $\omega_{\alpha,m}$ represents the eigenfrequency of the vibration

∇_m^2 is the linear differential operator defined by

$$\nabla_m^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} .$$

We now define the following quantities

$$\varepsilon = \frac{2\,E\,h^2}{3(1-\nu^2)(1+2k^2+\nu)a^4\mu\Omega^2}$$

$$\lambda_{\alpha,m}^2 = \frac{8\,\omega_{\alpha,m}^2}{(1+2k^2+\nu)\Omega^2}$$

$$k_1^2 = \frac{3+\nu}{1+2k^2+\nu}$$

$$k_2^2 = \frac{1+3\nu}{1+2k^2+\nu} .$$

The final expression for the differential equation for $\phi_{\alpha,m}(r)$ is thus given by

$$-\varepsilon\nabla_m^4\,\phi_{\alpha,m} + \frac{1}{r}\frac{d}{dr}\left[r(1-k_1^2r^2)\frac{d\phi_{\alpha,m}}{dr} - \frac{m^2}{r^2}\right.[1-k_2^2r^2]\phi_{\alpha,m}+\lambda_{\alpha,m}^2\phi_{\alpha,m} = 0 \tag{3.5}$$

with the following boundary conditions:

$$\left.\begin{aligned} \phi_{\alpha,m}(r) &= 0 \\ \frac{d\phi_{\alpha,m}}{dr}(r) &= 0 \end{aligned}\right\} \text{ when } r=1$$

Here, again, we notice that the small flexural rigidity of the disk combined with the high spin rate produces the small parameter ε. We also notice that the parameters k_1^2 and k_2^2 represent positive quantities less than one.

The problem has now been formulated in such a way that we are in a position where a procedure similar to the procedure used before can be used.

Expansion Valid in the Central Region

When $r \neq 0$, the expansion valid in the central area takes the form

$$\phi_{\alpha,m}(r,\varepsilon) = h_0(r) + \nu_1(\varepsilon)h_1(r) + \nu_2(\varepsilon)h_2(r)+\ldots$$

A similar asymptotic expansion is also used for the eigenvalue or

$$\lambda_{\alpha,m}^2(\varepsilon) = \lambda_0^2 + \kappa_1(\varepsilon)\lambda_1^2 + \kappa_2(\varepsilon)\lambda_2^2 +\ldots$$

Once more, the subscript α,m will be momentarily dropped in order to shorten the notation. It will be explicitly used later on when needed to differentiate the modes.

Introducing the asymptotic expansion valid for $\phi_{\alpha,m}$ and $\lambda_{\alpha,m}^2$ into the differential equation (3.5), we obtain the following expression:

$$- \varepsilon \nabla_m^4 (h_0 + \nu_1(\varepsilon) h_1 + \ldots)$$

$$+ \frac{1}{r}\frac{d}{dr}\left[r(1-k_1^2 r^2)\frac{dh_0}{dr}\right] + \nu_1(\varepsilon)\frac{1}{r}\frac{d}{dr}\left[r(1-k_1^2 r^2)\frac{dh_1}{dr}\right]$$

$$- \frac{m^2}{r^2}[1-k_2^2 r^2]h_0 - \nu_1(\varepsilon)\frac{m^2}{r^2}[1-k_2^2 r^2]h_1 + (\lambda_0^2 + \kappa_1(\varepsilon)\lambda_1^2 + \ldots)$$

$$(h_0(r) + \nu_1(\varepsilon) h_1 + \ldots) = 0 \quad .$$

The differential equation defining $h_0(r)$ is then given by:

$$\frac{1}{r}\frac{d}{dr}\left[r(1-k_1^2 r^2)\frac{dh_0}{dr}\right] - \frac{m^2}{r^2}[1 - k_2^2 r^2]h_0 + \lambda_0^2 h_0 = 0 \ .$$

This last expression can be written, when we introduce the new variable

$$k_1 r = r_1$$

as

$$\frac{1}{r_1}\frac{d}{dr_1}\left[r_1(1-r_1^2)\frac{dh_0}{dr_1}\right] - \frac{m^2}{r_1^2}[1-k_3^2 r_1^2]\, h_0 + \frac{\lambda_0^2}{k_1^2} h_0 = 0 \qquad (3.6)$$

where k_3^2 stands for $\dfrac{k_2^2}{k_1^2} = \dfrac{1+3\nu}{3+\nu}$.

Let us write down the solution of the previous equation under the form

$$h_0 = r_1^n H_0 \ .$$

By substituting the previous expression into the equation defining h_0(3.6) we can eliminate in the coefficient of H_0 the terms in r_1^{n-2}, by taking for n the solution of

$$n^2 - m^2 = 0$$

or

$$n = \pm m \ .$$

The solution $n = -m$ is to be eliminated for the singularity that such a solution introduces at the origin; we thus have

$$h_0 = r_1^m H_0 \quad .$$

The differential equation defining H_0 is then obtained

$$(1-r_1^2)\frac{d^2H_0}{dr_1^2} + \frac{1}{r_1}(1-r_1^2)\frac{dH_0}{dr_1} - 2r_1\frac{dH_0}{dr_1} + \frac{2m}{r_1}(1-r_1^2)\frac{dH_0}{dr_1} - \left(2m + m^2 - m^2 k_3^2 - \frac{\lambda_0^2}{k_1^2}\right) H_0 = 0$$

or

$$(1-r_1^2)\frac{d^2H_0}{dr_1^2} + \frac{1}{r_1}[(1+2m)-(3+2m)r_1^2]\frac{dH_0}{dr_1} - \left(2m+m^2-m^2k_3^2 - \frac{\lambda_0^2}{k_1^2}\right)H_0 = 0 \quad . \tag{3.7}$$

To bring this last equation to a form more familiar, a new change of variable is recommended and we define

$$r_1^2 = x.$$

We should emphasize here that x can only take on positive values lying in the interval between 0 and r_1^2 max $= k_1^2$ which is less than 1. With this last change of variables, the previous equation can be written under the form

$$x(1-x)\frac{d^2H_0}{dx^2} + [(m+1)-(m+2)x]\frac{dH_0}{dx} - \frac{1}{4}\left(2m+m^2-m^2k_3^2 - \frac{\lambda_0^2}{k_1^2}\right)H_0 = 0. \tag{3.8}$$

This last expression can be reduced to the hypergeometric equation which is written under the general form

$$x(1-x)\frac{d^2H_0}{dx^2} + [\gamma-(\alpha+\beta+1)x]\frac{dH_0}{dx} - \alpha\beta H_0 = 0$$

So we have to solve the following substitutions:

$$\gamma = 1 + m$$

$$\alpha + \beta = 1 + m$$

$$\alpha\beta = \frac{1}{4}\left(2m + m^2 - m^2\, k_3^2 - \frac{\lambda_0^2}{k_1^2}\right) .$$

The range of value for x lies within the range of convergence of the hypergeometric function.

For the previous equation, we know that the roots of the indicial equation at the regular singular point x=0, are given by 0 and $(1-\gamma)$. As a result, one of the solutions of the previous equation can be computed by considering

$$H_{10} = \sum_{n=0}^{\infty} a_n x^n \quad .$$

A direct substitution into the equation yields the identity

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} n(n-1)a_n x^n + \sum_{n=0}^{\infty} n\gamma\, a_n x^{n-1}$$

$$- \sum_{n=0}^{\infty} n(\alpha+\beta+1)a_n x^n - \sum_{n=0}^{\infty} \alpha\beta\, a_n x^n = 0.$$

We then group the corresponding summations and find

$$\sum_{n=0}^{\infty} n(\gamma+n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} [n^2 + n(\alpha+\beta) + \alpha\beta]a_n x^n = 0 \quad .$$

Now this may be written

$$\sum_{n=1}^{\infty} n(\gamma+n-1)a_n x^{n-1} - \sum_{n=0}^{\infty} (\alpha+n-1)(\beta+n-1)a_{n-1} x^{n-1} = 0 \quad .$$

From there, we find that the recursive relationship for all determination of the a_n's are

$$a_n = \frac{(\alpha+n-1)(\beta+n-1)}{(\gamma+n-1)\cdot n} a_{n-1}$$

for $n \geq 1$, with a_0 arbitrary. At this stage, we must insist on the fact that γ has to be different from zero or any negative integer to make sure that the previous recurrence is defined, but

$\gamma = 1 + m$ which represents always a positive quantity.

Finally, the solution of the previous equation is written with the help of the hypergeometric function

$$F(\alpha,\beta,\gamma;x) = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots +$$

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}\frac{x^n}{n!} + \ldots$$

under the form $H_{10} = a_0 F(\alpha,\beta,\gamma;x)$.

The solution we just developed represents one solution of the previous equation. This last solution is convergent for $|x| < 1$, which is our case of interest.

In our particular case, it turns out that the difference between the roots of the indicial equation at the point $x = 0$ is zero or a positive integer. As a result, the second solution generally used for the hypergeometric equation, more explicitly

$$x^{1-\gamma} F(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma;x)$$

is not linearly independent of the solution previously derived. However, it can be proved that the second linearly independent solution

is to be canceled. The derivation of the second solution is quite lengthy and is presented in Appendix II in order to lighten the text.

The final expression for $h_0(r)$ is then

$$h_0(r) = a_0 k_1^m \, r^m \, F(\alpha,\beta,\ 1+m;\ k_1^2 r^2)$$

or

$$h_0(r) = b_0 r^m \, F\left(\alpha,\beta,\ 1+m;\ \frac{3+\nu}{1+2k^2+\nu}\, r^2\right)$$

where

$$\begin{cases} \alpha+\beta = 1+m \\ \alpha\beta = \frac{1}{4}\left(2m + m^2 - m^2 k_3^2 - \frac{\lambda_0^2}{k_1^2}\right) \end{cases}$$

and

$$\frac{\lambda_0^2}{k_1^2} = \frac{8\,\lambda_0'^2}{(3+\nu)\Omega^2}$$

if we write

$$\omega_{\alpha,m}^2 = \lambda_0'^2 + \kappa_1(\varepsilon)\lambda_1'^2 + \kappa_2(\varepsilon)\lambda_2'^2 + \ldots$$

<u>Boundary Layer Near r=1</u>

We now have to study the boundary-layer near r=1 in order to pursue the problem.

The separation of variables being valid for the partial differential equation for the problem, the boundary layer problem is limited to radius dependence of the modes.

To study the boundary-layer expansion, another boundary layer coordinate must be selected, so that the higher derivative terms dominate the equations and are of the same order of magnitude. Let

$$\tilde{r} = \frac{1-r}{\phi(\varepsilon)} \qquad \text{where} \quad \phi(\varepsilon) \to 0 \qquad \text{when} \quad \varepsilon \to 0.$$

The corresponding asymptotic expansion valid near r=1 is taken as

$$\phi_{\alpha,m}(\tilde{r},\varepsilon) = \delta_0(\varepsilon)f_0(\tilde{r}) + \delta_1(\varepsilon)f_1(\tilde{r}) + \ldots$$

where the $\delta_i(\varepsilon)$ constitute an asymptotic sequence.

The linear differential operator ∇_m^2 becomes now

$$\nabla_m^2 = \frac{d^2}{\phi^2 d\tilde{r}^2} - \frac{1}{(1-\phi\tilde{r})}\frac{d}{\phi d\tilde{r}} - \frac{m^2}{(1-\phi\tilde{r})^2} .$$

The basic differential equation (3.5) becomes with the previous substitutions:

$$- \varepsilon\left[\frac{d^2}{\phi^2 d\tilde{r}^2} - \frac{1}{(1-\phi\tilde{r})}\frac{d}{\phi d\tilde{r}} - \frac{m^2}{(1-\phi\tilde{r})^2}\right]\left[\frac{d^2\phi_{\alpha,m}}{\phi^2 d\tilde{r}^2} - \frac{1}{(1-\phi\tilde{r})}\frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} - \frac{m^2\phi_{\alpha,m}}{(1-\phi\tilde{r})^2}\right] + \frac{1}{(1-\phi\tilde{r})}\frac{d}{\phi d\tilde{r}}\left[(1-\phi\tilde{r})[1-k_1^2(1-\phi\tilde{r})]^2\frac{d\phi_{\alpha,m}}{\phi d\tilde{r}}\right]$$

$$- \frac{m^2}{(1-\phi\tilde{r})^2}[1-k_2^2(1-\phi\tilde{r})^2]\phi_{\alpha,m} + \lambda^2\phi_{\alpha,m} = 0 \qquad (3.9)$$

or

$$- \varepsilon\left[\frac{1}{\phi^4}\frac{d^4\phi_{\alpha,m}}{d\tilde{r}^4} - \frac{2}{(1-\phi\tilde{r})}\frac{d^3\phi_{\alpha,m}}{\phi^3 d\tilde{r}^3} - \frac{d^2\phi_{\alpha,m}}{\phi^2 d\tilde{r}^2}\left(\frac{1}{(1-\phi\tilde{r})^2} + \frac{2m^2}{(1-\phi\tilde{r})^2}\right) - \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}}\left(\frac{1}{(1-\phi\tilde{r})^3} + \frac{2m^2}{(1-\phi\tilde{r})^3}\right) + \phi_{\alpha,m}\left(\frac{m^4}{(1-\phi\tilde{r})^4} - \frac{4m^2}{(1-\phi\tilde{r})^4}\right)\right]$$

$$+ \frac{1}{(1-\phi\tilde{r})\phi}\left[(1-\phi\tilde{r})\left(1-k_1^2(1-\phi\tilde{r})^2\right)\frac{d^2\phi_{\alpha,m}}{\phi d\tilde{r}^2} - \left(1-k_1^2(1-\phi\tilde{r})^2\right)\frac{d\phi_{\alpha,m}}{d\tilde{r}} + 2k_1^2(1-\phi\tilde{r})^2\frac{d\phi_{\alpha,m}}{d\tilde{r}}\right] - \frac{m^2}{(1-\phi\tilde{r})^2}\left(1 - k_2^2(1-\phi\tilde{r})^2\right)\phi_{\alpha,m} + \lambda^2\phi_{\alpha,m} = 0 \qquad (3.9')$$

The thickness of the boundary layer is determined by balancing the higher derivative terms or, if we assume that $1-k_1^2$ is not of an order of magnitude of ε, by considering:

$$-\frac{\varepsilon}{\phi^4} \approx \frac{1}{\phi^2}$$

or

$$\varepsilon = \phi^2 \quad \text{and} \quad \phi = \sqrt{\varepsilon} \quad .$$

The previous remark, about $1-k_1^2$, is considered now

$$1 - k_1^2 = 1 - \frac{3+\nu}{1+2k^2+\nu} = \frac{2(k^2-1)}{1+2k^2+\nu} \quad .$$

The previous assumption is of the same nature as the assumption we made while studying the taut rotating cable. We want the parameter k^2 to differ sufficiently enough from one, so that the difference $1-k_1^2$ differs from zero by an order of magnitude of ε.

Replacing in the last development $\phi_{\alpha,m}(\tilde{r},\varepsilon)$ by its expression defined previously, we obtain for the differential equations defining f_0, the dominant part in the development (3.9') or

$$-\frac{d^4 f_0}{d\tilde{r}^4} + (1-k_1^2)\frac{d^2 f_0}{d\tilde{r}^2} = 0 \quad .$$

The boundary conditions for the problem are

$$\left.\begin{aligned} f_0(\tilde{r}) &= 0 \\ \frac{df_0}{d\tilde{r}}(\tilde{r}) &= 0 \end{aligned}\right\} \quad \text{when } \tilde{r} = 0 \quad .$$

Defining

$$\frac{d^2 f_0}{d\tilde{r}^2} = f_0^*$$

and considering the fact that $1-k_1^2$ is a positive quantity, we obtain as a solution for f_0^*

$$f_0^* = C_1' \, e^{-\sqrt{1-k_1^2}\,\tilde{r}}$$

where the exponential growth has already been disregarded for its problem in the matching process.

Integrating the previous expression twice, and taking into account the boundary conditions for f_0, we obtain finally:

$$f_0 = \frac{C_1}{\sqrt{1-k_1^2}} \left[e^{-\sqrt{1-k_1^2}\,\tilde{r}} + \sqrt{1-k_1^2}\;\tilde{r} - 1 \right]$$

where the positive sign is always chosen in the previous square roots. We are now in a position where we can try a matching between the expansion valid in the central area and the boundary layer expansion. The equation defining the eigenvalue λ_0^2 should come out of this matching.

We need to expand the solution valid in the central area near r=1. We define the variable

$$r' = 1-r$$

and we have

$$h_0(\lambda_0'^2, 1-r') = h_0(\lambda_0'^2, 1) - \frac{dh_0}{dr}(\lambda_0'^2, 1)\, r' + \ldots$$

where

$$h_0(\lambda_0'^2, 1) = b_0 \, F\left(\alpha, \beta, 1+m; \frac{3+\nu}{1+2k^2+\nu}\right)$$

and

$$\frac{dh_0}{dr}(\lambda_0'^2,1) = b_0 m\, F\left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\right)$$

$$+\ b_0 \frac{2(3+\nu)}{1+2k^2+\nu} \frac{dF}{dx}(\alpha,\beta,1+m;\ x)\Bigg|_{x=\frac{3+\nu}{1+2k^2+\nu}} .$$

For matching, an intermediate limit suitable for matching near r=1 is given by r_η fixed:

$$r_\eta = \frac{r'}{\eta(\varepsilon)} \quad \text{where } r' \text{ is always positive.}$$

What we want in $\tilde{r} \to \infty$ when $\varepsilon \to 0$ and this is reached if

$$\frac{\eta}{\sqrt{\varepsilon}} \to \infty \quad \text{when} \quad \varepsilon \to 0 .$$

The behavior near r=1 takes the form:

$$\lim_{\substack{\varepsilon \to 0 \\ r_\eta \text{ fixed}}} \left\{ \begin{array}{l} h_0(\lambda_0'^2,1) - \frac{dh_0}{dr}(\lambda_0'^2,1)\,\eta r_\eta + \ldots \\ + \nu_1(\varepsilon)h_1(1) + \ldots - \delta_0(\varepsilon)\frac{C_1}{\sqrt{1-k_1^2}}\left(e^{-\sqrt{1-k_1^2}\frac{\eta r_\eta}{\sqrt{\varepsilon}}} + \sqrt{1-k_1^2}\frac{\eta r_\eta}{\sqrt{\varepsilon}} - 1\right) - \delta_1(\varepsilon) f_1\left(\frac{\eta r_\eta}{\sqrt{\varepsilon}}\right) + \ldots \end{array} \right\} = 0 .$$

The matching is accomplished if we take

$$h_0(\lambda_0'^2,1) = 0$$

$$\delta_0(\varepsilon) = \sqrt{\varepsilon} \quad \text{and} \quad -\frac{dh_0}{dr}(\lambda_0'^2,1) = C_1 .$$

The first equation is nothing else than the equation defining the eigenvalue $\lambda_0'^2$ of the problem or

$$F\left(\alpha,\ 1+m-\alpha,\ 1+m;\ \frac{3+\nu}{1+2k^2+\nu}\right) = 0 \ .$$

The second equation gives us

$$C_1 = -b_0 \frac{2(3+\nu)}{1+2k^2+\nu} \left.\frac{dF(\alpha,\beta,1+m;x)}{dx}\right|_{x\ =\ 3+\nu/1+2k^2+\nu}$$

Coming from the matching, we also have that

$$\nu_1(\varepsilon) = \sqrt{\varepsilon}$$

and

$$h_1(1) + \frac{C_1}{\sqrt{1-k_1^2}} = 0$$

or

$$h_1(1) = \frac{b_0}{\sqrt{1-k_1^2}} \frac{2(3+\nu)}{1+2k^2+\nu} \frac{dF}{dx}\left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\right) \ .$$

This information about h_1 enables us to obtain the differential equation defining h_1. Considering the development of the basic differential equation (3.5), we have:

$$\frac{1}{r}\frac{d}{dr}\left[r(1-k_1^2r^2)\frac{dh_1}{dr}\right] - \frac{m^2}{r^2}[1-k_2^2r^2]h_1 + \lambda_0^2 h_1 = -\lambda_1^2 h_0 \ . \qquad (3.10)$$

This last expression implies also the choice of $\sqrt{\varepsilon}$ for $\kappa_1(\varepsilon)$. This value has been considered for the same reason as in earlier development. There exists an orthogonality relationship between the modes — shown later — and this last constraint suggests as an order of magnitude of $\kappa_1(\varepsilon)$, the value $\sqrt{\varepsilon}$. We see, indeed, that for $\lambda_1^2 >> \sqrt{\varepsilon}$, the solution for h_1 would be the same as the one found for h_0 and the orthogonality would be violated.

In the differential equation (3.10), the only freedom left, to define the problem completely, lies in the determination of λ_1^2 and this constitutes our next task.

Orthogonality Relationship Between the Modes

The complete expression for the modes is given by

$$\phi_{\alpha,m}(r)\ \cos m\theta\ .$$

Let us now prove that the orthogonality relationship between two different modes is given by

$$\int_0^1 \int_2^{2\pi} r\ \phi_{\alpha,m}(r)\phi_{\beta,n}(r)\cos m\theta\ \cos n\theta\ dr d\theta = 0\ .$$

This last relationship is straightforward if we consider two modes for which angular dependences are different. The direct integration with respect to $d\theta$, for $m \neq n$ gives us directly

$$\int_0^{2\pi} \cos m\theta\ \cos n\theta\ d\theta = 0\ .$$

As a result, the case of main interest is given for two modes having the same angular dependence i.e. with m=n. Let us now prove that

$$\int_0^1 r\ \phi_{\alpha,m}(r)\phi_{\beta,m}(r)dr = 0\ .$$

The differential equations defining $\phi_{\alpha,m}$ and $\phi_{\beta,m}$ are respectively

$$-\varepsilon\ \nabla_m^4\ \phi_{\alpha,m} + \frac{1}{r}\ \frac{d}{dr}\left[r(1-k_1^2r^2)\ \frac{d\phi_{\alpha,m}}{dr}\right] - \frac{m^2}{r^2}\ [1-k_2^2r^2]\phi_{\alpha,m} + \lambda_{\alpha,m}^2\phi_{\alpha,m} = 0$$

$$-\varepsilon\ \nabla_m^4\ \phi_{\beta,m} + \frac{1}{r}\ \frac{d}{dr}\left[r(1-k_1^2r^2)\ \frac{d\phi_{\beta,m}}{dr}\right] - \frac{m^2}{r^2}\ [1-k_2^2r^2]\phi_{\beta,m} + \lambda_{\beta,m}^2\phi_{\beta,m} = 0$$

We now have to distinguish between the cases where m=0 and m≠0.

<u>Case A</u>: $m \neq 0$

The boundary conditions are given by
$$\left.\begin{aligned} \phi_{\alpha,m} &= 0 \\ \frac{d\phi_{\alpha,m}}{dr} &= 0 \end{aligned}\right\} \text{ for } r=1$$

and similar boundary conditions for $\phi_{\beta,m}$.

Developing the expression of $\nabla_m^4 \phi_{\alpha,m}$, we obtain:

$$\nabla_m^4 \phi_{\alpha,m} = \frac{d^4\phi_{\alpha,m}}{dr^4} + \frac{2}{r}\frac{d^3\phi_{\alpha,m}}{dr^3} - \frac{2m^2+1}{r^2}\frac{d^2\phi_{\alpha,m}}{dr^2} + \frac{2m^2+1}{r^3}\frac{d\phi_{\alpha,m}}{dr} + \frac{m^4-4m^2}{r^4}\phi_{\alpha,m} .$$

Taking the differential equation defining $\phi_{\alpha,m}$ and multiplying this relation by $r\,\phi_{\beta,m}$, an integration from 0 to 1 gives us

$$-\varepsilon\int_0^1 r\,\nabla_m^4\phi_{\alpha,m}\phi_{\beta,m}dr + \int_0^1 \frac{d}{dr}\left[r(1-k_1^2r^2)\frac{d\phi_{\alpha,m}}{dr}\right]\phi_{\beta,m}dr - \int_0^1 \frac{m^2}{r}[1-k_2^2r^2]\phi_{\alpha,m}\phi_{\beta,m}dr + \int_0^1 \lambda_{\alpha,m}^2\phi_{\alpha,m}\phi_{\beta,m}dr = 0$$

The first term can be developed as

$$\int_0^1 r\,\nabla_m^4\phi_{\alpha,m}\phi_{\beta,m}dr = \int_0^1 r\frac{d^4\phi_{\alpha,m}}{dr^4}\phi_{\beta,m}dr + \int_0^1 2\frac{d^3\phi_{\alpha,m}}{dr^3}\phi_{\beta,m}\,dr - \int_0^1 \frac{2m^2+1}{r}\frac{d^2\phi_{\alpha,m}}{dr^2}\phi_{\beta,m}dr + \int_0^1 \frac{2m^2+1}{r^2}\frac{d\phi_{\alpha,m}}{dr}\phi_{\beta,m}dr + \int_0^1 \frac{m^4-4m^2}{r^4}\phi_{\alpha,m}\phi_{\beta,m}\,dr .$$

Integrating by parts and using the boundary condition, the right hand side can be written

$$- \int_0^1 r \frac{d^3\phi_{\alpha,m}}{dr^3} \frac{d\phi_{\beta,m}}{dr} dr + \int_0^1 \frac{d^3\phi_{\alpha,m}}{dr^3} \phi_{\beta,m} dr$$

$$- (2m^2+1) \int_0^1 \frac{d}{dr}\left(\frac{1}{r} \frac{d\phi_{\alpha,m}}{dr}\right) \phi_{\beta,m} dr$$

$$+ \int_0^1 \frac{m^4-4m^2}{r^4} \phi_{\alpha,m}\phi_{\beta,m} dr \quad .$$

One remark should be added here. For the case where $m \neq 0$, the nature of the solution is such that there will always exist for those modes, a nodal diameter with a zero transverse displacement. But the existence of a nodal diameter of zero displacement implies, that the displacement of the origin should be zero and we thus have for the case where $m \neq 0$

$$\phi_{\alpha,m} = 0 = \phi_{\beta,m} \quad \text{for } r = 0 \ .$$

An integration by parts performed on the three first terms, combined with the use of the previous remark transforms the last expression into

$$\int_0^1 r \frac{d^2\phi_{\alpha,m}}{dr^2} \frac{d^2\phi_{\beta,m}}{dr^2} dr + (2m^2+1)\left(\frac{1}{r} \frac{d\phi_{\alpha,m}}{dr} \phi_{\beta,m}\right)\Bigg|_0$$

$$+ (2m^2+1) \int_0^1 \frac{1}{r} \frac{d\phi_{\alpha,m}}{dr} \frac{d\phi_{\beta,m}}{dr} dr$$

$$+ (m^4-4m^2)\int_0^1 \frac{1}{r^4} \phi_{\alpha,m}\phi_{\beta,m} dr \quad .$$

A similar derivation could be done starting with the differential equation defining $\phi_{\beta,m}$, and subtracting this similar expression from the expression we just found, we obtain

$$-\varepsilon\ [2m^2+1]\left[\frac{1}{r}\frac{d\phi_{\alpha,m}}{dr}\phi_{\beta,m}-\frac{1}{r}\frac{d\phi_{\beta,m}}{dr}\phi_{\alpha,m}\right]\Bigg|_0$$

$$+\left(\lambda^2_{\alpha,m}-\lambda^2_{\beta,m}\right)\int_0^1 r\ \phi_{\alpha,m}\phi_{\beta,m}=0 \quad .$$

Considering that $\phi_{\alpha,m}$ and $\phi_{\beta,m}$ are analytic in the neighborhood of the origin, we can expand $\phi_{\alpha,m}$ and $\phi_{\beta,m}$ in the following way

$$\phi_{\alpha,m}(r)=\phi_{\alpha,m}(o)+\frac{d\phi_{\alpha,m}}{dr}(o)\ r+\frac{d^2\phi_{\alpha,m}}{dr^2}(o)\ \frac{r^2}{2!}+\ldots$$

and

$$\phi_{\beta,m}(r)=\phi_{\beta,m}(o)+\frac{d\phi_{\beta,m}}{dr}(o)\ r+\frac{d^2\phi_{\beta,m}}{dr^2}(1)\ \frac{r^2}{2!}+\ldots$$

where the first terms are zero by virtue of the remark done before. We thus have

$$\frac{1}{r}\frac{d\phi_{\alpha,m}}{dr}\phi_{\beta,m}-\frac{1}{r}\frac{d\phi_{\beta,m}}{dr}\phi_{\alpha,m}=\frac{d\phi_{\alpha,m}}{dr}\left(\frac{d\phi_{\beta,m}}{dr}(o)+\frac{d^2\phi_{\beta,m}}{dr^2}(o)\frac{r}{2!}+\ldots\right)$$

$$-\frac{d\phi_{\beta,m}}{dr}\left(\frac{d\phi_{\alpha,m}}{dr}(o)+\frac{d^2\phi_{\alpha,m}}{dr^2}(o)\ \frac{r}{2!}+\ldots\right) .$$

This last expression evaluated at this origin gives us zero. We finally obtain the predicted orthogonality relationship

$$\int_0^1 r\ \phi_{\alpha,m}(r)\ \phi_{\beta,m}(r)\ dr=0 \quad .$$

<u>Case B</u>: $m = 0$

This case corresponds to radially symmetric vibrations, and as a result the physics of the problem introduces as value for the first derivative at the origin:

$$\frac{d\phi_{\alpha,0}}{dr} = \frac{d\phi_{\beta,0}}{dr} = 0 \qquad \text{when } r = 0 \quad .$$

A development similar to the one done before for $m \neq 0$, where the last remark has to be substituted gives us

$$- \varepsilon \left\{ r\, \phi_{\alpha,0} \frac{d}{dr}\left[\frac{1}{r}\frac{d\phi_{\beta,0}}{dr}\right] - r\, \phi_{\beta,0} \frac{d}{dr}\left[\frac{1}{r}\frac{d\phi_{\alpha,0}}{dr}\right]\right\}_0$$

$$+ (\lambda^2_{\alpha,0} - \lambda^2_{\beta,0}) \int_0^1 r\, \phi_{\alpha,0}\phi_{\beta,0}\, dr = 0 \; .$$

Let us consider the expression

$$r\, \phi_{\alpha,0} \frac{d}{dr}\left[\frac{1}{r}\frac{d\phi_{\beta,0}}{dr}\right] = \phi_{\alpha,0} \frac{d^2\phi_{\beta,0}}{dr^2} - \frac{1}{r}\phi_{\alpha,0}\frac{d\phi_{\beta,0}}{dr}$$

in the neighborhood of the origin. Considering that $\phi_{\beta,0}$ is analytic in the neighborhood of the origin, we can expand $\frac{d\phi_{\beta,0}}{dr}$ in the following way

$$\frac{d\phi_{\beta,0}}{dr}(r) = \frac{d\phi_{\beta,0}}{dr}(o) + \frac{d^2\phi_{\beta,0}}{dr^2}(o)r + \frac{d^3\phi_{\beta,0}}{dr^3}(o)\frac{r^2}{2!} + \ldots$$

Taking into account the remark done before about $\frac{d\phi_{\beta,0}}{dr}(o)$, we have

$$r\, \phi_{\alpha,0} \frac{d}{dr}\left[\frac{1}{r}\frac{d\phi_{\beta,0}}{dr}\right] = \phi_{\alpha,0}\frac{d^2\phi_{\beta,0}}{dr^2} - \phi_{\alpha,0}\left[\frac{d^2\phi_{\beta,0}}{dr^2}(o) + \frac{d^3\phi_{\beta,0}}{dr^3}(o)\frac{r}{2!} + \ldots\right]$$

Evaluated at the origin, the last expression becomes zero and we finally proved also that

$$\int_0^1 r\, \phi_{\alpha,0}(r)\phi_{\beta,0}(r)\, dr = 0$$

Now that an orthogonality relationship has been established for the modes, the substitution of the asymptotic expansions valid for each of them brings us the following relation for the zeroth order terms.

$$\int_0^1 \int_0^{2\Pi} r\, h_{0,\alpha,m}(r) h_{0,\beta,n}(r) \cos m\theta \cos n\theta \, dr d\theta = 0 \ .$$

Again, our main interest will be oriented towards the case where m=n. The case m≠n is, as mentioned before, automatically satisfied by performing the integration in θ first. The differential equation defining $h_{0,\alpha}$ and $h_{0,\beta}$ are given by:

$$\frac{1}{r} \frac{d}{dr} \left[r(1-k_1^2 r^2) \frac{dh_{0,\alpha}}{dr} \right] - \frac{m^2}{r^2} [1-k_2^2 r^2] \, h_{0,\alpha} + \lambda_{0,\alpha}^2 h_{0,\alpha} = 0$$

and

$$\frac{1}{r} \frac{d}{dr} \left[r(1-k_1^2 r^2) \frac{dh_{0,\beta}}{dr} \right] - \frac{m^2}{r^2} (1-k_2^2 r^2] \, h_{0,\beta} + \lambda_{0,\beta}^2 h_{0,\beta} = 0$$

or for r≠0

$$\frac{d}{dr} \left[r(1-k_1^2 r^2) \frac{dh_{0,\alpha}}{dr} \right] - \frac{m^2}{r^2} [1-k_2^2 r^2] h_{0,\alpha} + \lambda_{0,\alpha}^2 \, r \, h_{0,\alpha} = 0$$

and a similar expression for $h_{0,\beta}$. But these last expressions represent particular cases of the more general eigenvalue problem of the Sturm-Liouville type

$$\frac{d}{dr} \left[p(r) \frac{dh_{0,\alpha}}{dr} \right] - g(r) h_{0,\alpha} + \lambda_{0,\alpha}^2 \, r \, h_{0,\alpha} = 0.$$

If we now consider the boundary conditions of our problem or

$$h_{0,\alpha} = 0 \quad \text{when } r = 1 \quad ,$$

combined with

$$h_{0,\alpha} = 0 \quad \text{when } r = 0 \text{ for the case } m \neq 0$$

or

$$\frac{dh_{0,\alpha}}{dr} = 0 \quad \text{when } r = 0 \text{ for the case } m = 0,$$

our problem satisfies the requirements of the Sturm-Liouville problem. We can then deduce that for our problem, there exists a countable sequence of eigenvalues and a countable sequence of eigenfunctions satisfying our problem. In addition to that, we have that

$$\int_0^1 r\, h_{0,\alpha}(r) h_{0,\beta}\, dr = 0$$

and the orthogonality of the zero order is automatically satisfied.

The orthogonality relationship we derived before has to be completed by the relationship normalizing the modes. We then define

$$\int_0^1 \int_0^{2\Pi} r\, \phi^2_{\alpha,m}(r) \cos^2 m\theta\, d\theta = 1 \quad .$$

Integrating with respect to θ, we finally obtain

$$\int_0^1 r\, \phi^2_{\alpha,m}(r)\, dr = \frac{1}{\Pi} \quad .$$

Introducing in the last expression, the asymptotic expression used for $\phi_{\alpha,m}$, we have the following relations

$$\int_0^1 r\, h_0^2(r)\, dr = \frac{1}{\Pi}$$

$$\int_0^1 r\, h_0(r)\, h_1(r)\, dr = 0 \quad .$$

The first relationship will be used in order to define the constant b_0, still to be determined in the expression of h_0, and the second relationship will be used in order to determine the value of λ_1^2. We had

$$h_0(r) = b_0 r^m \, F\left(\alpha,\beta,1+m;\frac{3+\nu}{1+2k^2+\nu}\, r^2\right)$$

so we have

$$\int_0^1 r^{2m+1} \, b_0^2 \, F^2 \left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\, r^2\right) dr = \frac{1}{\Pi}$$

and

$$b_0^2 = \frac{1}{\Pi} \times \frac{1}{\displaystyle\int_0^1 r^{2m+1} F^2\left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\, r^2\right) dr} \quad .$$

In order to determine λ_1^2, let us consider the differential equation defining $h_1(r)$ (3.10) or

$$\frac{d}{dr}\left[r(1-k_1^2r^2)\,\frac{dh_1}{dr}\right] - \frac{m^2}{r}\,[1-k_2^2r^2]\,h_1 + r\,\lambda_0^2 h_1 = -\lambda_1^2 r\, h_0 \ .$$

Multiplying this last relation by h_0, and integrating by parts the first term, we have:

$$r(1-k_1^2r^2)\,\frac{dh_1}{dr}\,h_0\Big|_0^1 - \int_0^1 r(1-k_1^2r^2)\,\frac{dh_1}{dr}\,\frac{dh_0}{dr}\,dr - \int_0^1 \frac{m^2}{r}\,(1-k_2^2r^2)h_1h_0\,dr$$

$$= -\frac{\lambda_1^2}{\Pi} \ .$$

Considering the fact that $h_0(\lambda_0'^2,1) = 0$, and integrating by parts once more, we have

$$-\left[r(1-k_1^2r^2)\ \frac{dh_0}{dr}\ h_1\right]\Bigg|_0^1 + \int_0^1 h_1\ \frac{d}{dr}\left[r(1-k_1^2r^2)\ \frac{dh_0}{dr}\right] dr$$

$$- \int_0^1 \frac{m^2}{r}\ (1-k_2^2r^2)h_1h_0\ dr = -\ \frac{\lambda_1^2}{\Pi}\ .$$

The use of the differential equation (3.6) defining h_0, gives us

$$-(1-k_1^2)\ \frac{dh_0}{dr}\ h_1\Bigg|_1 - \int_0^1 r\ \lambda_0^2\ h_0\ h_1\ dr = -\ \frac{\lambda_1^2}{\Pi}$$

or

$$\lambda_1^2 = \Pi(1-k_1^2)\ \frac{dh_0}{dr}\ h_1\Bigg|_1$$

but

$$h_1(1) = -\ \frac{C_1}{\sqrt{1-k_1^2}} \quad \text{and} \quad \frac{dh_0}{dr}\ (1) = -\ C_1$$

so

$$\frac{dh_0}{dr}\ h_1\Bigg|_1 = \frac{C_1^2}{\sqrt{1-k_1^2}} = \frac{b_0^2}{\sqrt{1-k_1^2}}\left[\frac{2(3+\nu)}{1+2k^2+\nu}\ \frac{dF}{dx}\left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\right)\right]^2 .$$

Finally, we have

$$\lambda_1^2 = \sqrt{1-k_1^2}\ \frac{\left[\frac{2(3+\nu)}{1+2k^2+\nu}\ \frac{dF}{dx}\left(\alpha,\beta,1+m;\ \frac{3+\nu}{1+2k^2+\nu}\right)\right]^2}{\int_0^1 r^{2m+1}\ F^2\left(\alpha,\beta,1+m;\frac{3+\nu}{1+2k^2+\nu}\ r^2\right) dr}\ .$$

With the determination of λ_1^2, the obtention of $h_1(r)$ becomes an easy task and a well specified problem.

In order to complete the derivation done for this membrane, we should look at the problem that could arise at the origin. We know that the origin r=0 represents a singularity for the linear operator ∇_m^2 and, as a result, the fact of neglecting the contribution of this operator, close to the origin, might seem incorrect. When we looked at the reduced problem, we overlooked the term $-\varepsilon\, \nabla_m^4\, \phi_{\alpha,m}$ appearing in Equation (3.5). Close to the origin, this term could represent a main contribution. In Appendix III, we address ourselves to the problem near the origin and we find that the solution previously found for $r \neq 0$, is also valid through the origin. This result might be expected in the sense that there are no boundary conditions imposed at the origin and the solution found for $r \neq 0$ represents an expression that, when introduced into $-\varepsilon \nabla_m^4\, \phi_{\alpha,m}$ represents only a bounded function at the origin. Also, the presence of the singularity at the origin is simply originated by the choice of polar coordinates. For a rectangular plate described by cartesian coordinates, the center of the plate doesn't contain any singularity.

With this last remark, showing that the solution previously found is valid in the complete interior of the membrane, we have solved the proposed problem. To conclude this section, we summarize briefly some of the main results. The eigenfrequencies of the taut membrane with small flexural rigidity has been taken under the form

$$\omega_{\alpha,m}^2 = \frac{1+2k^2+\nu}{8}\, \Omega^2\, \lambda_{\alpha,m}^2 = \frac{1+2k^2+\nu}{8}\, \Omega^2 (\lambda_0^2 + \sqrt{\varepsilon}\, \lambda_1^2 + \ldots)$$

where

$$\varepsilon = \frac{2\, E\, h^2}{3(1-\nu^2)(1+2k^2+\nu)\, a^4 \mu \Omega^2} \quad .$$

The value of λ_0^2 is obtained by solving

$$F(\alpha_1\ 1+m - \alpha,\ 1+m;\ k_1^2) = 0$$

and

$$\alpha(1+m-\alpha) = \frac{1}{4}\left(2m + m^2 - m^2\ k_3^2 - \frac{\lambda_0^2}{k_1^2}\right).$$

The relation defining λ_1^2 is also repeated here

$$\lambda_1^2 = \sqrt{1-k_1^2}\ \frac{\left[2\ k_1^2\ \frac{dF}{dx}\ (\alpha,\ 1+m-\alpha,\ 1+m;\ k_1^2)\right]^2}{\int_0^1 r^{2m+1}\ F^2(\alpha,\ 1+m-\alpha,\ 1+m;\ k_1^2 r^2)dr}.$$

In these relations, the following definitions have been used

$$k_1^2 = \frac{3+\nu}{1+2k^2+\nu} \qquad \text{and} \qquad k_3^2 = \frac{1+3\nu}{3+\nu}.$$

3.3 Dynamics of a Rotating, Free, Elastic, Circular Membrane

We now end the chapter, 3, by considering another type of elementary structure. It represents the last analysis of modes undertaken in our research using continuous models. This time, the "object" under investigation is a rotating disk of small flexural rigidity, free along its edge. (Figure 10)

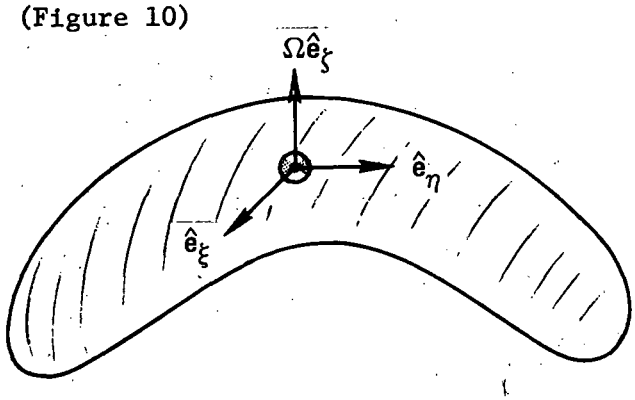

Figure 10. Rotating Free Membrane.

The basic physical assumptions made in this study are the same as those made earlier for the taut rotating membrane. This disk is assumed to rotate with constant angular velocity Ω around its axis of symmetry and the equations of the motion are given in a reference frame located at the center of mass of the disk when at steady state deformation and rotating with the system at the constant angular velocity Ω. In contrast with the previous example, the spin has a stabilizing effect on the structure and loosely speaking, the structural stiffness of the disk is increased by the sometimes called geometric stiffness, induced by spin. The origin of our angular variable is chosen such that no phase angle is needed in our equations. The generality of our problem is not affected by this assumption but the text is noticeably lightened. Even though the membrane will undergo a steady state radial displacement, we assume the physical characteristics of our system to remain constant throughout the whole disk.

Other classical assumptions, such as the planar state of stresses, are also adopted. There will be no variation in the radial and tangential stresses over the thickness of the membrane due to the assumption of small thickness. This last assumption merits one more comment related to the boundary conditions valid at the free edge. We have, indeed, to apply as boundary conditions at the free edge the Kirchoff condition and the zero bending condition. Let us develop those considerations for a while.

For the free edge, no restrictions are imposed on the deflection; however, there should be no bending moments, no twisting moments and no shear forces. There is no reason to disregard any of those as secondary effects. Kirchoff suggested the following way out of this difficulty.

Calling the bending moment M, the twisting moment H and the shear forces N, he considered the edge x = a of a plate as being acted upon by distributed torsional couples, as in Figure 11. Their

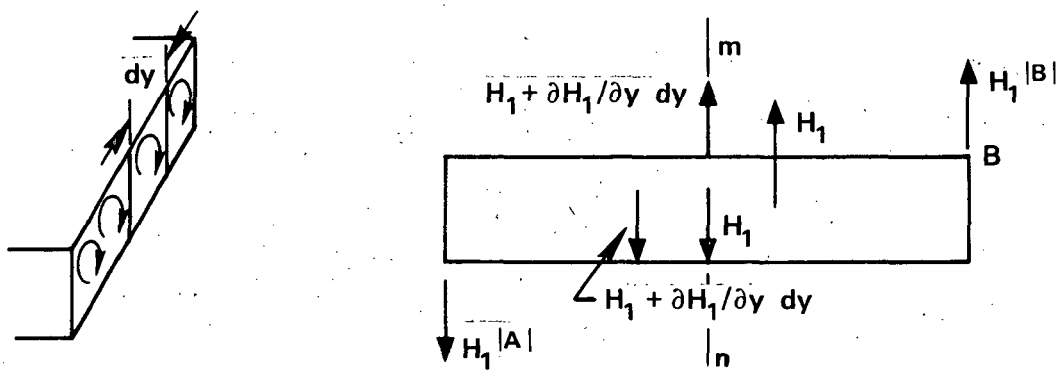

Figure 11. Kirchoff Condition.

distribution was assumed to be nonuniform, in general, and the magnitude of the moment of these couples per unit of length of the face, at a given point, is denoted by H_1. The magnitude of the moment per length dy of the face is obviously equal to $H_1 dy$. It is noted that from the viewpoint of statics, the distributed torsional couples are equivalent to a certain shearing force. Indeed, the couple with the moment $H_1 dy$ may be applied by means of two equal and opposite forces H_1 acting at the edges of an area of length dy. The couple $(H_1 + \frac{\partial H_1}{\partial y} dy)dy$ on the adjacent area of length dy may also be applied

as two opposite forces $H_1 + \frac{\partial H_1}{\partial y} dy$ with the arm dy. Having done this for all areas of the face concerned we see that the forces applied at points m and n on the boundary of two areas reduce to a single force $\frac{\partial H_1}{\partial y} dy$ per length dy. We observe, however, that there will remain two nonvanishing finite concentrated forces H_1^A and H_1^B at the edge A and B of the face.

Hence, we conclude that the distributed torsional couples of intensity H_1 are statically equivalent to the distributed shearing forces which has the intensity $\frac{\partial H_1}{\partial y}$ and two concentrated shearing forces H_1^A and H_1^B at the corners of the plate; if the contour is smooth, and without corners, these concentrated forces are absent. Proceeding from this, Kirchoff proposed that the three boundary conditions at the free edge be combined into two by equating to zero the bending moment M and the shearing force N and by adding to the latter the term $\partial H/\partial y$ which reflects the influence of the twisting moment H. We now arrive at the following two conditions at the free edge:

$$M = 0$$

$$N + \frac{\partial H}{\partial y} = 0 \ .$$

Written in polar coordinates, those previous boundary conditions become:

$$\frac{\partial^2 w}{\partial \rho^2} + \nu\left(\frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2}\right) = 0$$

$$\frac{\partial}{\partial \rho}\left[\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2}\right] + (1-\nu)\frac{\partial}{\rho\partial\theta}\left[\frac{\partial}{\partial \rho}\left(\frac{1}{\rho}\frac{\partial w}{\partial \theta}\right)\right] = 0$$

valid at the free edge of the disk, where ν is the Poisson modulus.

As stated previously we know that the dynamic equation for the transverse vibrations of a circular plate, subjected to radial and angular stresses is given in general form by

$$-\frac{E\,h^2}{12(1-\nu^2)}\nabla_\rho^4 w + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\sigma_\rho \rho \frac{\partial w}{\partial \rho}\right) + \frac{\sigma_\theta}{\rho^2}\frac{\partial^2 w}{\partial\theta^2} = \mu\frac{\partial^2 w}{\partial t^2} \qquad (3.11)$$

where σ_ρ and σ_θ represent respectively the radial and angular stresses

h stands for the thickness of the disk and

∇_ρ^2 is the Laplacian operator which expressed in polar coordinates is

$$\nabla_\rho^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}$$

μ is the disk mass per unit of volume.

For this case too, we found that the deformation equation represents a perturbation with respect to the steady state deformation, and once more, the radial displacement has to be determined first — a similar procedure was adopted in the previous case.

The equilibrium of the forces applied to an element of the disk along the radius is given by

$$\frac{d}{d\rho}(\rho\,\sigma_\rho) - \sigma_\theta + \mu\,\Omega^2\rho^2 = 0$$

where there is no dependence in the θ variable due to the symmetry of the problem. The use of the generalized Hooke's laws expressing the relationship between stress and strain, give us as differential equation for the steady state radial displacement u:

$$\rho^2\frac{d^2u}{d\rho^2} + \rho\frac{du}{d\rho} - u = -\frac{1-\nu^2}{E}\mu\,\Omega^2\rho^3 .$$

The general solution of this equation is

$$u = \frac{1}{E}\left[(1-\nu)C\rho - (1+\nu)\frac{C_1}{\rho} - \frac{1-\nu^2}{8}\mu\,\Omega^2\rho^3\right]$$

where C and C_1 are arbitrary constants.

For a disk without central hole, C_1 has to be zero, to keep the displacement bounded at the origin. The corresponding stress components are now found from:

$$\sigma_\rho = C - \frac{3+\nu}{8}\mu\,\Omega^2\rho^2$$

$$\sigma_\theta = C - \frac{1+3\nu}{8}\mu\,\Omega^2\rho^2 \ .$$

The constant C is determined from the condition existing at the periphery (ρ=a) of the disk. If there is no force applied there, we have

$$(\sigma_\rho)_{\rho=a} = 0$$

or

$$C = \frac{3+\nu}{9}\mu\,\Omega^2 a^2 \ ,$$

from which we have

$$\sigma_\rho = \frac{3+\nu}{8}\mu\,\Omega^2(a^2 - \rho^2)$$

$$\sigma_\theta = \frac{\mu\Omega^2}{8}\left[(3+\nu)a^2 - (1+3\nu)\rho^2\right] \ .$$

Let us now introduce the dimensionless variable, ρ = ar, so that Equation (3.11) takes the form:

$$-\frac{E\,h^2}{12a^4(1-\nu^2)}\nabla^4 w + \frac{1}{a^2 r}\frac{\partial}{\partial r}\left[r\,\sigma_r\frac{\partial w}{\partial r}\right] + \frac{\sigma_\theta}{a^2 r^2}\frac{\partial^2 w}{\partial\theta^2} = \mu\frac{\partial^2 w}{\partial t^2}$$

where

$$\sigma_r = \mu\,\Omega^2\,a^2\,\left(\frac{3+\nu}{8}\right)(1-r^2)$$

$$\sigma_\theta = \frac{\mu\,\Omega^2\,a^2}{8}\,[(3+\nu) - (1+3\nu)r^2] \quad .$$

Dividing through by $\mu\,\Omega^2$, we obtain

$$-\frac{E\,h^2}{12\mu\Omega^2 a^4(1-\nu^2)}\,\nabla^4 w + \frac{3+\nu}{8}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,[r(1-r^2)\frac{\partial w}{\partial r}\,]$$

$$+\frac{1}{8r^2}\,[(3+\nu)-(1+3\nu)r^2]\,\frac{\partial^2 w}{\partial\theta^2} = \frac{1}{\Omega^2}\,\frac{\partial^2 w}{\partial t^2} \quad .$$

This last equation is suitable for the use of separation of variables, providing that the given initial condition of the problem is also separable. We will then use an expression of the form

$$w = \phi_{\alpha,m}(r)e^{i(\omega_{\alpha,m}t + m\theta)}$$

where $\omega_{\alpha,m}$ is the eigenfrequency of the vibration.

In this last expression, no phase angle has been introduced for a reason explained earlier and this doesn't affect the generality of our problem.

The equation of main interest is given by the differential equation defining $\phi_{\alpha,m}(r)$. This last expression can be written

$$-\frac{E\,h^2}{12(1-\nu^2)a^4\mu\Omega^2}\,\nabla_m^4\,\phi_{\alpha,m} + \frac{3+\nu}{8r}\,\frac{d}{dr}\,[r(1-r^2)\,\frac{d\phi_{\alpha,m}}{dr}]$$

$$-\frac{m^2}{8r^2}\,[(3+\nu)-(1+3\nu)r^2]\phi_{\alpha,m} = -\frac{\omega_{\alpha,m}^2}{\Omega^2}\,\phi_{\alpha,m}$$

where ∇_m^2 is the linear differential operator defined by the following relation

$$\nabla_m^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} .$$

We now define the following quantities:

$$\varepsilon = \frac{2\,E\,h^2}{3(1-\nu^2)(3+\nu)a^4\mu\,\Omega^2}$$

$$\lambda_{\alpha,m}^2 = \frac{8\,\omega_{\alpha,m}^2}{(3+\nu)\Omega^2}$$

$$k^2 = \frac{1+3\nu}{3+\nu} .$$

The final expression for the differential equation for $\phi_{\alpha,m}(r)$ is then given by

$$-\varepsilon\nabla_m^4\,\phi_{\alpha,m} + \frac{1}{r}\frac{d}{dr}\left[r(1-r^2)\frac{d\phi_{\alpha,m}}{dr}\right] - \frac{m^2}{r^2}[1-k^2r^2]\phi_{\alpha,m} + \lambda_{\alpha,m}^2\phi_{\alpha,m} = 0. \qquad (3.12)$$

Applying the previous separation of variables to the boundary conditions of our problem, we obtain the boundary conditions valid for $\phi_{\alpha,m}(r)$ or

$$\frac{d^2\phi_{\alpha,m}}{dr^2} + \nu\left(\frac{1}{r}\frac{d\phi_{\alpha,m}}{dr} - \frac{m^2}{r^2}\phi_{\alpha,m}\right) = 0$$

$$\frac{d}{dr}\left[\frac{d^2\phi_{\alpha,m}}{dr^2} + \frac{1}{r}\frac{d\phi_{\alpha,m}}{dr} - \frac{m^2}{r^2}\phi_{\alpha,m}\right] - (1-\nu)\frac{m^2}{r}\frac{d}{dr}\left(\frac{\phi_{\alpha,m}}{r}\right) = 0$$

when $r = 1$.

Here, again, the small flexural rigidity of the disk is combined to the high spin rate to produce the small parameter ε. The problem

is now formulated in a way where a procedure similar to the one used before could be applied.

The operator ∇_m^2, defined earlier, contains a singularity at the origin r=0 and in the use of a perturbation method, some attention is required when considering the limit process. However, the solution found in the central area remains valid in the neighborhood of the origin. There are, indeed, no boundary conditions at the origin and by a procedure similar to the one adopted in Appendix III, we can show that no particular behavior occurs near the origin.

Expansion Valid in The Central Area

When $r \neq 0$, the expansion used for $\phi_{\alpha,m}(r)$ take the form:

$$\phi_{\alpha,m}(r,\varepsilon) = h_0(r) + \nu_1(\varepsilon)h_1(r) + \nu_2(\varepsilon)h_2(r)+\ldots$$

The subscripts α,m will be momentarily dropped, and used explicitly only when needed in order to differentiate between different modes. Introducing for the eigenvalue the expansion:

$$\lambda^2_{\alpha,m}(\varepsilon) = \lambda_0^2 + \kappa_1(\varepsilon)\lambda_1^2 + \kappa_2(\varepsilon)\lambda_2^2 +\ldots,$$

the basic differential equation for the mode (3.12) becomes:

$$\begin{aligned}
&-\varepsilon\, \nabla_m^4(h_0 + \nu_1(\varepsilon)h_1+\ldots) \\
&\qquad + \frac{1}{r}\frac{d}{dr}\left[r(1-r^2)\frac{dh_0}{dr}\right] + \nu_1(\varepsilon)\frac{1}{r}\frac{d}{dr}\left[r(1-r^2)\frac{dh_1}{dr}\right]+\ldots \\
&\qquad - \frac{m^2}{r^2}[1-k^2r^2]h_0 - \nu_1(\varepsilon)\frac{m^2}{r^2}[1-k^2r^2]h_1 \\
&\qquad +\left(\lambda_0^2 + \kappa_1(\varepsilon)\lambda_1^2+\ldots\right)\left(h_0 + \nu_1(\varepsilon)h_1 +\ldots\right) = 0 \quad .
\end{aligned}$$

The differential equation defining $h_0(r)$ is then given by:

$$\frac{1}{r}\frac{d}{dr}\left[r(1-r^2)\frac{dh_0}{dr}\right] - \frac{m^2}{r^2}(1-k^2r^2)h_0 + \lambda_0^2 h_0 = 0 \tag{3.13}$$

or

$$(1-r^2)\frac{d^2h_0}{dr^2} + \frac{1}{r}(1-3r^2)\frac{dh_0}{dr} - \frac{1}{r^2}[m^2 - (m^2k^2 + \lambda_0^2)r^2]h_0 = 0 .$$

The point r=0 is a regular singular point, for this last equation, and adequate for a solution in power series around r=0. Instead of transforming the previous equation to obtain a solution expressed in terms of the hypergeometric series, it is indicated to expand the solution directly under the form:

$$h_0(r) = \sum_{n=0}^{\infty} a_n r^{n+c}$$

where c is the index of the previous equation and a_0 an arbitrary constant.

Introducing the last expansion of $h_0(r)$ into the differential equation, (3.13), the indicial equation gives us

$$m^2 = c^2 .$$

Only the positive root is feasible for the problem, for its boundedness of the displacement at the origin, so c=m. Also, the recurrence relationship between the a_i's is easily obtained and we report the result

$$\frac{a_n}{a_{n-2}} = \frac{(n+m)(n+m-2)-(k^2m^2 + \lambda_0^2)}{n^2 + 2nm} .$$

By direct observation, we can see that the denominator is always different from zero and the solution $h_0(r)$ contains only odd or even powers of r. In order to pursue our study, it is interesting at this stage to look for a while at the boundary layer valid near the free edge of the disk. We need then to define a suitable boundary layer coordinate or

$$\tilde{r} = \frac{1-r}{\phi(\varepsilon)} \quad .$$

The operator ∇_m^2 becomes:

$$\nabla_m^2 = \frac{d^2}{\phi^2 d\tilde{r}^2} - \frac{1}{1-\phi\tilde{r}} \frac{d}{\phi d\tilde{r}} - \frac{m^2}{(1-\phi\tilde{r})^2} \quad .$$

So the basic differential equation for $\phi_{\alpha,m}(r)$ (3.12) becomes in the new boundary layer coordinate:

$$-\varepsilon\left[\frac{d^2}{\phi^2 d\tilde{r}^2} - \frac{1}{1-\phi\tilde{r}} \frac{d}{\phi d\tilde{r}} - \frac{m^2}{(1-\phi\tilde{r})^2}\right]\left[\frac{d^2\phi_{\alpha,m}}{\phi^2 d\tilde{r}^2} - \frac{1}{1-\phi\tilde{r}} \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} - \frac{m^2\phi_{\alpha,m}}{(1-\phi\tilde{r})^2}\right]$$

$$+ \frac{1}{1-\phi\tilde{r}} \frac{d}{\phi d\tilde{r}}\left[(1-\phi\tilde{r})[1 - (1-\phi\tilde{r})^2] \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}}\right]$$

$$- \frac{m^2}{(1-\phi\tilde{r})^2} [1-k^2(1-\phi\tilde{r})^2] \phi_{\alpha,m} + \lambda_{\alpha,m}^2 \phi_{\alpha,m} = 0 \qquad (3.14)$$

or

$$-\varepsilon\left[\frac{1}{\phi^4}\frac{d^4\phi_{\alpha,m}}{d\tilde{r}^4} - \frac{2}{(1-\phi\tilde{r})}\frac{d^3\phi_{\alpha,m}}{\phi^3 d\tilde{r}^3} - (2m^2+1)\frac{1}{(1-\phi\tilde{r})^2}\frac{d^2\phi_{\alpha,m}}{\phi^2 d\tilde{r}^2}\right.$$

$$\left. - (2m^2+1)\frac{1}{(1-\phi\tilde{r})^3}\frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} + (m^4-4m^2)\frac{\phi_{\alpha,m}}{(1-\phi\tilde{r})^4}\right]$$

$$+ \frac{1}{(1-\phi\tilde{r})^2}\frac{1}{\phi}\left[(3\phi^2\tilde{r}^2 - 6\phi\tilde{r} + 2)\frac{d\phi_{\alpha,m}}{d\tilde{r}}\right.$$

$$\left. + (\phi^2\tilde{r}^3 - 3\,\phi\tilde{r}^2 + 2\tilde{r})\frac{d^2\phi_{\alpha,m}}{d\tilde{r}^2}\right]$$

$$- \frac{m^2}{(1-\phi\tilde{r})^2}[1-k^2(1-\phi\tilde{r})^2]\phi_{\alpha,m} + \lambda^2_{\alpha,m}\phi_{\alpha,m} = 0 \quad .$$

The thickness of the boundary layer is then obtained by balancing

$$-\frac{\varepsilon}{\phi^4} \approx \frac{1}{\phi} \rightarrow \quad \phi = \varepsilon^{1/3} \quad .$$

If we define now the asymptotic expansion of $\phi_{\alpha,m}$ valid near the free edge in the following way

$$\phi_{\alpha,m}(\tilde{r},\varepsilon) = \delta_0(\varepsilon)f_0(\tilde{r}) + \delta_1(\varepsilon)f_1(\tilde{r}) + \ldots,$$

then the differential equation defining $f_0(\tilde{r})$ is given by

$$-\frac{d^4 f_0(\tilde{r})}{d\tilde{r}^4} + 2\frac{d\,f_0}{d\tilde{r}} + 2\tilde{r}\frac{d^2 f_0}{d\tilde{r}^2} = 0.$$

On the other hand, the boundary conditions become in the new boundary layer coordinate

$$\frac{d^2\phi_{\alpha,m}}{\phi^2\,d\tilde{r}^2} - \nu\left[\frac{1}{(1-\phi\tilde{r})}\frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} + \frac{m^2}{(1-\phi\tilde{r})^2}\phi_{\alpha,m}\right] = 0 \qquad (3.15)$$

and

$$- \frac{d^3\phi_{\alpha,m}}{\phi^3 d\tilde{r}^3} + \frac{1}{(1-\phi\tilde{r})^2} \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} - \frac{1}{(1-\phi\tilde{r})} \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} + \frac{2m^2}{(1-\phi\tilde{r})^3} \phi_{\alpha,m}$$

$$+ \frac{m^2}{(1-\phi\tilde{r})^2} \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} + (1-\nu) \frac{m^2}{(1-\phi\tilde{r})} \left[\frac{1}{(1-\phi\tilde{r})} \frac{d\phi_{\alpha,m}}{\phi d\tilde{r}} \right.$$

$$\left. + \frac{1}{(1-\phi\tilde{r})^2} \phi_{\alpha,m} \right] = 0 \qquad (3.16)$$

when $\tilde{r} = 0$.

For the first term of the expansion they become:

$$\left.\begin{aligned} \frac{d^2 f_0}{d\tilde{r}^2}(\tilde{r}) &= 0 \\ \frac{d^3 f_0}{d\tilde{r}^3}(\tilde{r}) &= 0 \end{aligned}\right\} \quad \text{when } \tilde{r} = 0 \quad .$$

The integration of the differential equation defining f_0 and the use of the boundary conditions just derived limit the solution for $f_0(\tilde{r})$ to a constant C_0. The determination of this last constant is determined through the matching process near the free edge. For matching, an intermediate limit suitable for matching near $r = 1$ is given by

$$r_\eta = \frac{1-r}{\eta(\varepsilon)} \qquad \text{where } r_\eta \text{ is fixed}$$

and $\eta(\varepsilon) \to 0$ when $\varepsilon \to 0$. What we want is $\tilde{r} \to \infty$ when $\varepsilon \to 0$, or

$$r_\eta = \frac{\varepsilon^{1/3}\,\tilde{r}}{\eta(\varepsilon)}$$

which implies $\eta/\varepsilon^{1/3} \to \infty$.

The behavior near r=1 takes the form:

$$\lim_{\substack{\varepsilon \to 0 \\ r_\eta \text{ fixed}}} \left\{ h_0(\eta r_\eta) + \nu_1(\varepsilon) h_1(\eta r_\eta) + \ldots - \delta_0(\varepsilon) f_0\left(\frac{\eta r_\eta}{\varepsilon^{1/3}}\right) - \delta_1(\varepsilon) f_1\left(\frac{\eta r_\eta}{\varepsilon^{1/3}}\right) + \ldots \right\} = 0$$

but $h_0(\eta r_\eta)$ can be written

$$h_0(\eta r_\eta) = h_0(1) - \frac{d}{dr} h_0(1)\, \eta r_\eta + \ldots$$

and $f_0(\tilde{r}) = C_0$. So we finally have:

$$\lim_{\substack{\varepsilon \to 0 \\ r_\eta \text{ fixed}}} \left\{ h_0(1) - \frac{d}{dr} h_0(1)\, \eta r_\eta + \ldots - \nu_1(\varepsilon) h_1(\eta r_\eta) + \ldots - \delta_0(\varepsilon) C_0 - \delta_1(\varepsilon) f_1\left(\frac{\eta r_\eta}{\varepsilon^{1/3}}\right) + \ldots \right\} = 0 \,. \tag{3.17}$$

The matching is then giving us

$$\delta_0(\varepsilon) = \varepsilon^0$$

and

$$h_0(1) = C_0 \,.$$

The determinations of $h_0(r)$ and λ_0^2 are now possible. The condition $h_0(1) = C_0$ — given through matching — appears, at first, to be unsufficient. But the consideration of the recurrence relationship between the a_i's shows that a sequence of value of λ_0^2 truncates the expansion for $h_0(r)$, namely:

$$(n+m)(n+m-2) - k^2 m^2 = \lambda_0^2 \,.$$

Only the sequence of values we just generated for λ_0^2 truncates the series and gives a bounded displacement near the edge. The value

$h_0(1)$ is indeed given by $\sum_{n=0}^{\infty} a_n$. By comparing the ratio of two consecutive terms of the last sequence to the ratio of two consecutive terms of the diverging sequence defined by

$$\frac{1}{p} + \frac{1}{p+2} + \frac{1}{p+4} \cdots ,$$

it is easy to prove that the ratio generated for the a_i's is greater than the ratio defined for the diverging sequence. The boundary condition for $h_0(r)$ is in fact

$$h_0(1) = \text{bounded} \quad .$$

It is concluded that for every value of λ_0^2 different from those truncating the series, the displacement along the edge is unbounded.

Having now completely defined the main contribution of the displacement valid in the central area, let us go back to the boundary layer expansion valid near the free edge. The consideration of the expression developed earlier for the matching suggests as an order of magnitude for $\delta_1(\varepsilon)$ the value $\varepsilon^{1/3}$.

Going back to the development of the differential equation defining the modes expressed in the boundary layer coordinate (3.14) we find that the differential equation defining f_1 is given by

$$-\frac{d^4 f_1}{d\tilde{r}^4} + 2\frac{df_1}{d\tilde{r}} + 2\tilde{r}\frac{d^2 f_1}{d\tilde{r}^2} = -2\frac{d^3 f_0}{d\tilde{r}^3} + 6\tilde{r}\frac{df_0}{d\tilde{r}} + 3\tilde{r}^2\frac{d^2 f_0}{d\tilde{r}^2}$$

$$+ [m^2 - k^2 m^2 - \lambda_0^2]\, f_0 \ .$$

Taking into account the expression already found for f_0 or

$$f_0(\tilde{r}) = h_0(1) \quad .$$

We have

$$-\frac{d^4 f_1}{d\tilde{r}^4} + 2\ \frac{df_1}{d\tilde{r}} + 2\tilde{r}\ \frac{d^2 f_1}{d\tilde{r}^2} = [m^2 - k^2 m^2 - \lambda_0^2] h_0(1). \qquad (3.18)$$

The expansion of the boundary conditions (3.15) and (3.16) expressed in the boundary layer coordinates also give us

$$\left.\begin{aligned} &\frac{d^2 f_1}{d\tilde{r}^2}(\tilde{r}) = 0 \\ &\frac{d^3 f_1}{d\tilde{r}^3} = 0 \end{aligned}\right. \qquad \text{when } \tilde{r} = 0 .$$

The integration of the previous linear differential equation and the use of the boundary conditions defined for f_1, gives us as a solution for f_1 the following expression:

$$f_1(\tilde{r}) = \frac{(m^2 - k^2 m^2 - \lambda_0^2)}{2}\ h_0(1)\ \tilde{r} + C_1 \quad .$$

Introducing the last expression into the matching process, we have

$$\lim_{\substack{\varepsilon \to 0 \\ r_\eta \text{ fixed}}} \left\{ -\frac{d}{dr} h_0(1)\ \eta r_\eta + \ldots + \nu_1(\varepsilon) h_1(\eta r_\eta) - \varepsilon^{1/3} \left[\frac{(m^2 - k^2 m^2 - \lambda_0^2)}{2}\ h_0(1)\ \frac{\eta r_\eta}{\varepsilon^{1/3}} + C_1 \right] + \ldots \right\} = 0$$

The matching process should determine the value of C_1. We observe directly that the expression

$$-\frac{d}{dr} h_0(1) - \frac{m^2 - k^2 m^2 - \lambda_0^2}{2}\ h_0(1)$$

is identically zero as seen by evaluating the differential equation defining $h_0(r)$, Equation (3.13), at the point r=1 or

$$- 2 \frac{dh_0}{dr} - [m^2 - m^2k^2 - \lambda_0^2] \, h_0 \Big|_1 = 0.$$

The determination of the constant C_1 seems to depend on the value of $\nu_1(\varepsilon)$.

Contrarily to our expectation, the value of $\nu_1(\varepsilon)$ is not of the order of magnitude of $\varepsilon^{1/3}$. The boundedness of the displacement along the rim of the disk and the orthogonality relationship — valid for this case too, as shown later — cancel the choice $\varepsilon^{1/3}$ for $\kappa_1(\varepsilon)$ and $\nu_1(\varepsilon)$ in their respective asymptotic expansion. As a result, the only feasible value for C_1 is

$$C_1 = 0 \ .$$

In Appendix IV, it is briefly presented why the choice $\kappa_1(\varepsilon) = \varepsilon^{1/3}$ in the asymptotic expansion of the eigenvalue has to be rejected. We now prove the orthogonality relationship between the modes. This last relationship is used in the determination of the next term of the asymptotic expansion valid for the eigenvalue. The orthogonality relationship between the modes will once more be given by

$$\int_0^1 \int_0^{2\Pi} r \, \phi_{\alpha,m} \, \phi_{\beta,n} \cos m\theta \cos n\theta \, dr d\theta = 0 \ .$$

It is obvious that, when $m \neq n$, the relationship previously written is satisfied, as a direct integration with respect to θ will give. The case of major interest is in fact given for the case where m=n. For that case, we thus have to prove that

$$\int_0^1 r \, \phi_{\alpha,m} \, \phi_{\beta,m} \, dr = 0 \ .$$

The differential equation defining $\phi_{\alpha,m}$ is given by

$$-\varepsilon\ \nabla_m^4\ \phi_{\alpha,m} + \frac{1}{r}\ \frac{d}{dr}\left[r(1-r^2)\ \frac{d\phi_{\alpha,m}}{dr}\right] - \frac{m^2}{r^2}\ [1-k^2r^2]\phi_{\alpha,m} + \lambda_{\alpha,m}^2\phi_{\alpha,m}=0$$

with the following boundary conditions:

$$\frac{d^2\phi_{\alpha,m}}{dr^2} + \nu\left(\frac{1}{r}\ \frac{d\phi_{\alpha,m}}{dr} - \frac{m^2}{r^2}\ \phi_{\alpha,m}\right) = 0$$

$$\frac{d}{dr}\left[\frac{d^2\phi_{\alpha,m}}{dr^2} + \frac{1}{r}\ \frac{d\phi_{\alpha,m}}{dr} - \frac{m^2}{r^2}\ \phi_{\alpha,m}\right] - (1-\nu)\ \frac{m^2}{r}\ \frac{d}{dr}\left(\frac{\phi_{\alpha,m}}{r}\right) = 0$$

when r=1. Similar expressions could be written for $\phi_{\beta,m}$. Taking the differential equation defining $\phi_{\alpha,m}$ and multiplying both sides by $r\phi_{\beta,m}$ and integrating, we have

$$-\varepsilon\ \int_0^1 r\ \nabla_m^4\ \phi_{\alpha,m}\phi_{\beta,m}\ dr + \int_0^1 \frac{d}{dr}\left[r(1-r^2)\ \frac{d\phi_{\alpha,m}}{dr}\right]\phi_{\beta,m}\ dr$$

$$- \int_0^1 \frac{m^2}{r}\ [1-k^2r^2]\phi_{\alpha,m}\phi_{\beta,m}\ dr$$

$$+ \int_0^1 \lambda_{\alpha,m}^2\ r\ \phi_{\alpha,m}\phi_{\beta,m}\ dr = 0 \quad .$$

Expanding the first term under the form

$$-\varepsilon\ \int_0^1 r\ \nabla_m^4\ \phi_{\alpha,m}\phi_{\beta,m}dr = -\varepsilon \int_0^1 r\ \phi_{\beta,m}\ \frac{d^2}{dr^2}\ [\nabla_m^2\ \phi_{\alpha,m}]\ dr$$

$$-\varepsilon \int_0^1 \phi_{\beta,m}\ \frac{d}{dr}\ [\nabla_m^2\phi_{\alpha,m}]\ dr$$

$$+\varepsilon m^2 \int_0^1 \frac{1}{r}\ \phi_{\beta,m}\ \nabla_m^2\phi_{\alpha,m}\ dr\ .$$

An integration by parts and the use of the boundary conditions transform the previous expression into

$$-\varepsilon(1-\nu)\ m^2 \frac{d}{dr}\left(\frac{\phi_{\alpha,m}}{r}\right)\phi_{\beta,m}\Bigg|_1 + \varepsilon \int_0^1 r \frac{d\phi_{\beta,m}}{dr} \frac{d}{dr}[\nabla_m^2\phi_{\alpha,m}]\ dr$$

$$+\varepsilon\ m^2 \int_0^1 \frac{\phi_{\beta,m}}{r}[\nabla_m^2\phi_{\alpha,m}]\ dr\ .$$

Integrating by parts once more, and considering the boundary conditions, we obtain for the previous expression

$$-\varepsilon(1-\nu)\left[-\frac{m^2}{r^2}\phi_{\alpha,m}\phi_{\beta,m} - \frac{d\phi_{\alpha,m}}{dr}\frac{d\phi_{\beta,m}}{dr} + \frac{m^2}{r}\phi_{\alpha,m}\frac{d\phi_{\beta,m}}{dr}\right.$$

$$\left.\left. + \frac{m^2}{r}\phi_{\beta,m}\frac{d\phi_{\alpha,m}}{dr}\right]\right|_1 -\varepsilon \int_0^1 r[\nabla_m^2\phi_{\alpha,m}][\nabla_m^2\phi_{\beta,m}]dr\ .$$

Introducing this last expression into the expression found previously, we observe a perfect symmetry between $\phi_{\alpha,m}$ and $\phi_{\beta,m}$. As a result, a similar procedure applied to the differential equation defining $\phi_{\beta,m}$ would give an expression that would differ only by the last term. Substracting both expressions, we are left with

$$(\lambda_{\alpha,m}^2 - \lambda_{\beta,m}^2)\int_0^1 r\ \phi_{\alpha,m}\phi_{\beta,m}\ dr = 0$$

which contains our required expression.

To complete the study of this problem, we will need the next term of the asymptotic expansion valid near the free edge. The matching process valid near the free edge suggests for $\delta_2(\varepsilon)$ the quantity $\varepsilon^{2/3}$.

Going back to the differential equation defining the modes, expressed in the boundary layer coordinates, Equation (3.14), we find that the differential equation for f_2 is given by:

$$-\frac{d^4 f_2}{d\tilde{r}^4} + 2\frac{df_2}{d\tilde{r}} + 2\tilde{r}\frac{d^2 f_2}{d\tilde{r}^2} = -2\frac{d^3 f_1}{d\tilde{r}^3} + 4\tilde{r}\frac{df_1}{d\tilde{r}} + 3\tilde{r}^2\frac{d^2 f_1}{d\tilde{r}^2}$$

$$+ [m^2 - k^2 m^2 - \lambda_0^2] f_1 - 3\tilde{r}^2\frac{df_0}{d\tilde{r}} + 2m^2\tilde{r}\, f_0 .$$

Taking into account the expressions already found for

$$f_0(\tilde{r}) = h_0(1)$$

and

$$f_1(\tilde{r}) = \frac{(m^2 - k^2 m^2 - \lambda_0^2)}{2} h_0(1)\tilde{r} ,$$

the last expression becomes:

$$-\frac{d^2 f_2}{d\tilde{r}^4} + 2\frac{df_2}{d\tilde{r}} + 2\tilde{r}\frac{d^2 f_2}{d\tilde{r}^2} = 2(m^2 - k^2 m^2 - \lambda_0^2)\, h_0(1)\tilde{r}$$

$$+ (m^2 - k^2 m^2 - \lambda_0^2)^2 \frac{h_0(1)}{2}\tilde{r} + 2m^2\tilde{r}\, h_0(1).$$

The expansion of the boundary conditions (3.15) and (3.16) expressed in the boundary layer coordinate give as boundary conditions for f_2, the following expressions:

$$\frac{d^2 f_2}{d\tilde{r}^2} - \nu\frac{df_1}{d\tilde{r}} - m^2\nu\, f_0 = 0$$

$$-\frac{d^3 f_2}{d\tilde{r}^3} + (1-\nu)m^2\frac{df_0}{d\tilde{r}} = 0 \qquad \text{when } \tilde{r} = 0.$$

Those last expressions can also be written:

$$\begin{cases} \dfrac{d^2 f_2}{d\tilde{r}^2} = \nu \dfrac{m^2 - k^2 m^2 - \lambda_0^2}{2} h_0(1) + m^2 \nu\, h_0(1) \\ \dfrac{d^3 f_2}{d\tilde{r}^3} = 0 \end{cases} \qquad \text{when } \tilde{r} = 0.$$

The equation defining f_2 can also be written

$$\frac{d^4 f_2}{d\tilde{r}^4} - 2 \frac{d}{d\tilde{r}} \left[\tilde{r} \frac{df_2}{d\tilde{r}} \right] = -2(m^2 - k^2 m^2 - \lambda_0^2) h_0(1) \tilde{r}$$

$$- (m^2 - k^2 m^2 - \lambda_0^2)^2 \frac{h_0(1)}{2} \tilde{r} + 2\tilde{r}\, m^2\, h_0(1) \ .$$

A particular solution is given by considering

$$f_{2p} = D_0\, \tilde{r}^2$$

where

$$D_0 = \frac{1}{4} (m^2 - k^2 m^2 - \lambda_0^2) h_0(1) + \frac{1}{16} (m^2 - k^2 m^2 - \lambda_0^2)^2 h_0(1) + \frac{m^2}{4} h_0(1) \ .$$

The solution to the homogeneous part is obtained, after a first integration, and the use of the second boundary condition by solving

$$\frac{d^3 f_{2h}}{d\tilde{r}^3} - 2 \left[\tilde{r} \frac{df_{2h}}{d\tilde{r}} \right] = 0 \ .$$

Defining $df_{2h}/d\tilde{r} = f_{2h}^*$, we have

$$\frac{d^2 f_{2h}^*}{d\tilde{r}^2} - 2\tilde{r}\, f_{2h}^* = 0.$$

The change of variables $r^* = 2^{1/3}\, \tilde{r}$ transforms the last equation into

$$\frac{d^2 f_{2h}}{dr^{*2}} - r^* f^*_{2h} = 0,$$

which general solution is expressed in terms of the Airy functions as

$$f^*_{2h} = D_1 \, Ai(r^*) + D_2 \, Bi(r^*).$$

The asymptotic expansions of the two Airy Integrals are given in the literature and show that $Bi(r^*)$ is inadequate for matching because of its exponential growth: we are thus left with

$$f^*_{2h} = D_1 \, Ai(r^*)$$

or

$$f^*_{2h} = D_1 \, Ai(2^{1/3} \tilde{r}).$$

The general solution for f_2 is:

$$f_2 = D_0 \tilde{r}^2 + D_1 \int_0^{\tilde{r}} Ai(2^{1/3}\xi) d\xi + D_3 \, .$$

The determination of D_1 is done by looking at the first boundary condition

$$\frac{d^2 f_2}{d\tilde{r}^2} = \nu \frac{m^2 - k^2 m^2 - \lambda_0^2}{2} \, h_0(1) + m^2 \nu \, h_0(1) \quad \text{when } \tilde{r} = 0$$

or

$$2D_0 + D_1 2^{1/3} \, Ai'(o) = \nu \frac{m^2 - k^2 m^2 - \lambda_0^2}{2} \, h_0(1) + m^2 \nu \, h_0(1).$$

From this last expression, D_1 is determined. The determination of D_3 is then obtained through the matching process by ensuring that the quantity

$$D_1 \int_0^{\tilde{r}} Ai(2^{1/3}\xi) d\xi + D_3$$

represents a transcendentally small term away from the boundary. The introduction of the expression found for $f_2(\tilde{r})$ into the limit process, considered in the matching, leaves us with the requirement:

$$\frac{1}{2}\frac{d^2}{dr^2}h_0(1) - D_0 = 0$$

where

$$D_0 = \frac{1}{4}(m^2-k^2m^2 - \lambda_0^2)h_0(1) + \frac{1}{16}(m^2-k^2m^2 - \lambda_0^2)^2 h_0(1) + \frac{m^2}{4}h_0(1).$$

The differential equation defining h_0, Equation (3.13), expressed in the r' variable becomes:

$$(2r'-r'^2)\frac{d^2h_0}{dr^2} + \frac{1}{1-r'}(-2+6r'-3r'^2)\frac{dh_0}{dr} - \frac{1}{(1-r')^2}[m^2-m^2k^2 - \lambda_0^2$$

$$+ 2(m^2k^2 + \lambda_0^2)r' - (m^2k^2 + \lambda_0^2)r'^2]h_0 = 0. \qquad (3.19)$$

Expanding d^2h_0/dr^2, dh_0/dr and h_0 in Taylor series in the neighborhood of $r' = 0$, we obtain the following identities:

$$-2\frac{dh_0}{dr}(1) - (m^2-m^2k^2 - \lambda_0^2)h_0(1) = 0$$

$$4\frac{d^2h_0}{dr^2}(1) + (m^2-m^2k^2 - \lambda_0^2 + 4)\frac{dh_0}{dr}(1) - 2m^2 h_0(1) = 0.$$

This last expression is also:

$$\frac{d^2h_0}{dr^2}(1) = \frac{m^2-m^2k^2 - \lambda_0^2}{2}h_0(1) + \frac{(m^2-m^2k^2 - \lambda_0^2)^2}{8}\frac{dh_0}{dr}(1)$$

$$+ \frac{m^2}{2}h_0(1)$$

or $d^2h_0/dr^2\,(1) = 2\,D_0$ which represents the predicted identity.

Having previously concluded the orthogonality relationship between the modes, we have to complete the specification of the mode by writing down their normality or

$$\int_0^1 \int_0^{2\Pi} r\ \phi^2_{\alpha,m} \cos^2 m\theta\ dr d\theta = 1 \quad .$$

A direct integration with respect to θ gives us

$$\int_0^1 r\ \phi^2_{\alpha,m}\ dr = \frac{1}{\Pi} \quad .$$

This relation represents the relation needed for the determination of a_0. Introducing the asymptotic expansion of $\phi_{\alpha,m}$, the previous equality becomes indeed

$$\int_0^1 r\ h_0^2\ dr = \frac{1}{\Pi}$$

or by writing the development of $h_0(r)$ under the form

$$h_0(r) = a_0 \sum_{n=0}^{\infty} a'_n\ r^{n+m} = a_0\ h'_0(r),$$

we obtain finally

$$a_0^2 = \frac{1}{\Pi \int_0^1 r\ h_0'^2(r) dr} \quad .$$

With the determination of $h_0(r)$ completed, we are now able to proceed and determine the value of λ_1^2. The orthogonality relationship between the different terms of the asymptotic expansion of the modes and the considerations introduced previously suggest for $\nu_1(\varepsilon)$ the

order of magnitude of ε. A suitable choice for $\kappa_1(\varepsilon)$ is also given by $\kappa_1(\varepsilon) = \varepsilon$.

The differential equation defining h_1 is then given by:

$$\frac{1}{r}\frac{d}{dr}\left[r(1-r^2)\frac{dh_1}{dr}\right] - \frac{m^2}{r^2}[1-k^2r^2]h_1 + \lambda_0^2 h_1 = \nabla_m^4 h_0 - \lambda_1^2 h_0 . \qquad (3.20)$$

Multiplying this equation by $r\, h_0(r)$ and integrating from 0 to 1, we obtain the following expression:

$$\int_0^1 h_0(r)\frac{d}{dr}\left[r(1-r^2)\frac{dh_1}{dr}\right]dr - \int_0^1 \frac{m^2}{r}(1-k^2r^2)h_1h_0\, dr + \lambda_0^2\int_0^1 r\, h_0h_1 dr$$
$$= \int_0^1 r\, \nabla_m^4\, h_0h_0\, dr - \frac{\lambda_1^2}{\Pi}$$

where the normality relationship of h_0 has been used. Integrating by parts the first term, we have:

$$h_0(r)r(1-r^2)\frac{dh_1}{dr}\Bigg|_0^1 - \int_0^1 r(1-r^2)\frac{dh_1}{dr}\frac{dh_0}{dr}dr - \int_0^1 \frac{m^2}{r}(1-k^2r^2)h_1h_0\, dr$$
$$+ \lambda_0^2\int_0^1 r\, h_0h_1 dr = \int_0^1 r\, \nabla_m^4\, h_0h_0\, dr - \frac{\lambda_1^2}{\Pi} .$$

Integrating by parts once more, and taking into account the differential equation defining h_0, Equation (3.13), we have:

$$-r(1-r^2)\frac{dh_0}{dr}h_1\Bigg|_0^1 + \int_0^1 h_1\left[\frac{m^2}{r}(1-k^2r^2)h_0 - r\,\lambda_0^2 h_0\right]dr$$
$$- \int_0^1 \frac{m^2}{r}(1-k^2r^2)h_1h_0 dr + \lambda_0^2\int_0^1 r\, h_0h_1 dr$$
$$= \int_0^1 r\, \nabla_m^4\, h_0h_0 dr - \frac{\lambda_1^2}{\Pi}$$

or

$$\lambda_1^2 = \Pi \int_0^1 r \, \nabla_m^4 \, h_0 h_0 \, dr \quad .$$

The previous determination of a_0^2 and this last relationship completely define the value of λ_1^2. In this problem, the flexural rigidity of the disk modifies the value of the eigenfrequencies only through an order of magnitude ε. In a procedure similar to the one used for the rotating cantilevered beam problem, the determination of λ_1^2 has been accomplished without the need of the orthogonality relationship.

Here again, the general expression valid for h_1 is given by an expression of the form

$$h_1 = \bar{h}_1 + k_1 h_0$$

where $\bar{h}_1$ is a particular solution which displacement at r=0 is zero and h_0 is the general solution of the homogeneous part. The determination of the constant k_1 is accomplished by considering the orthonormality condition. However, we have to be careful when doing this step. We already found that the boundary layer expansion contains a transcendentally small term in f_2, which order of magnitude is given by $\varepsilon^{2/3}$. The boundary layer being of thickness $\varepsilon^{1/3}$, we might conclude that the contribution of the TST in the integral appearing in the normality relationship gives us a contribution of order of magnitude ε. As a result, the uniformly valid expansion has to be considered in the normality relationship.

The uniformly valid expansion for $\phi_{\alpha,m}$ is given by

$$\phi_{\alpha,m} = h_0(r) + \varepsilon\, h_1(r) + \ldots + \varepsilon^{2/3} D_1 \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)\, d\xi + D_3' \right] + \ldots$$

where D_3 has been written $D_3 = D_1 D_3'$

$$\int_0^{\tilde{r}} Ai(2^{1/3}\xi)\, d\xi + D_3'$$

represents a transcendentally small quantity away from the boundary. The normality condition gives us then:

$$\int_0^1 r\, \phi_{\alpha,m}^2\, dr = \frac{1}{\Pi} = \int_0^1 r\, h_0^2\, dr + 2\varepsilon \int_0^1 r\, h_0 h_1 dr + \ldots$$

$$+ 2\varepsilon^{2/3} \int_0^1 r\, h_0\, D_1 \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)\, d\xi + D_3' \right] dr + \ldots$$

Let us concentrate on the last term:

$$D_1 \int_0^1 r\, h_0 \left[\int_0^{\tilde{r}} Ai(2^{1/2}\xi)\, d\xi + D_3' \right] dr \ .$$

The mixed appearance of r and $\tilde{r}$ has been kept on purpose as the rest of the development will show. We replace r by 1-r' in the previous expression or

$$D_1 \int_0^1 (1-r') h_0(1-r') \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)\, d\xi + D_3' \right] dr' \ .$$

The expression $\int_0^{\tilde{r}} Ai(2^{1/3}\xi)\, d\xi + D_3'$ represents a transcendentally small quantity away from the boundary, when considered in the r' variable and this remark suggests an expansion of $h_0(1-r')$ in Taylor series is the neighborhood of r' = 0.

$$D_1 \int_0^1 h_0(1) \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] dr' - D_1 \int_0^1 r' h_0(1) \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi \right.$$

$$\left. + D_3' \right] dr' - D_1 \int_0^1 \frac{d}{dr} h_0(1) r' \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] dr' + \ldots$$

Replacing now r' by $\varepsilon^{1/3}\,\tilde{r}$ and dr' by $\varepsilon^{1/3}d\tilde{r}$, the previous expression becomes

$$\varepsilon^{1/3} D_1 \left[h_0(1) \int_0^{1/\varepsilon^{1/3}} \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] d\tilde{r} \right.$$

$$-\varepsilon^{1/3} h_0(1) \int_0^{1/\varepsilon^{1/3}} \tilde{r} \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] d\tilde{r}$$

$$\left. -\varepsilon^{1/3} \frac{dh_0}{dr}(1) \int_0^1 \tilde{r} \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] d\tilde{r} + \ldots \right] .$$

The limit process applied to the normality conditions gives us then

$$\int_0^1 r\, h_0 h_1 dr + D_1\, h_0(1) \int_0^{\infty} \left[\int_0^{\tilde{r}} Ai(2^{1/3}\xi)d\xi + D_3' \right] d\tilde{r} = 0 .$$

Defining the integral of the transcendentally small quantity appearing in the last bracket as being D_4, we have

$$\int_0^1 r\, h_0 h_1\, dr + D_1 D_4\, h_0(1) = 0 \quad .$$

This last expression can now be used to determine the value of k_1.

$$\int_0^1 r\, h_0 \bar{h}_1 + k_1 \int_0^1 r\, h_0^2\, dr + D_1 D_4\, h_0(1) = 0$$

or

$$k_1 = -\Pi\left[\int_0^1 r\ h_0\bar{h}_1 + D_1D_4\ h_0(1)\right].$$

With the determination of k_1, the expression for h_1 is completely specified and we will then conclude here the study of the modal analysis of the last structure. Several properties that we mentioned for the previous problem can also be observed in this case, but for reason of shortening will not be developed here.

We conclude this chapter by recording briefly some results in the determination of the eigenfrequencies of the free rotating, circular membrane. The expansion found for the eigenfrequencies has the form

$$\omega^2_{\alpha,m} = \frac{3+\nu}{8}\,\Omega^2\ \lambda^2_{\alpha,m} = \frac{3+\nu}{8}\,\Omega^2(\lambda_0^2 + \lambda_1^2\ \varepsilon + \ldots)$$

where

$$\varepsilon = \frac{2\ E\ h^2}{3(1-\nu^2)(3+\nu)a^4\mu\ \Omega^2}\ .$$

The value of λ_0^2 is given by the sequence defined by

$$(n+m)(n+m-2) - k^2m^2 = \lambda_0^2$$

where

$$k^2 = \frac{1+3\nu}{3+\nu}\ ,$$

and

$$\lambda_1^2 = \Pi\int_0^1 r\ \nabla_m^4\ h_0h_0\ dr.$$

The obtention of h_0 results from

$$h_0(r) = \sum_{n=0}^{\infty} a_n r^{n+m}$$

where the truncated series is defined recursively by

$$\frac{a_n}{a_{n-2}} = \frac{(n+m)(n+m-2)-(k^2m^2 + \lambda_0^2)}{n^2 + 2mn} .$$

CHAPTER 4

LIAPUNOV STABILITY ANALYSIS

4.1 Introduction

Due to the particular orientation of this chapter, we should, at the outset, investigate the concept of stability, and state precisely what we mean by a stable motion for our problem. We will then in a first part review briefly the different ideas pertaining to the stability and cover to some extent a peculiarity appearing in the class of problem under investigation. In this first part, the concept of stability will be developed and the definition of nominal motion will become more specific. In the second part, the Liapunov stability analysis will be studied for flexible space craft when continuous models are used, and finally analytic criteria for the class of problems studied in chapters 2 and 3 will be obtained.

4.2 Concept of Stability

We might introduce this section on concept of stability by defining formally the notion of Liapunov stability.

If, for a given dynamical system, the equation of state

$$\mathring{X} = F(X, t)$$

admits the solution $X = 0$, and if for any $\varepsilon > 0$, there exists some $\delta > 0$ such that if the norm $|X(0)| \leq \delta$ then $|X(t)| < \varepsilon$ for all $t \geq 0$, then the motion characterized by $X = 0$ is said to be Liapunov stable; and if in addition

$$\lim_{t \to \infty} |X(t)| = 0$$

then the solution X = 0 is asymptotically stable. If X = 0 is not Liapunov stable, we will call it unstable.

One remark should be made about this definition: only local stability properties are examined. In order to explore the kind of instabilities occurring in the system, nonlinear simulations become necessary. The reason is that the characterization of stability is based on local properties of the motion, permitting no formal conclusions beyond the neighborhood of the nominal motion.

If we, now, consider the consequences of a small perturbation applied to a rigid body, initially rotating about a principal axis (major or minor) colinear with its inertial angular momentum vector, the subsequent motion is characterized by a rotation of its spin axis about the inertial direction defined by the new angular momentum vector — the sum of the initial angular momentum vector and its perturbation. The space craft is said to "wobble" and the angle between the spin axis and the new angular momentum vector is called the nutation angle or "wobble angle". Because the wobble angle resulting from a sufficiently small perturbation remains within any preassigned value, the motion is said to be Liapunov stable.

On the other hand, heuristic energy sink methods — based on the assumed presence of a hypothetical non-moveable energy dissipator, and applied to quasi rigid bodies — indicate a decrease in the wobble angle if the spin axis is the axis of maximum moment of inertia; we would like to define our state variables in such a way that we can call this motion asymptotically stable. Conversely, energy sink analysis shows that when the spin axis is the axis of minimum moment of inertia, the wobble angle

grows and the motion becomes unstable. These last observations are generally referred as "the greatest moment of inertia rule". The fact that in the perturbed motion of a stable quasi rigid body, the spin axis converges to the new angular momentum vector brings out an interesting point: the system is not "completely" damped when the motion is described in terms of coordinates measuring the deviation from the inertial orientation existing prior to perturbation. We will focus our attention on this last observation, by referring to a development by Barbera and Likins.[13]

In the application of asymptotic stability to our particular case, special attention is required in the choice of elements in the state variable X. If the state variables include, for instance, the deviation of the component of the angular velocity along the body spin axis, then the null solution will not be asymptotically stable for any vehicle configuration, since after perturbation, the vehicle will not return asymptotically to its nominal spin rate. For the same reason, we cannot include in X any deformations which vanish only in the unperturbed state, but we must define deformations relative to whatever deformed state corresponds to steady rotation about the body spin axis at the rate appropriate for the actual angular momentum, whether before or after perturbation. Similarly, we cannot use in the state variables any set of inertial attitude angles which are zero only when the body spin axis is colinear with the angular momentum vector prior to perturbation; the perturbed angular momentum will have in general a different orientation, and we wish to characterize the motion as asymptotically stable if the angular velocity vector of the central body approaches alignment

with the body spin axis and with the perturbed angular momentum vector. We must therefore choose attitude angles for the central body which vanish whenever the body spin axis is aligned with the actual angular momentum vector, whether before or after perturbation.

Our concept of stability is now well defined and the constraints on our definition of the state variable made more precise. We are in a position where two theorems, usually referred to as being elements of Liapunov's direct or second method, can be mentioned.

<u>Theorem 1</u>: The null solution $X(t) = 0$ of the differential equation $\overset{\circ}{X} = F(X)$ is asymptotically stable if there exists a function $L(X)$ in a region around the origin both positive definite and strictly decreasing for all solutions in that region except for $X \equiv 0$.

<u>Theorem 2</u>: The null solution $X(t) = 0$ of the differential equation $\overset{\circ}{X} = F(X)$ is unstable if there exists a function $L(X)$ in a region around the origin both negative semi-definite (or sign variable) and strictly decreasing for all solutions in that region except for $X \equiv 0$.

Although these theorems constitute usefuls tools for the determination of the necessary and sufficient conditions for asymptotic stability, they do not present procedures for the generation of a testing function. However, in a wide class of applications, the Hamiltonian is a convenient choice. More specifically, for "completely damped" systems, the Hamiltonian is a suitable testing function when the total energy of the system is free of explicit time dependence. For our purpose, the concept of "complete damping" implies a dissipation of energy for any possible motion, in the neighborhood of the origin of the coordinate space adopted, except for the nominal motion. However, as

mentioned in the preceding section, the damping of a freely spinning body with internal energy dissipation is not complete in terms of attitude angles measured with respect to the spin prior to perturbation, since the vehicle cannot return to its original state. As a result, for such systems, the Hamiltonian is not strictly decreasing in the neighborhood of the null solution and asymptotic stability cannot be proclaimed. In 1969, Pringle[14] provided a method to circumvent this problem. The procedure consists to constrain the attitude angles through the angular momentum integral so that they measure the deviation of the body spin axis from the instantaneous angular momentum vector, before or after perturbation from its nominal inertial orientation. The attitude angles are thus defined in such a way that complete damping is assured and asymptotic stability can be predicted. The requirement of complete damping also implies a judicious choice of deformation variables: they must vanish whenever the vehicle adopts a steady state configuration and rotates about the body spin axis (at whatever rate is appropriate with the actual angular momentum).

4.3 Stability Analysis

We specified in the previous section our objective and the path to follow in our stability study of flexible structures. In the following derivation, the analysis is kept as general as possible. We will restrict the class of problems covered by the study when needed and finally, specialize in a class of systems, including the flexible structures studied before, when analytic criteria will be developed.

We assume, at the outset, that the freely spinning flexible spacecraft under investigation is composed of two parts: a rigid part

and a flexible part. Later on, the flexible part will be assumed to lie in a plane perpendicular to the spin axis and passing through the center of mass of the whole structure when at steady-state deformation.

The kinetic energy of the system, denoted by T, is derived from the general expression:

$$2T = \int_{A,B} \dot{\underline{p}} \cdot \dot{\underline{p}} \, dm$$

where $\underline{p}$ is a inertial generic position vector and the capital letters A and B indicate that the integration is carried over the flexible appendage A and the rigid body B.

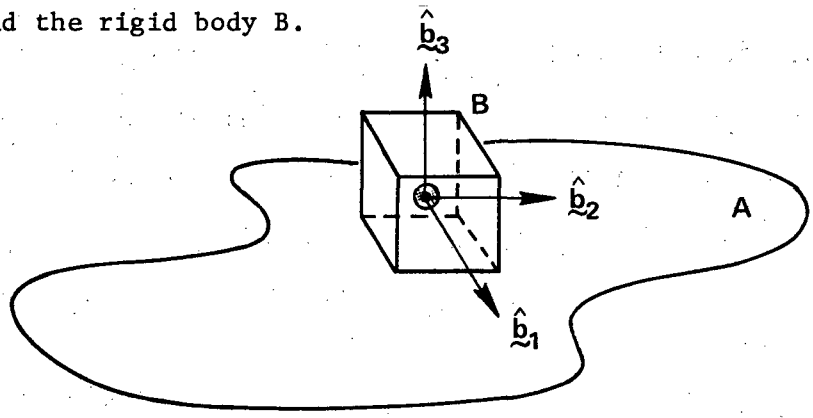

Figure 12. Freely Spinning Spacecraft.

In our force-free case, the system mass center is inertially fixed, and $\underline{p}$ can be written as the sum $\underline{c} + \underline{\rho}$ where $\underline{c}$ is the position vector directed from the system center of mass CM to a point N fixed in B, and $\underline{\rho}$ is a generic position vector directed from N. The point N is chosen to coincide with the CM when the structure is steadily spinning at the rate consistent for the actual angular momentum and hence elastically deformed by forces induced by spin. Thus 2T becomes

$$2T = \int_{A,B} (\dot{\underline{c}} + \dot{\underline{\rho}}) \cdot (\dot{\underline{c}} + \dot{\underline{\rho}})\ dm$$

$$= \dot{\underline{c}} \cdot \int_{A,B} (\dot{\underline{c}} + \dot{\underline{\rho}})\ dm + \int_{A,B} \dot{\underline{\rho}} \cdot (\dot{\underline{c}} + \dot{\underline{\rho}})\ dm$$

The first term vanishes by definition of mass center, i.e.,

$$\int_{A,B} (\underline{c} + \underline{\rho})\ dm = \underline{0} \quad \text{for every } t,$$

so

$$2T = \int_{A,B} \dot{\underline{\rho}} \cdot (\dot{\underline{c}} + \dot{\underline{\rho}})\ dm = \dot{\underline{c}} \int_{A,B} \dot{\underline{\rho}}\ dm + \int_{A,B} (\dot{\underline{\rho}} \cdot \dot{\underline{\rho}})\ dm\ .$$

By definition of the center of mass, we also have:

$$\underline{c}\ \mathcal{M} + \int_{A,B} \underline{\rho}\ dm = \underline{0} \text{ for every } t,$$

where $\mathcal{M}$ represents the total mass of the system.

The kinetic energy takes, then, the form:

$$2T = -\ (\dot{\underline{c}} \cdot \dot{\underline{c}})\ \mathcal{M} + \int_A (\dot{\underline{\rho}} \cdot \dot{\underline{\rho}})\ dm + \int_B (\dot{\underline{\rho}} \cdot \dot{\underline{\rho}})\ dm$$

or

$$2T = -\ (\dot{\underline{c}} \cdot \dot{\underline{c}})\ \mathcal{M} + \underline{\omega} \cdot \underline{I_B^N} \cdot \underline{\omega} + \int_A (\dot{\underline{\rho}} \cdot \dot{\underline{\rho}})\ dm \tag{4.1}$$

where $\underline{\omega}$ is the inertial angular velocity vector of vector basis $\{\underline{b}\}$ fixed in B and I_B^N is the inertia dyadic of body B about point N.

Expressing the right hand side of equation (1) in matrix notation with respect to the vector basis $\{\underline{b}\}$ fixed in body B, we obtain for the first term

$$\mathcal{M}(\dot{\underline{c}} \cdot \dot{\underline{c}}) = \mathcal{M}\,[\overset{\circ}{\underline{c}} + (\underline{\omega} \times \underline{c}) \cdot \overset{\circ}{\underline{c}} + (\underline{\omega} \times \underline{c})]$$

$$= \mathcal{M}\,[\overset{\circ}{\underline{c}} \cdot \overset{\circ}{\underline{c}} + 2\,\overset{\circ}{\underline{c}} \cdot (\underline{\omega} \times \underline{c}) + (\underline{\omega} \times \underline{c}) \cdot (\underline{\omega} \times \underline{c})]$$

$$= \mathcal{M}\,[\dot{c}^T \cdot \dot{c} - 2\dot{c}^T\,\tilde{c}\,\omega + \omega^T\,\tilde{c}^T\,\tilde{c}\omega]$$

where $(\circ)$ represents time differentiation with respect to the vector basis $\{\underline{b}\}$ fixed in B and the tilde $(\sim)$ operation is defined as

$$\tilde{c} \triangleq \begin{pmatrix} 0 & -c_z & c_y \\ c_z & 0 & -c_x \\ -c_y & c_x & 0 \end{pmatrix}$$

The last two definitions can be extended to any vector used in the sequel.

The second term becomes simply

$$\underline{\omega} \cdot \underline{I_B^N} \cdot \underline{\omega} = \omega^T\, I_B^N\, \omega,$$

where I_B^N is the inertia matrix in vector basis $\{\underline{b}\}$, corresponding to the dyadic $\underline{I_B^N}$.

To study the third term we express $\underline{\rho}$ as the summation of $\underline{\Gamma} + \underline{u}$, where $\underline{\Gamma}$ is the position vector from N fixed in B, to the element of mass dm in the "steady state" position. The "deformation" $\underline{u}$ measures the displacement relative to B of the element of mass dm from the location that it would occupy if the vehicle was rotating about the body spin axis at a steady state configuration at the rate appropriate for the angular momentum. With these definitions, we have successively:

$$\int_A \dot{\underline{\rho}} \cdot \dot{\underline{\rho}} \, dm = \int_A \dot{\underline{\Gamma}} + \dot{\underline{u}} \cdot \dot{\underline{\Gamma}} + \dot{\underline{u}} \, dm$$

$$= \int_A [\mathring{\underline{u}} + \underline{\omega} \times (\underline{\Gamma} + \underline{u}) \cdot \mathring{\underline{u}} + \underline{\omega} \times (\underline{\Gamma} + \underline{u})] \, dm$$

$$= \int_A (\mathring{\underline{u}} \cdot \mathring{\underline{u}}) \, dm + 2 \int_A \mathring{\underline{u}} \cdot [\underline{\omega} \times (\underline{\Gamma} + \underline{u})] \, dm$$

$$+ \int_A [\underline{\omega} \times (\underline{\Gamma} + \underline{u}) \cdot \underline{\omega} \times (\underline{\Gamma} + \underline{u})] \, dm$$

$$= \int_A \dot{u}^T \dot{u} \, dm + 2 \int_A \dot{u}^T \tilde{\omega} (\Gamma + u) dm + \int_A [(\widetilde{\Gamma + u}) \omega]^T (\widetilde{\Gamma + u}) \omega \, dm$$

$$= \int_A \dot{u}^T \dot{u} \, dm - 2 \int_A \dot{u}^T (\widetilde{\Gamma + u}) dm + \omega^T \left[\int_A (\widetilde{\Gamma + u})^T (\widetilde{\Gamma + u}) dm \right] \omega$$

The Hamiltonian can be written as

$$H = \frac{1}{2} \omega^T I_0^N \omega + \frac{1}{2} \int_A \dot{u}^T \dot{u} \, dm + \frac{1}{2} \omega^T \left[\int_A \tilde{\Gamma}^T \tilde{u} \, dm + \tilde{u}^T \tilde{\Gamma} \, dm \right] \omega$$

$$+ \frac{1}{2} \omega^T \left[\int_A \tilde{u}^T \tilde{u} \, dm \right] \omega - \left[\int_A \dot{u}^T (\tilde{\Gamma} + \tilde{u}) \, dm \right] \omega$$

$$- \frac{\mathcal{M}}{2} [\dot{c}^T \dot{c} - 2 \dot{c}^T \tilde{c} \, \omega + \omega^T \tilde{c}^T \tilde{c} \, \omega] + V + C_0 \tag{4.2}$$

where the system potential energy has been defined as V plus a constant.

We introduce the definition

$$I_0^N \triangleq I_B^N + \int_A \tilde{\Gamma}^T \tilde{\Gamma} \, dm$$

where the term $\int_A \tilde{\Gamma}^T \tilde{\Gamma} \, dm$ represents the inertia of the undeformed appendage. The expression I_0^N defines precisely the inertia of the complete system about point N when undeformed. If the vector basis $\{\underline{b}\}$ are assumed parallel to the principal axis, I_0^N is diagonal, i.e.

$$I_0^N = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

We are now in a position where we can compute the angular momentum integral. By definition, we have:

$$\begin{aligned}
\underline{h} &= \int_{A,B} (\underline{c} + \underline{\rho}) \times (\dot{\underline{c}} + \dot{\underline{\rho}})\, dm \\
&= \int_{A,B} (\underline{c} + \underline{\rho}) \times (\overset{\circ}{\underline{c}} + \overset{\circ}{\underline{\rho}})\, dm + \int_{A,B} (\underline{c} + \underline{\rho}) \times [\underline{\omega} \times (\underline{c} + \underline{\rho})]\, dm \\
&= \underline{c} \times \int_{A,B} (\overset{\circ}{\underline{c}} + \overset{\circ}{\underline{\rho}})\, dm - \overset{\circ}{\underline{c}} \times \int_{A,B} \underline{\rho}\, dm + \int_{A,B} \underline{\rho} \times \overset{\circ}{\underline{\rho}}\, dm \\
&+ \underline{c} \times \left[\underline{\omega} \times \int_{A,B} (\underline{c} + \underline{\rho})\, dm \right] + \int_{A,B} \underline{\rho} \times (\underline{\omega} \times \underline{\rho})\, dm \\
&- (\underline{\omega} \times \underline{c}) \times \int_{A,B} \underline{\rho}\, dm \,.
\end{aligned}$$

The first and fourth terms vanish by mass center definition. Defining the inertia dyadic of the complete system about point N, $\underline{I}^N$, we have

$$\underline{h} = \underline{I}^N \cdot \underline{\omega} + \mathcal{M}\, \dot{\underline{c}} \times \underline{c} + \int_A \underline{\rho} \times \overset{\circ}{\underline{\rho}}\, dm$$

The last two terms may be written in the vector basis $\{\underline{b}\}$ as

$$\mathcal{M}\, \dot{\underline{c}} \times \underline{c} = \mathcal{M}\, [\overset{\circ}{\underline{c}} + (\underline{\omega} \times \underline{c}) \times \underline{c}] = \{\underline{b}\}^T\, \mathcal{M}\, (\tilde{\dot{c}}\, c + \tilde{c}\, \tilde{c}\, \omega)$$

$$\int_A \underline{\rho} \times \overset{\circ}{\underline{\rho}}\, dm = \int_A (\underline{\Gamma} + \underline{u}) \times \overset{\circ}{\underline{u}}\, dm = \{\underline{b}\}^T \left[\int_A (\tilde{\Gamma}\, \dot{u} + \tilde{u}\, \dot{u})\, dm \right]$$

which allows the matrix representation of $\underline{h}$ in $\{\underline{b}\}$;

$${}^b h = I^N\, \omega + \int_A (\tilde{\Gamma} + \tilde{u})\, \dot{u}\, dm + \mathcal{M}(\tilde{\dot{c}}\, c + \tilde{c}\, \tilde{c}\, \omega).$$

In order to constrain the Hamiltonian through the angular momentum integral, we now have to solve the angular velocity components in terms of the attitude angles.

Let us define an inertial vector basis $\{\underline{n}\}$ and its transformation with respect to $\{\underline{b}\}$ as Θ, i.e.

$$\{\underline{n}\} = \Theta \{\underline{b}\}$$

such that $\hat{\underline{n}}_3$ is colinear with the angular momentum vector $\underline{h}$ after perturbation from its nominal inertial orientation. As the system is assumed torque free $\underline{h}$ must remain colinear with $\hat{\underline{n}}_3$; thus, subsequent to perturbation the following must prevail:

$$\underline{h} \cdot \hat{\underline{n}}_1 = 0$$

$$\underline{h} \cdot \hat{\underline{n}}_2 = 0$$

$$\underline{h} \cdot \hat{\underline{n}}_3 = h$$

where h is the magnitude of the perturbed angular momentum vector. With $\underline{h}$ written in vector basis $\{\underline{b}\}$ as prescribed before, the above equations become:

$$\Theta \left[I^N \omega + \int_A (\tilde{\Gamma} + u) \dot{u} \, dm + \mathcal{M} (\tilde{\dot{c}} \, c + \tilde{c} \, \tilde{c} \, \omega) \right] = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

or

$$\omega = \left[I^N + \mathcal{M} \, \tilde{c} \, \tilde{c} \right]^{-1} \left\{ \Theta^T \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \int_A (\tilde{\Gamma} + \tilde{u}) \dot{u} \, dm - \mathcal{M} \, \tilde{\dot{c}} \, c \right\}.$$

Let us, now, determine the transformation Θ. We have

$$\{\underline{b}\} = \Theta^T \{\underline{n}\}$$

We will use in the three rotations needed to define completely Θ, the sequence of rotations 1 - 2 - 3. As a result the matrix Θ^T may be written:[21]

$$\Theta^T = E - \tilde{\theta} + \frac{1}{2}\tilde{\theta}\,\tilde{\theta} + \frac{1}{2}\overline{\theta}\,\overline{U}\,\overline{\theta}$$

where E is the 3×3 identity matrix

$\tilde{\theta}$ is a skew-symmetric matrix defined as

$$\tilde{\theta} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}$$

$\overline{\theta}$ is the diagonal matrix defined as

$$\overline{\theta} = \begin{bmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{bmatrix}$$

$\overline{U}$ is the skew-symmetric matrix defined as

$$\overline{U} = \begin{bmatrix} 0 & +1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

With the matrix Θ^T, now well defined, the approximate expression for $\omega = (\omega_x, \omega_y, \omega_z)^T$ may be written as

$$\omega \approx \left[I^N + \mathcal{M}\,\tilde{c}\,\tilde{c}\right]^{-1} \left\{ \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \tilde{\theta}\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \int_A \tilde{\Gamma}\,\dot{u}\,dm \right.$$

$$\left. + \frac{1}{2}\left[\tilde{\theta}\,\tilde{\theta} + \overline{\theta}\,\overline{U}\,\overline{\theta}\right]\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \int_A \tilde{u}\,\dot{u}\,dm - \mathcal{M}\,\dot{\tilde{c}}\,c \right\} \qquad (4.3)$$

Terms of order higher than second have been disregarded since the stability analysis is only valid locally and the sign character of the Liapunov function can be determined in the neighborhood of the origin

from its quadratic approximation, as long as all coordinates are included.

The solution of (3) for the angular velocity components is then substituted into the Hamiltonian and limited to terms of order no higher than second.

The inertia matrix is defined for the flexible appendage with the $\{\underline{b}\}$ vector basis as

$$I_A^N = \int_A (\rho^T \rho \; E - \rho \; \rho^T) \; dm$$

where E is the 3×3 unit matrix. Replacing ρ by its value given in matrix form as

$$\rho = \Gamma + u$$

we have

$$I_A^N = \int_A (\Gamma^T + u^T) \; (\Gamma + u) \; E - (\Gamma + u) \; (\Gamma^T + u^T) \; dm$$

$$= \int_A (\Gamma^T \; \Gamma E - \Gamma\Gamma^T) \; dm + \int_A (u^T \Gamma E - u\Gamma^T) \; dm$$

$$+ \int_A (\Gamma^T \; u \; E - \Gamma \; u^T) \; dm + \int_A (u^T \; u \; E - u \; u^T) \; dm$$

The inertia matrix I^N of the complete matrix, computed with respect to the vector basis $\{\underline{b}\}$, consists of terms independent of deformation as well as terms both linear and second order in the deformations.

$$I^N = I_B^N + \int_A (\Gamma^T \Gamma E - \Gamma\Gamma^T) \; dm + \int_A (u^T \Gamma E - u\Gamma^T) \; dm + \int_A (\Gamma^T u E - \Gamma u^T) dm$$

$$+ \int_A (u^T \; u \; E - u \; u^T) \; dm$$

The sum of the first two terms has been defined earlier as I_0^N, prescribed to be principal and having as diagonal elements A,B and C. We also define

$$\int_A (u^T \Gamma E - u \Gamma^T)\, dm + \int_A (\Gamma^T u E - \Gamma u^T)\, dm \triangleq 2\Delta$$

$$\int_A (u^T u E - u u^T)\, dm \triangleq U$$

where we notice that Δ represents a symmetric matrix, so that

$$I^N = I_0^N + 2\,\Delta + U.$$

The definition of the center of mass or

$$\mathcal{M}\,\underline{c} + \int_{A,B} \underline{\rho}\, dm = 0$$

may be written as

$$\mathcal{M}\,\underline{c} + \int_A \underline{\rho}\, dm + \int_B \underline{\rho}\, dm = 0$$

$$\mathcal{M}\,\underline{c} + \int_A \underline{\Gamma}\, dm + \int_A \underline{u}\, dm + \int_B \underline{\rho}\, dm = 0.$$

By definition, the point $\underline{N}$ has been chosen as the center of mass of the complete system when the structure is undeformed. This allows us to consider

$$\int_A \underline{\Gamma}\, dm + \int_B \underline{\rho}\, dm = 0.$$

The vector $\underline{c}$ is then expressed with respect to the deformation $\underline{u}$ as

$$\underline{c} = -\frac{1}{\mathcal{M}} \int_A \underline{u}\, dm.$$

so that

$$\tilde{c} = -\frac{1}{\mathcal{M}} \int_A \tilde{u}\, dm \ .$$

The matrix $[I^N + \mathcal{M}\, \tilde{c}\, \tilde{c}]$ expands to:

$$I_0^N + 2\Delta + U + \frac{1}{\mathcal{M}}\left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right].$$

The inversion of the previous matrix becomes easy if we consider the fact that the matrix I_0^N is diagonal, and if we recognize the fact that only terms up to the second order are needed for the purpose of our derivation. We obtain the following expressions:

$$I^N + \mathcal{M}\, \tilde{c}\tilde{c} = I_0^N \left[E + 2I_0^{N^{-1}} \Delta + I_0^{N^{-1}} U + \frac{1}{\mathcal{M}} I_0^{N^{-1}}\left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right]\right]$$

so that

$$\left[I^N + \mathcal{M}\, \tilde{c}\tilde{c}\right]^{-1} =$$

$$\left[E + 2I_0^{N^{-1}} \Delta + I_0^{N^{-1}} U + \frac{1}{\mathcal{M}} I_0^{N^{-1}}\left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right]\right]^{-1} I_0^{N^{-1}} \ .$$

Taking now, into account the fact that the matrix Δ contains only first order terms in the assumed small variables, and similarly, the matrix U contains only second order terms in the same variables, we have:

$$\left[I^N + \mathcal{M}\, \tilde{c}\tilde{c}\right]^{-1} \approx I_0^{N-1} - 2\, I_0^{N-1}\, \Delta I_0^{N-1} - I_0^{N-1}\, U I_0^{N-1}$$

$$- \frac{1}{\mathcal{M}}\, I_0^{N-1}\left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right] I_0^{N-1} + 4\, I_0^{N-1} \Delta I_0^{N-1} \Delta I_0^{N-1}.$$

Introducing this last expansion into the expression previously found for the components of the angular velocity vector, and performing the indicated multiplication, we obtain as final expression, where only terms up to the second order have been kept

$$\omega \approx \left[I_0^{N^{-1}} - 2I_0^{N-1}\,\Delta I_0^{N^{-1}} - I_0^{N^{-1}}\,U I_0^{N^{-1}} - \frac{1}{\mathcal{M}}\, I_0^{N^{-1}} \left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right] I_0^{N^{-1}} \right.$$

$$\left. + 4\, I_0^{N^{-1}}\,\Delta I_0^{N^{-1}}\,\Delta I_0^{N^{-1}} \right] \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \left[I_0^{N^{-1}} - 2I_0^{N^{-1}}\,\Delta I_0^{N^{-1}} \right] \tilde{\theta} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

$$- \left[I_0^{N^{-1}} - 2I_0^{N^{-1}}\,\Delta I_0^{N^{-1}} \right] \int_A \tilde{\Gamma}\, \dot{u}\, dm + \frac{1}{2}\, I_0^{N^{-1}}\, [\tilde{\theta}\tilde{\theta} + \bar{\theta}\,\bar{U}\,\bar{\theta}] \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

$$- I_0^{N^{-1}} \int_A \tilde{u}\, \dot{u}\, dm - I_0^{N^{-1}}\, \mathcal{M}\, \tilde{\dot{c}}\, c.$$

When the expressions derived before for the angular velocity components are substituted into the Hamiltonian (4.2), a series of simplifications and combination of terms reduces the previous expression to:

$$H_c = \frac{1}{2}\,(00h) \left\{ I_0^{N^{-1}} - 2\, I_0^{N^{-1}}\,\Delta\, I_0^{N^{-1}} - 2\, I_0^{N^{-1}}\, \tilde{\theta} + 4\, I_0^{N^{-1}}\,\Delta\, I_0^{N^{-1}}\,\Delta\, I_0^{N^{-1}} \right.$$

$$- I_0^{N^{-1}}\, U\, I_0^{N^{-1}} - \frac{1}{\mathcal{M}}\, I_0^{N^{-1}} \left[\int_A \tilde{u}\, dm\right]\left[\int_A \tilde{u}\, dm\right] I_0^{N^{-1}} + 4\, I_0^{N^{-1}}\,\Delta\, I_0^{N^{-1}}\, \tilde{\theta}$$

$$\left. + I_0^{N^{-1}}\, [\tilde{\theta}\tilde{\theta} + \bar{\theta}\bar{U}\bar{\theta}] - \tilde{\theta}\, I_0^{N^{-1}}\, \tilde{\theta} \right\} \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}$$

$$+ \frac{1}{2} \left[\int_A \dot{u}^T\, \tilde{\Gamma}\, dm\right] I_0^{N^{-1}} \left[\int_A \tilde{\Gamma}\, \dot{u}\, dm\right] + \frac{1}{2} \int_A \dot{u}^T \dot{u}\, dm$$

$$- \frac{1}{2\mathcal{M}} \left[\int_A \dot{u}^T dm\right]\left[\int_A \dot{u}\, dm\right] + V + C_0 \qquad (4.4)$$

where H_c stands for the constrained Hamiltonian.

In this last expression, the potential energy must include the contribution of the steady state deflection of the flexible appendages induced by spin at a rate concordant with the angular momentum. In the following text, this spin rate will be identified as being ω_{SS}. This last observation expresses the fact that the expansion of the potential energy of deformation, in terms of the deformation variables, is not

accomplished with respect to the minimum energy of deformation state. As a result, the expansion consists not only of terms quadratic in the deformations, but also of terms linear in the deformation variables, plus a constant term. We limit the approximation of the potential energy of deformation to those terms because they represent a good indicator for the local behavior of the system.

$$U_d \approx L_0 + \int_A L_1^T \cdot u \, dm + \frac{1}{2} \int_A u^T K \, u \, dm$$

where U_d represents the energy of deformation, and L_1 and K are made of functions of position.

If we, now, consider our definition of u, the deformation with respect to the steady-state deformation induced by spin, we have to conclude that the equilibrium position is defined when the deformation variables are zero. But for a conservative system, the equilibrium position is obtained when the dynamic potential, composed of the energies of position and deformation and also of T_0– the part of the kinetic energy not a function of the time derivatives of the state variables of the system — is minimum with respect to the state variables, when evaluated at equilibrium. Only the deformation energy and the kinetic energy are functions of the deformation variables, and equilibrium has been defined when u and θ are zero and when the spin rate is given by ω_{SS}. The kinetic energy has been determined earlier and the minimization of $T_0 - U_d$ with respect to the deformation variables, evaluated at equilibrium, is equivalent to the requirement that the linear terms in the deformation variables u appearing in $T_0 - U_d$ accommodate themselves when the spin rate is ω_{SS} or

$$\frac{1}{2} \omega_{SS}{}^T \left[\int_A \tilde{\Gamma}^T \tilde{u} \, dm + \tilde{u}^T \tilde{\Gamma} \, dm \right] \omega_{SS} - \int_A L_1^T \cdot u \, dm = 0$$

The potential energy of deformation may then be written, after the observation that the terms in brackets are nothing else than 2Δ, in the following way

$$U_d \approx L_0 + \omega_{SS}{}^T \Delta \omega_{SS} + \frac{1}{2}\int_A u^T K u \, dm .$$

If we recognize the fact that ω_{SS} is $I_0^{N^{-1}}\begin{pmatrix}0\\0\\h\end{pmatrix}$ and notice that

$$(0\ 0\ h)\ I_0^{N^{-1}}\ \tilde{\theta}\begin{pmatrix}0\\0\\h\end{pmatrix} = 0 ,$$

we obtain the following expression for H_c:

$$H_c = \frac{1}{2}(0\ 0\ h)\left\{4I_0^{N^{-1}}\ \Delta I_0^{N^{-1}}\ \Delta I_0^{N^{-1}} - I_0^{N^{-1}}\ UI_0^{N^{-1}} - \frac{1}{\mathcal{M}}\ I_0^{N^{-1}}\left[\int_A \tilde{u}\ dm\right]\right.$$

$$\left. -\left[\int_A \tilde{u}\ dm\right] I_0^{N^{-1}} + 4I_0^{N^{-1}}\ \Delta I_0^{N^{-1}}\tilde{\theta} + I_0^{N^{-1}}\left[\tilde{\theta}\tilde{\theta} + \bar{\theta}\ \bar{U}\ \bar{\theta}\right] - \tilde{\theta}\ I_0^{N^{-1}}\tilde{\theta}\right\}\begin{pmatrix}0\\0\\h\end{pmatrix}$$

$$+ \frac{1}{2}\left[\int_A \dot{u}^T\tilde{\Gamma}\ dm\right] I_0^{N^{-1}}\left[\int_A \tilde{\Gamma}\ \dot{u}\ dm\right] + \frac{1}{2}\int_A \dot{u}^T\ \dot{u}\ dm - \frac{1}{2\mathcal{M}}\left[\int_A \dot{u}^T\ dm\right]\left[\int_A \dot{u}\ dm\right]$$

$$+ \frac{1}{2}\int_A u^T K\ u\ dm . \qquad (4.5)$$

This last expression contains only terms second order in the deformations and attitude angles. In the expansion of equation (4), several constant terms appear in addition to the arbitrary constant C_o previously identified. Since C_o is arbitrary, it has been chosen in such a way that the summation of the constant contributions is zero, allowing the Hamiltonian to vanish at the origin — a necessary requirement for a Liapunov function.

As H_c in the presence of complete damping is strictly decreasing, then by Theorem I the nominal motion is asymptotically stable for H_c

positive definite, and by Theorem II this motion is unstable for H_c either negative semi-definite or sign variable. We assume that energy is dissipated within the flexible appendage whenever time-varying deformations occur, as in the case for any physical object. We have carefully chosen our variables so that energy is dissipated for any admissible solution of the equations of the motion in the neighborhood of the nominal motion, except for the nominal motion itself, so we are assured of complete damping.

Since for asymptotic stability the complete function must be positive definite, then it is clear that the following must be satisfied for stability

$$\frac{1}{2} h^2 \theta_2^2 \frac{C-A}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{C-B}{BC} > 0 .$$

This result provides the familiar necessary stability criterion predicted by energy sink methods for spinning bodies having an internal energy dissipator, i, e.

$$C > A \quad \text{and} \quad C > B.$$

Thus, by inspection of the Hamiltonian we can formally conclude that, in the presence of damping, the spin axis must be the axis of maximum moment of inertia. However, our objective is to extract additional stability criteria, and this requires the determination of conditions for positive definite H_c.

Now, that the constrained Hamiltonian has been found in a completely general way, we are forced to consider more specific problems if we want to come up with analytic criteria. In order to introduce those particular case, some useful relationships have been derived in Appendix V. To obtain literal closed-form stability criteria we shall

restrict our flexible appendage model to lie in a plane containing the center of mass and normal to the spin axis.

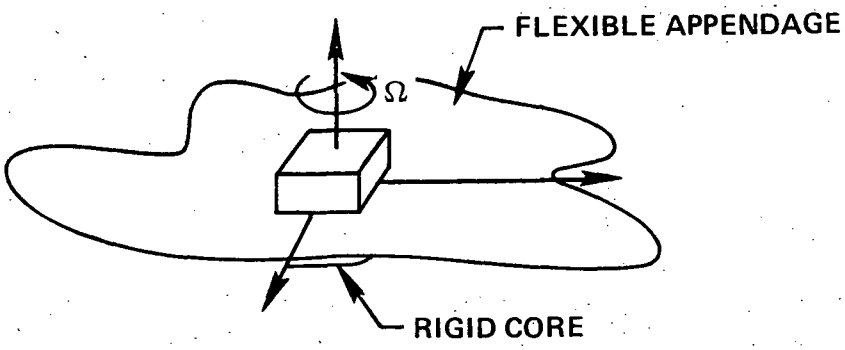

Figure 13. Restricted Model.

Following the definition of our particular case, we have to consider

$$\Gamma_3 = 0 \ .$$

Under this last assumption, the three scalar equations expressing the rotational motion of the rigid core are given by considering equation (1) in Appendix V for this particular case or:

$$0 = A\dot{\omega}_1 - \omega_{SS}\omega_2 \ (B-C) - 2 \ \Delta_{23} \ \omega_{SS}^2 + 2\dot{\Delta}_{13} \ \omega_{SS} \int_A \Gamma_1 \dot{u}_3 \ dm + \int_A \Gamma_2 \ddot{u}_3 \ dm$$

$$0 = B\dot{\omega}_2 + \omega_{SS} \ \omega_1 \ (A-C) + 2\Delta_{13} \ \omega_{SS}^2 + 2 \ \dot{\Delta}_{23} \ \omega_{SS} + \omega_{SS} \int_A \Gamma_2 \dot{u}_3 dm - \int_A \Gamma_1 \ddot{u}_3 \ dm$$

$$0 = C\dot{\omega}_3 + 2\dot{\Delta}_{23} \ \omega_{SS} - \int_A \Gamma_2 \ddot{u}_1 \ dm + \int_A \Gamma_1 \ddot{u}_2 \ dm$$

where we used the fact that the matrix I_0^N has been chosen to be diagonal and where the angular velocity vector at the "steady-state" deformation has only one non-zero component, ω_{SS}, along the spin axis.

Under the assumption of a flexible appendage lying in a plane orthogonal to the spin axis and passing through the center of mass, the kinetic energy may be written as:

$$T = T_1 + T_2,$$

where

$$T_1 = \frac{1}{2} A\omega_1^2 + \frac{1}{2} B\omega_2^2 + \frac{1}{2} \int_A \dot{u}_3^2 \, dm - \omega_{SS} \, \omega_1 \int_A \Gamma_1 \, u_3 \, dm - \omega_{SS} \, \omega_2 \int_A \Gamma_2 \, u_3 \, dm$$

$$+ \omega_1 \int_A \Gamma_2 \dot{u}_3 \, dm - \omega_2 \int_A \Gamma_1 \dot{u}_3 \, dm - \frac{1}{2\mathcal{M}} \left[\int_A \dot{u}_3 \, dm \right]^2$$

$$T_2 = \frac{1}{2} C\omega_{SS}^2 + C\omega_3 \, \omega_{SS} + \frac{1}{2} C\omega_3^2 + \frac{1}{2} \int_A \dot{u}_1^2 \, dm + \frac{1}{2} \int_A \dot{u}_2^2 \, dm$$

$$+ \frac{1}{2} \omega_{SS}^2 \int_A (u_1^2 + u_2^2) \, dm + \omega_{SS}^2 \int_A (\Gamma_2 \, u_2 + \Gamma_1 \, u_1) \, dm$$

$$+ 2\omega_3 \, \omega_{SS} \int_A (\Gamma_2 \, u_2 + \Gamma_1 \, u_1) dm - (\omega_3 + \omega_{SS}) \int_A \Gamma_2 \dot{u}_1 \, dm +$$

$$+ (\omega_3 + \omega_{SS}) \int_A \Gamma_1 \dot{u}_2 \, dm - \omega_{SS} \int_A (\dot{u}_1 \, u_2 - \dot{u}_2 \, u_1) \, dm - \frac{1}{2\mathcal{M}} \left\{ \left[\int_A \dot{u}_1 \, dm \right]^2 + \right.$$

$$\left. + \left[\int_A \dot{u}_2 \, dm \right]^2 \right\} + \frac{1}{\mathcal{M}} \left\{ \left[\int_A \dot{u}_1 \, dm \right] \left[\int_A u_2 \, dm \right] \omega_{SS} - \left[\int_A \dot{u}_2 \, dm \right] \left[\int_A u_1 \, dm \right] \omega_{SS} \right\}$$

$$- \frac{1}{2\mathcal{M}} \omega_{SS}^2 \left\{ \left[\int_A u_2 \, dm \right]^2 + \left[\int_A u_1 \, dm \right]^2 \right\}$$

If we, now, notice the fact that

$$2 \, \Delta_{13} = - \int_A \Gamma_1 \, u_3 \, dm$$

$$2 \, \Delta_{23} = - \int_A \Gamma_2 \, u_3 \, dm$$

$$2 \, \Delta_{33} = 2 \int_A (\Gamma_1 \, u_1 + \Gamma_2 \, u_2) \, dm,$$

it becomes obvious by looking at the previous set of equations that the equations describing the rotational motion of the central body separates. The equations expressing the change in the angular velocity components ω_1, ω_2 are coupled with the transverse vibration of the flexible appendage and form what is called the "wobbling" motion. Also the third component ω_3 is coupled only to the components u_1 and u_2 of the deformation and constitutes the "spinning" motion.

If the same separation is being observed in the deformation equations, we could conclude that the general motion is divided into two different types of motion. One, where we observe a coupling between transverse vibration and the first two components of the angular velocity, and the other where we find a coupling between the in-plane vibration and the angular velocity component along the spin axis. But we just observed that for our particular case (Γ_3=0) the kinetic energy can be expressed as the summation of T_1 and T_2 where in T_1 appear only the variables of the wobbling motion and in T_2 appear only the variables of the in-plane motion.

As a result the separation suggested before is effectively accomplished for our particular case if the potential energy of deformation is such that U_d can be written as the summation of U_1 and U_2 where the aforementioned separation prevails.

We will now assume that the in-plane and out-of-plane stiffness elements are uncoupled which is the same as expressing that the matrix K of the energy of deformation is partitioned such that

$$K = \begin{bmatrix} K_{11} & K_{12} & 0 \\ K_{21} & K_{22} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

Under this last assumption, the motion separates and the Hamiltonian itself can be written under the form:

$$H_c = H_1 + H_2$$

where H_1 contains the variables appearing in the wobbling motion and H_2 those constituting the spinning motion.

$$H_1 = \frac{1}{2}\frac{h^2}{C^2}\left\{\frac{1}{A}\left[\int_A \Gamma_1\, u_3\, dm\right]^2 + \frac{1}{B}\left[\int_A \Gamma_2\, u_3\, dm\right]^2\right\} - \frac{\theta_2 h^2}{AC}\int_A \Gamma_1\, u_3\, dm$$

$$+ \frac{h^2\theta_1}{BC}\int_A \Gamma_2\, u_3\, dm - \frac{1}{2}\theta_1^2\, h^2\left(\frac{B-C}{BC}\right) - \frac{1}{2}\theta_2^2\, h^2\left(\frac{A-C}{AC}\right)$$

$$- \frac{1}{2A}\left[\int_A \Gamma_2\, \dot{u}_3\, dm\right]^2 - \frac{1}{2B}\left[\int_A \Gamma_1\, \dot{u}_3\, dm\right]^2 + \frac{1}{2}\int_A \dot{u}_3^2\, dm$$

$$- \frac{1}{2\mathcal{M}}\left[\int_A \dot{u}_3\, dm\right]^2 + \frac{1}{2}\int_A K_{33}\, u_3^2\, dm\ ,$$

$$H_2 = \frac{2h^2}{C^2}\left\{\frac{1}{}\left[\int_A (u_1\, \Gamma_1 + u_2\, \Gamma_2)\, dm\right]^2\right\} - \frac{h^2}{2C^2}\int_A (u_1^2 + u_2^2)\, dm$$

$$+ \frac{1}{2\mathcal{M}}\frac{h^2}{C^2}\left\{\left[\int_A u_2\, dm\right]^2 + \left[\int_A u_1\, dm\right]^2\right\} - \frac{1}{2C}\left[\int (\Gamma_1\, \dot{u}_2 - \Gamma_2\, \dot{u}_1)\, dm\right]^2$$

$$+ \frac{1}{2}\int_A (\dot{u}_1^2 + \dot{u}_2^2)\, dm - \frac{1}{2\mathcal{M}}\left\{\left[\int_A \dot{u}_1\, dm\right]^2 + \left[\int_A \dot{u}_2\, dm\right]^2\right\}$$

$$+ \frac{1}{2}\int_A (u_1^2\, K_{11} + 2u_1\, u_2\, K_{12} + u_2^2\, K_{22})\, dm$$

We have proven that $H = H_1 + H_2$ where the variables appearing in H_1 are uncoupled to those appearing in H_2 through the equations of the motion. We conclude that if both H_1 and H_2 are positive definite, we are assured of asymptotic stability, and if either H_1 or H_2 is sign variable or negative definite, the nominal motion is unstable.

Experience suggests that stability criteria emanating from H_2 are of little practical importance; the variables in H_2 describe deformation in the plane of the appendage, which is normal to the spin axis, and generally those criteria express simply that the structure has to be sufficiently stiff to avoid destruction by "centrifugal forces." We will thus consider the case where the stiffness elements orthogonal to the spin axis are assumed infinitely large ($u_1 = u_2 = 0$) so that the structure is allowed to vibrate only in the u_3 direction. The cited assumptions allow stability criteria extracted from H_1 to establish characteristics of the entire system. To formulate the restricted Hamiltonian in a more useful form, we have to introduce some new notions.

The class of problems covered by all the previous assumptions includes as particular cases all the problems and particular structures we consider in chapters 2 and 3. We now consider that the deformations are expressed in terms of modes of the structures; this last consideration will enable us to be more specific about the stiffness K_{33} .

The deformation will be expressed in linear approximation as a linear combination of modes

$$u_3 = \sum_{\nu=1}^{N} \beta_\nu \phi_\nu .$$

We will become more explicit later, concerning the exact definition and choice of modes. Let us just remark that we introduce in the last definition N new independent generalized coordinates — the β_ν's. The expression ϕ_ν is related to the shape of the "displacement" in the mode ν and is thus defined as being a function of position only, the real

deformation for this mode being then equal to the product of ϕ_ν by the harmonic function of time β_ν.

Introducing the last definition into the energy of deformation, we obtain:

$$U_d \approx L_0 + \omega_{SS}^2 \Delta_{33} + \frac{1}{2} \int_A \sum_\nu \beta_\nu \phi_\nu K_{33} \sum_\mu \beta_\mu \phi_\mu \, dm \ .$$

For our particular case, $\Delta_{33} = 0$, so the previous expression may be written:

$$U_d \approx L_0 + \frac{1}{2} \beta^T \overline{K} \beta$$

where we define β as being the $N \times 1$ vector

$$\beta \triangleq \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_\nu \\ \vdots \\ \beta_N \end{bmatrix}$$

and $\overline{K}$ as being the symmetric $N \times N$ matrix defined by

$$\overline{K} \triangleq [k_{\nu\mu}] \triangleq \int_A \phi_\nu K_{33} \phi_u \, dm \ .$$

Expanding the deformation u_3 in terms of the previous N modes, the equations of deformation β_ν are readily obtained in their Lagrangian form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\beta}_\nu}\right) - \frac{\partial T}{\partial \beta_\nu} + \frac{\partial U_d}{\partial \beta_\nu} = 0$$

where T and U_d are the kinetic and potential energies.

For this particular case ($\Gamma_3 = 0$, $u_1 = 0$, $u_2 = 0$) the kinetic energy is reduced to the following combination:

$$T = \frac{1}{2} A\omega_1^2 + \frac{1}{2} B\omega_2^2 + \frac{1}{2} \int_A \dot{u}_3^2 \, dm - \omega_{SS} \, \omega_1 \int_A \Gamma_1 \, u_3 \, dm - \omega_{SS} \, \omega_2 \int_A \Gamma_2 \, u_3 \, dm$$

$$+ \omega_1 \int_A \Gamma_2 \, \dot{u}_3 \, dm - \omega_2 \int_A \Gamma_1 \, \dot{u}_3 \, dm - \frac{1}{2\mathcal{M}} \left[\int_A \dot{u}_3 \, dm \right]^2 + \frac{1}{2} C\omega_{SS}$$

$$+ C\omega_3 \, \omega_{SS} + \frac{1}{2} C\omega_3^2 \, .$$

Expressed in terms of the variables β_ν, we have,

$$u_3 = \beta^T \phi$$

where we define the $N \times 1$ vector ϕ as

$$\phi \triangleq \left[\phi_1 \cdots \phi_\nu \cdots \phi_N \right]^T .$$

We also have

$$\int_A \dot{u}_3^2 \, dm = \int_A \dot{\beta}^T \phi\phi^T \dot{\beta} \, dm = \dot{\beta}^T \left[\int_A \phi\phi^T \, dm \right] \dot{\beta} \triangleq \dot{\beta}^T M \dot{\beta}$$

$$\int_A \Gamma_1 \, u_3 \, dm = \int_A \Gamma_1 \, \beta^T \phi \, dm = \beta^T \int_A \Gamma_1 \, \phi \, dm \triangleq \beta^T \Lambda_1$$

$$\int_A \Gamma_2 \, u_3 \, dm = \int_A \Gamma_2 \, \beta^T \phi \, dm = \beta^T \int_A \Gamma_2 \, \phi \, dm \triangleq \beta^T \Lambda_2$$

$$\int_A \Gamma_2 \, \dot{u}_3 \, dm = \dot{\beta}^T \Lambda_2$$

$$\int_A \Gamma_1 \, \dot{u}_3 \, dm = \dot{\beta}^T \Lambda_1$$

$$\left[\int_A \dot{u}_3 \, dm \right]^2 = \left[\dot{\beta}^T \int_A \phi \, dm \right]^2 = \dot{\beta}^T \left[\int_A \phi \, dm \right] \left[\int_A \phi^T \, dm \right] \dot{\beta} = \dot{\beta}^T m \, \dot{\beta}$$

where the matrices M and m are defined as being the following $N \times N$ symmetric matrices

$$\int_A \phi\phi^T \, dm \triangleq M$$

$$\left[\int_A \phi \, dm\right]\left[\int_A \phi^T \, dm\right] \triangleq m$$

and the N × 1 vectors Λ_1 and Λ_2 by

$$\int_A \Gamma_1 \, \phi \, dm \triangleq \Lambda_1$$

$$\int_A \Gamma_2 \, \phi \, dm \triangleq \Lambda_2 \, .$$

The kinetic energy then becomes:

$$T = \frac{1}{2} A\omega_1^2 + \frac{1}{2} B\omega_2^2 + \frac{1}{2} \dot{\beta}^T M\dot{\beta} - \omega_{SS} \, \omega_1 \, \beta^T \Lambda_1 - \omega_{SS} \, \omega_2 \, \beta^T \Lambda_2$$
$$+ \omega_1 \, \dot{\beta}^T \Lambda_2 - \omega_2 \, \dot{\beta}^T \Lambda_1 - \frac{1}{2\mathcal{M}} \dot{\beta}^T m \, \dot{\beta} + \frac{1}{2} C\omega_{SS}^2 + C\omega_3 \, \omega_{SS} + \frac{1}{2} C\omega_3^2 \, .$$

The deformation equations can, now, be written in a completely explicit form:

$$M \, \ddot{\beta} - \frac{1}{\mathcal{M}} m \, \ddot{\beta} + \overline{K} \, \beta = \dot{\omega}_2 \, \Lambda_1 - \dot{\omega}_1 \, \Lambda_2 - \dot{\omega}_1 \, \omega_{SS} \, \Lambda_1 - \omega_2 \, \omega_{SS} \, \Lambda_2$$

If we add to this set of equations the rotational equations of the central rigid body under the assumptions $\Gamma_3 = 0$, $u_1 = 0$, $u_2 = 0$ we get

$$0 = A\dot{\omega}_1 - \omega_2 \, \omega_{SS} \, (B-C) + \omega_{SS}^2 \int_A \Gamma_2 \, u_3 \, dm + \int_A \Gamma_2 \, \ddot{u}_3 \, dm$$

$$0 = B\dot{\omega}_2 + \omega_1 \, \omega_{SS} \, (A-C) - \omega_{SS}^2 \int_A \Gamma_1 \, u_3 \, dm - \int_A \Gamma_1 \, \ddot{u}_3 \, dm$$

$$0 = C\dot{\omega}_3$$

or, introducing some of the previous definitions:

$$0 = A\dot{\omega}_1 - \omega_2 \, \omega_{SS} \, (B-C) + \omega_{SS}^2 \, \beta^T \Lambda_2 + \ddot{\beta}^T \Lambda_2$$

$$0 = B\dot{\omega}_2 + \omega_1 \, \omega_{SS} \, (A-C) - \omega_{SS}^2 \, \beta^T \Lambda_1 - \ddot{\beta}^T \Lambda_1$$

In order to study the stability of the previous system, we now need to define how the modes have to be computed. In order to do so, we have to keep in mind that we are interested in obtaining a system which is suitable for a mathematical analysis. Similarly, for the common procedure used in the "hybrid-coordinates" approach of flexible space craft and developed to some extent by P.W. Likins,[22], we have to use some practical simplifications in order to obtain the solution.

Even though, in the equations of the deformation, the forcing terms, linear in the ω's, are in fact coupled to the flexible appendage deformation variables, β_ν, through the central body motion, we define our eigenmodes as being obtained only by the consideration of the homogeneous part of the equations in β's. Unfortunately, the modes are still coupled, through the motion of the center of mass, motion resulting from the appendage deformation. The center of mass motion is imbedded in the matrix m of those equations. In order to decouple the equations, we must add the assumption that the eigenmodes are defined when the system's center of mass is assumed fixed, but the structure deformed by the deformations induced by the steady-state spin.

This last assumption could appear too restrictive but the coupling between the rotational equations and the deformation equations is obtained through the vectors Λ_1 and Λ_2. For a symmetric structure, with homogeneous physical characteristics, only antisymmetric modes introduce nonzero elements in the vectors Λ_1 and Λ_2. So, we remark that the coupling between the two sets of equations is obtained by means of antisymmetric modes and those, precisely, do not affect the motion of the center of mass.

We could also, as several authors did, consider that the rotational motion of the central body of a symmetric vehicle is only affected by the anti-symmetric modes, and from there consider in the deformation variables, only those corresponding to anti-symmetric modes. With this approach, the term $\frac{1}{\mathcal{M}}$ m $\ddot{\beta}$ would not be present in our equation.

In what follows, we focus attention on symmetric vehicles, in order to take advantage of the possibility of separating eigenmodes into symmetric and anti-symmetric classes.

With the modes completely specified, we know that if we let the structure vibrate in one of its modes, the frequency of the vibration would be given by ω_ν^2, the eigenfrequency corresponding to the aforementioned mode, and we will anticipate a periodic motion (no damping has been introduced when deriving the eigenmodes and eigenfrequencies.) For an orthonormal system of modes, the matrix M becomes the unity matrix E and we thus obtained as equations for the deformation variables

$$E\,\ddot{\beta}^* + \overline{K}\,\beta^* = 0$$

where the variables β_ν^* have been substituted for β_ν in order to differentiate them from the equations obtained with the forcing terms.

This last equation may also be written with the substitution

$$\overline{K} = \Omega^2 = \begin{bmatrix} \omega_1^2 & & & & 0 \\ & \ddots & & & \\ & & \omega_\nu^2 & & \\ & & & \ddots & \\ 0 & & & & \omega_N^2 \end{bmatrix}$$

where the $\overline{K}$ matrix becomes a diagonal matrix the elements of which are composed of the squares of the modal frequencies. Those frequencies are

the loaded natural frequencies of the appendage accounting for the fact that the stiffness elements are modified by spin. That is, the tension in the structure induced by spin alters the appendage stiffness characteristics.

With all the previous assumptions and new definitions, we are in a position where stability criteria could be extracted from H_1 and where the stability characteristics of the entire system could be established. Introducing the following definitions

$$\left[\int_A \Gamma_1 \, u_3 \, dm\right]^2 = \left[\beta^T \Lambda_1\right]^2 \triangleq \beta^T \Pi_1 \beta$$

$$\left[\int_A \Gamma_2 \, u_3 \, dm\right]^2 = \left[\beta^T \Lambda_2\right]^2 \triangleq \beta^T \Pi_2 \beta$$

where the matrices Π_1 and Π_2 are the $N \times N$ symmetric matrices satisfying

$$\Pi_1 \triangleq \Lambda_1 \Lambda_1^T$$

$$\Pi_2 \triangleq \Lambda_2 \Lambda_2^T$$

the general expression for the Hamiltonian H_1 becomes in a matrix notation

$$H_1 = \frac{1}{2} h^2 \theta_2^2 \frac{C-A}{AC} + \frac{1}{2} h^2 \theta_1^2 \frac{C-B}{BC} - \frac{h^2\theta_2}{AC} \beta^T\Lambda_1 + \frac{h^2\theta_1}{BC} \beta^T\Lambda_2$$

$$+ \frac{1}{2} \beta^T \left\{\Omega^2 + \frac{h^2}{AC^2} \Pi_1 + \frac{h^2}{BC^2} \Pi_2\right\} \beta$$

$$+ \frac{1}{2} \dot{\beta}^T \left\{E - \frac{1}{A} \Pi_2 - \frac{1}{B} \Pi_1 - \frac{1}{2\mathcal{M}} m\right\} \dot{\beta}$$

We can now write the stability criteria as being given by the positive definiteness of H_1 or

$$H_1 > 0$$

where it is to be understood that > 0 means positive for all values of θ_1, θ_2, β and $\dot{\beta}$ in the neighborhood of the origin $\theta_1 = \theta_2 = \beta_\nu = \dot{\beta}_\nu = 0$, except equal to zero at the origin itself.

If we, now, group the eigenmodes into two categories, the first one including symmetric modes and the second one regrouping the antisymmetric modes, it is straightforward to see that in the development of H_1, the symmetric modes contribute to H_1, only through

$$\frac{1}{2}\,\beta_S{}^T \left\{ \Omega_S{}^2 \right\} \beta_S + \frac{1}{2}\,\dot{\beta}_S{}^T \left\{ E_S - \frac{1}{2\mathcal{M}}\, m_S \right\} \dot{\beta}_S$$

where the subscript "S" has been introduced to distinguish this part from the rest of the development of H_1. This last observation is in fact a consequence of the remark made earlier: only anti-symmetric modes have nonzero contribution to the vectors Λ_1 and Λ_2. It is evident that the first term is always strictly greater than zero. In order to prove that the second term

$$\frac{1}{2}\,\dot{\beta}_S{}^T \left\{ E_S - \frac{1}{2\mathcal{M}}\, m_S \right\} \dot{\beta}_S$$

is always strictly greater than zero, let us consider the following expression

$$\int_A u_{3S}^2 \, dm - \frac{1}{2\mathcal{M}} \left[\int_A u_{3S} \, dm \right]^2$$

where u_{3S} stands for the symmetric part of the deformation u_3. Applying the Scharwz inequality.

$$|P(v,w)|^2 \leq P(v,v).P(w,w)$$

to the term $\left[\int_A u_{3S} \, dm \right]^2$, we obtain

$$\left[\int_A u_{3S}\ dm\right]^2 \leq \int_A u_{3S}^2\ dm\ .\ \int_A dm = m_A \int_A u_{3S}^2\ dm$$

where m_A stands for the mass of the flexible appendage only.

As a result, we have

$$\int_A u_{3S}^2\ dm - \frac{1}{2\mathcal{M}}\left[\int_A u_{3S}\ dm\right]^2 \geq \left(1 - \frac{m_A}{2\mathcal{M}}\right) \int_A u_{3S}^2\ dm$$

but

$$1 - \frac{m_A}{2\mathcal{M}} = \frac{2\mathcal{M} - m_A}{2\mathcal{M}} = \frac{m_A + 2m_B}{\mathcal{M}} > 0$$

where $\mathcal{M}$ has been written as the summation of m_A, the mass of the flexible appendage, plus m_B, the mass of the rigid core.

From the last observation, we conclude

$$\int_A u_{3S}^2\ dm - \frac{1}{2\mathcal{M}}\left[\int_A u_{3S}\ dm\right]^2 > 0$$

Expressing the deformation u_{3S} in terms of the symmetric mode, we have

$$u_{3S} = \sum_{\nu_S=0}^{N_S} \beta_{\nu S}\ \phi_{\nu S},$$

The last inequality becomes:

$$\beta_S^T \left\{ E_S - \frac{1}{2\mathcal{M}} m_S \right\} \beta_S > 0$$

We conclude from the last relationship that the matrix

$$E_S - \frac{1}{2\mathcal{M}} m_S$$

is positive definite and from these that

$$\frac{1}{2}\ \dot{\beta}_S^T \left\{ E_S - \frac{1}{2\mathcal{M}} m_S \right\} \dot{\beta}_S > 0$$

So we just proved that the variables of deformation corresponding to symmetric modes separate from the remaining terms and that their contribution to the Hamiltonian H_1 is always strictly greater than zero. The stability criterion becomes finally

$$H_1 = \frac{1}{2} h^2 \theta_2^2 \left(\frac{C-A}{AC}\right) + \frac{1}{2} h^2 \theta_1^2 \left(\frac{C-B}{BC}\right) - \frac{h^2\theta_2}{AC} \beta^T \Lambda_1 + \frac{h^2\theta_1}{BC} \beta^T \Lambda_2$$

$$+ \frac{1}{2} \beta^T \left\{ \Omega^2 + \frac{h^2}{AC^2} \Pi_1 + \frac{h^2}{BC^2} \Pi_2 \right\} \beta$$

$$+ \frac{1}{2} \dot{\beta}^T \left\{ E - \frac{1}{A} \Pi_2 - \frac{1}{B} \Pi_1 \right\} \dot{\beta} > 0$$

where only the contributions of anti-symmetric modes have to be taken into account. We observe that in the last inequality the matrix m is no more present.

In the last development of the Hamiltonian, the last term is uncoupled from the remaining terms. We conclude directly from here a requirement for the asymptotic stability of our system, i.e., the matrix

$$E - \frac{1}{A} \Pi_2 - \frac{1}{B} \Pi_1$$

has to be positive definite. The author believes that due to physical properties of realizable systems, the previous matrix is always positive definite. He was however unable to prove this last statement for a general case, but for a wide class of flexible spacecrafts, it is verified.

For the class of flexible spacecraft where the rigid core presents an inertia symmetric with respect to the spin axis, or where the principal axis of the rigid core coincide with those of the flexible appendages, the principal axis of the entire system are the same as the principal axis of the flexible appendage alone. If, together with this last

property, the flexible appendage presents a symmetry not only with respect to the center of mass but also with respect to the principal axis of inertia, then the previous statement is indeed true.

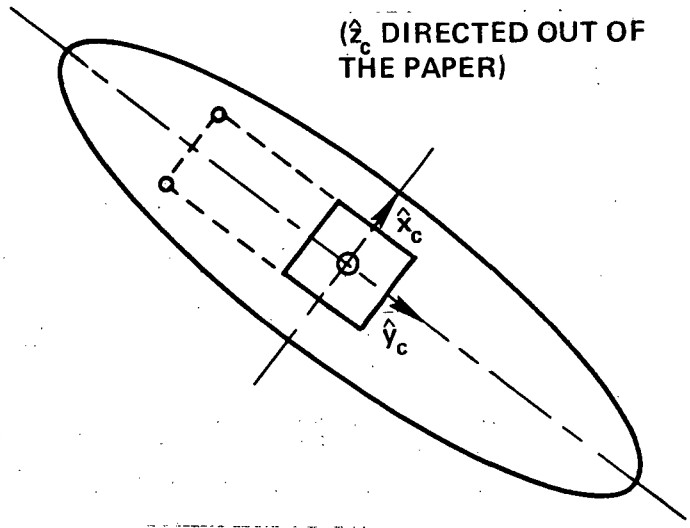

Figure 14. Particular Case.

Let us consider the following example where all the previous requirements are met. The figure represents the body when spinning at steady state. From previous consideration, we know that only anti-symmetric modes are of interest. We, now, consider in the flexible part two elements of mass dm of coordinates (Γ_1, Γ_2) and $(-\Gamma_1, \Gamma_2)$ when at steady-state deformation. Because of the assumed symmetry of the flexible appendage, such a pair always exists.

We now decompose the anti-symmetric part of the deformation — our main interest — into two parts:

$$u_{3A}(\Gamma_1, \Gamma_2) = u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2) + u_{3A}^{\Gamma_2}(\Gamma_1, \Gamma_2)$$

where the deformations $u_{3A}^{\Gamma_1}$ and $u_{3A}^{\Gamma_2}$ are defined in the following way

$$u_{3A}(\Gamma_1, \Gamma_2) + u_{3A}(-\Gamma_1, \Gamma_2) = 2u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2)$$

and $$u_{3A}(\Gamma_1, \Gamma_2) - u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2) = u_{3A}^{\Gamma_2}(\Gamma_1, \Gamma_2)$$

We, also, have then

$$u_{3A}(-\Gamma_1, \Gamma_2) = u_{3A}^{\Gamma_1}(-\Gamma_1, \Gamma_2) + u_{3A}^{\Gamma_2}(-\Gamma_1, \Gamma_2)$$

with $u_{3A}^{\Gamma_1}(-\Gamma_1, \Gamma_2) = u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2)$

and $u_{3A}^{\Gamma_2}(-\Gamma_1, \Gamma_2) = - u_{3A}^{\Gamma_2}(+\Gamma_1, \Gamma_2)$

We then have for the two elements of mass dm, of coordinate (Γ_1, Γ_2) and $(-\Gamma_1, \Gamma_2)$

$$\Gamma_1 u_{3A}(\Gamma_1, \Gamma_2)\, dm - \Gamma_1 u_{3A}(-\Gamma_1, \Gamma_2)\, dm =$$

$$\Big(\Gamma_1 u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2) + \Gamma_1 u_{3A}^{\Gamma_2}(\Gamma_1, \Gamma_2) - \Gamma_1 u_{3A}^{\Gamma_1}(-\Gamma_1, \Gamma_2)$$

$$- \Gamma_1 u_{3A}^{\Gamma_2}(-\Gamma_1, \Gamma_2)\Big)\, dm = 2\Gamma_1 u_{3A}^{\Gamma_2}(\Gamma_1, \Gamma_2)\, dm$$

and similarly

$$\Gamma_2 u_{3A}(\Gamma_1 \Gamma_2)\, dm + \Gamma_2 u_{3A}(-\Gamma_1, \Gamma_2)\, dm = 2\Gamma_2 u_{3A}^{\Gamma_1}(\Gamma_1, \Gamma_2)\, dm$$

From there, we may conclude

$$\int_A \Gamma_1 u_{3A}\, dm = \int_A \Gamma_1 u_{3A}^{\Gamma_2}\, dm$$

$$\int_A \Gamma_2 u_{3A}\, dm = \int_A \Gamma_2 u_{3A}^{\Gamma_1}\, dm$$

Let us now consider the expression:

$$\left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A u_{3A}^2 \, dm\right] - \left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2 \, u_{3A} \, dm\right]^2$$
$$- \left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A \Gamma_1 \, u_{3A} \, dm\right]^2 \tag{4.6}$$

We have

$$\int_A u_{3A}^2 \, dm = \int_A \left(\left(u_{3A}^{\Gamma_1}\right)^2 + \left(u_{3A}^{\Gamma_2}\right)^2 + 2 \, u_{3A}^{\Gamma_1} \, u_{3A}^{\Gamma_2}\right) dm$$

$$= \int_A \left(u_{3A}^{\Gamma_1}\right)^2 dm + \int_A \left(u_{3A}^{\Gamma_2}\right)^2 dm$$

where we observed $\int u_{3A}^{\Gamma_1} \, u_{3A}^{\Gamma_2} \, dm = 0$

Introducing the equality previously found, the expression (4.6) is also equal to

$$\left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A \left(u_{3A}^{\Gamma_1}\right)^2 dm + \int_A \left(u_{3A}^{\Gamma_2}\right)^2\right] dm$$
$$- \left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2 \, u_{3A}^{\Gamma_1} \, dm\right]^2 - \left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A \Gamma_1 \, u_{3A}^{\Gamma_2} \, dm\right]^2$$

Applying the Scharwz inequality, we have simultaneously

$$\left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A \left(u_{3A}^{\Gamma_1}\right)^2 dm\right] \geq \left[\int_A \Gamma_2 \, u_{3A}^{\Gamma_1} \, dm\right]^2$$
$$\left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \left(u_{3A}^{\Gamma_2}\right)^2 dm\right] \geq \left[\int \Gamma_1 \, u_{3A}^{\Gamma_2} \, dm\right]^2$$

Introducing those two inequalities into the development of the expression (4.6), we conclude that for the particular case into investigation, we have for every anti-symmetric deformation u_{3A}, the following inequality

$$\left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A u_{3A}^2 \, dm\right] - \left[\int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2 \, u_{3A} \, dm\right]^2$$

$$- \left[\int_A \Gamma_2^2 \, dm\right]\left[\int_A \Gamma_1 \, u_{3A} \, dm\right]^2 \geq 0 \, .$$

With this last inequality, we are now able to conclude that for every anti-symmetric deformation u_{3A}, the following inequality is true

$$AB \int_A u_{3A}^2 \, dm - B\left[\int_A \Gamma_2 \, u_{3A} \, dm\right]^2 - A\left[\int_A \Gamma_1 \, u_{3A} \, dm\right]^2 > 0 \qquad (4.7)$$

where A, and B are the inertia of the total system. By definition, we have

$$A = A^1 + \int_A \Gamma_2^2 \, dm$$

$$B = B^1 + \int_A \Gamma_1^2 \, dm$$

where A^1 and B^1 are the inertia of the rigid core.

Expanding the left hand side of the expression (4.7), we have

$$\left[B^1 + \int_A \Gamma_1^2 \, dm\right]\left[A^1 + \int_A \Gamma_2^2 \, dm\right]\int_A u_{3A}^2 \, dm - \left[B^1 + \int_A \Gamma_1^2 \, dm\right]\left[\int_A \Gamma_2 \, u_{3A} \, dm\right]^2$$

$$- \left[A^1 + \int_A \Gamma_2^2 \, dm\right]\left[\int_A \Gamma_1 \, u_{3A} \, dm\right]^2$$

The use of the Scharwz inequality and the consideration of the previous inequality enables us to conclude the inequality (4.7), after the observation that for any physical realisable system we have

$$A^1 \, B^1 \int_A u_{3A}^2 \, dm > 0$$

So we just proved that for the class of problems where the principal axis of the flexible appendages and of the entire system coincide, we have

$$AB \int_A u_{3A}^2 \, dm - B \left[\int_A \Gamma_2 \, u_{3A} \, dm \right]^2 - A \left[\int_A \Gamma_1 \, u_{3A} \, dm \right]^2 > 0$$

Expressing u_{3A} in terms of the anti-symmetric modes, we have

$$u_{3A} = \sum_1^{N_A} \beta_{\nu A} \, \phi_{\nu A}$$

or

$$AB \, \beta^T E\beta - B\beta^T \Pi_2 \beta - A\beta^T \Pi_1 \beta > 0$$

Dividing through by AB, we obtain finally:

$$\beta^T \left\{ E - \frac{1}{A} \Pi_2 - \frac{1}{B} \Pi_1 \right\} \beta > 0$$

We thus conclude that the matrix

$$E - \frac{1}{A} \Pi_2 - \frac{1}{B} \Pi_1$$

is positive definite.

The requirements of positive definiteness of H_1 are now reduced to the consideration of

$$H_1' = \frac{1}{2} h^2 \theta_2^2 \left(\frac{C-A}{CA} \right) + \frac{1}{2} h^2 \theta_1^2 \left(\frac{C-B}{BC} \right) - \frac{h^2 \theta_2}{AC} \beta^T \Lambda_1 + \frac{h^2 \theta_1}{BC} \beta^T \Lambda_2$$

$$+ \frac{1}{2} \beta^T \left\{ \Omega^2 + \frac{h^2}{AC^2} \Pi_1 + \frac{h^2}{BC^2} \Pi_2 \right\} \beta > 0$$

In matrix notation, we have

$$(\theta_1\ \theta_2\ \beta^T)\begin{pmatrix} \frac{h^2(C-B)}{2BC} & 0 & \frac{h^2}{2BC}\Lambda_2^T \\ 0 & \frac{h^2(C-A)}{2AC} & -\frac{h^2}{2AC}\Lambda_1^T \\ \frac{h^2}{2BC}\Lambda_2 - \frac{h^2}{2AC}\Lambda_1 & & \frac{1}{2}\left(\Omega^2 + \frac{h^2}{AC^2}\Pi_1 + \frac{h^2}{BC^2}\Pi_2\right) \end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \beta\end{pmatrix} > 0$$

The N deformation variables can now be truncated to a single mode, identified by index 1; thus the total number of coordinates is reduced to three. Accordingly, the N ×1 vectors Λ_1 and Λ_2 reduce to scalars. Similarly the N × N matrices Π_1 and Π_2 reduce to scalars. Implementing the above truncation allows the stability condition to be written as

$$(\theta_1\ \theta_2\ \beta_1)\begin{pmatrix} \frac{h^2}{2}\frac{(C-B)}{BC} & 0 & \frac{h^2}{2BC}\Lambda_2 \\ 0 & \frac{h^2}{2}\frac{(C-A)}{AC} & -\frac{h^2}{2AC}\Lambda_1 \\ \frac{h^2}{2BC}\Lambda_2 & -\frac{h^2}{2AC}\Lambda_1 & \frac{1}{2}\left(\omega_1^2 + \frac{h^2}{AC^2}\Lambda_1^2 + \frac{h^2}{BC^2}\Lambda_2^2\right) \end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\\ \beta_1\end{pmatrix} > 0$$

where $\Lambda_1 = \int_A \Gamma_1\ \phi_1\ dm$

$$\Lambda_2 = \int_A \Gamma_2\ \phi_1\ dm$$

$$\Omega^2 = \omega_1^2$$

$$\Pi_1 = \Lambda_1^2 = \left[\int_A \Gamma_1\ \phi_1\ dm\right]^2$$

$$\Pi_2 = \Lambda_2^2 = \left[\int_A \Gamma_2\ \phi_1\ dm\right]^2$$

The sign character of the above quadratic function is determined by testing the sign character of its corresponding symmetrix matrix. Sylvester's Theorem assures that the necessary and sufficient conditions for the previous matrix to be positive definite is that all principal diagonal minors be simultaneously positive. If this test fails, H_1' is not positive definite and is either negative semidefinite (or sign variable), implying instability, or positive semidefinite. If we exclude this latter limiting case (as for an axisymmetric vehicle, with C = B or C = A), necessary and sufficient conditions for asymptotic stability of the restricted planar appendage model are given by:

$$\frac{h^2}{2}\ \frac{(C-B)}{BC} > 0$$

$$\frac{h^2}{2}\ \frac{(C-A)}{AC} > 0$$

$$\frac{h^2}{2}\ \frac{(C-B)}{BC}\left\{\frac{h^2}{2}\ \frac{(C-A)}{AC}\left[\frac{\omega_1^2}{2} + \frac{h^2}{2AC^2}\ \Lambda_1^2 + \frac{h^2}{2BC^2}\ \Lambda_2^2\right] - \frac{h^4}{4}\ \frac{\Lambda_1^2}{A^2C^2}\right\}$$
$$- \frac{h^2}{2}\ \frac{\Lambda_2}{BC}\left[\frac{h^4}{4}\ \frac{\Lambda_2}{BC}\ \frac{(C-A)}{AC}\right] > 0 \ .$$

The combination of the first two conditions, as predicted by energy sink methods, requires that the spin axis be the axis of maximum moment of inertia, i.e.

$$C > A \quad \text{and} \quad C > B \ .$$

The requirement of the third condition emerges and can be written

$$\omega_1^2 > \frac{h^2\ \Lambda_1^2}{C^2(C-A)} + \frac{h^2\Lambda_2^2}{C^2(C-B)} \ .$$

By replacing h by its zeroth order approximation $C\,\bar{\Omega}$, where $\bar{\Omega}$ is the nominal spin frequency, the above condition simplifies to the following

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\Lambda_1^2}{C-A} + \frac{\Lambda_2^2}{C-B} \quad .$$

Thus a stability criterion arises which explicitly bounds the first modal frequency of the structure.

To conclude this derivation and to demonstrate the interrelationship between Chapter 4 and the preceding chapters, it might be useful to work out a simple example. An elementary but meaningful example is given by the beam pair configuration, as in Figure 15.

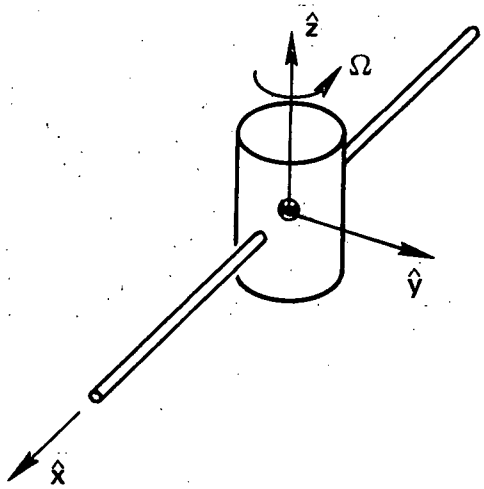

Figure 15. Simple Example.

For this particular structure, we have $\Gamma_3 = \Gamma_2 = 0$. The stability criteria for asymptotic stability of spin are given by C > A and C > B and also

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\Lambda_1^2}{C-A} + \frac{\Lambda_2^2}{C-B} \quad .$$

The consideration of $\Gamma_2 = 0$ reduces the last criterion to

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\Lambda_1^2}{C-A} \quad .$$

Let us concentrate on the first mode, as our single mode. From our previous derivation, we conclude that only the anti-symmetric modes are relevant to our study. From the derivation accomplished in Chapter 2, we have the following results, when considering the first mode.

$$\left(\frac{\omega_1}{\Omega}\right)^2 = 1 + 2.12\sqrt{\varepsilon} \quad ,$$

also

$$\phi_1 = h_0(x) + \sqrt{\varepsilon}\ h_1(x) + \ldots$$

For the first mode, $h_0(x)$ represents a linear function $k_0 x$, so we obtain for Λ_1 the following expressions

$$\Lambda_1 = \int_A \Gamma_1 \phi_1 \, dm = 2 \int_0^L \Gamma_1 \phi_1 \, dm$$

where the anti-symmetry of the deformation has been used. For a uniform beam, we have $dm = \mu\, dx$, so the previous expression becomes

$$\Lambda_1 = 2\mu \int_0^L x\, h_0(x) dx + 2\mu\sqrt{\varepsilon} \int_0^L x\, h_1(x) dx + \ldots$$

For the first mode, the orthogonality relationship between h_0 and h_1, cancels the second term and we are left with

$$\Lambda_1 = 2\mu \int_0^L k_0 x^2\, dx = \frac{2}{3}\mu\ k_0\, L^3.$$

The modes have been defined in such a way that

$$\int_A \phi_1^2 \, dm = 1$$

or

$$2 \int_0^L k_0^2 \, x^2 \, \mu \, dx = 1$$

so

$$k_0^2 = \frac{3}{2} \frac{1}{\mu L^3} \quad .$$

The final expression for Λ_1 is then

$$\Lambda_1^2 = \frac{2}{3} \mu \, L^3 = \frac{2}{3} \, m_R L^2$$

where $m_R = \mu \, L$ stands for the mass of one beam. The final expression for the last criterion is thus

$$1 + 2.12 \sqrt{\varepsilon} > \frac{2}{3} \frac{m_R L^2}{C-A} \quad .$$

If we observe that the inertia of a uniformly distributed beam about the core is given by $\frac{1}{3} m_R L^2$, the last criterion is also

$$1 + 2.12 \sqrt{\varepsilon} > \frac{\text{(Inertia of the two beams)}}{C-A}$$

This last result has to be compared with the stability criteria developed by L. Meirovitch and R.A. Calico, Reference (23), where they obtained stability criteria for a spacecraft characterized by a rigid core having attached to it flexible booms. Their results, when applied to radial rods only, become

$$\left(\frac{\omega_1}{\Omega}\right)^2 > \frac{\text{(Inertia of the rods)}}{C-A}$$

which shows a perfect analogy with the result of Chapter 4.

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CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

In the work developed, two different problems have been approached: A modal analysis, covered in Chapters 2 and 3, and an attitude stability study, presented in Chapter 4. Each part represents a different aspect to investigate for the preliminary design of flexible appendages to be attached to a spinning spacecraft; they complement each other under the general question of stability.

In the first and more extended part, general inferences concerning the magnitude and character of the influence of spin on the natural frequencies and mode shapes of some rotating structures are done. One of the objectives of this part was also to address ourselves to the level of sophistication that a derivation using a continuous model for an elastic appendage attached to a rigid base which is constrained to rotate with a constant angular speed Ω about a body axis, fixed in inertial space, could bring us. We decided to try to get away from more standard procedures — based on energy consideration and transforming the eigenvalue problem to a minimization one — by approaching them by a method unfamiliar to those particular applications. The method of matched asymptotic expansions has been proved to present a very powerful tool for the particular problems we looked at. Even though, we didn't attempt to consider all the meaningful structures where a singular perturbation method could be appropriate, we believe that we cover several problems where the usefulness of the method has been found. In contrast with the classical Rayleigh-Ritz

or Galerkin approaches, the accuracy of the perturbation method is determined by the number of terms taken in the various expansions and may be improved step by step by the investigator without the necessity of repeating the analysis with an augmented number of terms in the expansion of the solution.

The literal stability criteria developed in the last part of this present work represent anticipated results in the search for closed form conditions for attitude stability of spinning flexible spacecraft. They are, in fact, a natural extension of those derived in a similar study where another way of modeling the flexible part of the spacecraft was adopted. They represent (except for a few recognizable singular cases) necessary and sufficient conditions for stability for any spacecraft characterized by the planar appendage model, such as a spacecraft containing solar panels and/or radial booms. More precisely, they are necessary and sufficient for that portion of the system representative of the wobbling motion. In terms of the composite motion, these conditions can only be classified as necessary. Also this analysis does not reveal the system behavior at any time, but only the quality of the motion in the neighborhood of the dynamic equilibrium configuration consisting of the body spinning at a high angular velocity about the spin axis.

Although the results of this study could be used in preliminary design of flexible appendiages to be attached to spinning spacecraft, we have to be very cautious in extending the previous work to more general types of configuration. We mentioned earlier that a modal analysis of an idealized elastic structure on a rotating base requires

the derivation of the linearized equations of small vibrations of the mathematical model from its steady state of deformation, induced by spin, and the transformation of these equations into a system of uncoupled equations of motion representing each normal modes. For a continuum model, the number of normal modes could be infinite. For a completely general structure, this step represents a major obstacle. Even in the very elementary structure consisting of a spinning central body with a cantilevered beam oriented in the direction of the spin axis, the influence of the basis rotation is manifested in the form of centripetal acceleration and Coriolis acceleration. The latter term couples the equations, and provides an obstacle to modal analysis except for the particular case where the linear density is constant throughout the beam and both transverse inertias of the cross section of the beam are the same constant. And, even in this last case, the modes are given by a complex function.

It is to circumvent the last obstacles, that several authors consider in their modal analysis an attractive alternative, employing for the flexible appendage the coordinates which are normal mode coordinates when the basis is inertially fixed. Although this last shortcut may be acceptable when the motion is a small perturbation from a rigid body displacement at a relatively slow spin, it is more difficult to justify for a general case. They also look, at the outset, at the spinning stiffness matrix $\bar{K}$ as composed essentially of the summation of the nonspinning stiffness $\hat{K}$ and a matrix representing the contribution of spin — sometimes referred to as the geometric stiffness. But here again, they face another source of problems. In

general, the eigenmodes of the nonrotating structure are not orthogonal for the rotating structure; as a result, the spinning stiffness matrix is no longer diagonal for this last choice of eigenmodes. As a result, the nonspinning stiffness $\hat{K}$ is well defined and made of the unloaded natural eigenfrequencies along the diagonal, but there is no real physical basis to determine the correction to add to the last $\hat{K}$ matrix in order to obtain the complete stiffness matrix and this last effect, configuration dependent, may or may not seriously modify the stiffness property of the structure.

In view of those several difficulties, we might conclude that the development of elastic continuum models could yield very useful results when applied to a small class of special cases, but this approach lacks the general utility and tractability of distributed-mass finite element models, since in the latter case, the governing equations are always linear, constant coefficient ordinary differential equations. Finally, the generalization of the previous results can only be considered for that small class of problems of elastic models where the steady state deformation can be solved first, and where the vibration equations are obtained by a linearization procedure; and even in this case implementation of the general theory may be very difficult.

REFERENCES

1. Yntema, R.T. "Simplified Procedures and Charts for the Rapid Estimation of Bending Frequencies of Rotating Beams," Langley Aeronautical Laboratory, Langley Field, Virginia, Technical Note 3459 (NACA-TN-3459), June 1955.

2. Renard, M.L. and J.E. Rakowski. "Equatorial Vibrations of a Long Flexible Boom on a Spin-Stabilized Satellite of Non-zero Radius," Proceedings of the 20th Congress of the International Astronautical Federation, Mar del Plata, Argentina, 1969, Pergamon Press, PWN - Polish Scientific Publishers, Poland, 1969, pp. 35-52.

3. Hughes, P.C. and J.C. Fung. "Liapunov Stability of Spinning Satellites with Long, Flexible Appendages," Celestial Mechanics, Vol. 4, No. 3/4, pp. 295-308, 1971.

4. Lamb, H. and R.V. Southwell. "The Vibrations of a Spinning Disk," Proceedings of the Royal Society of London, London, July 1, 1921, The Royal Society, London, September 1921, Series A, pp. 272-280.

5. Simmonds, J.G. "The Transverse Vibrations of a Flat Spinning Membrane," Journal of the Aeronautical Sciences, Vol. 29, No. 1, pp. 16-18, January 1962.

6. Eversman, W. "Some Equilibrium and Free-Vibration Problems Associated with Centrifugally Stabilized Disk and Shell Structures," Wichita State University, Wichita, Kansas, Report on Contract NGR 17-003-004(NASA CR-1178), December 1968.

7. Eversman, W. and R.O. Dodson. "Free Vibration of a Centrally Clamped Spinning Circular Disk," AIAA J., Vol. 7, No. 10, pp. 2010-2012, October 1969.

8. Cole, Julian D. Perturbation Methods in Applied Mathematics, Blaisdell, Waltham, Massachusetts, 1968.

9. Boyce, W.E. and G. Handelman. "Vibrations of Rotating Beams with Tip Mass," Zeitschrift fur angewandte Mathematik und Physic, Vol. 12, Fasc. 5, pp. 369-391, September 25, 1961.

10. Moser, J. "Singular Perturbation of Eigenvalue Problems for Linear Differential Equations of Even Order," Communication of Pure and Applied Mathematics, Vol. 8, No. 2, pp. 251-278, May 1955.

11. Abel, J.M. and W.C. Kerr. "Transverse Vibrations of a Rotating Counterweighted Cable of Small Flexural Rigidity," AIAA J., Vol. 9, No. 12, pp. 2326-2332, December 1971.

REFERENCES (Cont'd)

12. Chobotov, V. "Gravity Gradient Excitation of a Rotating Cable-Counterweight Space Station in Orbit," Journal of Applied Mechanics, Vol. 30, Serie E, No. 4, pp. 547-554, December 1963.

13. Barbera, F.J. and P.W. Likins. "Liapunov Stability Analysis of Spinning Flexible Spacecraft," AIAA J., To appear, March 1973.

14. Pringle, R. Jr. "Stability of the Force-Free Motions of a Dual-Spin Spacecraft," AIAA J., Vol. 7, No. 6, pp. 1054-1063, June 1969.

15. Likins, P.W., F.J. Barbera, and V. Baddeley. "Mathematical Modeling of Spinning Elastic Bodies for Modal Analysis," submitted to AIAA Journal.

16. Meirovitch, Leonard. Analytical Methods in Vibrations, Macmillan, New York, 1967.

17. Rainville, Earl D. Intermediate Courses in Differential Equations, J. Wiley and Sons, Inc., New York, 1943.

18. Luke, Yudell L. Integrals of Bessel Functions, McGraw-Hill, New York, 1962.

19. Vigneron, F.R. "Stability of a Freely Spinning Satellite of Crossed-Dipole Configuration," Canadian Aeronautics and Space Institute Transactions, Vol. 3, No. 1, pp. 8-19, March 1970.

20. Prescott, John. Applied Elasticity, Dover Publications, New York, 1946.

21. Willems, P.Y. "Attitude Stability of Deformable Satellites," Proceedings of the International Conference on Attitude Control and Stabilization of Satellites, Paris, France, October 8-11, 1968, Centre National d' Etudes Spatiales, 1969, pp. 219-251.

22. Likins, P.W. "Dynamics and Control of Flexible Space Vehicles," Jet Propulsion Laboratory, Pasadena, California, Technical Report 32-1329, Revision 1, January 1970.

23. Meirovitch, L. and R.A. Calico. "The Stability of Motion of Force-free Spinning Satellites with Flexible Appendages," AAS/AIAA Astrodynamics Specialists Conference, Ft. Lauderdale, Florida, August 17-19, 1971 (AAS Paper 71-345).

APPENDIX I

ORTHOGONALITY OF THE MODES

Our objective is to prove that the values previously found for λ_0^2 and λ_1^2, also satisfy:

$$\int_{-k^{-1}}^{k^{-1}} \phi_\alpha \phi_\beta \, dx = 0 \qquad \text{for } \alpha \neq \beta \, .$$

The asymptotic expansions for the modes are given by:

$$\phi_\alpha(x,\varepsilon) = h_{0\alpha}(x) + \sqrt{\varepsilon} \; h_{1\alpha}(x) + \ldots$$

$$\phi_\beta(x,\varepsilon) = h_{0\beta}(x) + \sqrt{\varepsilon} \; h_{1\beta}(x) + \ldots$$

The problem becomes equivalent to:

$$\int_{-k^{-1}}^{k^{-1}} h_{0\alpha} h_{0\beta} \, dx = 0$$

$$\int_{-k^{-1}}^{k^{-1}} (h_{0\alpha} h_{1\beta} + h_{0\beta} h_{1\alpha}) dx = 0$$

where $h_{0\alpha}$ and $h_{1\alpha}$ satisfy

$$[(1-x^2)h'_{0\alpha}]' + \lambda_{0\alpha}^2 \, h_{0\alpha} = 0$$

$$[(1-x^2)h'_{1\alpha}]' + \lambda_{0\alpha}^2 \, h_{1\alpha} = -\lambda_{1\alpha}^2 \, h_{0\alpha} \, ,$$

and similar expressions for $h_{0\beta}$ and $h_{1\beta}$.

The first equality is easy to show. Multiplying the differential equation defining $h_{0\alpha}$ by $h_{0\beta}$ and integrating over the whole span, we have:

$$\int_{-k^{-1}}^{k^{-1}} [(1-x^2)h'_{0\alpha}]' \, h_{0\beta} \, dx + \lambda_{0\alpha}^2 \int_{-k^{-1}}^{k^{-1}} h_{0\alpha} h_{0\beta} \, dx = 0.$$

After an integration by parts, the use of the transcendental equation defining $\lambda_{0\alpha}^2$ and $\lambda_{0\beta}^2$, gives us:

$$\int_{-k^{-1}}^{k^{-1}} (1-x^2) h'_{0\alpha} h'_{0\beta} dx = \lambda_{0\alpha}^2 \int_{-k^{-1}}^{k^{-1}} h_{0\alpha} h_{0\beta} \, dx \, .$$

By symmetry the previous equation is also equal to $\lambda_{0\beta}^2 \int_{-k^{-1}}^{k^{-1}} h_{0\alpha} h_{0\beta} \, dx$, from where the first equality results.

The equality

$$\int_{-k^{-1}}^{k^{-1}} (h_{0\alpha} h_{1\beta} + h_{0\beta} h_{1\alpha}) dx = 0$$

is more difficult to prove. The differential equation defining $h_{1\alpha}$ is first multiplied by $h_{0\beta}$ and integrated over the whole span. The orthogonality of $h_{0\alpha}$ and $h_{0\beta}$ is then used and we have:

$$\int_{-k^{-1}}^{k^{-1}} [(1-x^2)h'_{1\alpha}]' \, h_{0\beta} \, dx + \lambda_{0\alpha}^2 \int_{-k^{-1}}^{k^{-1}} h_{1\alpha} h_{0\beta} \, dx = 0.$$

Integrating the first term by parts and using the boundary conditions on $h_{0\beta}$, we have

$$-\int_{-k^{-1}}^{k^{-1}} (1-x^2) h'_{1\alpha} h'_{0\beta} dx + \lambda_{0\alpha}^2 \int_{-k^{-1}}^{k^{-1}} h_{1\alpha} h_{0\beta} \, dx = 0.$$

Integrating by parts once more, and after the use of the differential equation defining $h_{0\beta}$, we have:

$$-(1-x^2)h'_{0\beta}h_{1\alpha}\Big|_{-k^{-1}}^{k^{-1}} + (\lambda_{0\alpha}^2 - \lambda_{0\beta}^2)\int_{-k^{-1}}^{k^{-1}} h_{1\alpha}h_{0\beta}dx = 0 .$$

A similar expression can be obtained starting from the differential equation defining $h_{1\beta}$ or

$$(1-x^2)h'_{0\alpha}h_{1\beta}\Big|_{-k^{-1}}^{k^{-1}} + (\lambda_{0\alpha}^2 - \lambda_{0\beta}^2)\int_{-k^{-1}}^{k^{-1}} h_{1\beta}h_{0\alpha}\, dx = 0 .$$

Adding up those two relations, we have

$$(1-x^2)h'_{0\alpha}h_{1\beta}\Big|_{-k^{-1}}^{k^{-1}} - (1-x^2)h'_{0\beta}h_{1\alpha}\Big|_{-k^{-1}}^{k^{-1}}$$

$$+ (\lambda_{0\alpha}^2 - \lambda_{0\beta}^2)\int_{-k^{-1}}^{k^{-1}} (h_{1\beta}h_{0\alpha} + h_{1\alpha}h_{0\beta})dx = 0 .$$

The two first terms can be written:

$$(1-k^{-2})[h'_{0\alpha}(k^{-1})h_{1\beta}(k^{-1}) - h'_{0\alpha}(-k^{-1})h_{1\beta}(-k^{-1}) - h'_{0\beta}(k^{-1})h_{1\alpha}(k^{-1})$$
$$- h'_{0\beta}(-k^{-1})h_{1\alpha}(-k^{-1})],$$

and by direct substitution, this parenthesis comes out to be identically zero. We then conclude

$$\int_{-1^{-1}}^{k^{-1}} (h_{1\beta}h_{0\alpha} + h_{1\alpha}h_{0\beta})\, dx = 0.$$

APPENDIX II

SOLUTION TO THE REDUCED EQUATION OF THE TAUT MEMBRANE

In the consideration of the second linearly independent solution of the previous equation, two cases have to be considered.

CASE A: Both roots of the indicial equation are the same and equal to zero or

$$1 - \gamma = 0 \quad \text{or} \quad \gamma = 1.$$

This case corresponds physically to m=0 or to radially symmetric vibrations of our clamped disk. To obtain the second linearly independent solution, we shall assume that

$$H_c = x^c + \sum_{n=1}^{\infty} b_n x^{n+c}$$

where c is retained as a parameter. The last expression does not satisfy the differential equation defining H_0, but if we write the differential equation defining H_0 or

$$x(1-x)\frac{d^2H_0}{dx^2} + (1-2x)\frac{dH_0}{dx} + \frac{1}{4}\frac{\lambda_0^2}{k_1^2}H_0 = 0$$

under the form

$$\mathscr{L}_{H_0}[H_0] = 0$$

where $\mathscr{L}_{H_0}$ represents the linear differential operator

$$\mathscr{L}_{H_0} \equiv x(1-x)\frac{d^2}{dx^2} + (1-2x)\frac{d}{dx} + \frac{1}{4}\frac{\lambda_0^2}{k_1^2} \quad ,$$

we have

$$\mathscr{L}_{H_0}[H_c] = c^2x^{c-1} - c^2x^2 + b_1(c+1)cx^c + \frac{\lambda_0^2}{k_1^2}x^c - cx^c + b_1(c+1)x^c$$

$$+ \sum_{n=2}^{\infty}\Big[b_n(n+c)(n+c-1) - b_{n-1}(n+c-1)(n+c-2) + b_n(n+c)$$

$$-2\, b_{n-1}(n+c-1) + \frac{\lambda_0^2}{4k_1^2}\, b_{n-1}\Big]\, x^{n+c-1} \; .$$

We are now able to make all but the first term of the right hand member of the above equation vanish without selecting c. To do this, we must take

$$-c^2 + b_1(c+1)c + \frac{\lambda_0^2}{4k_1^2} - c + b_1(c+1) = 0$$

and

$$b_n(n+c)^2 - b_{n-1}(n+c-1)(n+c) + b_{n-1}\,\frac{\lambda_0^2}{4k_1^2} = 0 \; .$$

So far, we have found an expression

$$H_c = x^c + \sum_{n=1}^{\infty} b_n\, x^{n+c}$$

that reduces the previous expression to

$$\mathscr{L}_{H_0}[H_c] = c^2x^{c-1} \; ,$$

no matter what value c takes on. But this last expression not only shows that the case c=0 is solution of the equation defining H_0, but also (dH_c/dc) evaluated at c=0. The solution, found before, corresponds to the case where c=0 so our interest lies mainly in finding

$$\frac{dH_c}{dc} \quad \text{for} \quad c=0,$$

or

$$\frac{dH_c}{dc} = x^2 \log x + \sum_{n=1}^{\infty} b_n x^{n+c} \log x + \sum_{n=1}^{\infty} \frac{\partial b_n}{\partial c} x^{n+c} .$$

We finally obtain the second linearly independent solution

$$H_{20} = \left(1 + \sum_{n=1}^{\infty} b_n \Big|_{c=0} x^n\right) \log x + \sum_{n=1}^{\infty} \frac{\partial b_n}{\partial c} \Big|_{c=0} x^n$$

or

$$H_{20} = F(\alpha,\beta,\gamma;x) \log x + \sum_{n=1}^{\infty} \frac{\partial b_n}{\partial c} \Big|_{c=0} x^n .$$

The complete solution is then given by:

$$H_0 = a_0 F(\alpha,\beta,\gamma,x) + b_0 F(\alpha,\beta,\gamma;x) \log x + b_0 \sum_{n=1}^{\infty} \frac{\partial b_n}{\partial c} \Big|_{c=0} x^n .$$

Due to the singularity at the origin, we have to take $b_0 = 0$, and the solution reduces to:

$$H_0 = a_0 F(\alpha,\beta,\gamma;x) .$$

<u>Case B:</u> In this case, the roots of the indicial equation are different and from there, m is different from zero. In order to determine the other solution, two different cases are to be considered depending on the values of α and β.

a) α or β is an integer between 1 and m. Let us try a particular solution starting with the index -m, or

$$H_{02} = \sum_{n=0}^{\infty} a_n x^{n-m} .$$

By substituting the last expression into the differential equation defining H_0, we obtain as a coefficient for a_0, the indicial equation

which is satisfied because the particular solution starts with the index -m. The other part gives us the recurrence relationship between the a_n's or more explicitly

$$a_n = \frac{(n-\alpha)(n-\beta)}{(n-m)n} \quad a_{n-1} \; .$$

By looking at this particular solution, and knowing that m is different from zero, but taking on positive integer value, the only way such a solution could exist is if the numerator of a_m is also equal to zero, but

$$a_m = \frac{(m-\alpha)(m-\beta)}{(n-m)m} \times \frac{(m-\alpha-1)(m-\beta-1)}{(m-1-m)(m-1)} \quad \ldots a_0 \quad .$$

The last particular solution is then possible only if α or β satisfy one of the following equality:

α is an integer between 1 and m
β is an integer between 1 and m.

If α or β is an integer between 1 and m, one of the a_i's, in the previous recurrence relationship becomes zero and the particular solution is represented by a truncated serie. This particular solution combined with the result found earlier represents the general solution of the equation. Unfortunately, this last solution H_{02} has to be disregarded for its singularity at the origin.

b) If α and β are not integers lying between 1 and m, then the second linearly independent solution for H_0 can be taken under the form

$$H_{02} = H_{01} \log x + H_{0c} \; .$$

Introducing this last expression into the differential equation defining H_0, we are left with

$$x(1-x)\frac{d^2H_{0c}}{dx^2} + [\gamma - (\alpha+\beta+1)x]\frac{dH_{0c}}{dx} - \alpha\beta\, H_{0c} + 2(1-x)\frac{dH_{01}}{dx}$$
$$- \frac{(1-x)}{x} H_{01} + [\gamma-(\alpha+\beta+1)x]\frac{H_{01}}{x} = 0 .$$

We now consider for H_{0c} a solution of the form

$$H_{0c} = \sum_{n=0}^{\infty} d_n x^{n+c} \quad \text{where } d_0 \neq 0 .$$

Introducing for H_{01}, the solution we found before, we have for the equation defining d_0, two choices:

1) $c(c-1)d_0 + \gamma\, c\, d_0 - a_0 + \gamma\, a_0 = 0$ and $c=0$

where the terms in d_0 compensate for the term in a_0 or

2) $c(c-1)d_0 + \gamma\, c\, d_0 = 0$ and $c < 1$

where the terms in d_0 balance themselves. The first case is impossible as can be seen directly and we are thus left with

$$c - 1 + \gamma = 0$$

or

$$c = -m .$$

Knowing that $c = -m$, we could now compute the recurrence relationship between the d_i's. But once more, the solution H_{02} is to be disregarded for its singularity at the origin, and from there, the solution found before in Chapter 3, represents the general solution of our equation.

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APPENDIX III

BEHAVIOR NEAR THE ORIGIN

In order to shorten the text, the behavior of the solution near r=0 has been introduced as an Appendix. It is indeed deduced that the solution previously found is valid throughout the neighborhood of the origin and doesn't present any singularity. As explained earlier, the reason to consider the neighborhood of the origin is justified for the singularity of the operator ∇_m^2 at the origin.

Boundary Layer Expansion Valid Near r=0

Close to r=0, let us define a suitable boundary layer variable or

$$r^* = \frac{r}{\sigma(\varepsilon)} \quad \text{where} \quad \sigma(\varepsilon) \to 0 \quad \text{when} \quad \varepsilon \to 0 \quad .$$

The corresponding asymptotic expansion valid near r=0 is taken as

$$\phi_{\alpha,m}(r,\varepsilon) = \mu_0(\varepsilon)\ g_0(r^*) + \mu_1(\varepsilon)\ g_1(r^*) + \ldots$$

In the new variables, the operator ∇_m^2 becomes:

$$\nabla_m^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} = \frac{d^2}{\sigma^2 dr^{*2}} + \frac{1}{\sigma^2 r^*}\ \frac{d}{dr^*} - \frac{m^2}{\sigma^2 r^{*2}} = \frac{1}{\sigma^2}\ \nabla_m^{*2}$$

where ∇_m^{*2} stands for

$$\frac{d^2}{dr^{*2}} + \frac{1}{r^*}\frac{d}{dr^*} - \frac{m^2}{r^{*2}} \quad .$$

The complete equation for the deformation becomes, with the use of the previous definition:

$$-\varepsilon\left[\frac{1}{\sigma^4}\nabla_m^{*2}\nabla_m^{*2}\left(\mu_0(\varepsilon)g_0(r^*)+\mu_1(\varepsilon)g_1(r^*)+\ldots\right)\right]$$

$$+\frac{1}{\sigma^2}\frac{d}{r^*dr^*}\left[r^*(1-k_1^2\sigma^2r^{*2})\frac{d}{dr^*}\left(\mu_0(\varepsilon)g_0(r^*)+\mu_1(\varepsilon)g_1(r^*)+\ldots\right)\right]$$

$$-\frac{m^2}{\sigma^2r^{*2}}[1-k_2^2\sigma^2r^{*2}]\left(\mu_0(\varepsilon)g_0(r^*)+\mu_1(\varepsilon)g_1(r^*)+\ldots\right)$$

$$+(\lambda_0^2+\sqrt{\varepsilon}\,\lambda_1^2+\ldots)\left(\mu_0(\varepsilon)g_0(r^*)+\mu_1(\varepsilon)g_1(r^*)+\ldots\right)=0 \quad .$$

The thickness of the boundary layer is obtained by considering:

$$\frac{\varepsilon}{\sigma^4}\approx\frac{1}{\sigma^2} \quad \text{or} \quad \sigma=\sqrt{\varepsilon} \quad \text{and} \quad r^*=\frac{r}{\sqrt{\varepsilon}} \quad .$$

The dominant boundary layer equation is then obtained by considering the terms of order $\mu_0(\varepsilon)/\varepsilon$ in the previous development or

$$-\nabla_m^{*2}\nabla_m^{*2}g_0(r^*)+\frac{1}{r^*}\frac{d}{dr^*}g_0(r^*)+\frac{d^2g_0(r^*)}{dr^{*2}}-\frac{m^2}{r^{*2}}g_0(r^*)=0 \quad .$$

This expression can be written shortly:

$$-\nabla_m^{*2}\nabla_m^{*2}g_0(r^*)+\nabla_m^{*2}g_0(r^*)=0 \; .$$

Writing this last expression in terms of the operator ∇_m^{*2}, we have

$$\nabla_m^{*2}(\nabla_m^{*2}-1)g_0(r^*)=0$$

or

$$(\nabla_m^{*2}-1)\nabla_m^{*2}g_0(r^*)=0 \quad .$$

We observe a permutativity between the two operators $(\nabla_m^{*2}-1)$ and ∇_m^{*2}. As a result, we know from the theory of linear differential equation, that the obtention of the four linearly independent solutions of the

previous equation can be done by solving the two systems

$$\begin{cases} (\nabla_m^{*2} - 1)\ g_0(r^*) = 0 \\ \nabla_m^{*2}\ g_0(r^*) = 0\ . \end{cases}$$

The general solution of the complete equation is then obtained by taking a linear combination of the four solutions generated. The determination of the constants appearing in the general solution are then determined through the matching process near the origin. Let us look first at the solutions of

$$(\nabla_m^{*2} - 1)g_0(r^*) = 0 \quad \text{or}$$

$$\frac{d^2}{dr^{*2}}\, g_0 + \frac{1}{r^*}\ \frac{d}{dr}\, g_0(r^*) - \frac{m^2}{r^{*2}}\, g_0(r^*) - g_0(r^*) = 0\ .$$

This equation is a Bessel equation which solution is given in terms of the modified Bessel functions I_n and K_n. But both of these solutions have to be canceled, one for its exponential growth and the other for introducing an unbounded displacement at the origin. The other independent solutions are determined in solving

$$\nabla_m^{*2}\ g_0(r^*) = 0.$$

This equation can easily be integrated by the use of the substitution $g_0(r^*) = r^{*k}$ leading to the characteristic equation, with two roots or

$$k^2 = m^2 \quad \text{or} \quad k = \pm\ m.$$

The solution $k = -m$ introduces a singularity at the origin and will therefore be neglected and we are left with

$$g_0(r^*) = C_0 r^{*m}\ .$$

This expression represents precisely what the solution found in the central area becomes in the neighborhood of the origin and the determination of the constant C_0 is obtained directly.

Due to the fact that there is no boundary condition at the origin and that both solutions found for $(\nabla_m^2 - 1)g_0(r^*) = 0$ have to be canceled for being improper for matching, we deduce that the solution found earlier is valid in the neighborhood of the origin, and that, for this problem, there is no need for a boundary layer at the center of the membrane.

For the particular case where m=0, the previous solution has to be revised. The equation $\nabla_m^{*2}\, g_0(r^*) = 0$ becomes

$$\frac{d^2 g_0}{dr^{*2}} + \frac{1}{r^*}\,\frac{dg_0}{dr^*} = 0$$

or

$$g_0 = C_0 \ln r^* + C_1 .$$

Again, the solution in $\ln r^*$ has to be rejected for its singularity at the origin, and we are left with

$$g_0(r^*) = C_1,$$

which represents the displacement of the membrane at the origin. The determination of C_1 is again obtained by matching with the solution found previously. Once more, there is no need for a boundary layer in the neighborhood of the origin. We conclude then, by saying that the solution found earlier is valid in all the central area of our problem.

APPENDIX IV

DETERMINATION OF THE CONSTANT C_1

In the matching process valid near r=1, it seems natural to consider as an order of magnitude for $\nu_1(\varepsilon)$ the value

$$\nu_1(\varepsilon) = \varepsilon^{1/3} \quad .$$

Going back to the expansion done for the differential equation defining $\phi_{\alpha,m}$, an indicated choice for $\kappa_1(\varepsilon)$ is given by $\kappa_1(\varepsilon) = \varepsilon^{1/3}$. The differential equation defining $h_1(r)$ is then given by the following expression

$$(1-r^2)\frac{d^2h_1}{dr^2} + \frac{1}{r}(1-3r^2)\frac{dh_1}{dr} - \frac{1}{r^2}[m^2-(m^2k^2 + \lambda_0^2)r^2]h_1 = -\lambda_1^2 h_0.$$

We can see from a straightforward observation that the homogeneous part of the previous equation is the same as the differential equation defining h_0, and that h_0 appears as a forcing terms. As a result, the eigenfrequency λ_1^2 of the previous equation, can be obtained once more by considering only finite displacement at the rim of the disk. Let us express:

$$h_1 = \sum_{n=0}^{\infty} b_n r^{n+m} \quad .$$

The index of the last equation is obviously the same as the one found for h_0. We then have if we replace h_0, by its development

$$h_0 = \sum_{n=0}^{\infty} a_n r^{n+m} \quad ,$$

where all a_i's are known.

$$\sum_{n=2}^{\infty} [(n+m)(n+m-1)b_n - (n+m-2)(n+m-3)b_{n-2} + (n+m)b_n$$

$$- 3(n+m-2)b_{n-2} - m^2 b_n + (k^2 m^2 + \lambda_0^2) b_{n-2}] r^{n+m-2}$$

$$= - \lambda_1^2 \sum_{n=2}^{\infty} a_{n-2} r^{n+m-2}$$

or

$$b_n = \frac{(n+m-2)(n+m) - (k^2 m^2 + \lambda_0^2)}{n^2 + 2mn} b_{n-2} - \frac{\lambda_1^2}{n^2 + 2mn} a_{n-2} .$$

The recurrence relationship contains two terms, one has the same ratio as the one found for the a_i's and the second term is coming from the forcing terms. But if λ_0^2 is such that the serie defined for h_0 is truncated starting at $a_{n'}$, we have that $a_{n'+2}, \ldots = 0$. But in the expansion of h_1, the term

$$b_{n'} = - \frac{\lambda_1^2}{(n'^2 + 2n'm)} a_{n'-2}$$

represents a nonzero contribution.

As a result, we conclude that the rest of the expansion for h_1 represents a diverging expression for r=1, as it was proven for h_0. We, then have to conclude that $\kappa_1(\varepsilon)$ is not of an order of magnitude of $\varepsilon^{1/3}$, and by the orthogonality relationship between the terms of the asymptotic expansion valid for the mode, that $\nu_1(\varepsilon)$ is not $\varepsilon^{1/3}$. This result justifies the choice

$$C_1 = 0.$$

APPENDIX V

EQUATIONS FOR THE ROTATIONAL MOTION

In this appendix, we will rederive some basic equations whose consideration are necessary for the mentioned particular case. First of all, let us derive the equations of motion for the attitude angles of the central body, for a completely general problem.

In the derivation done in Chapter 4, we obtained the following expression for the angular momentum vector with respect to the center of mass.

$$\underline{h} = \underline{I}^N \cdot \underline{\omega} + \mathcal{M}\dot{\underline{c}} \times \underline{c} + \int_A \underline{\rho} \times \dot{\underline{\rho}}\, dm \ .$$

The equation defining the angular velocity vector for a freely spinning flexible spacecraft is then obtained by expressing the nullity of the applied torque acting about the center of mass or

$$\dot{\underline{h}} = \underline{0} = \underline{I}^N \cdot \dot{\underline{\omega}} + \underline{\omega} \times \underline{I}^N \cdot \underline{\omega} + \overset{\circ}{\underline{I}}{}^N \cdot \underline{\omega} + \underline{\omega} \times \int_A \underline{\rho} \times \dot{\underline{\rho}}\, dm + \int_A \underline{\rho} \times \ddot{\underline{\rho}}\, dm$$

where we limited our derivation to the linearized equation of the motion. Written in matrix form, the previous equation becomes

$$0 = I^N \dot{\omega} + \tilde{\omega}\, I^N \omega + \overset{\circ}{I}{}^N \omega + \tilde{\omega} \int_A \tilde{\rho}\, \dot{\rho}\, dm + \int_A \tilde{\rho}\ddot{\rho}\, dm \ .$$

The two last terms can be expanded in the following way:

$$\int_A \tilde{\rho}\, \ddot{\rho}\, dm = \int_A \tilde{\Gamma}\, \dot{u}\, dm + \int_A \tilde{u}\, \dot{u}\, dm$$

$$\int_A \tilde{\rho}\, \ddot{\rho}\, dm = \int_A \tilde{\Gamma}\, \ddot{u}\, dm + \int_A \tilde{u}\, \ddot{u}\, dm \ .$$

Expanding similarly the matrix of inertia of the complete system into

$$I^N = I_0^N + 2\Delta + U$$

where 2Δ represents the first order terms of the matrix of inertia and similarly, U represents the second order terms.

The linearized equation of the rotational motion are given by

$$0 = I_0^N \dot{\omega} + \tilde{\omega} I_0^N \omega + 2\,\tilde{\omega}_{SS}\,\Delta\,\omega_{SS} + 2\,\overset{\circ}{\Delta}\,\omega_{SS} + \tilde{\omega}_{SS} \int_A \tilde{\Gamma}\,\dot{u}\,dm + \int_A \tilde{\Gamma}\,\ddot{u}\,dm \qquad (V.1)$$

where ω_{SS} has been introduced to express the angular velocity vector when the system is spinning at equilibrium at a rate compatible with the angular momentum.