HEAT CONDUCTION ERRORS AND TIME LAG IN CRYOGENIC THERMOMETER INSTALLATIONS

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ABSTRACT

Installation practices are recommended that will increase rate of heat exchange between the thermometric sensing element and the cryogenic fluid and that will reduce the rate of undesired heat transfer to higher-temperature objects. Formulas and numerical data are given that help to estimate the magnitude of heat-conduction errors and of time lag in response.

INTRODUCTION

There are instances in cryogenic engineering where high accuracy is required in temperature measurement or where thermometric time lag must be estimated. An example of the need for accuracy occurs in the measurement of fluid density. An effective way of determining the density of a monomolecular cryogenic liquid, like hydrogen, nitrogen, oxygen, or methane, is to measure its pressure and temperature. Pressure measurement is rarely a problem because liquid density is only a weak function of pressure if one is well away from the critical point. However, the density of a cryogenic liquid depends strongly on its temperature. The following table shows the temperature accuracy required to determine within 0.1 percent the density of four liquids at their normal (1-bar) boiling points.

<table>
<thead>
<tr>
<th>Liquid</th>
<th>H₂</th>
<th>N₂</th>
<th>O₂</th>
<th>CH₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>ΔT for 0.1% error in ρ, K</td>
<td>0.06</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

In the case of gases, less than 0.1 percent error in absolute pressure and temperature can be tolerated if no more than 0.1 percent inaccuracy in density is to be achieved.

When a complex automatic control system is being designed, allowance must be made for the lag in response of the thermometer. Making this lag negligible may be precluded by the need for mechanical ruggedness and reliability. Nor is zero lag always desirable. When a high-speed sampled data system is used with a fluid whose temperature may fluctuate randomly, too fast a response may prove dangerous; integration over an appreciable part of the time between samplings may actually improve accuracy and control stability.

This paper is directed only to a narrow range of situations; namely, those where the fluid exists in a single phase, and where the thermometer bulb temperature is near to the fluid temperature. These are the situations where high accuracy is most likely to be of major interest. Response time when there is nucleate boiling, as when a warm thermometer bulb is plunged into a liquid, has been well illustrated by Miller and Flynn(1).

The specific problems to be treated are (1) to determine what is required to keep the error in steady-state temperature measurement less than an arbitrarily-prescribed amount and (2) to estimate the lag in response to temperature changes occurring under prescribed conditions. The first problem arises because there is an undesirable flow of heat to the thermometer from higher-temperature sources, at the same time that there is a transfer of heat between thermometer and fluid; the equilibrium temperature reached and indicated by the thermometer is the result of the balance between these two rates of heat transfer. The second problem arises because the heat capacity of the thermometer prevents its instantaneous response to changes in fluid temperature. The rate of response depends on the balance between the rate at which heat can be stored in the thermometer and the rate at which heat can be transferred between the thermometer and the fluid. Thus, both steady-state error and time lag can be reduced by improving the rate of heat transfer between fluid and thermometer and reducing the rate of conduction of heat from the ambient atmosphere to the thermometer. This paper first treats briefly the mechanical configurations that would help achieve this goal; next, the quantitative magnitude of steady-state error is estimated in those situations that are amenable to simple analysis; finally, the time lag is estimated in similar situations.

The heat transfer computations to be made are based on the data presented in classical texts(2),(3) on heat transfer which summarize and collate the work of many experimenters. The heat transfer computations may easily be in error by 30 percent, not because of comparable uncertainty in the original data, but because the premises on which the original

(1) Superior numbers refer to similarly-numbered references at the end of this paper.
data were based are not fully met by the practical situation. Some additional factors that influence a practical situation are free-stream turbulence, distortion of the velocity profile by end effects and pipe wall interference, and deviation of actual thermometer design from the assumed, idealized thermometer model. The uncertainties in computed results are not serious if they err on the conservative side, because they permit realization of the goal described as problem (1), to yield an error less than a prescribed amount. Furthermore, a 30 percent error in an error that itself is only of the order of a few percent is indeed nonsignificant. Appreciable errors in dynamical quantities like time constant and time lag also are usually acceptable, and cause negligible reduction in utility. Consequently, the use of two or three significant figures in some of the formulas presented herein is often for convenience of computation, and does not necessarily imply a commensurate accuracy.

INSTALLATION PRACTICES

Desirable installation practices are those which expose maximum length of the thermometer bulb and its support to the fluid, and produce the longest possible path for heat conduction from the atmosphere. The need for adequate mechanical strength and vibration resistance is an important constraint. The detailed design can be left to the ingenuity of the engineer, but Fig. 1 illustrates some approaches. The sketches are conceptual and not necessarily to scale. The shaded portion of the thermometer bulb represents the temperature-sensitive portion. The design of Fig. 1(a) is satisfactory only if the pipe wall is at the same temperature as the fluid, and both pipe wall and thermometer head are very well insulated. Figures 1(b), (c), and (d) show improvements which involve no change in direction of the fluid. Figures 1(e) and (f) show designs applicable to locations where there is a change in fluid direction.

Conduction along the wire leads can be obviated by the technique shown in Fig. 2. The leads are wrapped around the outside of the pipe for a distance of at least 100 wire diameters before they are led to the outside atmosphere. If cementing is inconvenient, kraft-paper-backed pressure-sensitive adhesive tape may be used to hold the wires firmly against the pipe, provided the tape is applied at room temperature to a clean, dry pipe surface. (Plastic-backed tape may not be used.) The tape should not be handled when it is cold. If the wires have no covering other than enamel, a single layer of thin writing paper may be used between wires and pipe to improve electrical insulation.

If the pipe is vacuum jacketed, the thermometer head should not be in contact with the outer pipe jacket that contains the vacuum. The outer pipe should hold only a cover plate that permits access to the thermometer, and the necessary electrical terminals. This type of construction also accommodates relative longitudinal motion between the pipes. Wires to the electrical terminals should be routed from the thermometer bulb in the manner shown in Fig. 2.

If there is doubt about the temperature of the thermometer head, an auxiliary thermometer may be attached to the head. A thermocouple is adequate for this purpose, and usually is more convenient to install than a resistance thermometer.

An installation in a storage tank, where heat transfer is principally by natural convection, requires that the support for the sensitive portion of the thermometer be exceptionally long, with much of it exposed to the cryogenic fluid. Some conceptual designs are shown in Fig. 3. Because of stratification, accuracy of local temperature measurement may require that the thermometer bulb and part of the stem lie in a horizontal plane; the stem may be curved in order to increase the length exposed to the fluid (Fig. 3(a)). In another design, the temperature-sensitive element may take the form of a bare-wire thermocouple or resistance wire supported from two prongs of low thermal conductance (Figs. 3(b) and (c)).

PRINCIPAL SYMBOLS

The following symbols are used in the subsequent portion of this paper.

- m: mass of sensing element
- m_2: mass of bulb wall
- P_Gr: Grashof number, \( D^3 \rho g \beta \Delta T \mu^2 \)
- N_u: Nusselt number, \( hD/k_f \)
- P_Re: Prandtl number, \( c_f \mu/k_f \)
- T: electric power input
- T_a, T_b: fluid temperature
- T_w: bulb wall temperature
- T_e: temperature error
- t: time
- X: distance
- \( \beta \): coefficient of volumetric expansion
- \( \epsilon \): natural base of logarithms
- \( \eta \): conductance parameter, Eq. (4)
- \( \mu \): viscosity
- \( \nu \): kinematic viscosity
Lumped-element circuit model. A simplified diagram of conductance if, conductance correction factor. However, the leads from the sensing element have longitudinal thermal conductance kA/L. Flow through a medium, shaded, which has a thermal conductance between bulb wall and thermometer head, in order to maintain the lumped-element model of Fig. 5. Heat capacities enter only into the computation of time lag, and not into the computation of steady-state errors. Electrical power delivered to the sensing element, when the latter is a resistance thermometer, is denoted by P.

Distributed-element considerations. The lumped-element model works well with respect to all components except the portion of the bulb wall over the distance Lg. The bulb wall temperature Tw(x) varies significantly over the distance 0 ≤ x ≤ Lg, so that some decision must be made on the definition of Tw in Fig. 5, if the convenience of Fig. 5 is to be utilized. The arbitrary decision we shall make is that Tw in Fig. 5 shall be the value of Tc(x) as computed by more rigorous analysis. This decision is conservative, because the effective value of Tw is likely to lie closer to the desired fluid temperature Tj. The concomitant specification of K4 will be discussed in the next section.

Thermometer model

Terms. The subsequent analysis will make use of the concepts of thermal capacity and thermal conductance. The thermal capacity of an object is a measure of its ability to store heat. It is equal to the product of the mass of the object by its specific heat; its dimensions are energy per unit temperature difference. The thermal conductance of an object or medium is a measure of its ability to transmit heat. A slab of material of thermal conductivity k, cross sectional area A, and length L has a longitudinal thermal conductance kA/L.

Thermal conductance is thus analogous to the electrical conductance of a resistor, and it will be symbolized by a resistor in the diagram of heat flow. The dimensions of thermal conductance are power per unit temperature difference.

Mechanical model. Figure 4 shows the generalized model that will be assumed for the thermometer and its installation. Specific deviations from this model will be discussed as the need arises. The "sensing element" is the resistance-thermometer wire or the thermocouple junction that is responsive to temperature. The bulb is assumed to be a thin-walled, closed-end metallic tube whose thermal conductance in a radial direction is substantially infinite. The exchange of heat between sensing element and bulb wall is assumed to take place through a medium, shaded, which has a thermal conductance K1. This exchange of heat is assumed to take place principally over the length L1. However, a greater length L2 of the bulb is exposed to the moving fluid whose temperature Tp is desired. The thermal conductance between bulb wall and fluid is K2; it may be thought of as the thermal conductance of the fluid boundary layer around the bulb. The total length of bulb, from bulb tip to head, is L2. The length L2 - Lg lies in the stagnant fluid and will conservatively be assumed to exchange negligible heat with the surrounding fluid. The temperature of the thermometer head is Th. However, the leads from the sensing element are assumed to terminate at a source whose temperature is Tb. Temperature Tb may be room temperature if a construction like that of Fig. 2 is not feasible, but it will probably be close to Ta, or lower, if the construction of Fig. 2 can be used effectively.

Lumped-element circuit model. A simplified diagram of the heat flow, shown in Fig. 5. The sensing element, at temperature Tj, exchanges heat with the bulb wall, at temperature Tw, through a medium of conductance K3. Longitudinal flow of heat along the bulb wall occurs through conductance K4. Flow of heat to the sensing element, along the connecting wires, is through conductance K5. Sensing element heat capacity is mCj; bulb wall heat capacity is mLgCg. The heat capacities of the media whose conductance is K1, K2, or K4 are assumed zero, or to have been apportioned among sensing element, bulb, and thermometer head, in order to maintain the lumped-element model of Fig. 5. Heat capacities enter only into the computation of time lag, and not into the computation of steady-state errors. Electrical power delivered to the sensing element, when the latter is a resistance thermometer, is denoted by P.

STEADY-STATE ACCURACY

In the steady state, if K5 << K1, (K3 + K4) << K2, the temperature indication is

\[
T = T_f + \frac{k_3 + k_4}{k_1} (T_f - T_a) + (T_f - T_p) \frac{K_3}{K_4}
\]

(1)

The condition for adequate accuracy is that the quantity in the braces shall be less than the maximum error in temperature measurement that can be tolerated. The term proportional to P is the "self-heating error."

It remains to estimate K1, K2, K3, and K4. These will depend on properties of the fluid and of materials of construction. Table I lists some pertinent properties of four cryogenic liquids at their 1-atmosphere boiling points, and of water at 300 K for reference. Table II lists the thermal conductivities of some solids at various temperatures. These data will be found useful for numerical calculations. They have been taken from Scott(4), the WADD Compendium(5), and the AIP handbook(6).

Estimation of K1. In general, the value of K1 cannot be easily estimated, since it is an internal constructional characteristic of the thermometer. An experimental determination may be necessary; a possible technique will be described in the discussion of time-lag. In some special cases, however, an estimation of K1 is possible:

K1 may be taken as infinite in the following cases:

1. a bare-wire thermocouple, without enclosing bulb;
2. a thermocouple grounded to the tip of the bulb;
3. a resistance-thermometer element, without enclosing bulb, in which bare or enameled wires are in direct contact with the fluid.

\[ P \]

\[ k \]

\[ \tau_e \]

\[ \psi \]

\[ \rho \]

\[ \tau_f \]

\[ \tau_i \]
If the wires of the sensing element are covered by a layer of glass or ceramic, which, in turn, is in intimate contact with the fluid or with the inner wall of the enclosing bulb, the value of $K_4$ may be approximated by $\pi d_c L_c k_c / 4 h_4$, where $d_c$ and $L_c$ are the respective diameter and length of the glass- or ceramic-covered assembly, and $k_c$ and $h_c$ are, respectively, the thermal conductivity and radial thickness of the ceramic.

Estimation of $K_2$. A conservative representation of $K_2$ is $\pi h m_3$ where $h$ is the fluid heat transfer coefficient; it is expressible in terms of the Nusselt number as

$$h = \frac{k_f N_u}{D} \quad (2)$$

Estimation of $K_3$. The thermal conductance of each wire lead is $\pi d_l^2 m_3 / 4 h_4$ where $L_4$ is the length to the point where $T_b$ occurs and $k_m$ is the average thermal conductivity over the range $T_f$ to $T_b$. Reference (5) lists values of the thermal conductivity integral $\int_{T_f}^{T_b} (T_f - T) dT$ for various materials. For No. 30 B&S Gage wires 10 cm long, with $T_b = 300 K$, $T_f = 90 K$, the value of $K_3(T_b - T_f)$, in watts, is 0.042 and 0.0018 for copper and manganin, respectively.

An alternative computation of $K_3$ would replace $L_4$ by $L_3$, and $T_b$ by $T_a$, on the assumption that the temperature of the thermometer head was communicated to the wires at the point where they pass through the gas tight insulation. Conservatively, one would use whichever method gave the larger temperature error.

If the sensing element is supported by an axial strut anchored at the thermometer head, the temperature error due to lead conductance must be augmented by the error due to strut conductance. This error is $(T_a - T_f) k_s A_s / L_3$, where $A_s$ is the cross sectional area of the support and $k_s$ its average thermal conductivity over the interval $T_f$ to $T_a$. Reference (5) lists the thermal conductivity integral for several plastics and ceramics.

Estimation of $K_4$. The thermal conductance of the bulb wall over the distance $L_3 - L_2$ may be taken as $\pi d_c k_m / (L_3 - L_2)$ where $b$ is wall thickness and $k_m$ is the average conductivity over the interval $T_f$ to $T_b$. Reference (5) lists the value of $(T_a - T_f) k_m / 4 l_4$ for various metals. This computation of $K_4$ is conservative, because it neglects the favorable effect of radial heat conduction through the liquid between the bulb wall and the extension tube.

The thermal conductance of the bulb wall over the distance $L_2$ cannot be stated so simply. Because of the interrelationship between boundary-layer heat transfer and axially-directed conduction, it is more accurate to discuss the value of the ratio $K_4/K_2$ that occurs in Eq. (1).

Estimation of $K_4/K_2$. This ratio must be determined by solving the differential equation of balance between radially-directed convection and axially-directed conduction. To make the problem tractable, it will be assumed that $K_4 = 0$. Later, the correction for the presence of $K_4$ will be indicated. The heat-balance equation is

$$\frac{1}{\eta^2} \cdot \frac{d^2 T}{dx^2} + \frac{1}{\eta} \cdot \frac{dT}{dx} = 0 \quad (3)$$

where

$$\eta^2 = \frac{h}{k_m L_j} = \frac{K_f}{k_m L_j}$$

$T_w = T_f + (T_a - T_f) \psi$, where

$$\psi = \cos \eta x / (\cos \eta L_3 + \eta (L_3 - L_2) \sin \eta L_2) \quad (5)$$

For $\eta L_2 > 4$,

$$\psi = (1 + \eta (L_3 - L_2) \eta L_2) \eta L_2 \cos \eta x / \cos \eta L_2 \quad (6)$$

The quantity $(T_b - T_f) \psi$ in Eq. (5) represents the amount by which bulb wall temperature $T_w$ differs from the desired fluid temperature $T_f$. The difference increases as $x$ increases. If we assume that we are interested in $T_w$ only over the distance $0 < x < L_1$, then the maximum difference occurs when $x = L_1$, at which point

$$\psi = \psi(L_1) = \psi_1 \quad (7)$$

Comparison of Eqs. (1) and (5) leads to the conclusion that taking

$$K_4/K_2 = \psi_1 \quad (8)$$

would provide a conservative estimate of the error due to $K_4$. (A more exact computation leads to the fact that, if $1 < \psi_1 < 10$, then $\psi(0.75 L_1)$ would be a more accurate choice than $\psi(L_1)$, but the more conservative formula of Eq. (9) appears simpler and more desirable.)

To correct for the presence of $K_4$, it would be necessary to multiply the right-hand side of Eq. (4) by $1 + K_4/K_2$. This complication is usually not worthwhile, since its neglect makes the estimate of $K_4/K_2$ more conservative (i.e., larger in value).

Table III lists values of $1/\psi_1$ for various combinations of $L_1/L_2$, $L_3/L_2$, and $nL_2$. Linear interpolation between rows is adequate, but logarithmic interpolation is required between columns. For example, if $L_1/L_2 = 0.90$, $L_3/L_2 = 2$, $nL_2 = 6$, then the value of $1/\psi_1$ is

$$\text{antilog} \left[ \frac{1.0 - 0.9}{1.0 - 0.75} (\log 31.4 - \log 7) + \log 7 \right]$$

The reciprocal of $\psi_1$ has been tabulated in order to obviate negative logarithms in interpolation.
Table III indicates the desirability of having the
length \( L_1 \) only a fraction of the length \( L_2 \) that
is exposed to the fluid stream. Increases in the ratio
\( L_2/L_1 \) also produce distinctive gains. The
column for \( L_2/L_1 = 0 \) represents the case of a
bare-wire thermocouple or of an enclosed thermo-
couple grounded to the tip of the bulb.

High values of \( \eta L_2 \) are desirable. The means of
achieving this goal are to make \( L_2 \) large and \( Db 
small, and to keep \( k_m \) small by using a bulb ma-
terial like austenitic stainless steel.

Nusselt number determination. The value of \( N_{Nu} \)
to be used in Eq. (4) will be determined by the
conditions of the fluid. The following information is
derived from the extensive reviews provided by
McAdaras(2) and Jakob(3).

(1) For fluids flowing transversely to cylin-
ders,
\[
N_{Nu} = 0.95 N^{1/3} Pr^{0.31+0.037 \log N_{Re}} \text{ (1a)}
\]

or
\[
N_{Nu} = 0.32 + 0.48 N^{1/3} Pr^{0.52} \text{ (1b)}
\]

\[
N_{Nu} = 0.27 N^{1/3} Pr^{0.60} \text{ (1c)}
\]

\[
1000 \leq N_{Re} \leq 50000
\]

Equation (1a) is useful when \( N_{Re} \) is known and
\( N_{Nu} \) is to be determined. Equations (1b) and
(1c) are useful in the reverse situation.

Figure 6 gives values of the dimensional quantity
\[ N_{Nu} k_f/k_m = \eta^2 Db \]
as a function of the quantity \( D_u \), for several cryo-
genic liquids at their 1-atmosphere boiling points,
and for water at 300 \( K \), on the assumption that \( k_m \)
is that of austenitic stainless steel.

The normal sequence of computing the error due to
\( k_f \) would be to compute, in turn, \( N_{Re}, N_{Nu} \)
(Eq. (10)), \( \eta^2 \) (Eq. (4)), and \( \psi_1 \) (Eqs. (6) to (8)), and
then to substitute the value of \( K_f/K_g \) (Eq. (9))
into Eq. (1). However, where Fig. 6 can be used, \( \eta^2 \)
ought directly, without having to compute \( N_{Re} \) and
\( N_{Nu} \). Table III then yields the value of \( \psi_1 = K_f/k_m \), without having to use Eqs.
(6) to (8). Thus, one may proceed from Fig. 6 to
Table III to Eq. (1).

(2) For fluids flowing parallel to a cylinder,
the value of \( N_{Nu} \) in Eq. (10) and the ordinate in
Fig. 6 should each be divided by 1.6.

(3) For natural-convection situations, with
bulb horizontal, an iterative approximation of \( N_{Nu} \)
is necessary. For an initially assumed temperature
difference \( 6T \) between bulb wall and fluid, one
successively computes the product of Grashof and
Prandtl numbers
\[
X = N_{Gr} N_{Pr} = (D^5 \beta g \rho \alpha, 2 \cdot 57)/(\mu k_f), \text{ (11)}
\]
the Nusselt number
\[
N_{Nu} = 1.16 X^{-0.13+0.0091 \log X}, \text{ (12)}
\]
and the value of \( \eta^2 \) from Eq. (4). Table III then
gives the value of
\[
\frac{1}{\psi_1} = \frac{T_a - T_f}{6T} \text{ (13)}
\]

from which a new value of \( 6T \) is obtained. The
iterative process (Eqs. (11) to (13)) is repeated
until the result becomes reasonably convergent or
moderately oscillatory. For convenience in com-
puting \( X \), Table I lists the values of the dimen-
sional quantity
\[
\beta g \rho \alpha, 2 / \mu k_f = N_{Gr} N_{Pr} / (D^5, 6T)
\]

for several liquids at their 1-atmosphere boiling
points; \( g \) is the standard acceleration of gravity.

(4) No definitive, generalized heat transfer
data exist for the case where the thermometer bulb
is protected by a perforated metallic tube that
permits internal circulation of fluid. It is rec-
ommended that the temperature of the outer tube
first be computed on the assumption that the fluid
passes only over the outside of the tube; length
\( L_1 \) is to be taken as the length of the inner bulb.
Then, the temperature of the inner bulb is to be
computed, assuming a fluid velocity that represents
the best guess based on perforation size and inter-
nal flow impedance, and assuming a fluid tempera-
ture equal to the computed temperature of the outer
tube.

(5) If boiling is anticipated at the surface
of the thermometer bulb, the most conservative ap-
proach is to assume that the fluid is a gas possess-
ing the temperature and linear velocity of the liq-
uid, but the density and transport properties of
the gas.

TIME LAG

Theory

Internal and external time constants. In many prac-
tical applications, the dynamic behavior of the
thermometer model of Figs. 4 and 5 is conveniently
described in terms of two parameters: an internal
time constant \( \tau_i \) and an external time constant \( \tau_e \).
The internal time constant is an intrinsic
property of the thermometer that can be listed with
its other properties like dimensions, mass, and sen-
sitivity. The external time constant is an extrin-
sic property that depends on the characteristics of
the fluid stream as well as on some properties of
the thermometer bulb. The reason why this subdi-
vision into two parameters is possible is that (a) to
a first and often adequate approximation, the tem-
perature of the bulb wall constitutes the boundary
condition for the solution of the differential equa-
tion of heat transfer between the bulb wall and the

5
sensing element, and (b) to a first and often adequate approximation, the bulb wall temperature will be determined principally by the balance between the rate of heat transfer between bulb and fluid and the rate of heat storage in the bulb wall. The experimental method of determining internal time constant, which will be described later on, provides a check on whether the two-time constant model is a reasonable approximation.

The physical process of communication of temperature from the fluid to the sensing element may be described in the following way. Suppose that a small step change \( \Delta T_f \) occurs in the fluid temperature \( T_f \). The bulb, of mass \( m_2 \) and specific heat \( c_2 \), will respond like a first-order system with external time constant \( \tau_e \), in accordance with the equation

\[
\tau_e \frac{dT_w}{dt} + T_w = T_f + \Delta T_f .
\]

If \( K_3 \) and \( K_4 \) are momentarily neglected to simplify this analysis, the value of \( \tau_e \) is

\[
\tau_e = \frac{m_2 c_2}{K_2} = \frac{m_2 c_2}{mL_4} ,
\]

where \( m_2 \) is of the order of mass of the length \( L_1 \) of the bulb; consequently

\[
\tau_e = \rho_2 c_2 L_1 .
\]

The sensitivity element, within the bounding wall, responds to the exponential change in bulb wall temperature like a first-order system with internal time constant \( \tau_i \), in accordance with the equation

\[
\tau_i \frac{dT}{dt} + T = T_w ,
\]

where

\[
\tau_i = \frac{m_1 c_1}{K} ,
\]

The response, which represents the simultaneous solution of Eqs. (14) and (17), is of the form shown in Fig. 8. The distinctive difference between Figs. 7 and 8 is that, in the latter figure, the slope at the origin is zero, rather than a maximum. The slope in Fig. 8 does not reach a maximum until the lapse of a time that does not exceed \( 0.5(\tau_i + \tau_e) \). The tangent at the point of inflection intersects the asymptote at a time that is \( \tau_i + \tau_e \) greater than the time at which the inflection occurred. The time at which 63 percent of the excursion is completed does not exceed \( 1.08(\tau_i + \tau_e) \), and the time at which 90 percent of the excursion is completed does not exceed \( 2(\tau_i + \tau_e) \).

If there had been no abrupt (step) change in \( T_f \), but \( T_f \) were fluctuating mildly, the response \( T \) would lag behind \( T_f \) by an amount substantially equal to \( \tau_i + \tau_e \).

Departures from the simplified model. The simple model presented heretofore permits estimation of time lags from two time constants: an internal one, \( \tau_i \), which may be determinable experimentally, and an external one, \( \tau_e \), which may be computed. The assumption on which this two-time constant concept is based is tantamount to the assumption that \( K_1 \ll K_2 \). The quantitative data presented above, and the corresponding values of time lag that appear in Fig. 8, are rigorously applicable only to this condition. A rigorous solution applicable to all conditions would require replacement of \( \tau_i \) as defined by Eq. (18) by the quantity \( \tau_i(K_1 + K_2)/K_2 \) in all the various expressions that appear in Fig. 8. The consequence of this more exact representation is that the quantity to replace \( \tau_i \) ranges from \( \tau_i \) where \( K_1 \ll K_2 \), to \( 2\tau_i \) where \( K_1 = K_2 \).

When \( K_1 \gg K_2 \), the two-time constant concept is unnecessary, because the thermometer will behave like a simple first-order system with the single time constant \( (m_1 c_1 + m_2 c_2)/K_2 \).

Effect of conduction. If the effects of \( K_3 \) and \( K_4 \) are included in the preceding description, the following changes occur:

1. Time constant \( m_1 c_1/K_3 \) is changed to \( m_1 c_1/(K_1 + K_3) \);

2. Time constant \( m_2 c_2/K_4 \) is changed to \( m_2 c_2(K_1 + K_3)/K_2 K_4 \) where

\[
F = 1 + K_3/K_1 + (K_3 + K_4)/K_2 + K_2 K_4/K_1 K_2
\]

A more exact computation of the conduction correction for \( \tau_e \) is derived by solving the Fourier equation

\[
\frac{1}{\tau_e^2} \frac{\partial^2 T_w}{\partial x^2} = \frac{2\tau_e}{\tau_i} \frac{\partial T_w}{\partial t} + T_w - T_f
\]

where \( \tau_e \) is given by Eq. (16). The computation, which is quite lengthy, has been treated in detail for the case of a bare thermocouple wire (7); the nature of the conclusion there reached applies also to the case of the thermometer bulb. Substantially, the conclusion is that the effect of the presence of \( K_4 \) is to reduce the time constant \( \tau_e \) to the value \( \tau_e(1 - \psi_1) \), where \( \psi_1 \) is given by Eqs. (7), (8), and (9).

Experimental determination of internal time constant. Usually, \( \tau_i \) must be determined experimentally. Such determination is often feasible in the case of a resistance thermometer; it may be difficult in the case of a thermocouple. The determination makes use of the self-heating effect of the thermometer, and involves a reversal of the usual sequence of responses that was described above, wherein \( T_f \) affected \( T_w \), which then affected \( T \). In this experimental determination, the bulb wall is held at substantially constant temperature and the sensing element is directly exposed to a step change in heat input by abruptly increasing power \( P \) (Fig. 5) by an amount sufficient to raise sensing element temperature by perhaps a degree. The sequence of responses is now as follows: the sensing element, being directly subjected to a step change
in input power, responds in the manner shown in
Fig. 7, except that now the associated time constant
is $\tau_1$, the bulb wall, on the other hand, now being
the second mass in the chain of heat transmission,
responds as shown in Fig. 8. The response of the bulb
wall is not of interest here and will not be
discussed in detail.

The experimental technique works well when the
thermometer bulb is so massive that the time con-
stant $\tau_1$ is large compared to the time constant $\tau_t$. Then, the response of the sensing element to the
step change in $P$ is almost completed before the
bulb wall temperature begins to show apprecia-
table response to the increase in $P$.

The technique will be clarified by a description of
the actual experiment, which will be given next.

**Mechanical arrangement.** A suitable arrangement of
apparatus is shown in Fig. 9. A 4-liter dewar holds
the cryogenic liquid. The dewar is carefully
covered and insulated at the top, except for essen-
tial penetrations, to prevent entry of room air.

The thermometer bulb is immersed in the liquid,
which is maintained in circulation by a motor-driven
stirrer. The motor is well removed from the liquid;
the connecting shaft and the thermometer support
are of low thermal conductance.

**Electrical arrangement for resistance thermometer.**
An appropriate circuit is shown in Fig. 10. A Kelvin-bridge circuit is shown since the increased
accuracy or sensitivity provided by a multiple
bridge is usually unnecessary. Bridge unbalance
is recorded by a strip-chart recorder of linear re-
sponse and adequate response speed. First, with
switch $S$ in position 1, and with $R_1$ so high
that there is negligible self-heating, the bridge
is balanced manually. Switch $S$ is then thrown to
position 2, whereby electric power, limited solely
by $R_2$, is increased sufficiently to produce a
major steady-state deflection of the unbalance re-
corder. The Record of temperature rise of the bulb
may then be used to determine $\tau_1$, by the methods
shown in Fig. 7.

**Quantitative relations for a resistance thermometer.**
Figure 11 shows a record obtained with a Myers(9)
design of platinum resistance thermometer as modi-
fied for cryogenic use(10) by replacing the glass
bulb with a platinum one. (Note the change of time
scale in this record.) A record of this form fol-
lows the equation

$$T = T_0 + (P/K_1)[1 - \exp(-t/\tau_1)]$$

$$+ (P/K_2)[1 - \exp(-t/\tau_2)]$$  \hspace{1cm} (21)

if $K_3 < K_1$, and $(K_3 + K_4) < K_2$. If the first
exponential clearly seems to have an asymptote
(even if the "asymptote" has a slight positive
slope rather than being horizontal), the intercept
of this asymptote on the ordinate at $t = 0$ repre-
sents $P/K_3$, so that $K_1$ can be estimated.

In the above example, $\tau_2$ was much larger than $\tau_1$.
If $\tau_1$ and $\tau_2$ are of comparable magnitudes, the
response will not be of an exact exponential form,
although its shape will resemble that of an expo-
ential. However, a tangent at the origin will in-
tersect the asymptote at a time $m_1/c_1(K_1 + K_2)$
and the ordinate of this asymptote will be

$$P/K_1 + (P/K_2).$$

By use of the above technique, a Myers design of
cryogenic thermometer(9)(10) was found to have a
time constant of 8 seconds at the boiling point of
liquid nitrogen. A germanium resistance ther-
ometer of the design sketched by Sinclair, et al.,(11)
wherin heat presumably is conducted to the sensing
element principally by the leads, had an internal
time constant of 0.3 second at 77 K.

**Electrical arrangement for thermocouple.** In the
case of a thermocouple, the delivery of adequate
electrical power to the thermocouple is not always
possible. In some cases, if the thermocouple wires
have high resistivity, the circuit of Fig. 12 does
permit determination of $\tau_1$. However, none of the
quantitative relations involving $P$, as given above
for a resistance thermometer, are applicable. Only
$\tau_1$ itself can be determined.

The circuit of Fig. 12 applies a step change of a-c
heat input to the thermocouple. This a-c. is kept
from the thermocouple-emf measuring circuit by cir-
cuit elements that have very high normal-mode re-
jection for a-c. To minimize spurious thermal emf's, the junctions $J$ between copper-heating-
circuit wires and the thermocouple wires should
preferably be made without a break in the thermo-
couple wires and should be well immersed in the
stirred cryogenic liquid; the junctions should also
be as close as possible to the thermocouple junc-
tion.

The capacitors $C$ must be high-quality low-leakage
paper dielectric types; several hundred microfarads
may be required to pass the necessary heating cur-
cent. Since full-scale recorder deflection may
represent only one or two degrees, a suppression
circuit must be used to balance out the normal po-
tential difference generated when switch $S$ is in
position 1.

By use of the above technique, an ungrounded thermo-
couple of No. 24 B&S gage Chromel-Alumel wire,
swaged into a 2 mm i.d. sheath, was found to have
an internal time constant of 2 seconds at a tempera-
ture of 77 K.

**CONCLUDING REMARKS**

The information that has been presented may be used
either to establish an upper limit to the error
that might be made in steady-state temperature mea-
surement or to estimate the probable magnitude of
the error. In the latter case, the estimate may be
used as a correction to the temperature measurement,
in order to improve accuracy. However, even after
application of the correction, the probable error
that remains may be between 20 and 50 percent of
the original error. It is therefore desirable to
reduce the steady-state error as much as possible.
This will generally be found to be fairly easy in
the case of liquids flowing through a pipe; it will
be more difficult in the case of stagnant gases in
a tank. In the latter case, it will therefore be particularly desirable to choose so simple a design of the thermometer and its installation that a rather accurate computation of steady-state correction can be made. This design may entail auxiliary direct measurement of $T_a$ and $T_b$.

Deviations from the design of the thermometer model considered here may be used to reduce conduction errors. An obvious improvement, for fluids that are free of abrasive particles, is the use of bleed holes in the bulb wall, not only opposite the sensing element but also nearer to the thermometer head, in order to promote forced cooling of the stem. The use of bleed holes is a common commercial practice.

The method of representing the time response of a thermometer in terms of an internal time constant and an external time constant is a convenient one, particularly in the case of a resistance thermometer, because then the internal time constant can be determined by a simple experiment. It is desirable to perform this experiment at cryogenic temperature, because of the dependence of thermal conductivity and other pertinent properties on temperature. However, even an experiment with water at room temperature may suffice to establish an order of magnitude of the internal time constant. The external time constant can generally be well estimated from established fluid heat transfer relationships; if it cannot be estimated, it will generally be possible at least to estimate an upper bound to its magnitude.

The two-time-constant approach is unnecessary in those thermometer designs, like the bare-wire sensing element directly exposed to the fluid, in which a simple first-order differential equation adequately describes the behavior of the thermometer. In such cases, the computation of an external time constant, based on established heat transfer laws, suffices.

REFERENCES


| Table I. - Properties of Various Liquids |
|---------------------|----|----|----|----|
| Liquid              | N<sub>2</sub> at 20.4 K | N<sub>2</sub> at 77 K | O<sub>2</sub> at 90 K | CH<sub>4</sub> at 112 K | H<sub>2</sub>O at 300 K |
| ρ, g/cc             | 0.071 | 0.81 | 1.14 | 0.424 | 1.00 |
| c, J/(g K)          | 2.5  | 2.05 | 1.71 | 3.47  | 4.16 |
| λ, mW/(cm K)        | 1.19 | 1.40 | 1.50 | 1.92  | 6.18 |
| μ<sub>1</sub>, cP    | 0.0338 | 0.158 | 0.163 | 0.118 | 0.86 |
| ν, cS               | 0.0194 | 0.00106 | 0.00160 | 0.00277 | 0.0086 |
| β, K<sup>-1</sup>    | 0.0162 | 0.0059 | 0.0047 | 0.00340 | 0.00027 |
| N<sub>Fr</sub>      | 1.11  | 2.24  | 2.08  | 2.12  | 5.78 |
| N<sub>T</sub>       | 1.04  | 1.32  | 1.26  | 1.28  | 1.80 |

<p>| Table II. - Thermal Conductivity of Some Solids |
|---------------------|---------------------|</p>
<table>
<thead>
<tr>
<th>Solids</th>
<th>Temperature, K</th>
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<tr>
<td>300 Ser, stainless steel</td>
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<td>Manganese</td>
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<td>Constantan</td>
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<td>Platinum</td>
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<td>Iron</td>
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<td>Copper, elec. tough pitch</td>
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<td>Quartz, glass</td>
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<td>$L_3/L_2$</td>
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Fig. 1. Representative thermometer installations.

Fig. 2. Method of reducing lead-conduction error.
Fig. 3. Thermometer supports in a storage tank.

Fig. 4. Thermometer model.
Fig. 5. Heat flow diagram.

Fig. 6. $N_u U/k_m \cdot \eta^2 D_b$ as a function of $D_u$, for water at 300K and for cryogenic liquids at their normal boiling points; austenitic stainless steel bulb.
Fig. 7. Response of first-order system to a step change.

Fig. 8. Response of first-order system to an exponential change.
Fig. 9. Experimental determination of internal time constant. Mechanical arrangement.

Fig. 10. Internal time constant determination. Circuit for resistance thermometer.
Fig. 11. Typical record of time constant determination.
Type 8164 bulb.

Fig. 12. Internal time constant determination.
Circuit for thermocouple.