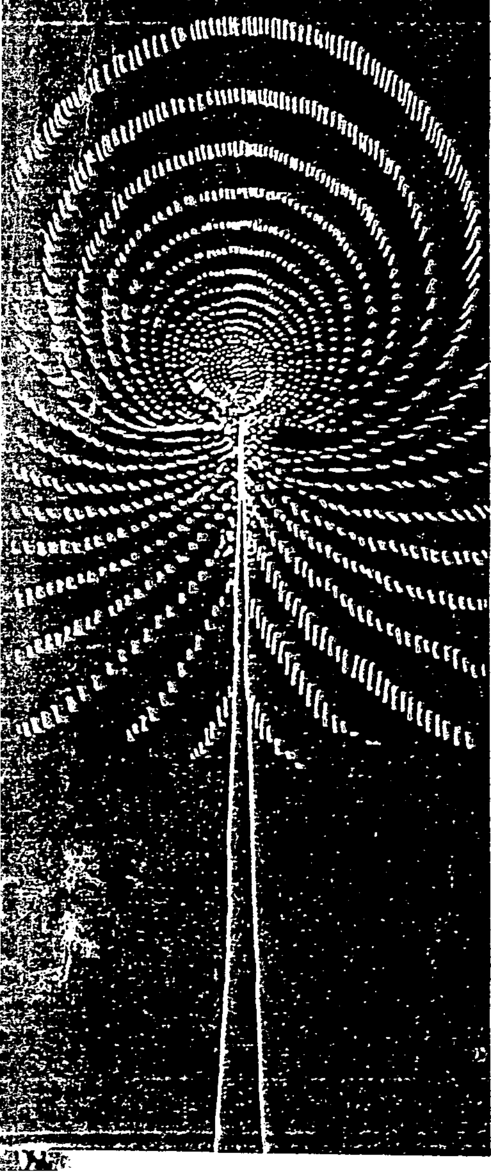
IFSM-73-41

INSTITUTE OF FRACTURE AND SOLID MECHANICS



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Technical Report NASA TR-73-10

May 1973

National Aeronautics and Space Administration
Grant NGR-39-007-011

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Abstract

The plane interaction problem for a circular elastic inclusion imbedded into an elastic matrix which contains an arbitrarily oriented crack is considered. Using the existing solutions for the edge dislocations [6] as Green's functions, first the general problem of a through crack in the form of an arbitrary smooth arc located in the matrix in the vicinity of the inclusion is formulated. The integral equations for the line crack are then obtained as a system of singular integral equations with simple Cauchy kernels. The singular behavior of the stresses around the crack tips is examined and the expressions for the stress intensity factors representing the strength of the stress singularities are obtained in terms of the asymptotic values of the density functions of the integral equations. The problem is solved for various typical crack orientations and the corresponding stress intensity factors are given.

1. INTRODUCTION

In fracture studies of ceramics and other composite materials it is generally conjectured that the fracture of the solid will initiate at and will propagate from a "dominant flaw". This may be a manufacturing flaw, it may be caused by residual stresses or some other type of loading before the part is put into use, or it may result from the growth of a "micro flaw" due to cyclic nature of the operating stresses. In some cases it may be possible to detect such flaws by using nondestructive testing techniques.

*This work was supported by the National Science Foundation under the Grant GK-11977 and by the National Aeronautics and Space Administration under the Grant NGR 39-007-011.

More often, in studies relating to structural integrity and reliability one simply assumes their existence. Thus, preliminary to the application of the relevant fracture theory, one needs to solve the mechanics problem for the composite medium consisting of inclusions and/or pores and the surrounding elastic matrix which contains the flaw. Generally the flaw is assumed to be an internal crack the size of which is of the same order of magnitude as that of the inclusions. The exact elasticity treatment of the three dimensional problem with a regular or a random array of elastic inclusions imbedded into an elastic matrix containing an internal crack appears to be hopelessly complicated. However, one may have some idea about the response of the composite solid if the solution of the related two-dimensional elasticity problem were to be available.

In this paper such an idealized problem will be considered. It will be assumed that the elastic matrix contains only "sparsely" distributed inclusions. Hence the mechanical interaction is primarily between an isolated inclusion or a hole and a line crack arbitrarily located in the neighborhood of the inclusion in the surrounding elastic matrix which is assumed to be infinitely large. The body will be assumed to be in a state of plane strain or generalized plane stress. The special cases of this problem in which the geometry of the medium and the external loads contain a plane of symmetry were considered in [1-4]. [4] also includes the solution of the problems for two collinear cracks (one in the matrix, one in the inclusion), for a crack terminating at the inclusion-matrix interface, and for a crack crossing the interface.

2. THE INTEGRAL EQUATIONS

As in many crack problems, the solution of the present problem too may be obtained through the superposition of two solutions. The first refers to the simple problem of a circular elastic inclusion inserted into a matrix without the crack. This problem is solved under the given system of external loads. In the second problem only the stress disturbance due to the existence of the crack in the matrix is considered. In this problem the only external loads are the crack surface tractions which are equal in magnitude and opposite in sign to the stresses obtained in the first problem along the line which is the presumed location of the crack. The nonhomogeneous medium may be subjected to an arbitrary set of external loads (including quasi-static thermal loads) applied to the matrix and the inclusion. However, it is assumed that the dimensions of the matrix are sufficiently large so that in the second problem the interaction between the outer boundary of the matrix and the crack-inclusion combination may be neglected. Thus, in the second problem, which contains the singular part of the solution, the matrix will be assumed as being infinite. It will henceforth be assumed that the solution of the first problem is known. For example, if the matrix is infinitely large and is subjected to biaxial stresses at infinity, this solution may be obtained by adding the results for two perpendicular uniaxial stresses given in [5], namely

$$\sigma_{rr} = \frac{\sigma_o}{2}\left[1 - \frac{\gamma R^2}{r^2} + \left(1 - \frac{2\beta R^2}{r^2} - \frac{3\delta R^4}{r^4}\right)\cos 2\theta\right] ,$$

$$\sigma_{\theta\theta} = \frac{\sigma_o}{2}\left[1 + \frac{\gamma R^2}{r^2} - \left(1 - \frac{3\delta R^4}{r^4}\right)\cos 2\theta\right] ,$$

$$\sigma_{r\theta} = -\frac{\sigma_o}{2}\left(1 + \frac{\beta R^2}{r^2} + \frac{3\delta R^4}{r^4}\right)\sin 2\theta \ , \qquad (1.a\text{-}c)$$

$$\beta = -\frac{2(\mu_2-\mu_1)}{\mu_1+\kappa_1\mu_2} \ , \qquad \gamma = \frac{\mu_1(\kappa_2-1) - \mu_2(\kappa_1-1)}{2\mu_2 + \mu_1(\kappa_2-1)} \ ,$$

$$\delta = \frac{\mu_2-\mu_1}{\mu_1+\kappa_1\mu_2} \ , \qquad (2.a\text{-}c)$$

where σ_o is the uniaxial stress at infinity, R is the radius of the inclusion, r and θ are the polar coordinates (r is measured from the center of the inclusion, θ is measured from the direction of loading, σ_o), and μ_1,κ_1 and μ_2,κ_2 are, respectively, the elastic constants of the matrix and the inclusion.

The integral equations for the stress disturbance problem may be obtained by using the solution for the edge dislocations given in [6] as the Green's functions. Referring to Figure 1(a), let the matrix 1 contain two edge dislocations at the point ($x=c_o$, $y=0$) with Burgers vectors b_x and b_y. Following [6], the stresses at a point P(x,y) in the matrix may be expressed as

$$\frac{\pi(\kappa_1+1)}{\mu_1}\sigma^1_{yy}(x,y,c_o) = h_{yy1}(x,y,c_o)b_x + h_{yy2}(x,y,c_o)b_y \ ,$$

$$\frac{\pi(\kappa_1+1)}{\mu_1}\sigma^1_{xx}(x,y,c_o) = h_{xx1}(x,y,c_o)b_x + h_{xx2}(x,y,c_o)b_y \ ,$$

$$\frac{\pi(\kappa_1+1)}{\mu_1}\sigma^1_{xy}(x,y,c_o) = h_{xy1}(x,y,c_o)b_x + h_{xy2}(x,y,c_o)b_y \ ,$$

$$(3.a\text{-}c)$$

where

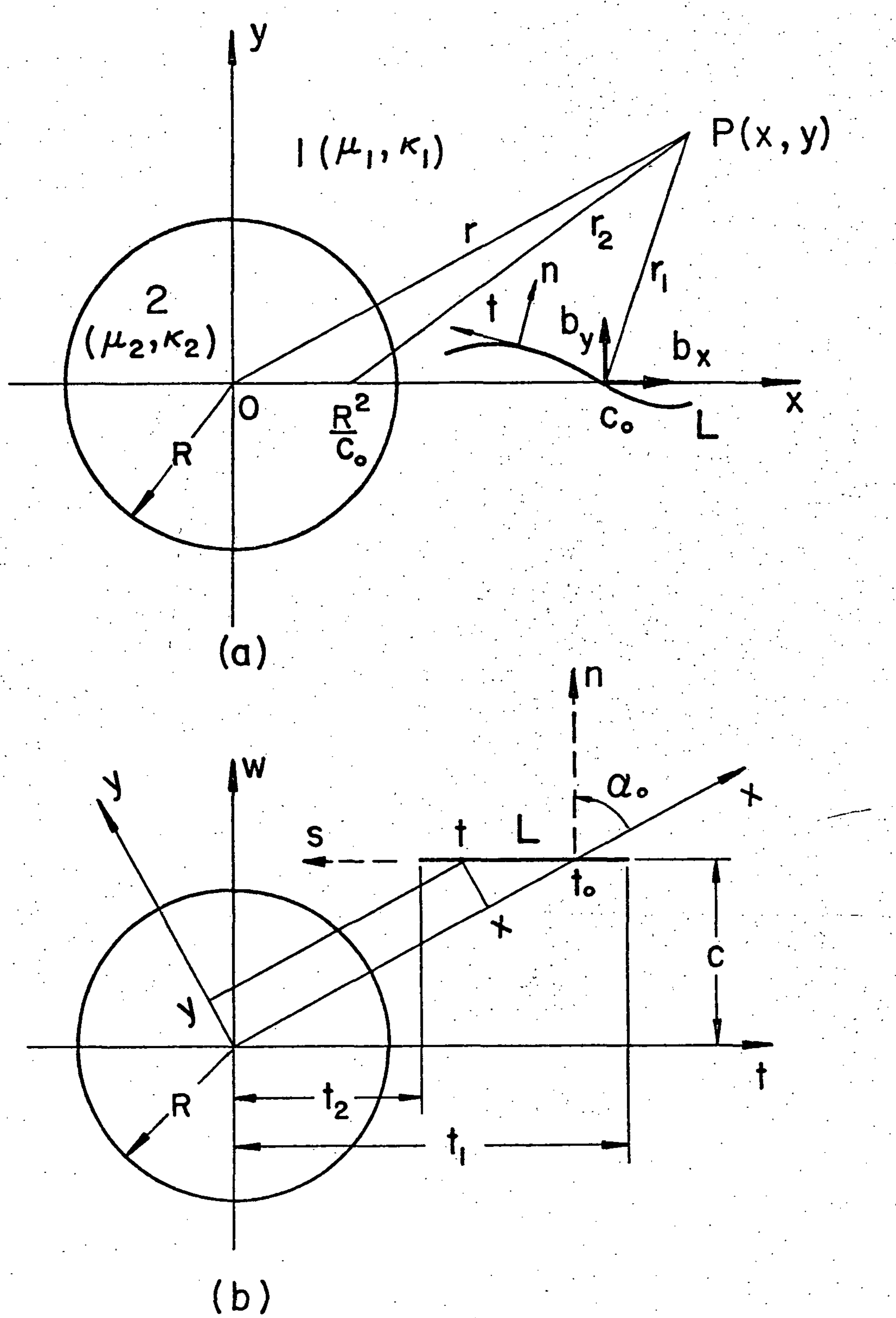

Figure 1. Geometry showing the dislocations b_x and b_y, and the crack L in the neighborhood of the inclusion 2.

$$h_{yy1} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left(3A - B - 4A\frac{x_2^2}{r_2^2}\right)\frac{y}{r_2^2} - \left(3A - B - 4A\frac{x^2}{r^2}\right)\frac{y}{r^2}$$

$$- 2A\frac{\beta^2-1}{\beta^3}\frac{R}{r_2^2}\left[-\frac{6x_2y}{r_2^2} + \frac{8x_2^3y}{r_2^4} + \frac{\beta^2-1}{\beta}\left(1 - \frac{4x_2^2}{r_2^2}\right)\frac{Ry}{r_2^2}\right]$$

$$+ \frac{B-A}{\beta}\frac{2xyR}{r^4} + 2A\left(1 - \frac{4x^2}{r^2}\right)\frac{R^2y}{r^4},$$

$$h_{xx1} = -2\left(1 + \frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left[B + A\left(1 + \frac{4x_2^2}{r_2^2}\right)\right]\frac{y}{r_2^2}$$

$$- \left[B + A\left(1 + \frac{4x^2}{r^2}\right)\right]\frac{y}{r^2} - 2\frac{B-A}{\beta}\frac{xyR}{r^4} - 2A\left(1 - \frac{4x^2}{r^2}\right)\frac{R^2y}{r^4}$$

$$- A\frac{\beta^2-1}{\beta^3}\left[\left(1 - \frac{4x_2^2}{r_2^2}\right)\frac{2x_2}{r_2^2} - \frac{\beta^2-1}{\beta}\left(1 - \frac{4x_2^2}{r_2^2}\right)\frac{R}{r_2^2}\right]\frac{2yR}{r_2^2},$$

$$h_{xy1} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} + \left(3A - B - 4A\frac{x_2^2}{r_2^2}\right)\frac{x_2}{r_2^2}$$

$$- 2A\frac{\beta^2-1}{\beta^3}\left[1 - \frac{8x_2^2}{r_2^2} + \frac{8x_2^4}{r_2^4} + \frac{\beta^2-1}{\beta}\left(3 - 4\frac{x_2^2}{r_2^2}\right)\frac{Rx_2}{r_2^2}\right]\frac{R}{r_2^2}$$

$$- \left(3A - B - 4A\frac{x^2}{r^2}\right)\frac{x}{r^2} - (B-A)\frac{1}{\beta}\left(1 - 2\frac{x^2}{r^2}\right)\frac{R}{r^2}$$

$$+ 2A\left(3 - 4\frac{x^2}{r^2}\right)\frac{R^2x}{r^4},$$

$$h_{yy2} = 2\left(3 - \frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} - \left(5A + B - 4A\frac{x_2^2}{r_2^2}\right)\frac{x_2}{r_2^2}$$

$$+ \left(5A + B - 4A\frac{x^2}{r^2}\right)\frac{x}{r^2} - A\frac{\beta^2-1}{\beta^3}\left[2(2-\beta^2) - 4(5-\beta^2)\frac{x_2^2}{r_2^2}\right.$$

$$\left. + 16\frac{x_2^4}{r_2^4} + \frac{2(\beta^2-1)}{\beta}\left(3 - 4\frac{x_2^2}{r_2^2}\right)\frac{Rx_2}{r_2^2}\right]\frac{R}{r_2^2} - 2A\left(3 - \frac{4x^2}{r^2}\right)\frac{R^2x}{r^4}$$

$$- \frac{1}{\beta}\left(1 - \frac{2x^2}{r^2}\right)\frac{R}{r^2}\left[A(2\beta^2-1) + M(1+\kappa_2) - 1\right] ,$$

$$h_{xx2} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{x_1}{r_1^2} + \left[B + A\left(1 - \frac{4x_2^2}{r_2^2}\right)\right]\frac{x_2}{r_2^2} - \left[B + A\left(1 - \frac{4x^2}{r^2}\right)\right]\frac{x}{r^2}$$

$$- A\frac{\beta^2-1}{\beta^3}\left[2\beta^2\left(1 - \frac{2x_2^2}{r_2^2}\right) + 4\left(3 - 4\frac{x_2^2}{r_2^2}\right)\frac{x_2^2}{r_2^2}\right.$$

$$\left. - \frac{2(\beta^2-1)}{\beta}\left(3 - 4\frac{x_2^2}{r_2^2}\right)\frac{Rx_2}{r_2^2}\right]\frac{R}{r_2^2}$$

$$+ \left[A(2\beta^2-1) + M(1+\kappa_2) - 1\right]\frac{1}{\beta}\left(1 - \frac{2x^2}{r^2}\right)\frac{R}{r^2} + 2A\left(3 - \frac{4x^2}{r^2}\right)\frac{R^2x}{r^4} ,$$

$$h_{xy2} = -2\left(1 - \frac{2x_1^2}{r_1^2}\right)\frac{y}{r_1^2} + \left[B + A\left(1 - \frac{4x_2^2}{r_2^2}\right)\right]\frac{y}{r_2^2} - \left[B + A\left(1 - \frac{4x^2}{r^2}\right)\right]\frac{y}{r^2}$$

$$+ 2A\frac{\beta^2-1}{\beta^3}\left[\left(2\beta^2 - 4 + 8\frac{x_2^2}{r_2^2}\right)\frac{x_2y}{r_2^2} + \frac{\beta^2-1}{\beta}\left(1 - \frac{4x_2^2}{r_2^2}\right)\frac{Ry}{r_2^2}\right]\frac{R}{r_2^2}$$

$$- \left[A(2\beta^2-1) + M(1+\kappa_2) - 1\right]\frac{1}{\beta}\frac{2xyR}{r^4} + 2A\left(1 - \frac{4x^2}{r^2}\right)\frac{R^2y}{r^4}] ;$$

(4.a-f)

$$\beta = \frac{c}{R}, \qquad A = \frac{1-m}{1+m\kappa_1}, \qquad B = \frac{\kappa_2 - m\kappa_1}{\kappa_2 + m}, \qquad m = \frac{\mu_2}{\mu_1},$$

$$M = \frac{m(1+\kappa_1)}{(m+\kappa_2)(\kappa_2 - 1 + 2m)}; \tag{5}$$

$$x_1 = x - c_o, \qquad x_2 = x - \frac{R^2}{c_o}, \qquad r^2 = x^2 + y^2,$$

$$r_1^2 = (x-c_o)^2 + y^2, \qquad r_2^2 = (x - \frac{R^2}{c_o})^2 + y^2. \tag{6}$$

Let us now assume that the matrix contains a cut along a smooth arc L going through the point $(c_o,0)$ (Figure 1a). Let $\sigma_n(s)$ and $\sigma_t(s)$, $(s \in L)$ be the normal and tangential components of the stress vector on L in the composite without the cut and subjected to the given set of external loads, where s is the arc length measured on L (i.e., the solution of the first problem). Let the point $(c_o,0)$ correspond to $s = s_o$ on L. Once L is specified, from (3) we may also obtain the normal and tangential stresses on L due to the dislocations b_x and b_y only as follows:

$$\sigma_n^b(s,s_o) = h_{n1}(s,s_o)b_x + h_{n2}(s,s_o)b_y,$$

$$\sigma_t^b(s,s_o) = h_{t1}(s,s_o)b_x + h_{t2}(s,s_o)b_y, \tag{7.a,b}$$

where h_{ni} and h_{ti}, $(i=1,2)$ may be obtained from (4) by using the coordinates on L. If n and t refer to the positive normal and tangent to L at s (Figure 1a) and if $\alpha = \alpha(s)$ is the angle between the x-axis and the normal n, noting that (3) is also valid for $(x,y) \in L$, σ_n^b and σ_t^b are given by

$$\sigma_n^b = \sigma_{xx}^1 \cos^2\alpha + \sigma_{yy}^1 \sin^2\alpha + 2\sigma_{xy}^1 \sin\alpha\cos\alpha,$$

$$\sigma_t^b = (\sigma_{yy}^1 - \sigma_{xx}^1)\sin\alpha\cos\alpha + \sigma_{xy}^1(\cos^2\alpha - \sin^2\alpha). \tag{8.a,b}$$

Substituting $x = x(s)$, $y = y(s)$ from the equation of L, (3), (4), (7) and (8) give h_{ni} and h_{ti}, $(i=1,2)$.

If we now assume that b_x and b_y are continuously distributed (unknown) functions (of s_0) on L, the stress disturbance problem (the second problem) may be formulated as

$$-\sigma_n(s) = \int_L [h_{n1}(s,s_0)b_x(s_0) + h_{n2}(s,s_0)b_y(s_0)]ds_0 \ ,$$

$$-\sigma_t(s) = \int_L [h_{t1}(s,s_0)b_x(s_0) + h_{t2}(s,s_0)b_y(s_0)]ds_0 \ . \qquad (9.a,b)$$

The single-valuedness condition of displacement vector requires that the density functions of the system of integral equations (9) satisfy the following relations:

$$\int_L b_x(s_0)ds_0 = 0 \ , \qquad \int_L b_y(s_0)ds_0 = 0 \ . \qquad (10.a,b)$$

Superimposed on the solution of the corresponding "first problem", in principle (9) and (10) give the solution to any problem of an arbitrarily oriented arc-shaped crack in the matrix.

In particular if the crack is along a straight arc L shown in Figure 1b the transformations would be considerably simplified. Let (t,w) be the fixed coordinate system and the end points of the cut be at (t_1,c) and (t_2,c) (Figure 1b). Note that on L

$$\alpha = \alpha_0, \quad t = -s, \quad \sigma_{ww} = \sigma_{nn} = \sigma_n^b, \quad \sigma_{wt} = -\sigma_{ns} = -\sigma_t^b \ . \qquad (11)$$

The relations and quantities necessary to express the kernels in (9) explicitly in terms of t and t_0 may then be written as

$$\sin\alpha = \frac{t_0}{\sqrt{c^2 + t_0^2}} \ , \qquad \cos\alpha = \frac{c}{\sqrt{c^2 + t_0^2}} \ ,$$

$$x = t \sin\alpha + c \cos\alpha \ , \qquad y = c \sin\alpha - t \cos\alpha \ ,$$

$$x_1 = \frac{t_o(t - t_o)}{\sqrt{c^2 + t_o^2}} , \qquad x_2 = \frac{tt_o + c^2 - R^2}{\sqrt{c^2 + t_o^2}} , \qquad r^2 = c^2 + t^2 ,$$

$$r_1^2 = (t_o - t)^2 , \qquad r_2^2 = \frac{(tt_o + c^2 - R^2)^2 + c^2(t_o - t)^2}{c^2 + t_o^2} . \qquad (12)$$

If we now let the stresses on L obtained from the solution of the problem without the crack be

$$-\sigma_{ww}(t,c) = p_1(t) , \qquad -\sigma_{wt}(t,c) = p_2(t) , \qquad (13)$$

and define

$$b_x(s_o) = -f_1(t_o) , \qquad b_y(s_o) = -f_2(t_o) , \qquad (14)$$

the system of integral equations (9) and the conditions (10) may be expressed as

$$\frac{\pi(\kappa_1 + 1)}{2\mu_1} p_i(t) = \int_{t_2}^{t_1} \sum_{j=1}^{2} K_{ij}(t,t_o) f_j(t_o) dt_o ,$$

$$(i=1,2;\ t_2<t<t_1) . \qquad (15)$$

$$\int_{t_2}^{t_1} f_i(t)dt = 0 , \qquad (i=1,2) . \qquad (16)$$

It is not difficult to show that at $t = t_o$ the kernels $K_{ij}(t,t_o)$ have Cauchy type singularities. In fact, examining the terms containing x_1/r_1^2 and y/r_1^2 in (4) we find

$$\frac{x_1}{r_1^2} = -\left(\frac{t_o}{\sqrt{c^2 + t_o^2}}\right) \frac{1}{t_o - t} , \qquad \frac{y}{r_1^2} = \left(\frac{c}{\sqrt{c^2 + t_o^2}}\right) \frac{1}{t_o - t} , \qquad (17)$$

whereas all the remaining terms can be shown to be bounded in the closed interval $t_2 \le (t,t_o) \le t_1$. Separating the singular parts of the kernels, (15) may be written as

$$\frac{\pi(\kappa_1+1)}{2\mu_1} p_1(t) = \int_{t_2}^{t_1} f_1(t_o) \frac{c}{\sqrt{c^2+t_o^2}} \frac{dt_o}{t_o - t} + \int_{t_2}^{t_1} k_{11}(t,t_o) f_1(t_o) dt_o$$

$$+ \int_{t_2}^{t_1} f_2(t_o) \frac{t_o}{\sqrt{c^2+t_o^2}} \frac{dt_o}{t_o - t} + \int_{t_2}^{t_1} k_{12}(t,t_o) f_2(t_o) dt_o \ ,$$

$$\frac{\pi(\kappa_1+1)}{2\mu_1} p_2(t) = \int_{t_2}^{t_1} f_1(t_o) \frac{t_o}{\sqrt{c^2+t_o^2}} \frac{dt_o}{t_o - t} + \int_{t_2}^{t_1} k_{21}(t,t_o) f_1(t_o) dt_o$$

$$- \int_{t_2}^{t_1} f_2(t_o) \frac{c}{\sqrt{c^2+t_o^2}} \frac{dt_o}{t_o - t} + \int_{t_2}^{t_1} k_{22}(t,t_o) f_2(t_o) dt_o \ ,$$

$$(t_2<t<t_1), \qquad (18.a,b)$$

where $k_{ij}(t,t_o)$, $(i,j=1,2)$ are bounded functions in the closed interval $(t_2 \leq (t_o,t) \leq t_1)$. For the homogeneous medium $\mu_1 = \mu_2$, $\kappa_1 = \kappa_2$, $m=1$, $A=0=B$, $M=1/(1+\kappa_1)$. Consequently, k_{ij} may be shown to vanish. Thus, defining t,w-components of the dislocations by

$$b_x \sin\alpha - b_y \cos\alpha = b_t = -g_1(t_o) \ ,$$

$$b_x \cos\alpha + b_y \sin\alpha = b_w = -g_2(t_o) \ , \qquad (19.a,b)$$

the integral equations (18) may be expressed as

$$\frac{\pi(\kappa_1+1)}{2\mu_1} p_1(t) = \int_{t_2}^{t_1} \frac{g_2(t_o)}{t_o - t} dt_o \ ,$$

$$\frac{\pi(\kappa_1+1)}{2\mu_1} p_2(t) = \int_{t_2}^{t_1} \frac{g_1(t_o)}{t_o - t} dt_o \ , \qquad (t_1<t<t_2). \qquad (20.a,b)$$

(20) are the well-known integral equations for a straight crack in an infinite homogeneous medium in which the normal and tangential crack surface tractions p_1 and p_2 are the only external loads and

the density functions g_1 and g_2 are related to the crack surface displacements by [7]

$$g_1(t) = \frac{\partial}{\partial t}(u_t^+ - u_t^-) , \qquad g_2(t) = \frac{\partial}{\partial t}(u_w^+ - u_w^-) , \qquad (21.a,b)$$

where $u_t^+ = u_t(t,c+0)$, $u_t^- = u_t(t,c-0)$, etc.

3. STRESS INTENSITY FACTORS

Following [8] it may be shown that the indexes of the singular integral equations (18.a) and (18.b) are +1. Hence, the general solution will contain two arbitrary constants which may be determined from the single-valuedness conditions (16). Also, the fundamental functions of the system are

$$w_1(t) = w_2(t) = (t - t_2)^{-1/2}(t_1 - t)^{-1/2} . \qquad (22)$$

Hence the solution of (18) may be expressed as [8]

$$f_i(t) = w_i(t)F_i(t) , \qquad (i=1,2;\ t_2<t<t_1), \qquad (23)$$

where F_1 and F_2 are Hölder-continuous in the closed interval $[t_2,t_1]$. In order to investigate the singular behavior of the stresses around the end points of the crack, we note that the expressions (18) give the normal and the tangential components of the stresses in the perturbation problem outside as well as inside the cut L, i.e.,

$$\sigma_{ww}^2(t,c) = p_1(t) , \qquad \sigma_{tw}^2(t,c) = p_2(t) , \qquad (24.a,b)$$

for all t in the matrix. Let us now define the following sectionally holomorphic functions

$$\phi_{kj}(z) = \frac{1}{\pi}\int_{t_2}^{t_1} \frac{a_{kj}(t_o)f_j(t_o)}{t_o - z}\,dt_o$$

$$= \frac{i}{\pi}\int_{t_2}^{t_1} \frac{a_{kj}(t_o)F_j(t_o)}{(t_o - z)[(t_o - t_1)(t_o - t_2)]^{1/2}}\,dt_o\,,$$

$$(k,j = 1,2), \qquad (25)$$

where

$$a_{11} = -a_{22} = \frac{c}{\sqrt{c^2 + t_o^2}}\,, \qquad a_{12} = a_{21} = \frac{t_o}{\sqrt{c^2 + t_o^2}}\,. \qquad (26)$$

Following [8] the asymptotic behavior of the Cauchy integral (25) may be expressed as

$$\phi_{kj}(z) = ia_{kj}(t_2)F_j(t_2)(t_1 - t_2)^{-1/2}(z - t_2)^{-1/2}$$

$$- a_{kj}(t_1)F_j(t_1)(t_1 - t_2)^{-1/2}(z - t_1)^{-1/2} + \phi_{okj}(z)\,,$$

$$(27)$$

where around t_1 and t_2 ϕ_{okj} is generally bounded or at most has a singularity which is weaker than that of $(z - t_j)^{-1/2}$. Since $\phi_{kj}(z)$ is holomorphic outside the cut we may write

$$\frac{1}{\pi}\int_{t_2}^{t_1} \frac{a_{kj}(t_o)f_j(t_o)}{t_o - t}\,dt_o = \phi_{kj}(t)$$

$$= \frac{a_{kj}(t_2)F_j(t_2)}{(t_1 - t_2)^{1/2}(t_2 - t)^{1/2}} - \frac{a_{kj}(t_1)F_j(t_1)}{(t_1 - t_2)^{1/2}(t - t_1)^{1/2}} + \phi_{okj}(t)\,,$$

$$(t<t_2,\ t>t_1). \qquad (28)$$

From (18) and (28) it is seen that around the crack tips $t=t_1$ and $t=t_2$ the stresses p_1 and p_2 will have the conventional square-root singularity. If we now define the "stress intensity factors" as follows:

$$k_1(t_1) = \lim_{t\to t_1} \sqrt{2(t - t_1)}\, p_1(t) ,$$

$$k_2(t_1) = \lim_{t\to t_1} \sqrt{2(t - t_1)}\, p_2(t) ,$$

$$k_1(t_2) = \lim_{t\to t_2} \sqrt{2(t_2 - t)}\, p_1(t) ,$$

$$k_2(t_2) = \lim_{t\to t_2} \sqrt{2(t_2 - t)}\, p_2(t) , \qquad (29.a\text{-}d)$$

and let

$$a = (t_2 - t_1)/2 , \qquad (30)$$

noting that

$$\int_{t_2}^{t_1} k_{ij}(t,t_o) f_j(t_o) dt_o = \text{finite} , \quad (-\infty<t<\infty) , \qquad (31)$$

we find

$$k_1(t_1) = -\frac{2\mu_1}{1+\kappa_1} \frac{1}{\sqrt{a(c^2 + t_1^2)}} [cF_1(t_1) + t_1 F_2(t_1)] ,$$

$$k_2(t_1) = -\frac{2\mu_1}{1+\kappa_1} \frac{1}{\sqrt{a(c^2 + t_1^2)}} [t_1 F_1(t_1) - cF_2(t_1)] ,$$

$$k_1(t_2) = \frac{2\mu_1}{1+\kappa_1} \frac{1}{\sqrt{a(c^2 + t_2^2)}} [cF_1(t_2) + t_2 F_2(t_2)] ,$$

$$k_2(t_2) = \frac{2\mu_1}{1+\kappa_1} \frac{1}{\sqrt{a(c^2 + t_2^2)}} [t_2 F_1(t_2) - cF_2(t_2)] , \qquad (32.a\text{-}d)$$

where F_1 and F_2 are the bounded functions defined by (23). It should be obersved that if the transformation (19) is made before solving the integral equations, repeating the foregoing analysis or directly from (32) it is seen that in terms of the new unknown functions g_1 and g_2 the stress intensity factors may be expressed as

$$k_1(t_1) = -\lim_{t \to t_1} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(t_1 - t)}\, g_2(t) ,$$

$$k_2(t_1) = -\lim_{t \to t_1} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(t_1 - t)}\, g_1(t) ,$$

$$k_1(t_2) = \lim_{t \to t_2} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(t - t_2)}\, g_2(t) ,$$

$$k_2(t_2) = \lim_{t \to t_2} \frac{2\mu_1}{1+\kappa_1} \sqrt{2(t - t_2)}\, g_1(t) , \qquad (33.a\text{-}d)$$

where g_1 and g_2 are the derivatives of the crack surface displacements (see (21)).

4. NUMERICAL RESULTS

Once the tractions p_1 and p_2 are specified, the system of singular integral equations may be solved in a straightforward manner (see [9] for an effective numerical technique). After solving these equations, all the desired field quantities may be evaluated by means of definite integrals with appropriate kernels and f_1 and f_2 as the density functions. Since the main interest in this paper is in the fracture of composite materials the numerical results will be presented only for the stress intensity factors defined by (29) and evaluated from (32). The results are given for a uniaxial stress σ_0 at infinity (see the insert in Figures 2-11). The calculated results are shown in Tables 1-7 and

Figures 2-11. The stress intensity factor ratios, k_{ij}, $(i,j=1,2)$ shown in the tables and the figures are normalized with respect to $\sigma_0\sqrt{a}$ which is the stress intensity factor in a uniaxially stressed infinite plane containing a crack of length 2a perpendicular to the direction of loading. Thus

$$k_{ij} = \frac{k_i(t_j)}{\sigma_0\sqrt{a}} , \qquad (i,j=1,2) , \tag{34}$$

where $k_i(t_j)$ is defined by (29). The length parameters a, b, c, and R are shown in the insert of Figures 2-11. In the numerical analysis it is assumed that the crack is perpendicular to the direction of external loading, σ_0. Only two material combinations have been considered. The first refers to a circular hole (i.e., $\mu_2 = 0$) and in the second it is assumed that

$$(\mu_2/\mu_1) = 23 , \qquad \kappa_1 = 1.6 , \qquad \kappa_2 = 1.8 \tag{35}$$

which roughly corresponds to a metallic inclusion imbedded into an epoxy-type matrix.

Table 1 and Figure 2 show the results for a radial crack perpendicular to σ_0. Note that as the crack tip t_2 approaches the hole k_{12} goes to infinity and as it approaches the inclusion-matrix interface (where $\mu_2 > \mu_1$) k_{12} goes to zero. This problem of a crack terminating at and going through the interface was extensively studied in [10, 11, and 4]. It should also be noted that for $\mu_2 = 0$ and $b/a = 3$, k_{11} will be finite (see [4]).

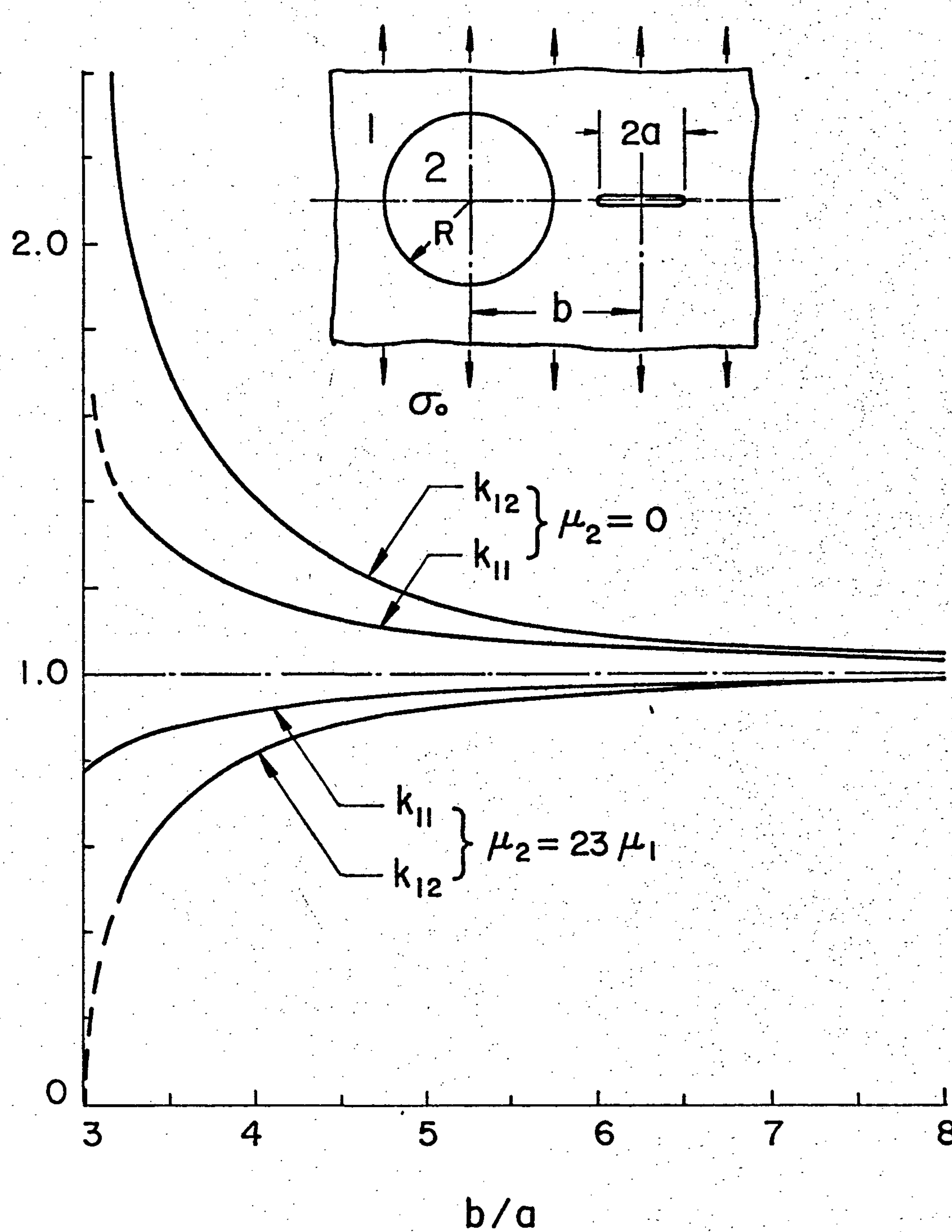

Figure 2. The stress intensity factors for a symmetrically located radial crack ($R/a = 2$, $c = 0$).

Table 1. The stress intensity factors for a symmetrically located radial crack ($R/a = 2$, $c = 0$).

	$\mu_2 = 0$		$\mu_2 = 23\mu_1$	
b/a	k_{11}	k_{12}	k_{11}	k_{12}
3.0		$\to \infty$		$\to 0$
3.2	1.417	2.274	0.827	0.467
3.5	1.290	1.722	0.874	0.671
4	1.188	1.394	0.918	0.821
5	1.102	1.174	0.957	0.924
6	1.065	1.099	0.973	0.959
8	1.033	1.045	0.987	0.982
∞	1.0	1.0	1.0	1.0

Table 2. Stress intensity factors for a symmetrically located "tangential crack" perpendicular to the load ($b = 0$, $R = 2a$).

	$\mu_2/\mu_1 = 0$		$\mu_2/\mu_1 = 23$	
c/a	$k_{11} = k_{12}$	$k_{21} = -k_{22}$	$k_{11} = k_{12}$	$k_{21} = -k_{22}$
2.1	0.0626	-0.170	1.201	0.105
2.5	0.260	-0.101	1.196	0.171
3	0.395	-0.0644	1.196	0.202
4	0.574	-0.0175	1.180	0.180
6	0.768	0.0274	1.110	0.109
10	0.906	0.0361	1.046	0.0567
∞	1.0	0	1.0	0

Table 3. Stress intensity factors for a crack perpendicular to the external load ($R = 2a$, $b = 3a$).

	$\mu_2 = 0$				$\mu_2 = 23\mu_1$			
c/a	k_{11}	k_{12}	k_{21}	k_{22}	k_{11}	k_{12}	k_{21}	k_{22}
0		$\to \infty$	$\to 0$	$\to 0$		$\to 0$	$\to 0$	$\to 0$
0.3	1.607	4.267	0.072	-0.391	0.784	0.225	-0.004	0.072
0.5	1.544	3.070	0.113	-0.412	0.792	0.341	-0.006	0.101
1.0	1.430	1.969	0.177	-0.316	0.817	0.613	-0.005	0.057
1.5	1.357	1.552	0.184	-0.241	0.839	0.763	0.008	-0.007
2	1.299	1.337	0.150	-0.243	0.860	0.845	0.034	-0.021
3	1.189	1.083	0.052	-0.282	0.905	0.953	0.089	-0.001
4	1.091	0.957	0.011	-0.261	0.951	1.014	0.117	0.002
8	0.955	0.911	-0.014	-0.114	1.020	1.043	0.088	-0.026
∞	1.0	1.0	0	0	1.0	1.0	0	0

Table 4. Stress intensity factors for a crack perpendicular to the external load ($R = 2a$, $c = a$).

	$\mu_2 = 0$				$\mu_2 = 23\mu_1$			
b/a	k_{11}	k_{12}	k_{21}	k_{22}	k_{11}	k_{12}	k_{21}	k_{22}
$\sqrt{3}/2$		$\to \infty$		$\to -\infty$		$\to 0$		$\to 0$
2.8	1.562	2.700	0.206	-0.790	0.781	0.483	0.002	0.167
3.0	1.430	1.970	0.177	-0.316	0.817	0.613	-0.005	0.057
3.5	1.274	1.548	0.134	-0.047	0.878	0.752	-0.012	-0.047
4	1.193	1.364	0.107	0.023	0.914	0.833	-0.012	-0.068
6	1.070	1.109	0.048	0.032	0.970	0.952	-0.006	-0.041
8	1.036	1.049	0.025	0.015	0.985	0.980	-0.002	-0.021
10	1.022	1.028	0.014	0.007	0.991	0.989	-0.0004	-0.012
∞	1.0	1.0	0	0	1.0	1.0	0	0

Table 5. Stress intensity factors for a crack perpendicular to the external load ($c = R$, $b-a = 0.2R$; a = constant, R variable, or R = constant, a variable).

	$\mu_2 = 0$				$\mu_2 = 23\mu_1$			
R/a	k_{11}	k_{12}	k_{21}	k_{22}	k_{11}	k_{12}	k_{21}	k_{22}
5	0.366	0.0582	-0.102	-0.0210	1.120	1.096	0.0256	0.124
3	0.882	0.257	-0.0774	-0.349	0.984	1.063	0.139	0.150
2	1.247	0.600	0.0935	-0.826	0.868	1.022	0.172	0.162
1.5	1.354	0.926	0.246	-1.226	0.829	0.990	0.168	0.162
1.0	1.312	1.384	0.400	-1.719	0.839	0.951	0.165	0.153
0.75	1.225	1.679	0.453	-1.972	0.866	0.929	0.176	0.144
0.5	1.092	2.180	0.486	-2.239	0.899	0.909	0.194	0.129

Table 6. Stress intensity factors for a crack perpendicular to the external load ($R = 2a$, $c = 2.2a$).

	$\mu_2 = 0$				$\mu_2 = 23\mu_1$			
b/a	k_{11}	k_{12}	k_{21}	k_{22}	k_{11}	k_{12}	k_{21}	k_{22}
0	0.146	0.146	-0.113	0.113	1.199	1.199	0.127	-0.127
0.5	0.549	0.008	-0.124	-0.101	1.096	1.214	0.202	-0.026
1.0	0.952	0.196	-0.043	-0.372	0.972	1.156	0.212	0.031
2	1.315	0.926	0.092	-0.526	0.847	0.994	0.122	0.035
4	1.188	1.268	0.123	-0.060	0.911	0.871	0.011	-0.066
6	1.084	1.126	0.0765	0.025	0.961	0.941	-0.0016	-0.060

Table 7. Stress intensity factors for a crack perpendicular to the external load ($c = R/2$).

$\mu_2 = 0$					
R/a	b/a	k_{11}	k_{12}	k_{21}	k_{22}
5	5.5	1.866	2.426	8.22×10^{-4}	-0.816
3	3.7	1.721	2.668	0.148	-0.791
2	2.8	1.562	2.862	0.206	-0.727
1.5	2.35	1.448	2.993	0.217	-0.645
1.0	1.9	1.309	3.139	0.205	-0.472
0.75	1.675	1.232	3.185	0.188	-0.319
0.5	1.45	1.153	3.120	0.165	-0.0976
$\mu_2 = 23\mu_1$					
5	5.5	0.654	0.508	0.050	0.025
3	3.7	0.716	0.483	0.010	0.178
2	2.8	0.781	0.469	0.002	0.156
1.5	2.35	0.827	0.463	0.008	0.139
1.0	1.9	0.881	0.460	0.026	0.113
0.75	1.675	0.911	0.464	0.040	0.094
0.5	1.45	0.940	0.481	0.056	0.062

Table 2 and Figure 3 show the results for the other symmetric crack geometry, namely, $b = 0$ and c variable. In this case it is seen that the stress intensity factor ratio k_{11} for the cleavage mode may be greater than 1 for the stiffer inclusion and may be considerably less than 1 for the hole. This and the similar results observed in some of the subsequent figures may at first appear to be somewhat paradoxical. However, if one considers the distribution of the stresses obtained from (1) in the absence of the crack which is shown in Figures 4 and 5, the explanation for these trends would be clear. The stresses given in Figures 4 and 5 are related to the input functions in (18) by

$$p_1(t) = -\sigma_{yy}^1(t,c) \ , \qquad p_2(t) = -\sigma_{xy}^1(t,c) \ . \tag{36}$$

Other results which may be of interest in the applications are given in Tables 3-7 and Figures 6-11. Since the shear components, k_{2j}, $(j=1,2)$ of the stress intensity factor are not zero, in a brittle or quasi-brittle matrix the crack propagation would not be expected to be in the plane of the crack. From the results given in this paper, it is not difficult to show that, generally for the crack tip near the interface, the crack would propagate towards the interface if $\mu_2 = 0$ or $\mu_2 < \mu_1$, and away from it if $\mu_2 > \mu_1$. A quantitative model for this phenomenon was discussed in [12].

REFERENCES:

1. Isida, M., "On the Determination of Stress Intensity Factors for Some Common Structural Problems", J. Engng. Fracture Mechanics, Vol. 2, 1970, pp. 61-79.

2. Atkinson, C., "The Interaction Between a Crack and an Inclusion", Int. J. Engng. Sci., Vol. 10, 1972, pp. 127-136.

3. Bhargava, R.D. and Bhargava, R.R., "Elastic Circular Inclusion in an Infinite Plane Containing Two Cracks", Int. J. Engng. Sci., Vol. 11, 1973, pp. 437-449.

4. Erdogan, F. and Gupta, G.D., "The Inclusion Problem with a Crack Crossing the Boundary", NASA Project Report NGR 39-007-011, March 1973.

5. Muskhelishvili, N.I., "Some Basic Problems of the Mathematical Theory of Elasticity", P. Noordhoff Ltd., Groningen, Holland, 1953.

6. Dundurs, J. and Mura, T., "Interaction Between an Edge Dislocation and a Circular Inclusion", J. Mech. Phys. Solids, Vol. 12, 1964, pp. 177-189.

7. Erdogan, F. and Gupta, G.D., "Stress Analysis of Multi-Layered Composites with a Flaw", Int. J. Solids, Structures, Vol. 7, 1971, pp. 39-61.

8, Muskhelishvili, N.I., "Singular Integral Equations", P. Noordhoff Ltd., Groningen, Holland, 1953.

9. Erdogan, F. and Gupta, G.D., "On the Numerical Solution of Singular Integral Equations", Quarterly of Applied Mathematics, Vol. 29, 1972, pp. 525-534.

10. Cook, T.S. and Erdogan, F., "Stresses in Bonded Materials with a Crack Perpendicular to the Interface", Int. J. Engng. Sci., Vol. 10, 1972, pp. 667-697.

11. Erdogan, F. and Biricikoglu, V., "Two Bonded Half Planes with a Crack Going Through the Interface", Int. J. Engng. Sci., 1973 (to appear).

12. Erdogan, F. and Sih, G.C., "On the Crack Extension in Plates Under Plane Loading and Transverse Shear", J. Basic Engng., Trans. ASME, Vol. 85, Series D, 1963, pp. 519-526.

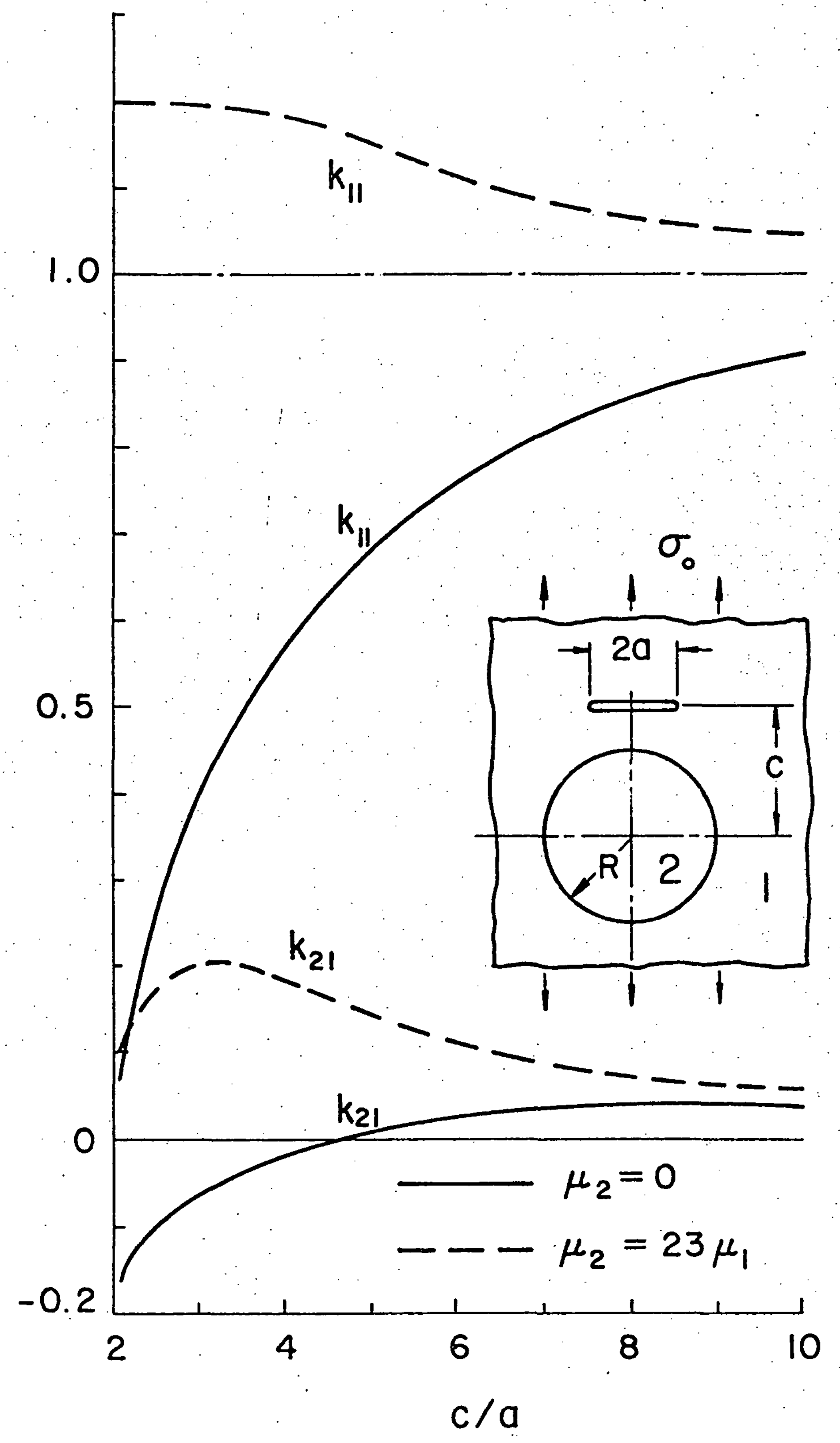

Figure 3. The stress intensity factors for a symmetrically located "tangential crack" perpendicular to the load (b = 0, R = 2a).

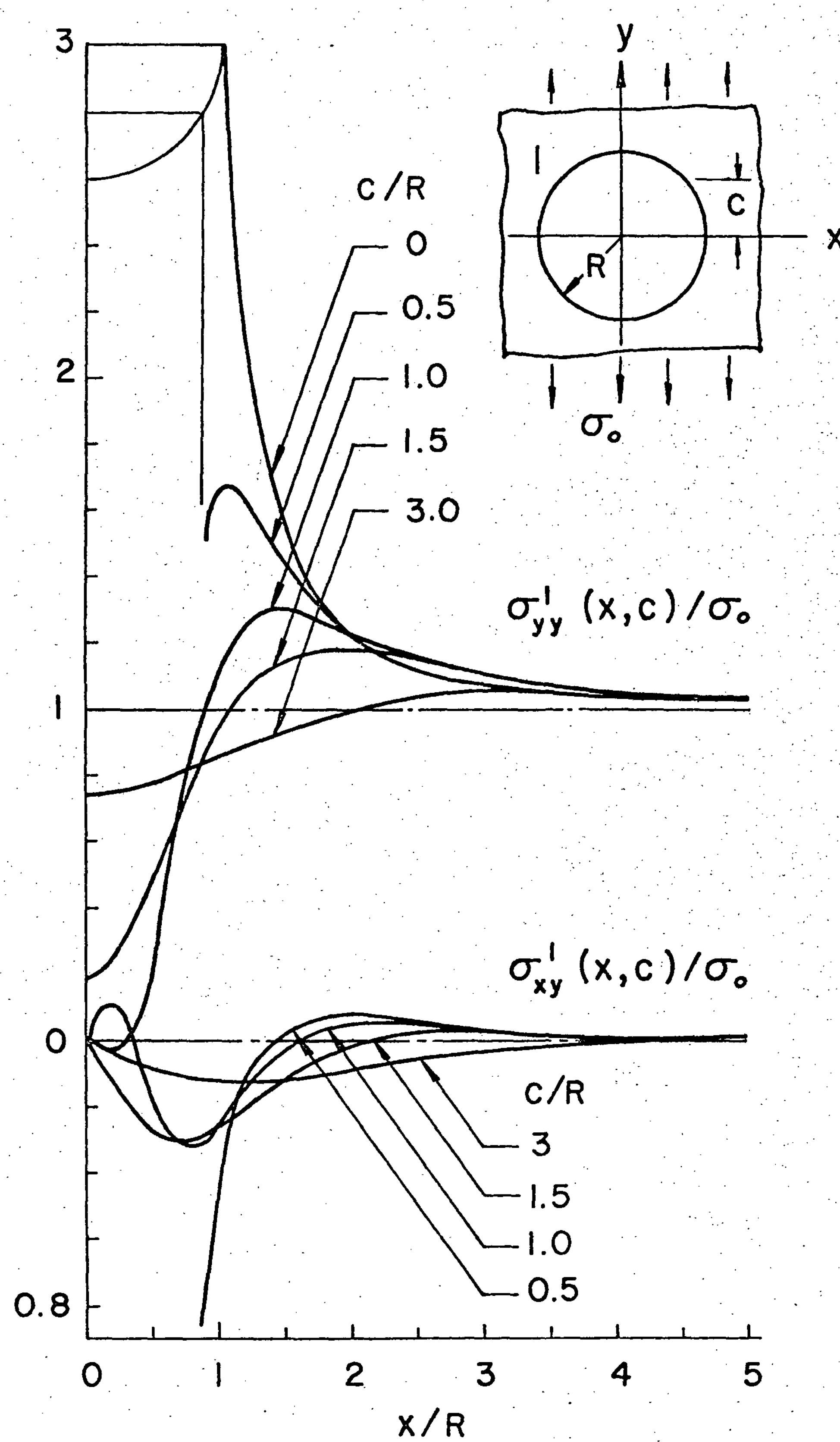

Figure 4. The stress distribution in a plate with a circular hole.

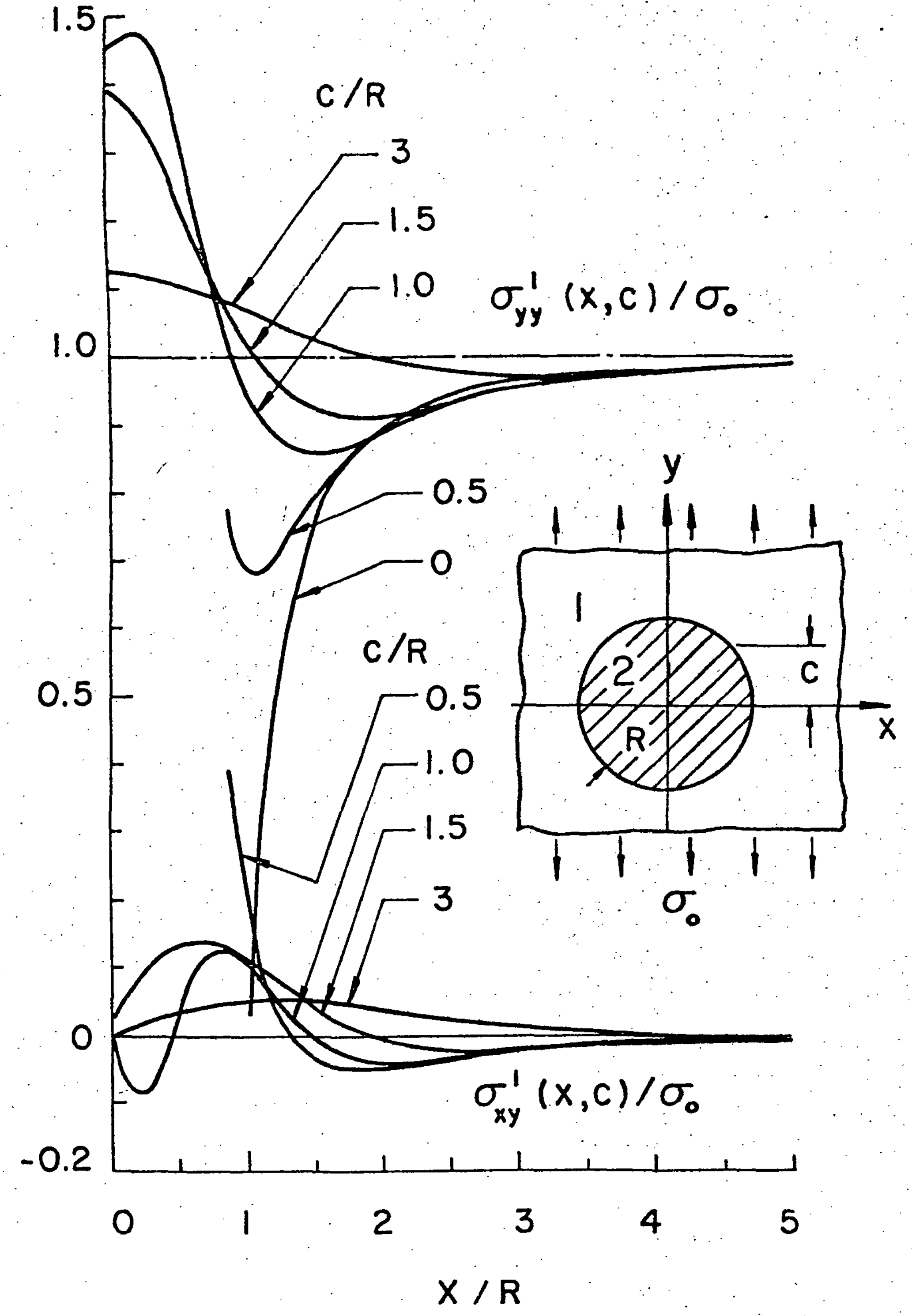

Figure 5. The stress distribution in a plate with a circular elastic inclusion ($\mu_2 = 23\mu_1$, $\kappa_1 = 1.6$, $\kappa_2 = 1.8$).

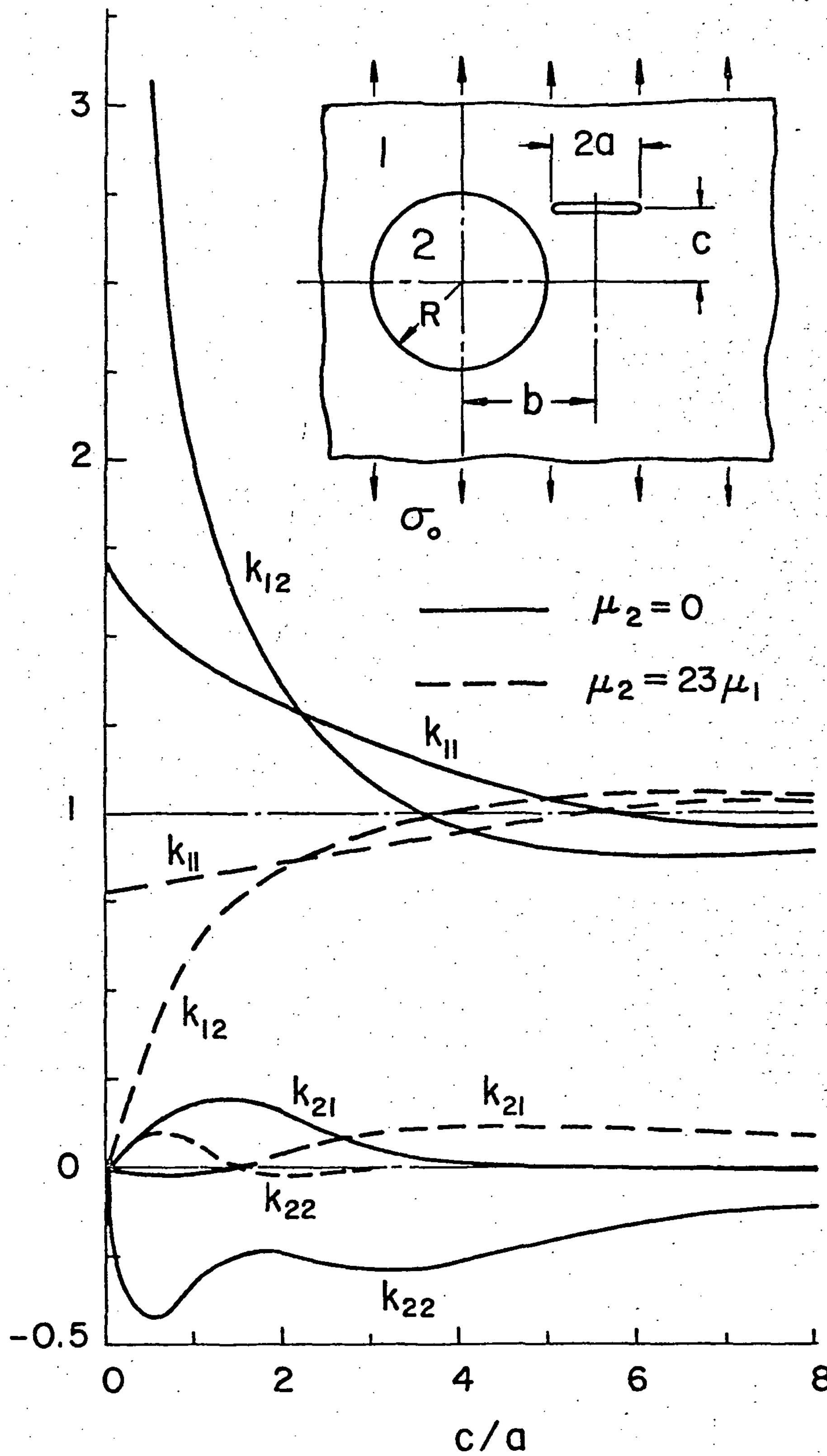

Figure 6. Stress intensity factors for a crack perpendicular to the external load (R = 2a, b = 3a).

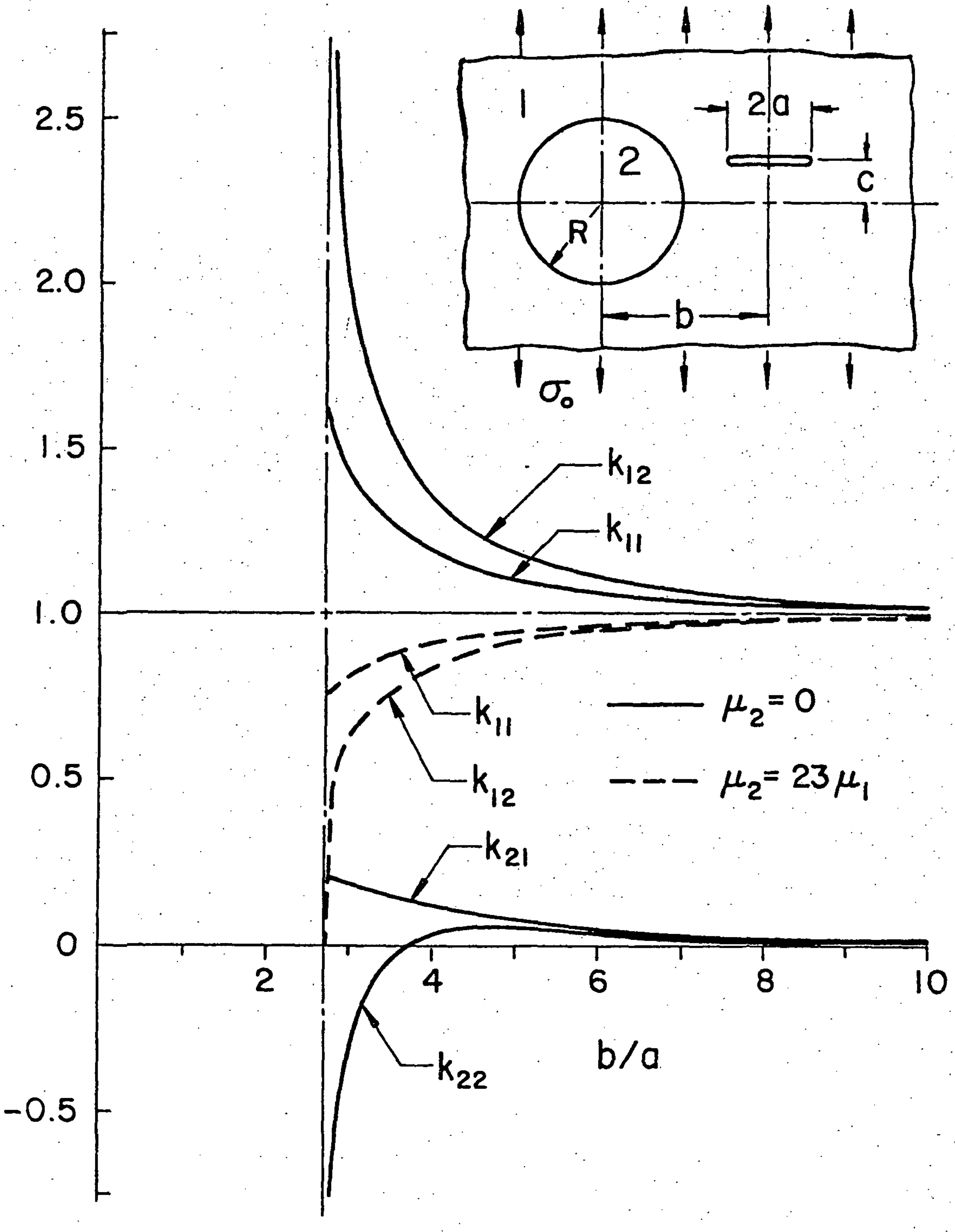

Figure 7. Stress intensity factors for a crack perpendicular to the external load (R = 2a, c = a).

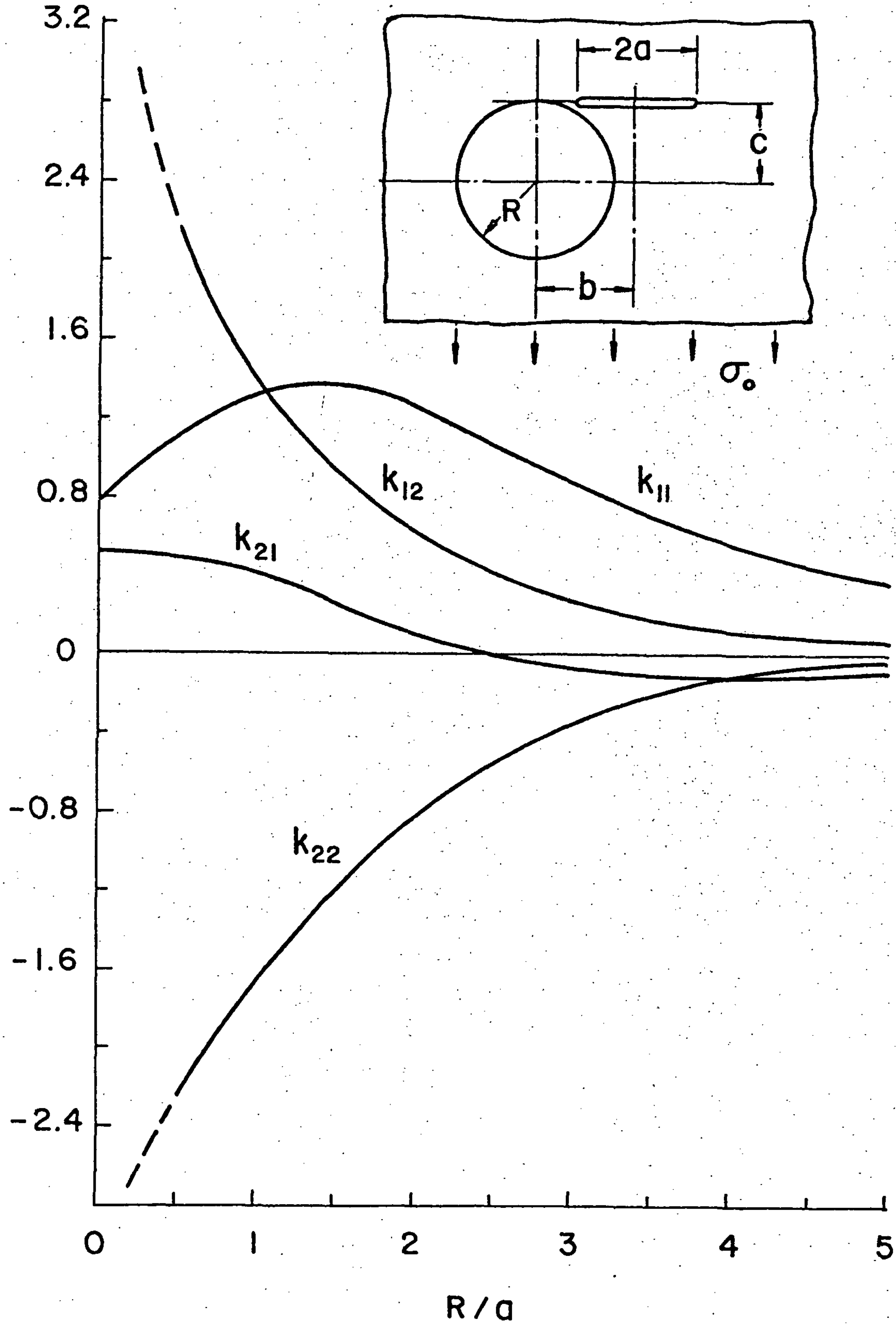

Figure 8. Stress intensity factors for a crack in the matrix containing a circular hole ($\mu_2 = 0$, $c = R$, $b - a = 0.2R$, $a =$ constant).

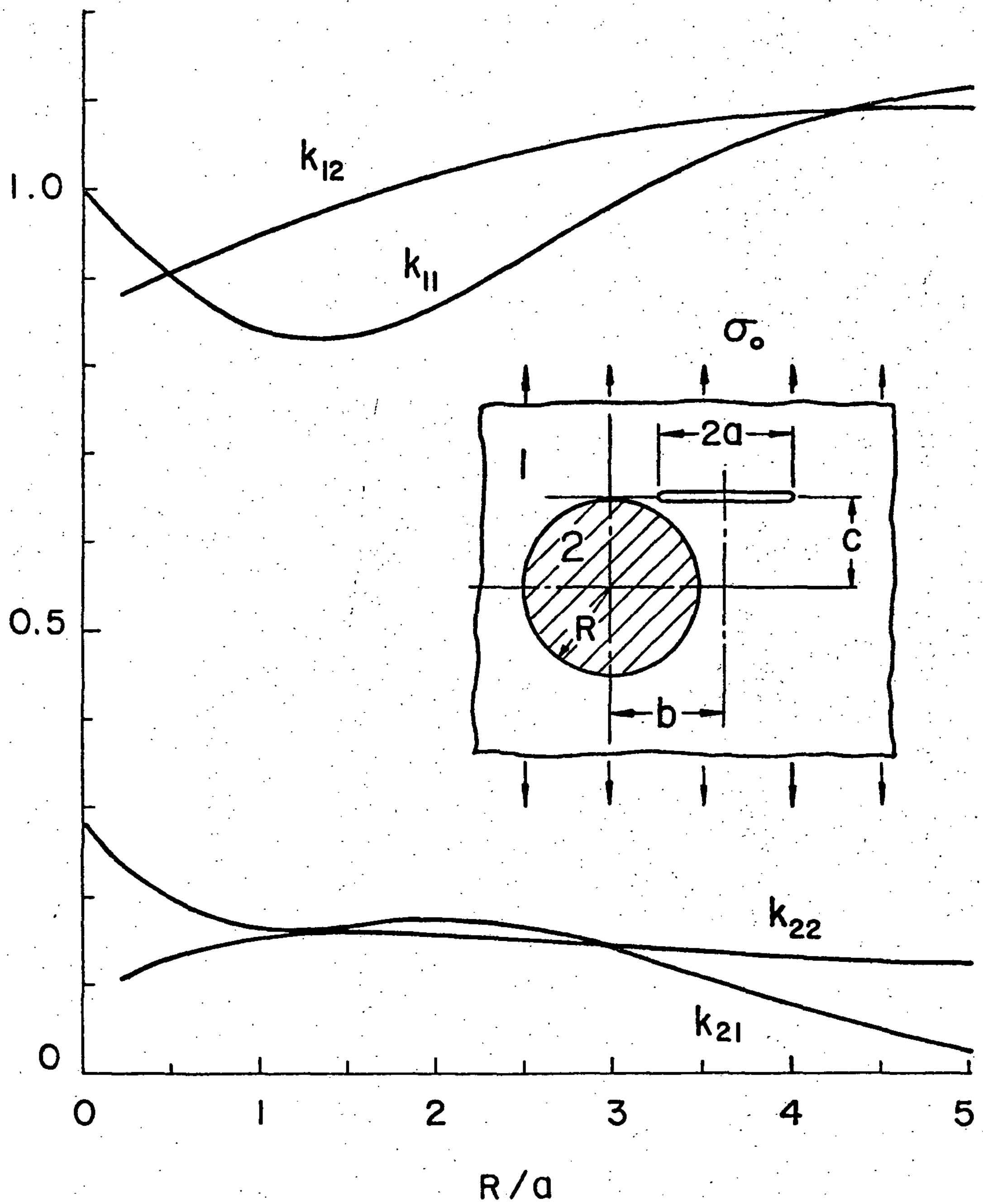

Figure 9. Stress intensity factors for a crack in the matrix containing an elastic inclusion ($\mu_2 = 23\mu_1$, $c = R$, $b-a = 0.2R$, $a = \text{constant}$).

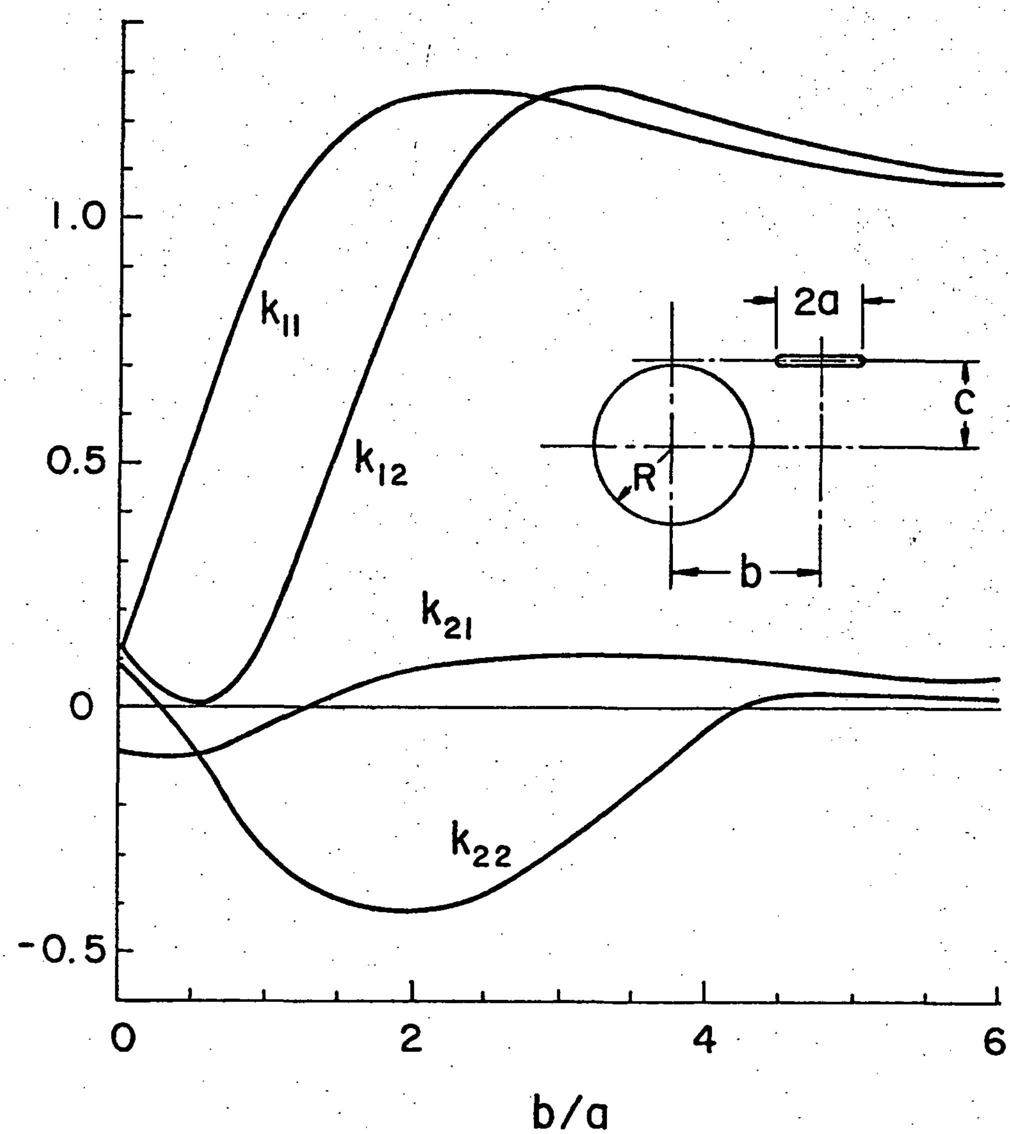

Figure 10. Stress intensity factors for a crack in the matrix containing a circular hole ($\mu_2 = 0$, $c = 2.2a$, $R = 2a$).

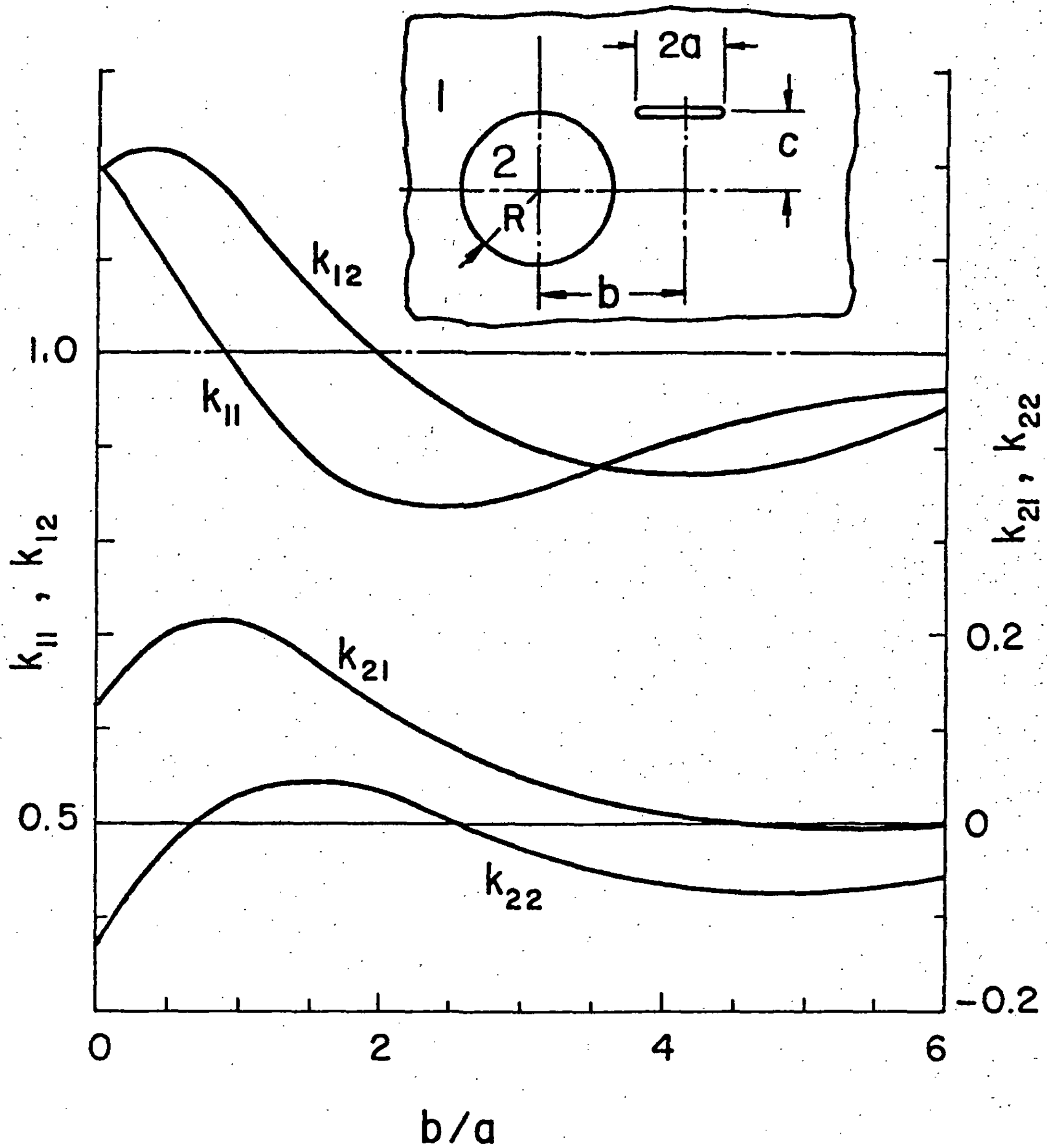

Figure 11. Stress intensity factors for a crack in the matrix containing an elastic inclusion ($\mu_2 = 23\mu_1$, $c = 2.2a$, $R = 2a$).