CONFORMAL MAPPING TECHNIQUE FOR TWO-DIMENSIONAL POROUS MEDIA AND JET IMPINGEMENT HEAT TRANSFER

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CONFORMAL MAPPING TECHNIQUE FOR TWO-DIMENSIONAL POROUS MEDIA AND JET IMPINGEMENT HEAT TRANSFER

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Abstract

Transpiration cooling and liquid metals both provide highly effective heat transfer. Using Darcy's law in porous media, and the inviscid approximation for liquid metals, the local fluid velocity in these flows equals the gradient of a potential. The energy equation and flow region are simplified when transformed into potential plane coordinates. In these coordinates the present problems are reduced to heat conduction solutions which are mapped into the physical geometry. Results are obtained for a porous region with simultaneously prescribed surface temperature and heat flux, heat transfer in a two-dimensional porous bed, and heat transfer for two liquid metal slot jets impinging on a heated plate.

NOMENCLATURE

- Characteristic dimension; half width of jet or coolant slot, \( m \)
- Specific heat of fluid at constant pressure, kcal/kg K
- Volume flow rate per unit depth, \( m^2/s \)
- Thermal conductivity of porous matrix, \( k_m \), or liquid metal \( k \), kcal/m s K
- Outward unit normal vector
- Static pressure, N/m²
- Total amount of heat transferred, kcal/s
- Surface; coolant exit from porous medium; half-spacing between jets, \( m \)
- Dimensionless temperature, \( (t - t_{\infty})/(t_{\infty} - t_0) \) for porous bed
- Temperature, K
- x and y velocity components, m/s
- Vector velocity, m/s; \( \mathbf{U} = U(F/2b) \)
- Dimensionless coordinates, \( x/b, y/b \)
- Jet stagnation point
- Rectangular coordinates, m
- Thermal diffusivity, \( \kappa / \rho C_p \), m²/s
- Permeability of porous medium, m²
- Fluid viscosity, \( \mu \), \( \mu_2 \), kg/m s
- Fluid density, kg/m³
- Dimensionless potential, \( \varphi / F; \varphi_0 = \varphi_0 / F \)
- Velocity potential, for porous medium
- Stream function, \( \psi \), \( \psi = (P_0 - P) / \mu \); \( \varphi_0 = (P_0 - P_b) / \mu_1 \)
- Flow plane coordinates as shown in Fig. 4(b).
- Dimensionless dimensionless, \( \psi = \psi / F \)
- Dimensionless coordinate, \( \beta = \beta / b \)

Subscripts

- Reference value
- On surface where coolant exits from porous medium
- Wall
- At reservoir condition
- On surface where coolant enters porous medium
- At wall 1 or 2

INTRODUCTION

In connection with advanced power producing devices, the heat transfer engineer must contend with high heat fluxes arising from increasing temperatures of working fluids to raise thermal efficiency, or from high operating temperatures required in devices such as fusion reactors. Two means of effective cooling are transpiration-cooled porous walls and liquid metal coolants. A common feature of these flows is that locally the fluid velocity is equal to the gradient of a velocity potential. This is true for Darcy flow in porous media, and for the inviscid flow approximation in liquid-metal heat transfer analyses. As a consequence the energy equation acquires a simplified form when transformed into a "natural" system of potential and streamline coordinates. For the present problems the solutions in the potential plane are reduced to heat conduction solutions which are mapped into the physical system.

The use of the potential plane was developed for two- and three-dimensional porous cooling in [1] and [2], and for liquid metal heat transfer for flow across cylinders in [3]. The technique is further developed here to obtain: (1) the shape of a two-dimensional porous region to provide a desired surface temperature for a specified surface heat flux, (2) the heat transfer for a two-dimensional porous bed with specified boundary temperatures, and (3) the temperature distribution on a uniformly heated plate cooled by impinging liquid metal jets.

ANALYSIS

Figure 1 shows a two-dimensional porous region with coolant supplied through a slot. The upper surface receives a uniform heat flux and is to be maintained at a specified temperature. The shape of the surface is to be found to meet these restrictions. The geometry is in dimensionless form in Fig. 4(a) and as will be discussed, maps into potential plane coordinates as shown in Fig. 4(b). Figure 2 shows a porous region between solid walls \( S_1 \) and \( S_2 \) with specified temperature distributions. It is desired to obtain the total heat transferred to the flow. Figure 5 shows the dimensionless form and the mapping into the potential plane. The third situation analyzed is the impingement heat transfer of the parallel liquid metal slot jets in Fig. 3. The dimensionless form and mapping into the potential plane are in Fig. 6.

Governing Equations

The solutions obtained here are for constant prop-
For heat transfer in a porous medium it is assumed that the coolant and porous matrix are in sufficiently good thermal contact that locally they are at a common temperature. With these limitations the heat transfer in the porous medium or jet is governed by the continuity and energy equations,
\[ \nabla \cdot \vec{u} = 0 \]  
\[ \nabla^2 T = \rho c_p \nabla T = 0 \]
In the porous medium the velocity is governed by Darcy's equation,
\[ \vec{u} = -\frac{k}{
abla \phi \right) \]  
while in the liquid metal flow (assumed irrotational and inviscid) the velocity can be written as the gradient of a potential
\[ \vec{u} = \nabla \phi \]

Boundary Conditions

The thermal boundary conditions are discussed in relation to each problem so only the pressure boundary conditions are given now. The static pressure drop as the fluid accelerates from the reservoir to the coolant inlet is usually small compared with the pressure drop in the porous medium. Hence, along \( S_0 \) in Figs. 1 and 2
\[ P = P_0 = \text{constant on } S_0 \]  
The coolant exit face is also at a specified constant pressure
\[ P = P_e = \text{constant on } S \]  
Similarly the inlet and outlet of the jet are at constant pressure.

Equations and Boundary Conditions in Dimensionless Form

Let \( F \) be the volume flow rate, per unit depth normal to the \( x,y \) plane, through the porous medium or in one of the jets. For the porous medium \( F \) is found in the analysis in terms of the pressure difference \( P_0 - P_e \) while for the jet it is specified as \( 2b \psi_e \) (Fig. 3). A characteristic velocity is \( F/2b \) as \( 2b \) is used as a reference length in Figs. 1 and 2. Using dimensionless variables Eqs. (1), (2), and (4) become
\[ \nabla \cdot \vec{u} = 0 \]  
\[ \nabla^2 T = \text{Pe} \nabla T = 0 \]  
\[ \vec{u} = \nabla \phi \]
From the definitions of \( \phi \) and \( \psi \), conditions (5) and (6) become
\[ \phi = 0, \psi = \text{constant on } S_0 \]  
\[ \phi = \phi_e = \text{constant on } S \]  

Transformation into Potential and Stream Function Coordinates

Conditions (10) and (11) show that the inlet and outlet faces of the porous media are at constant potential. From Eqs. (7) and (9)
\[ \nabla \cdot \vec{u} = \nabla \cdot \nabla \phi = \nabla^2 \phi = 0 \]  
so that in the flow regions the potential satisfies Laplace's equation. Orthogonal to the \( \phi \) lines...
E.S.

(a) DIMENSIONLESS COORDINATES WITH BOUNDARY CONDITIONS IN TERMS OF A POTENTIAL

\[ \phi = \frac{1}{P_0 a} \]

(b) REGION IN POTENTIAL PLANE.

Fig. 4 POROUS LEADING-EDGE REGION WITH COOLANT SUPPLIED THROUGH SLOT

(a) REGION AND POTENTIAL BOUNDARY CONDITIONS IN DIMENSIONLESS PHYSICAL PLANE.

(b) REGION IN POTENTIAL PLANE.

Fig. 5 POROUS BED WITH SPECIFIED WALL TEMPERATURE VARIATIONS

are streamlines that satisfy

\[ \nabla^2 \Psi = 0 \]  

The impervious boundaries \( S_3 \) and \( S_5 \) in Fig. 4(a), and \( S_1 \) and \( S_2 \) in Fig. 5(a) are streamlines. For the jet in Fig. 5(a) 12 and 45 are free streamlines, and the flow must be along the line of symmetry and the plate so \( \psi_2 = 0 \) is along a streamline. The dividing streamline 38 terminates at the stagnation point. These conditions cause the flow regions to map into simple rectangles and strips in potential-stream function coordinates in Figs. 4(b), 5(b), and 6(b). This geometric simplification as well as simplifications in the differential equation and boundary conditions make it convenient to

obtain solutions in the potential plane and then conformally map the results into physical coordinates.

The pressure boundary conditions are in terms of the potential in Eqs. (10) and (11). To express the energy equation in terms of potential coordinates, Eq. (9) is used to eliminate \( U \) from Eq. (8). The resulting equation has the same form as Eq. (16) of [1], and in [1] the details are given of the transformation into the potential plane. Using Eq. (26) of [1] yields the energy equation,

\[ \frac{\partial^2 
abla^2}{\partial \phi^2} - \frac{\partial^2 T}{\partial \phi^2} - \rho_0 c \frac{\partial \phi}{\partial x} = 0 \]  

This is the same equation as for convective heat transfer in a parallel plate channel with uniform flow in the \( \phi \) direction and with the channel width in the \( T \) direction.

From the definition of the stream function the difference between the upper and lower streamlines in Figs. 4(b), 5(b), and 6(b) is the volume flow rate between them. Since \( \Psi = \Psi_1 / \Psi_2 \), the range of \( \Psi \) is unity. When the geometries in Figs. 4 and 5 are mapped into the potential plane the range in \( \Phi \) is 0 to \( \Phi \). Thus the determination of \( \Phi \) from the mapping yields the volume flow rate,

\[ F = \Phi \frac{\Phi_1}{\Phi_2} (P_0 - P_a) \frac{K}{\mu} \]  

R.S.
Leading Edge Region with Unknown Surface Shape

In Fig. 1 a porous leading edge region is held by 
two insulated supports. The upper surface is sub-
jected to a uniform heat flux and it is desired to 
maintain this surface at a uniform temperature \(t_s\) 
set by design considerations. By not having any 
portion of the surface below \(t_s\), the surface is 
not locally over cooled and coolant is not wasted. 
Thus the boundary conditions along \(S\) are both 
uniform heat flux and uniform temperature, and the 
shape of \(S\) is to be determined. Along the inlet 
face \(S_0\), conservation of energy requires that any 
ergy conducted out of the porous medium be 
acquired by the incoming fluid. Summarizing these 
conditions:

\[
\begin{align*}
\frac{k_n}{t_s} - \frac{q_s}{t_s} = \text{constant} \quad & \text{on } S, \quad (16a) \\
\frac{\alpha}{t_s} = \text{constant} \quad & \text{on } S. \quad (16b)
\end{align*}
\]

In dimensionless form these become

\[
\begin{align*}
\frac{\hat{h}_n}{t_s} - \frac{\hat{q}_s}{t_s} = 1 \quad & \text{on } S, \quad (18a) \\
\frac{\hat{C}}{T_s} = \text{constant} \quad & \text{on } S, \quad (18b) \\
\frac{\hat{\alpha}}{t_s} - \text{Pe } \frac{\hat{\alpha}}{T_s} - \hat{u} \frac{\hat{X}}{Y} \quad & \text{on } S_0. \quad (19)
\end{align*}
\]

Since \(S_0\) is a line of constant potential \(\hat{\alpha} = \text{constant}\) and Eq. (15) becomes

\[
\frac{\alpha}{t_s} - \text{Pe } \frac{\alpha}{T_s} = \text{constant} \quad \text{on } S_0 \quad (20)
\]

This problem can be reduced to a heat conduction 
solution which will yield the shape of \(S\). The 
surface \(S\) has potential and temperature both uni-
form. This suggests trying a solution where \(T\) is 
only a function of \(\phi\). Then Eq. (14) becomes

\[
\left(\frac{dT}{d\phi}\right)^2 - \text{Pe } \frac{dT}{d\phi} = 0 \quad (21)
\]

Integrating gives

\[
\frac{dT}{d\phi} - \text{Pe } T = C_1 \quad (22)
\]

To obtain \(C_1\) apply the boundary condition (20) at 
\(S_0\) where \(\phi = 0\). Using \(V_T = \text{Pe } \frac{dT}{d\phi}\) reduces this to

\[
\frac{dT}{d\phi}|_{\phi=0} = \text{Pe } T|_{\phi=0} \quad (23)
\]

Applying Eq. (23) to Eq. (22) gives \(C_1 = 0\). Then 
by integrating Eq. (22) from \(0\) to \(\phi_0\)

\[
T(\phi) = T_{fe} = T_{fe} \quad (24)
\]

There still remains to satisfy the uniform heat 
flux along \(S\), condition (18a). Using \(V_T = \text{Pe } \frac{dT}{d\phi}\) and from Eq. (22) \(\frac{dT}{d\phi} = \text{Pe } T_s\), yields 
the condition for \(\phi_0\)

\[
\hat{h}_n - \hat{q}_s = 1/(\text{Pe } T_s) \quad \text{on } S. \quad (25)
\]

The boundary conditions for \(\phi\) are summarized in 
Fig. 4(a) and from Eq. (12) \(\phi\) is a solution to 
Laplace's equation. The shape of surface \(S\) has 
to be determined such that \(\phi\) will be constant and 
have a constant normal derivative along \(S\). As 
shown in [4] and [5] this type of heat conduction 
problem can be solved by conformal mapping by using 
an auxiliary potential derivative plane. The freez-
ing problem [4] depends on the same boundary value 
problem as the present case. The conformal 
transformations are given in [4] for mapping Fig. 4(b) 
into 4(a); to apply these results here requires 
only a change in notation. Results are given in 
Fig. 7.

Two-Dimensional Heat Transfer in Porous Bed

When the porous bed in Fig. 5(a) is mapped into 
Fig. 5(b) the problem becomes convective heat trans-
sfer to uniform flow in a parallel-plate channel of 
unit width. Assuming the conformal mapping is known 
between Figs. 5(a) and 5(b), the \(T(x,y)\) along \(S_1\) 
and \(S_2\) are transformed to yield the boundary 
conditions in the potential plane

\[
T_1(\phi) - \text{Pe } T_2(\phi) = 0 \quad (26a)
\]

\[
T_2(\phi) = T_2(\phi) = 0 \quad (26b)
\]

For convective heat transfer in a channel the axial 
conduction term can be neglected unless \(\text{Pe}\) is less 
than about 10 so for most situations Eq. (14) can 
be approximated as

\[
\frac{\partial^2 T}{\partial \phi^2} - \text{Pe } \frac{\partial T}{\partial \phi} = 0 \quad (27)
\]

Without axial conduction the fluid and solid at the 
entrance of the bed are at the reservoir temperature,

\[
T = 0 \quad \text{on } S_0 \quad (28)
\]

Equation (27) has the same form as the one dimen-
sional transient conduction equation. The solution 
in the potential plane is found by using results for 
a slab of constant thickness initially at zero tem-
perature and then heated in a time dependent manner 
as given by conditions (26). From Eq. (1) on 
page 103 of [6] the temperature distribution is 
given by

\[
T(\phi, \lambda) = \frac{1}{\text{Pe}} \sum_{n=1}^{\infty} \left[ \frac{\text{Pe}}{\lambda^2} \right] e^{-\frac{\text{Pe}}{\lambda^2}} \sin \pi \phi \lambda \quad (29)
\]

Since there is the same volume flow in each \(d\phi\) 
layer, the average exit temperature from the bed is

\[
T(\phi) = \int_0^1 T(\phi, \lambda) d\lambda \quad (30)
\]

The total heat transferred by the bed to the fluid is

\[
Q_{tot} = \text{Pe } \int_0^1 T(\phi, \lambda) d\lambda \quad (31)
\]

The \(Q\) is obtained from Eq. (15) by using the \(\phi_0\) 
found in the mapping.

Heat Transfer for Two Impinging Jets

As given in [7] the impinging jet in Fig. 6(a) maps 
into the potential region in Fig. 6(b); this repre-
sents a uniform flow in the \(\phi\) direction. With 
axial conduction neglected the heat transfer can be 
computed as convection to two separate flows in 
channels \(h_1\) and \(h_2\) wide. The heat losses along 
the free streamlines are neglected. The normals to 
the free streamlines are along the \(\phi\) direction so that

\[
(\partial T/\partial \phi) = 0 \quad \text{on } S_1 \quad (32)
\]
From symmetry about streamline 67
\[ (\partial / \partial y) = 0 \quad \phi, \psi \text{ on } 67 \]  
(35)
The plate 789 is heated with a uniform flux \( q_w = -k \partial T/\partial y \) or
\[ (\partial T/\partial y) = -1 \quad \chi, \gamma \text{ on } 789 \]  
(34)
This must be transformed into the potential plane.
Using
\[ \partial T/\partial y = (\partial T/\partial y)(\partial \phi/\partial y) + (\partial T/\partial \psi)(\partial \psi/\partial y) \]
and the relations along 789 that \( y = 0 \) and \( \partial / \partial y = 0 \) (since \( v = 0 \)) yield
\[ -1 = (\partial T/\partial y)|_{T=0} (\partial \phi/\partial y)|_{\psi=1} \]  
(35)
From Eq. (9) and the Cauchy-Riemann equations, \( U = \partial \phi / \partial y \). Condition (35) becomes
\[ (\partial T/\partial y) = -1/U(\psi) \quad \phi, \psi \text{ on } 789 \]  
(36)
The boundary conditions are summarized in Fig. 6.(b). The velocity \( U(\psi) \) along 789 needed in Eq. (36) is given by the conformal mapping solution for impinging jets in [7].

With these boundary conditions the channels of width \( h_1 \) and \( h_2 \) have nonuniform heat addition along 78 and 89, and the other boundaries are insulated. Since Eq. (27) has the same form as the transient conduction equation, the condition (36) can be thought of as a heat flux variation with time. In [6] the transient temperature solution is given for suddenly imposed heating at one surface of a slab with the other surface insulated. The temperature variation at the heated surface for an imposed flux of unit magnitude is \( G(\psi) \). By superposition, for a variable heat flux,
\[ T_w(\psi) = \int_0^\infty \frac{1}{U(\psi - \tau)} d\tau \]  
(37)
Differentiating the \( G(\psi) \) given in [6], page 112, the temperature distribution along the heated plate becomes,
\[ T_w(\psi) = \frac{t_w - t_m}{\omega} e^{-1/2 \int_0^\psi \frac{1}{U(\psi - \tau)} d\tau} \sum_{m=0}^\infty \left[ e^{-\text{Pe} \psi h_1^2 \tau/\tau} - e^{-\text{Pe} (m+1) \psi h_1^2 \tau/\tau} \right] \]  
(38)
where \( h_1 = h_1 \) for \( 0 \leq X < X_0 \) and \( h_1 = h_2 \) for \( X_0 \leq X \leq X_1 \), and \( h_1 \) and \( h_2 \) are in [7]. The mapping relations in [7] are used to transform \( T_w(\psi) \) to \( T_w(X) \). Typical temperature distributions are in Fig. 8.

DISCUSSION
The shapes of a porous cooled region are shown in Fig. 7(a) that simultaneously satisfy the conditions of uniform heat flux and temperature along the coolant exit face. The governing parameter involves the overall temperature and pressure differences, and the surface heat flux. If the allowable surface temperature \( t_s \) is increased, less coolant flow is required. The region thickness is increased thereby reducing the flow since the pressure difference is fixed. If the surface heat flux \( q_s \) is increased, a greater flow is required which corresponds to a thinner region. The coolant flow is given in Fig. 7(b).

REFERENCES